

Elliptic Virtual Structure Constants and Gromov–Witten Invariants for Complete Intersections in Weighted Projective Space

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Abstract. In this paper, we generalize our formalism of the elliptic virtual structure constants to hypersurfaces and complete intersections within certain weighted projective spaces possessing a single Kähler class.

Key words: mirror symmetry; elliptic Gromov–Witten invariants; weighted projective space; residue integral

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1 Introduction

In this paper, we generalize the definition of elliptic virtual structure constants for projective hypersurfaces, as presented in [7], to encompass hypersurfaces and complete intersections in specific weighted projective spaces with one Kähler class. We then compute their genus 1 (elliptic) Gromov–Witten invariants. The guideline for this generalization stems from previous works [5, 8] of our group, which detail the construction of genus 0 virtual structure constants for the K3 surface in the weighted projective space $P(1, 1, 1, 3)$. However, akin to the situation in [7], our definition lacks a rigorous geometrical construction of the expected moduli space of quasimaps from an elliptic curve to weighted projective spaces. Consequently, we only explicitly write down the integrands of the residue integrals associated with the graph types (i), (ii), (iii), and (iv) introduced in [7] (the necessary graph types remain the same as for projective hypersurfaces). A primary motivation for this generalization is to validate our conjecture by comparing the results obtained from our formalism for certain complex 3-dimensional Calabi–Yau hypersurfaces in weighted projective spaces with one Kähler class against the corresponding results derived from the original BCOV formalism [1]. Let $P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m)$ denote a complete intersection of degree k_1, \dots, k_m homogeneous polynomials within the weighted projective space $P(a_1, a_2, \dots, a_N)$ with a single Kähler class. The significant findings resulting from this generalization can be summarized in the following two points:

(i) The nontrivial part:

$$\left(-\frac{N-1}{N} \frac{1}{w^N} - \frac{N+1}{N} \frac{1}{(z_0)^N} \right),$$

which appeared in the integrand associated with the type (iii) graph (see [7]), is modified

to

$$\left(-\frac{N-m}{N} \frac{1}{w^N} - \frac{N+m}{N} \frac{1}{(z_0)^N} \right).$$

(ii) The symmetric factor:

$$R_{N,k}(d) := \left(\frac{N-1}{2} \right) \frac{1}{d} - \left(N - \frac{1}{k} \right) \frac{1}{d^2},$$

in [7] associated with the type (iv) graph is modified to

$$R(d) := \left(\prod_{i=1}^N a_i \right) \left(\left(\frac{N-m}{2} \right) \frac{1}{d} - \left(\sum_{j=1}^N \frac{1}{a_j} - \sum_{l=1}^m \frac{1}{k_l} \right) \frac{1}{d^2} \right).$$

The remaining generalizations follow straightforwardly from the results presented in [7] and our prior findings concerning the K3 surface in the weighted projective space $P(1, 1, 1, 3)$ [5, 8].

This paper unfolds as follows: First, in Section 2, we lay the groundwork by reviewing essential properties of weighted projective spaces and the complete intersections they contain, drawing upon the established literature [4, 10, 11]. Moving on to Section 3, we then elucidate the theoretical framework we employ – our formalism detailed in [5] – for the computation of Gromov–Witten invariants through the lens of virtual structure constants. Subsequently, Section 4 provides a precise definition of the elliptic virtual structure constant. This definition takes the form of a sum of residue integrals, where each integrand corresponds to one of the four graph types originally introduced in [7]. In Section 5, we put our formalism to the test by presenting a range of numerical results. We begin by examining the genus 1 Gromov–Witten invariants of Fano hypersurfaces within specific weighted projective spaces. Following this, we turn our attention to Calabi–Yau 3-folds residing in the weighted projective space previously studied in [1]. Notably, our findings regarding the count of elliptic (and rational) curves for these Calabi–Yau manifolds are consistent with those reported in [1]. We then extend our analysis to complete intersections within standard projective space and, finally, to complete intersections in various weighted projective spaces. Lastly, for the interested reader, Appendix A offers a formal proof of a proposition concerning the vanishing of residue integrals associated with type (iii) graphs, specifically in the context of Calabi–Yau manifolds.

2 Complete intersection in weighted projective space

In this section, we summarize the properties of weighted projective space and complete intersections within it. First, we introduce the weighted projective space $P(a_1, a_2, \dots, a_N)$, where a_1, a_2, \dots, a_N are positive integers.

From this point onward, we restrict a_1 to be 1 and assume that the greatest common divisor of a_2, a_3, \dots, a_N is 1. Let (x^1, x^2, \dots, x^N) be the coordinates on \mathbb{C}^N . We define an action of the multiplicative group of complex numbers, \mathbb{C}^\times , on $\mathbb{C}^N \setminus \{\mathbf{0}\}$ as follows:

$$\lambda \cdot (x^1, x^2, \dots, x^N) = (\lambda x^1, \lambda^{a_2} x^2, \dots, \lambda^{a_N} x^N), \quad \lambda \in \mathbb{C}^\times.$$

Then, $P(a_1, a_2, \dots, a_N)$ is defined as the orbit space of this \mathbb{C}^\times action,

$$P(a_1, a_2, \dots, a_N) = (\mathbb{C}^N \setminus \{\mathbf{0}\}) / \mathbb{C}^\times.$$

We denote a point in $P(a_1, a_2, \dots, a_N)$ represented by $(x_1, x_2, \dots, x_N) \in \mathbb{C}^N \setminus \{\mathbf{0}\}$ as $(x_1 : x_2 : \dots : x_N)$.

In general, $P(a_1, a_2, \dots, a_N)$ can have singularities. Information about its singular locus can be obtained from the toric construction of the weighted projective space. It is constructed as a toric variety associated with a polytope $\Delta^* \subset \mathbb{R}^{N-1}$. The polytope Δ^* is given as the convex hull of the following N points in \mathbb{R}^{N-1} :

$$v_1 = (-a_2, -a_3, \dots, -a_N), \quad v_i = e_{i-1}, \quad i = 2, \dots, N,$$

where e_{i-1} is the $(i-1)$ -th standard basis vector of \mathbb{R}^{N-1} . Let $\langle v_{i_1}, \dots, v_{i_m} \rangle_{\mathbb{R}_{\geq 0}}$, $1 \leq i_1 < i_2 < \dots < i_m \leq N$, be the m -dimensional cone in \mathbb{R}^{N-1} defined as follows:

$$\langle v_{i_1}, \dots, v_{i_m} \rangle_{\mathbb{R}_{\geq 0}} := \left\{ \sum_{j=1}^m u_j v_{i_j} \in \mathbb{R}^{N-1} \mid u_1, \dots, u_m \geq 0 \right\}.$$

The conventional rule for identifying the singular locus is as follows:

- (i) The locus $\{ (x_1 : \dots : x_N) \in P(a_1, \dots, a_N) \mid x_{i_1} = x_{i_2} = \dots = x_{i_m} = 0 \}$ is singular if and only if the cone $\langle v_{i_1}, \dots, v_{i_m} \rangle_{\mathbb{R}_{\geq 0}}$ contains integral points that cannot be expressed as an integral linear combination of $v_{i_1}, v_{i_2}, \dots, v_{i_m}$.

Let us consider $P(1, 1, 1, 2)$ as an example. The corresponding polytope Δ^* is the convex hull of the following four points in \mathbb{R}^3 :

$$v_1 = (-1, -1, -2), \quad v_2 = (1, 0, 0), \quad v_3 = (0, 1, 0), \quad v_4 = (0, 0, 1).$$

We can easily observe that only the cone $\langle v_1, v_2, v_3 \rangle_{\mathbb{R}_{\geq 0}}$ contains integral points that cannot be expressed as an integral linear combination of v_1, v_2 , and v_3 (for example, $(0, 0, -1)$). Therefore, the singular locus of $P(1, 1, 1, 2)$ is given by the point $(0 : 0 : 0 : 1) \in P(1, 1, 1, 2)$. The singular locus of the weighted projective spaces treated in this paper is always a single point.

On the other hand, we have the following exact sequence:

$$\mathbf{0} \rightarrow \mathbb{C}^{N-1} \rightarrow \bigoplus_{j=1}^N \mathbb{C} v_j \rightarrow CH^1(P(a_1, a_2, \dots, a_N)) \rightarrow \mathbf{0},$$

where $CH^1(M)$ is the degree 1 Chow ring of an algebraic variety M (if M is non-singular, it equals $H^{1,1}(M, \mathbb{C})$). Hence, $\dim_{\mathbb{C}}(CH^1(P(a_1, a_2, \dots, a_N)))$ is 1. In the cases of hypersurfaces and complete intersections in the weighted projective spaces treated in this paper, we can make them non-singular by avoiding the singular points of the weighted projective spaces through an appropriate choice of defining equations. Therefore, the resulting hypersurface or complete intersection M becomes non-singular and has a single Kähler class.

Next, we introduce the concept of a non-singular complete intersection in the weighted projective space $P(a_1, a_2, \dots, a_N)$. Let F_1, F_2, \dots, F_m be weighted homogeneous polynomials with degrees k_1, k_2, \dots, k_m , respectively. The degree k_i hypersurface in the weighted projective space is defined as the zero locus of F_i in $P(a_1, a_2, \dots, a_N)$, and we denote it by $P(a_1, a_2, \dots, a_N \mid k_i)$.

In what follows, we assume that the hypersurface is well-formed and quasi-smooth. In particular, when it is Cartier, its degree is divisible by the relevant weights. If some a_i coincides with k_1 , then for a general weighted complete intersection the corresponding factor can be eliminated without affecting the Gromov–Witten invariants.

The complete intersection $P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m)$ is defined by

$$P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m) = P(a_1, a_2, \dots, a_N \mid k_1) \cap \dots \cap P(a_1, a_2, \dots, a_N \mid k_m)$$

where k_i , $i = 1, \dots, m$, are positive integers and a_j divides k_i for all i and j .

Lastly, we introduce the Calabi–Yau threefolds treated in this paper. In Section 5, we compute the genus 1 Gromov–Witten invariants for the following three types of Calabi–Yau hypersurfaces, which were also used as examples in [10]:

$$\begin{aligned} P(1, 1, 1, 1, 2 \mid 6) &= \left\{ \sum_{i=1}^4 (x^i)^6 + 2(x^5)^3 = 0 \right\}, \\ P(1, 1, 1, 1, 4 \mid 8) &= \left\{ \sum_{i=1}^4 (x^i)^8 + 4(x^5)^2 = 0 \right\}, \\ P(1, 1, 1, 2, 5 \mid 10) &= \left\{ \sum_{i=1}^3 (x^i)^{10} + 2(x^4)^5 + 5(x^5)^2 = 0 \right\}. \end{aligned}$$

3 Setup of our computation

In this section, we outline our strategy to compute genus 1 Gromov–Witten invariants of the non-singular complete intersection $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$ in the weighted projective space $P(a_1, \dots, a_N)$. Let h be the hyperplane (Kähler) class of $P(a_1, \dots, a_N)$. We also denote the restriction of this hyperplane class to the complete intersection by the same symbol h . We then introduce the genus g Gromov–Witten invariants

$$\left\langle \prod_{a=2}^{N-m-1} (\mathcal{O}_{h^a})^{n_a} \right\rangle_{g,d}$$

of $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$, which represent the intersection number of the moduli space of holomorphic maps from genus g stable curves Σ_g with $\sum_{a=2}^{N-m-1} n_a$ marked points to $P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m)$ of degree d . We denote this moduli space by

$$\overline{M}_{g, \sum_{a=2}^{N-m-1} n_a} (P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m), d).$$

Here, \mathcal{O}_{h^a} represents the insertion of the pullback of the cohomology class h^a by the evaluation map

$$\begin{aligned} \text{ev}_i: \overline{M}_{g, \sum_{a=2}^{N-m-1} n_a} (P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m), d) \\ \rightarrow P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m), \end{aligned}$$

at the i -th marked point $z_i \in \Sigma_g$, $i = 1, 2, \dots, \sum_{a=2}^{N-m-1} n_a$. The complex dimension of the moduli space is given by

$$N - m - 1 + \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d + (3g - 3) + \sum_{a=2}^{N-m-1} n_a.$$

Therefore, $\langle \prod_{a=2}^{N-m-1} (\mathcal{O}_{h^a})^{n_a} \rangle_{g,d}$ is non-zero only if the following condition is satisfied:

$$\begin{aligned} N - m - 1 + \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d + (3g - 3) + \sum_{a=2}^{N-m-1} n_a &= \sum_{a=2}^{N-m-1} n_a a \\ \iff N - m - 1 + \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d + (3g - 3) &= \sum_{a=2}^{N-m-1} n_a (a - 1). \end{aligned} \quad (3.1)$$

Next, we introduce the genus 0 multi-point virtual structure constants

$$w \left(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \prod_{j=0}^{N-m-1} (\mathcal{O}_{h^j})^{n_j} \right)_{0,d}$$

of the same complete intersection. To simplify the notation, we define the following polynomials:

$$e_k(x, y) := \prod_{i=0}^k (ix + (k-i)y), \quad w_a(x, y) := \frac{x^a - y^a}{x - y},$$

$$q(x, y) := \prod_{i=1}^N \prod_{j=1}^{a_i-1} (jx + (a_i - j)y).$$

If $a_i = 1$ for some i , then the term $\prod_{j=1}^{a_i-1} (jx + (a_i - j)y)$ is defined to be 1. With these definitions, the multi-point virtual structure constant for $d \geq 1$ is given by

$$\begin{aligned} & w \left(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p} \right)_{0,d} \\ &= \frac{1}{(2\pi\sqrt{-1})^{d+1}} \oint_{C_{z_0}} dz_0 \oint_{C_{z_1}} dz_1 \cdots \oint_{C_{z_d}} dz_d \frac{(z_0)^a (z_d)^b}{(\prod_{i=1}^N a_i)^{d+1} \prod_{i=0}^d (z_i)^N} \\ & \times \prod_{j=1}^d \left(\frac{\prod_{p=1}^m e_{k_p}(z_{j-1}, z_j)}{(\prod_{p=1}^m k_p z_j) (2z_j - z_{j-1} - z_{j+1})} \right) \left(\prod_{p=1}^m k_p z_d \right) \frac{1}{\prod_{j=1}^d q(z_{j-1}, z_j)} \\ & \times \left(\prod_{l=0}^{N-m-1} \left(\sum_{i=1}^d w_l(z_{i-1}, z_i) \right)^{n_l} \right). \end{aligned}$$

Here, $\frac{1}{2\pi\sqrt{-1}} \oint_{C_{z_i}} dz_i$ represents taking the residue at $z_i = 0$ if $i = 0$ or $i = d$, and at $z_i = \frac{z_{i-1} + z_{i+1}}{2}$ or $z_i = 0$ if $i = 1, 2, \dots, d-1$. From this definition, we can see that it obeys the following selection rule:

$$\begin{aligned} & w \left(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \prod_{j=0}^{N-m-1} (\mathcal{O}_{h^j})^{n_j} \right)_{0,d} \neq 0 \\ & \Rightarrow \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d + N - m - 2 = a + b + \sum_{j=0}^{N-m-1} n_j (j - 1). \end{aligned}$$

Since $w_0(x, y) = 0$ and $w_1(x, y) = 1$, the following condition follows:

$$w \left(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p} \right)_{0,d} = \delta_{n_0,0} \cdot d^{n_1} w \left(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \prod_{p=2}^{N-m-1} (\mathcal{O}_{h^p})^{n_p} \right)_{0,d}. \quad (3.2)$$

If $d = 0$, the multi-point virtual constants vanish except for the following cases:

$$w(\mathcal{O}_{h^a} \mathcal{O}_{h^b} \mid \mathcal{O}_{h^c})_{0,0} = \frac{\prod_{l=1}^m k_l}{\prod_{i=1}^N a_i} \delta_{a+b+c, N-m-1}. \quad (3.3)$$

We introduce two types of perturbed two-point functions: $w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_0(x^*)$ and $\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_0(t^*)$. Here, t^* and x^* denote deformation variables $t^0, t^1, t^2, \dots, t^{N-m-1}$ and $x^0, x^1, x^2, \dots, x^{N-m-1}$,

respectively. These functions are generating functions of the multi-point virtual structure constants and the Gromov–Witten invariants, respectively. The first one, $w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^*)$, is defined as follows:

$$\begin{aligned} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^0, x^1, x^2, \dots, x^{N-m-1}) \\ = \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=0}^{\infty} w\left(\mathcal{O}_{h^a}\mathcal{O}_{h^b} \left| \prod_{q=0}^{N-m-1} (\mathcal{O}_{h^q})^{n_q} \right.\right)_{0,d} \prod_{q=0}^{N-m-1} \frac{(x^q)^{n_q}}{n_q!}. \end{aligned}$$

By using the properties (3.2) and (3.3), we can rewrite the right-hand side as follows:

$$\begin{aligned} w(\mathcal{O}_{h^a}\mathcal{O}_{h^b})_0(x^0, x^1, x^2, \dots, x^{N-m-1}) &= \frac{\prod_{l=1}^m k_l}{\prod_{i=1}^N a_i} (x^1)^{N-m-1-a-b} \\ &+ \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} e^{dx^1} w\left(\mathcal{O}_{h^a}\mathcal{O}_{h^b} \left| \prod_{q=2}^{N-m-1} (\mathcal{O}_{h^q})^{n_q} \right.\right)_{0,d} \prod_{q=2}^{N-m-1} \frac{(x^q)^{n_q}}{n_q!}. \end{aligned} \quad (3.4)$$

The second one, $\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_0(t^*)$, is defined by

$$\begin{aligned} \langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \rangle_0(t^0, t^1, t^2, \dots, t^{N-m-1}) &= \frac{\prod_{l=1}^m k_l}{\prod_{i=1}^N a_i} (t^1)^{N-m-1-a-b} \\ &+ \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} e^{dt^1} \left\langle \mathcal{O}_{h^a}\mathcal{O}_{h^b} \left| \prod_{q=2}^{N-m-1} (\mathcal{O}_{h^q})^{n_q} \right. \right\rangle_{0,d} \prod_{q=2}^{N-m-1} \frac{(t^q)^{n_q}}{n_q!}. \end{aligned}$$

At this stage, we rewrite (3.1), the topological selection rule for the Gromov–Witten invariants of $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$ for the specific $g = 0, 1$ cases:

$$\begin{aligned} \left\langle \prod_{a=0}^{N-m-1} (\mathcal{O}_{h^a})^{n_a} \right\rangle_{0,d} \neq 0 &\Rightarrow \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d + N - m - 2 = \sum_{j=0}^{N-m-1} n_j(j-1), \\ \left\langle \prod_{a=0}^{N-m-1} (\mathcal{O}_{h^a})^{n_a} \right\rangle_{1,d} \neq 0 &\Rightarrow \left(\sum_{i=1}^N a_i - \sum_{j=1}^m k_j \right) d = \sum_{j=0}^{N-m-1} n_j(j-1). \end{aligned}$$

With these setups, we define the mirror maps $t^p(x^*)$, $p = 0, 1, 2, \dots, N-m-1$, using the first type of the two-point perturbed function as follows:

$$t^p(x^0, x^1, \dots, x^{N-m-1}) := \frac{\prod_{i=1}^N a_i}{\prod_{l=1}^m k_l} w(\mathcal{O}_{h^{N-m-1-p}}\mathcal{O}_1)_0(x^*), \quad p = 0, 1, 2, \dots, N-m-1.$$

We can see from (3.4) that it has the following structure:

$$\begin{aligned} t^p(x^0, x^1, \dots, x^{N-m-1}) &= x^p + \frac{\prod_{i=1}^N a_i}{\prod_{l=1}^m k_l} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} e^{dx^1} \\ &\quad \times w\left(\mathcal{O}_{h^{N-m-1-p}}\mathcal{O}_1 \left| \prod_{q=2}^{N-m-1} (\mathcal{O}_{h^q})^{n_q} \right.\right)_{0,d} \prod_{q=2}^{N-m-1} \frac{(x^q)^{n_q}}{n_q!}, \end{aligned}$$

which enables us to invert the mirror map

$$x^p = x^p(t^0, t^1, \dots, t^{N-m-1}), \quad p = 0, 1, 2, \dots, N-m-1.$$

In this paper, we use the following conjecture [9] to compute the genus 0 Gromov–Witten invariants, which has already been proved in [6] for the case of projective hypersurfaces.

Conjecture 3.1.

$$\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_0(t^0, t^1, t^2, \dots, t^{N-m-1}) = w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_0(x^0(t^*), x^1(t^*), x^2(t^*), \dots, x^{N-m-1}(t^*)).$$

Lastly, we introduce the generating function for genus 1 Gromov–Witten invariants. In this paper, we also introduce the generating function of the elliptic virtual structure constants $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,d}$, whose general definition will be given in the next section, to compute the genus 1 Gromov–Witten invariants. In the $d = 0$ case, $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,0}$ vanishes except for the following case:

$$\begin{aligned} w(\mathcal{O}_h)_{1,0} &:= \langle \mathcal{O}_h \rangle_{1,0} = -\frac{1}{24} \int_M c_{N-m-2}(P(a_1, \dots, a_N \mid k_1, \dots, k_m))h \\ &= -\frac{\prod_{l=1}^m k_l}{24 \prod_{j=1}^N a_j} c_{N-m-2}. \end{aligned}$$

Here, h is the Kähler class of $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$. $c_{N-m-2}(P(a_1, \dots, a_N \mid k_1, \dots, k_m))$ is the second top Chern class of $P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m)$. c_{N-m-2} in the right-hand side is given by the following formula:

$$c(P(a_1, \dots, a_N \mid k_1, \dots, k_m)) = \frac{\prod_{j=1}^N (1 + ta_j h)}{\prod_{l=1}^m (1 + tk_l h)} =: \sum_{j=0}^{N-m-1} c_j h^j.$$

If $d \geq 1$, $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,d}$ obeys the following conditions:

- (i) $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,d} \neq 0 \Rightarrow (\sum_{i=1}^N a_i - \sum_{j=1}^m k_j)d = \sum_{j=0}^{N-m-1} n_j(j-1)$.
- (ii) $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,d} = \delta_{n_0,0} d^{n_1} w(\prod_{p=2}^{N-m-1} (\mathcal{O}_{h^p})^{n_p})_{1,d}$.

We then introduce the generating function of $w(\prod_{a=0}^{N-m-1} (\mathcal{O}_{h^a})^{n_a})_{1,d}$,

$$\begin{aligned} F_1^B(x^0, x^1, \dots, x^{N-m-1}) &= \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} w\left(\prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p}\right)_{1,d} \prod_{q=0}^{N-m-1} \frac{(x^q)^{n_q}}{n_q!} \\ &= -\frac{\prod_{l=1}^m k_l}{24 \prod_{i=1}^N a_i} c_{N-m-2} x^1 \\ &\quad + \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} e^{dx^1} w\left(\prod_{p=2}^{N-m-1} (\mathcal{O}_{h^p})^{n_p}\right)_{1,d} \prod_{q=2}^{N-m-1} \frac{(x^q)^{n_q}}{n_q!}, \end{aligned}$$

and the generating function of genus one Gromov–Witten invariants

$$\begin{aligned} F_1^A(t^0, t^1, \dots, t^{N-m-1}) &= \sum_{n_0=0}^{\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} \left\langle \prod_{p=0}^{N-m-1} (\mathcal{O}_{h^p})^{n_p} \right\rangle_{1,d} \prod_{q=0}^{N-m-1} \frac{(t^q)^{n_q}}{n_q!} \\ &= -\frac{\prod_{l=1}^m k_l}{24 \prod_{i=1}^N a_i} c_{N-m-2} t^1 \\ &\quad + \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \cdots \sum_{n_{N-m-1}=0}^{\infty} \sum_{d=1}^{\infty} e^{dt^1} \left\langle \prod_{p=2}^{N-m-1} (\mathcal{O}_{h^p})^{n_p} \right\rangle_{1,d} \prod_{q=2}^{N-m-1} \frac{(t^q)^{n_q}}{n_q!}. \end{aligned}$$

Our conjecture used for computing genus 1 Gromov–Witten invariants of the non-singular complete intersection $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$ is given as follows.

Conjecture 3.2.

$$F_1^A(t^0, t^1, \dots, t^{N-m-1}) = F_1^B(x^0(t^*), x^1(t^*), x^2(t^*), \dots, x^{N-m-1}(t^*)).$$

Remark 3.3. In the case when $a_1 = a_2 = \dots = a_N = 1$ and $m = 1$, these expressions reduce to the ones used in our previous paper [7].

4 Elliptic virtual structure constants**4.1 Graphs for elliptic virtual structure constants [7]**

The elliptic virtual structure constants $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{hp})^{n_p})_{1,d}$ are defined by the summation of residues of integrands associated with graphs. In this section, we briefly introduce the four types of graphs used in their definition. For more detailed explanations or figures of the graphs, we recommend that readers refer to [7].

For preparation, we introduce the partition of a positive integer d ,

$$\sigma = (d_1, d_2, \dots, d_{l(\sigma)}), \quad d_1 \leq d_2 \leq d_3 \leq \dots \leq d_{l(\sigma)}, \quad \sum_{i=1}^{l(\sigma)} d_i = d.$$

We call $l(\sigma)$ the length of the partition σ . Let P_d be the set of partitions of the positive integer d :

$$P_d := \left\{ \sigma = (d_1, d_2, \dots, d_{l(\sigma)}) \mid d_1 \leq d_2 \leq d_3 \leq \dots \leq d_{l(\sigma)}, \sum_{i=1}^{l(\sigma)} d_i = d \right\}.$$

We then define the symmetry factor associated with $\sigma \in P_d$:

$$\text{Sym}(\sigma) = \frac{(l(\sigma) - 1)!}{\prod_{i=1}^{l(\sigma)} \text{mul}(\sigma, i)!}, \quad \sigma \in P_d.$$

Here, $\text{mul}(\sigma, i)$ denotes the multiplicity of i , $1 \leq i \leq d$, in σ .

The graphs used in the definition of $w(\prod_{p=0}^{N-m-1} (\mathcal{O}_{hp})^{n_p})_{1,d}$ are constructed using edges and three types of vertices:

- (i) normal vertex,
- (ii) elliptic vertex,
- (iii) cluster vertex of degree d .

A single edge is assigned a degree of 1. A normal vertex, an elliptic vertex, and a cluster vertex of degree d are assigned degrees of 0, 0, and d , respectively.

Next, we introduce the following four types of graphs:

- (i) star graph associated with $\sigma \in P_d$,
- (ii) loop graph with d edges and d normal vertices ($d \geq 2$),
- (iii) star graph associated with $\sigma \in P_{d-f}$, $1 \leq f \leq d-1$, having a cluster vertex of degree d as its center,
- (iv) graph consisting of a single cluster vertex of degree d .

In this paper, we denote the sets of graphs of type (i) and (iii) by $\text{Graph}_d^{(i)}$ and $\text{Graph}_d^{(iii)}$, respectively. We can easily see that the type (ii) graph and the type (iv) graph of degree d are unique, and we denote these graphs by Γ_d^{loop} and Γ_d^{point} , respectively. We also denote by f_Γ the integrand associated with the graph Γ , which will be defined in the next subsection. With these setups, the elliptic virtual structure constant $w(\prod_{a=0}^{N-m-1} (\mathcal{O}_{ha})^{n_a})_{1,d}$ is defined as follows.

Definition 4.1.

$$\begin{aligned}
 w \left(\prod_{a=0}^{N-m-1} (\mathcal{O}_{h^a})^{n_a} \right)_{1,d} &:= \sum_{\Gamma \in \text{Graph}_d^{(i)}} \text{Res}(f_\Gamma) + \text{Res}(f_{\Gamma_d^{\text{loop}}}) \\
 &+ \sum_{\Gamma \in \text{Graph}_d^{(iii)}} \text{Res}(f_\Gamma) + \text{Res}(f_{\Gamma_d^{\text{point}}}).
 \end{aligned}$$

Here, $\text{Res}(f_\Gamma)$ denotes the procedure of taking the residue of f_Γ , which will also be explained in the next subsection.

4.2 Residue integrals for elliptic virtual structure constants

In this subsection, we explicitly write down the integrand f_Γ associated with the graph Γ . First, we recall the total Chern class of the complete intersection $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$, as given in [3]:

$$c(P(a_1, \dots, a_N \mid k_1, \dots, k_m)) = \frac{\prod_{j=1}^N (1 + ta_j h)}{\prod_{l=1}^m (1 + tk_l h)} =: \sum_{j=0}^{N-m-1} c_j h^j.$$

Then, we define $c_T(z)$ as follows:

$$c_T(z) := c_{N-m-1} z.$$

The integrand f_Γ associated with a graph Γ is defined as follows: For a type (i) star graph of degree d associated with a partition $\sigma = (d_1, d_2, \dots, d_l) \in P_d$, we prepare $d + 1$ variables z_0 and $z_{i,j}$ (where $1 \leq i \leq l$, $1 \leq j \leq d_j$). Here, z_0 is associated with the elliptic vertex, and $z_{i,j}$ is associated with a normal vertex. Then, f_Γ for this graph is given by

$$\begin{aligned}
 &\frac{\text{Sym}(\sigma)}{24(\prod_{i=1}^N a_i)^{d+1} (z_0)^N} \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i} \frac{1}{(z_{i,j})^N} \right) \right) \frac{c_T(z_0) \prod_{a=2}^{N-m-1} \left(\sum_{i=1}^l \sum_{j=1}^{d_i} w_a(z_{i,j-1}, z_{i,j}) \right)^{n_a}}{\prod_{l=1}^m (k_l z_0)^{l-1}} \\
 &\times \left(\prod_{i=1}^l \frac{\prod_{l=1}^m e_{k_l}(z_0, z_{i,1})}{q(z_0, z_{i,1})(z_{i,1} - z_0)} \right) \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i-1} \frac{\prod_{l=1}^m e_{k_l}(z_{i,j}, z_{i,j+1})}{q(z_{i,j}, z_{i,j+1})(2z_{i,j} - z_{i,j-1} - z_{i,j+1}) \prod_{l=1}^m k_l z_{i,j}} \right) \right).
 \end{aligned}$$

The integrand associated with the type (ii) loop graph $f_{\Gamma_d^{\text{loop}}}$ is given by

$$\begin{aligned}
 &\frac{1}{2d(\prod_{i=1}^N a_i)^d} \left(\prod_{i=1}^d \frac{\prod_{l=1}^m e_{k_l}(z_i, z_{i+1})}{q(z_i, z_{i+1})(2z_i - z_{i-1} - z_{i+1}) \prod_{l=1}^m k_l z_i} \right) \left(\prod_{i=1}^d \frac{1}{(z_i)^N} \right) \\
 &\times \prod_{a=2}^{N-m-1} \left(\sum_{i=1}^d w_a(z_i, z_{i+1}) \right)^{n_a}.
 \end{aligned}$$

The integrand associated with the type (iii) star graph of degree d having a cluster vertex of degree f ($1 \leq f \leq d - 1$) and the partition $\sigma = (d_1, d_2, \dots, d_l) \in P_{d-f}$ is given by

$$\begin{aligned}
 &\frac{\text{Sym}(\sigma)}{24(\prod_{i=1}^N a_i)^d (z_0)^{N(f-1)}} \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i} \frac{1}{(z_{i,j})^N} \right) \right) \left(-\frac{N-m}{N} \frac{1}{w^N} - \frac{N+m}{N} \frac{1}{(z_0)^N} \right) \\
 &\times \frac{1}{(w - z_0)^2 q(w, z_0) (q(z_0, z_0))^{f-1}} \left(\prod_{p=1}^m \frac{1}{(k_p z_0)^{l-1}} \frac{e_{k_p}(w, z_0) (e_{k_p}(z_0, z_0))^{f-1}}{k_p w (k_p z_0)^f} \right)
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\prod_{i=1}^l \frac{\prod_{p=1}^m e_{k_p}(z_0, z_{i,1})}{q(z_0, z_{i,1})(z_{i,1} - z_0)} \right) \\
& \times \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i-1} \frac{\prod_{p=1}^m e_{k_p}(z_{i,j}, z_{i,j+1})}{q(z_{i,j}, z_{i,j+1})(2z_{i,j} - z_{i,j-1} - z_{i,j+1})(\prod_{p=1}^m k_p z_{i,j})} \right) \right) \\
& \times \prod_{a=2}^{N-m-1} \left(w_a(w, z_0) + (f-1)w_a(z_0, z_0) + \sum_{i=1}^l \sum_{j=1}^{d_i} w_a(z_{i,j-1}, z_{i,j}) \right)^{n_a}.
\end{aligned}$$

The integrand $f_{\Gamma^{\text{point}}}$ is given by

$$\frac{R(d)}{24(\prod_{i=1}^N a_i)^{d+1}(z_0)^{Nd+1}(q(z_0, z_0))^d} \left(\prod_{l=1}^m \left(\frac{e_{k_l}(z_0, z_0)}{k_l z_0} \right)^d \right) \left(\prod_{a=2}^{N-m-1} (dw_a(z_0, z_0))^{n_a} \right),$$

where the symmetric factor $R(d)$ is defined as follows:

$$R(d) = \left(\prod_{i=1}^N a_i \right) \left(\left(\frac{N-m}{2} \right) \frac{1}{d} - \left(\sum_{j=1}^N \frac{1}{a_j} - \sum_{l=1}^m \frac{1}{k_l} \right) \frac{1}{d^2} \right).$$

Lastly, we explain how to take the residue of f_{Γ} , according to our previous work [7].

Definition 4.2. For each type of graph, the residue operation $\text{Res}: f_{\Gamma} \rightarrow \mathbb{R}$ is defined as follows:

- (i) First, we take the residue of f_{Γ} at $z_0 = 0$. Next, we take the residue of the resulting function at $z_{i,j} = 0$ and $z_{i,j} = \frac{z_{i,j+1} - z_{i,j-1}}{2}$, and sum them sequentially in ascending order of j (for $1 \leq j \leq d_i - 1$). Lastly, we take the residue of the resulting function at $z_{i,d_i} = 0$. The order among different i 's does not matter.
- (ii) We take the residue of f_{Γ} at $z_i = 0$ and $z_i = \frac{z_{i+1} - z_{i-1}}{2}$, and sum them sequentially in ascending order of i , $1 \leq i \leq d$.
- (iii) First, we take the residue of f_{Γ} at $w = z_0$. Then, the remaining process is the same as in the case of type (i).
- (iv) We take the residue of f_{Γ} at $z_0 = 0$.

5 Numerical tests for various examples

In this section, we present the numerical results for several $P(a_1, \dots, a_N \mid k_1, \dots, k_m)$'s obtained from Conjecture 3.2. Our examples consist of three types: degree k hypersurfaces $P(a_1, a_2, \dots, a_N \mid k)$, complete intersections in the projective space, and general complete intersections $P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m)$. For simplicity, let us denote e^{x^1} and e^{t^1} by q and Q , respectively.

5.1 Examples of Fano hypersurfaces

Below, we present the results for $P(1, 1, 1, 2 \mid 4)$, $P(1, 1, 1, 1, 2 \mid 2)$, and $P(1, 1, 1, 1, 2 \mid 4)$. First, we show the results for $P(1, 1, 1, 2 \mid 4)$, a degree 4 hypersurface in $P(1, 1, 1, 2)$. This is a complex two-dimensional manifold. The mirror maps are given by

$$\begin{aligned}
t^0 &= 12q + x^0 + 696q^2x^2 + 85344q^3(x^2)^2 + \dots, \\
t^1 &= x^1 + 52qx^2 + 4752q^2(x^2)^2 + \frac{2193344}{3}q^3(x^2)^3 + \dots, \\
t^2 &= x^2 + 52q(x^2)^2 + \frac{22960}{3}q^2(x^2)^3 + 1471808q^3(x^2)^4 + \dots.
\end{aligned}$$

Their inversions, $x^0(t^*)$, $x^1(t^*)$, and $x^2(t^*)$, are given as follows:

$$x^0 = t^0 - 12Q - 72Q^2t^2 - 864Q^3(t^2)^2 + \dots, \quad (5.1)$$

$$x^1 = t^1 - 52Qt^2 + 656Q^2(t^2)^2 - \frac{34720Q^3(t^2)^3}{3} + \dots, \quad (5.2)$$

$$x^2 = t^2 - 52Q(t^2)^2 + \frac{1376Q^2(t^2)^3}{3} - \frac{167648Q^3(t^2)^4}{3} + \dots. \quad (5.3)$$

The generating function F_1^B is given by

$$F_1^B = -\frac{x^1}{12} - \frac{13}{3}qx^2 - 394q^2(x^2)^2 - \frac{543920}{9}q^3(x^2)^3 + \dots.$$

Following Conjecture 3.2, we obtain the generating function F_1^A by substituting (5.1), (5.2), and (5.3) into F_1^B :

$$F_1^A = -\frac{t^1}{12} + 2Q^2(t^2)^2 + \frac{224Q^3(t^2)^3}{3} + \dots.$$

Then, we obtain the genus 1 Gromov–Witten invariants $\langle (\mathcal{O}_{h^2})^d \rangle_{1,d}$ as follows:

$$\langle \mathcal{O}_{h^2} \rangle_{1,1} = 0, \quad \langle (\mathcal{O}_{h^2})^2 \rangle_{1,1} = 2 \times 2! = 4, \quad \langle (\mathcal{O}_{h^2})^3 \rangle_{1,1} = \frac{224}{3} \times 3! = 448.$$

Next, we present the results for $P(1, 1, 1, 1, 2 \mid 2)$, the degree 2 hypersurface in $P(1, 1, 1, 1, 2)$. The mirror maps are given by

$$\begin{aligned} t^0 &= x^0 + q \left(\frac{1}{6}(x^2)^3 + x^2x^3 \right) \\ &\quad + q^2 \left(\frac{32}{315}(x^2)^7 + \frac{9}{5}(x^2)^5x^3 + \frac{23}{3}(x^2)^3(x^3)^2 + \frac{13}{2}x^2(x^3)^3 \right) + \dots, \\ t^1 &= x^1 + q \left(\frac{(x^2)^4}{6} + \frac{3}{2}(x^2)^2x^3 + (x^3)^2 \right) \\ &\quad + q^2 \left(\frac{163(x^2)^8}{1260} + \frac{121}{45}(x^2)^6x^3 + \frac{89}{6}(x^2)^4(x^3)^2 + \frac{64}{3}(x^2)^2(x^3)^3 + \frac{11(x^3)^4}{3} \right) + \dots, \\ t^2 &= x^2 + q \left(\frac{(x^2)^5}{12} + \frac{7}{6}(x^2)^3x^3 + \frac{5}{2}x^2(x^3)^2 \right) \\ &\quad + q^2 \left(\frac{191(x^2)^9}{2268} + \frac{193}{90}(x^2)^7x^3 + 16(x^2)^5(x^3)^2 + \frac{229}{6}(x^2)^3(x^3)^3 + \frac{62}{3}x^2(x^3)^4 \right) + \dots, \\ t^3 &= x^3 + q \left(\frac{(x^2)^6}{36} + \frac{7}{12}(x^2)^4x^3 + \frac{5}{2}(x^2)^2(x^3)^2 + \frac{7}{6}(x^3)^3 \right) \\ &\quad + q^2 \left(\frac{106(x^2)^{10}}{2835} + \frac{71}{60}(x^2)^8x^3 + \frac{47}{4}(x^2)^6(x^3)^2 + \frac{1511}{36}(x^2)^4(x^3)^3 + 45(x^2)^2(x^3)^4 \right. \\ &\quad \left. + \frac{19(x^3)^5}{3} \right) + \dots. \end{aligned}$$

Inverting these mirror maps, we obtain

$$\begin{aligned}
x^0 &= t^0 + Q \left(-\frac{(t^2)^3}{6} - t^2 t^3 \right) \\
&\quad + Q^2 \left(-\frac{11(t^2)^7}{2520} - \frac{2(t^2)^5 t^3}{15} - \frac{13(t^2)^3 (t^3)^2}{12} - \frac{11(t^2)(t^3)^3}{6} \right) + \dots, \\
x^1 &= t^1 + Q \left(-\frac{(t^2)^4}{6} - \frac{3(t^2)^2 t^3}{2} - (t^3)^2 \right) \\
&\quad + Q^2 \left(-\frac{11(t^2)^8}{2520} - \frac{83(t^2)^6 t^3}{360} - \frac{13(t^2)^4 (t^3)^2}{6} - \frac{49(t^2)^2 (t^3)^3}{12} - \frac{(t^3)^4}{3} \right) + \dots, \\
x^2 &= t^2 + Q \left(-\frac{(t^2)^5}{12} - \frac{7(t^2)^3 t^3}{6} - \frac{5t^2 (t^3)^2}{2} \right) \\
&\quad + Q^2 \left(-\frac{29(t^2)^9}{9072} - \frac{41(t^2)^7 t^3}{180} - \frac{31(t^2)^5 (t^3)^2}{12} - \frac{139(t^2)^3 (t^3)^3}{18} - \frac{73t^2 (t^3)^4}{12} \right) + \dots, \\
x^3 &= t^3 + Q \left(-\frac{(t^2)^6}{36} - \frac{7(t^2)^4 t^3}{12} - \frac{5(t^2)^2 (t^3)^2}{2} - \frac{7(t^3)^3}{6} \right) \\
&\quad + Q^2 \left(-\frac{121(t^2)^{10}}{45360} - \frac{127(t^2)^8 t^3}{720} - \frac{173(t^2)^6 (t^3)^2}{72} - \frac{95(t^2)^4 (t^3)^3}{9} - \frac{41(t^2)^2 (t^3)^4}{3} \right. \\
&\quad \left. - \frac{13(t^3)^5}{12} \right) + \dots.
\end{aligned}$$

The generating function F_1^B is given by

$$\begin{aligned}
F_1^B &= -\frac{x^1}{4} + q \left(-\frac{7}{144}(x^2)^4 - \frac{5}{12}(x^2)^2 x^3 - \frac{7(x^3)^2}{24} \right) + q^2 \left(-\frac{4541(x^2)^8}{120960} - \frac{1097(x^2)^6 x^3}{1440} \right. \\
&\quad \left. - \frac{33}{8}(x^2)^4 (x^3)^2 - \frac{853}{144}(x^2)^2 (x^3)^3 - \frac{19(x^3)^4}{18} \right) + \dots.
\end{aligned}$$

In the same way as the previous example, we obtain the generating function F_1^A by substituting the inversions of the mirror maps into F_1^B ,

$$\begin{aligned}
F_1^A &= -\frac{t^1}{4} + Q \left(-\frac{(t^2)^4}{144} - \frac{(t^2)^2 t^3}{24} - \frac{(t^3)^2}{24} \right) \\
&\quad + Q^2 \left(-\frac{23(t^2)^8}{40320} - \frac{(t^2)^6 t^3}{160} - \frac{(t^2)^4 (t^3)^2}{48} - \frac{(t^2)^2 (t^3)^3}{48} \right) + \dots.
\end{aligned}$$

We present numerical results of the genus 1 Gromov–Witten invariants of this example in Table 1. $N_{d,a,b}^0$, $N_{d,a,b}^1$, and $w_{a,b}$ denote $\langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{0,d}$, $\langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{1,d}$, and $w((\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b)_{1,d}$, respectively. Since $P(1, 1, 1, 1, k \mid k)$ is deformation equivalent to $P(1, 1, 1, 1) = CP^3$, these results agree with the results presented in [2].

Lastly, we consider the case of genus 1 Gromov–Witten invariants for $P(1, 1, 1, 1, 2 \mid 4)$, a degree 4 hypersurface in $P(1, 1, 1, 1, 2)$. The mirror maps are given by

$$t^0 = x^0 + 12qx^2 + q^2(848(x^2)^3 + 2160x^2x^3) + \dots,$$

Table 1. $P(1, 1, 1, 1, 2)$, $k = 2$.

d	(a, b)	$N_{d,a,b}^0$	$N_{d,a,b}^1$	$\frac{2*d-1}{12}N_{d,a,b}^0 + N_{d,a,b}^1$	$w_{a,b}$
1	(0,2)	1	$-\frac{1}{12}$	0	$-\frac{7}{12}$
1	(2,1)	1	$-\frac{1}{12}$	0	$-\frac{5}{6}$
1	(4,0)	2	$-\frac{1}{6}$	0	$-\frac{7}{6}$
2	(0,4)	0	0	0	$-\frac{76}{3}$
2	(2,3)	1	$-\frac{1}{4}$	0	$-\frac{853}{12}$
2	(4,2)	4	-1	0	-198
2	(6,1)	18	$-\frac{9}{2}$	0	$-\frac{1097}{2}$
2	(8,0)	92	-23	0	$-\frac{4541}{3}$
3	(0,6)	1	$-\frac{5}{12}$	0	$-\frac{19959}{4}$
3	(2,5)	5	$-\frac{25}{12}$	0	$-\frac{62338}{3}$
3	(4,4)	30	$-\frac{25}{2}$	0	$-\frac{516827}{6}$
3	(6,3)	190	$-\frac{469}{6}$	1	$-\frac{1068442}{3}$
3	(8,2)	1312	$-\frac{1598}{3}$	14	$-\frac{4408330}{3}$
3	(10,1)	9864	-3960	150	$-\frac{18159922}{3}$
3	(12,0)	80160	-31900	1500	$-\frac{74719852}{3}$
4	(0,8)	4	$-\frac{4}{3}$	1	$-\frac{7111330}{3}$
4	(2,7)	58	$-\frac{179}{6}$	4	$-\frac{26141813}{2}$
4	(4,6)	480	-248	32	-71830274
4	(6,5)	4000	$-\frac{6070}{3}$	310	$-\frac{1182256279}{3}$
4	(8,4)	35104	$-\frac{51772}{3}$	3220	-2159333004
4	(10,3)	327888	-156594	34674	$-\frac{35458691818}{3}$
4	(12,2)	3259680	-1515824	385656	$-\frac{193936379144}{3}$
4	(14,1)	34382544	-15620216	4436268	-353359995764
4	(16,0)	383306880	-170763640	52832040	-1930689790136
5	(0,10)	105	$-\frac{147}{4}$	42	$-\frac{8363354113}{4}$
5	(2,9)	1265	$-\frac{2379}{4}$	354	$-\frac{28682135389}{2}$
5	(4,8)	13354	$-\frac{13047}{2}$	3492	$-\frac{196198477325}{2}$
5	(6,7)	139098	$-\frac{132549}{2}$	38049	$-\frac{2010681907978}{3}$
5	(8,6)	1492616	-677808	441654	$-\frac{13724961403006}{3}$
5	(10,5)	16744080	-7179606	5378454	$-\frac{93619004917238}{3}$
5	(12,4)	197240400	-79637976	68292324	-212735629674372
5	(14,3)	2440235712	-928521900	901654884	$-\frac{4348697671027760}{3}$
5	(16,2)	31658432256	-11385660384	12358163808	-9873859605646752
5	(18,1)	429750191232	-146713008096	175599635328	$-\frac{201722432909390752}{3}$
5	(20,0)	6089786376960	-1984020394752	2583319387968	$-\frac{1373530281059327936}{3}$

$$t^1 = x^1 + q(32(x^2)^2 + 52x^3) + q^2\left(\frac{10568(x^2)^4}{3} + 16272(x^2)^2x^3 + 6416(x^3)^2\right) + \dots,$$

$$t^2 = x^2 + q(28(x^2)^3 + 116x^2x^3) + q^2\left(\frac{22936(x^2)^5}{5} + \frac{96064}{3}(x^2)^3x^3 + 33552x^2(x^3)^2\right) + \dots,$$

$$t^3 = x^3 + q(14(x^2)^4 + 116(x^2)^2x^3 + 84(x^3)^2) \\ + q^2\left(\frac{17064(x^2)^6}{5} + \frac{105904}{3}(x^2)^4x^3 + 74144(x^2)^2(x^3)^2 + 17808(x^3)^3\right) + \dots,$$

and their inversions are given as follows:

$$x^0 = t^0 - 12Qt^2 + Q^2(-128(t^2)^3 - 144t^2t^3) + \dots, \\ x^1 = t^1 + Q(-32(t^2)^2 - 52t^3) + Q^2\left(\frac{64(t^2)^4}{3} + 512(t^2)^2t^3 + 656(t^3)^2\right) + \dots, \\ x^2 = t^2 + Q(-28(t^2)^3 - 116t^2t^3) + Q^2\left(\frac{1424(t^2)^5}{5} - \frac{1216(t^2)^3t^3}{3} - 4320t^2(t^3)^2\right) + \dots, \\ x^3 = t^3 + Q(-14(t^2)^4 - 116(t^2)^2t^3 - 84(t^3)^2) \\ + Q^2\left(\frac{1136(t^2)^6}{5} - \frac{6184(t^2)^4t^3}{3} - 9280(t^2)^2(t^3)^2 + 672(t^3)^3\right) + \dots.$$

Then, we present the generating functions F_1^B and F_1^A as follows:

$$F_1^B = -\frac{x^1}{2} + q(-16(x^2)^2 - 26x^3) \\ + q^2\left(-\frac{16096}{9}(x^2)^4 - 8184(x^2)^2x^3 - 3212(x^3)^2\right) + \dots,$$

and

$$F_1^A = -\frac{t^1}{2} + Q^2\left(-\frac{244(t^2)^4}{9} - 48(t^2)^2t^3 - 4(t^3)^2\right) + \dots.$$

We present the genus 1 Gromov–Witten invariants of this example in Table 2.

5.2 Example of Calabi–Yau threefolds

In this subsection, we present the results for Calabi–Yau threefolds. When $\sum_{i=1}^N a_i = k$ (meaning the first Chern class is zero), the hypersurface $P(a_1, a_2, \dots, a_{N-1}, a_N \mid k)$ becomes a Calabi–Yau manifold. The elliptic virtual structure constants for Calabi–Yau manifolds consist of contributions from type (i), (ii), and (iv) graphs only, because the following proposition holds.

Proposition 5.1 ([7]). *When the complete intersection is a Calabi–Yau manifold, for any positive d and any $\Gamma \in \text{Graph}_d^{(\text{iii})}$, $\text{Res}(f_\Gamma)$ vanishes.*

The proof of this proposition will be given in Appendix A. In [1], the genus 1 Gromov–Witten invariants for the Calabi–Yau 3-folds $P(1, 1, 1, 1, 2 \mid 6)$, $P(1, 1, 1, 1, 4 \mid 8)$, and $P(1, 1, 1, 2, 5 \mid 10)$ were computed. In the following, we present the numerical results for these three examples obtained using our formalism. The mirror map and its inversion for the degree 6 hypersurface in $P(1, 1, 1, 1, 2)$ are given as follows:

$$t = x + 2772q + 9545850q^2 + 53054643120q^3 + 362147606012925q^4 + \dots, \\ x = t - 2772Q - 1861866Q^2 - 5621359992Q^3 - 20982861018549Q^4 + \dots.$$

Table 2. $P(1, 1, 1, 1, 2)$, $k = 4$.

d	(a, b)	$N_{d,a,b}^0$	$N_{d,a,b}^1$	$\frac{d-1}{12}N_{d,a,b}^0 + N_{d,a,b}^1$	$w_{a,b}$
1	(0,1)	24	0	0	-26
1	(2,0)	80	0	0	-32
2	(0,2)	144	-8	4	-6424
2	(2,1)	1248	-96	8	-16368
2	(4,0)	8192	$-\frac{1952}{3}$	32	$-\frac{128768}{3}$
3	(0,3)	3456	-128	448	-4177344
3	(2,2)	48384	-5888	2176	-16223616
3	(4,1)	491520	-65792	16128	$-\frac{193136768}{3}$
3	(6,0)	5242880	$-\frac{2206720}{3}$	138240	$-\frac{770396416}{3}$
4	(0,4)	165888	17664	59136	-4606798080
4	(2,3)	3207168	-279552	522240	-24060080640
4	(4,2)	44826624	-5371904	5834752	$-\frac{383428919296}{3}$
4	(6,1)	631504896	-85794816	72081408	$-\frac{2046661990400}{3}$
4	(8,0)	9330229248	-1381306368	951250944	$-\frac{10951138650112}{3}$
5	(0,5)	12441600	5320704	9467904	-7269250486272
5	(2,4)	306892800	22683648	124981248	-47699671228416
5	(4,3)	5506596864	36335616	1871867904	-317367889719296
5	(6,2)	97146372096	-2452013056	29930110976	$-\frac{6361733957066752}{3}$
5	(8,1)	1761381187584	-82586042368	504541020160	$-\frac{42609768014790656}{3}$
5	(10,0)	33262843985920	$-\frac{6573683900416}{3}$	8896386695168	$-\frac{285858107179958272}{3}$

Since the mirror maps in Calabi–Yau manifolds involve only t^1 and x^1 , we denote t^1 and x^1 by t and x , respectively. The generating functions F_1^B and F_1^A are given by

$$F_1^B = -\frac{7}{4}x - 4194q - 14373450q^2 - 80082321984q^3 - 547479376081866q^4 + \dots,$$

$$F_1^A = -\frac{7}{4}t + 657Q + \frac{1021167}{2}Q^2 + 1136816358Q^3 + \frac{18625762314603}{4}Q^4 + \dots.$$

The numerical results for the degree 8 hypersurface in $P(1, 1, 1, 1, 4)$ are given as follows:

$$t = x + 15808q + 303422880q^2 + \frac{28300071331840}{3}q^3 + 360758676442805200q^4 + \dots,$$

$$x = t - 15808Q - 53530016Q^2 - \frac{2907870121984}{3}Q^3 - 20199602025147344Q^4 + \dots,$$

$$F_1^B = -\frac{11}{6}x - \frac{79568}{3}q - \frac{1519889680}{3}q^2 - \frac{141843428301824}{9}q^3$$

$$- \frac{1807425012871005968}{3}q^4 + \dots,$$

$$F_1^A = -\frac{11}{6}t + \frac{7376}{3}Q + 10778784Q^2 + \frac{1260969572864}{9}Q^3 + 3738046766828024Q^4 + \dots.$$

Lastly, we present the numerical results for the degree 10 hypersurface in $P(1, 1, 1, 2, 5)$:

$$t = x + 179520q + 41513527200q^2 + 15647390855936000q^3$$

$$+ 7272953267875497090000q^4 + \dots,$$

$$\begin{aligned}
x &= t - 179520Q - 9286096800Q^2 - 1968068105216000Q^3 \\
&\quad - 523504041681831810000Q^4 + \dots, \\
F_1^B &= -\frac{17}{12}x - \frac{704320}{3}q - \frac{162228419200}{3}q^2 - \frac{182678675797888000}{9}q^3 \\
&\quad - \frac{28070398497758257040000}{3}q^4 + \dots, \\
F_1^A &= -\frac{17}{12}t + \frac{58640}{3}Q + \frac{3677018600}{3}Q^2 + \frac{2727182857856000}{9}Q^3 \\
&\quad + 132217958645787677500Q^4 + \dots.
\end{aligned}$$

At this stage, we briefly review how to compute the number of elliptic curves from these F_1^A . In general, the perturbed two-point function $\langle \mathcal{O}_h \mathcal{O}_h \rangle(t)$ for a Calabi–Yau threefold given as a hypersurface in weighted projective space has the structure

$$\langle \mathcal{O}_h \mathcal{O}_h \rangle(t) = \frac{kt}{\prod_{i=1}^N a_i} + \sum_{d=1}^{\infty} Q^d \langle \mathcal{O}_h \mathcal{O}_h \rangle_{0,d}.$$

Let n_d be the number of degree d rational curves in the Calabi–Yau threefold. Then, n_d is determined from the relation

$$\frac{d}{dt} \langle \mathcal{O}_h \mathcal{O}_h \rangle(t) = \frac{k}{\prod_{i=1}^N a_i} + \sum_{d=1}^{\infty} \frac{n_d d^3 Q^d}{1 - Q^d}. \quad (5.4)$$

Let m_d be the number of degree d elliptic curves in the Calabi–Yau threefold. According to [1], this m_d is determined from the following relation:

$$F_1^A = -\frac{kc_{N-3}}{24 \prod_{i=1}^N a_i} t - \sum_{d=1}^{\infty} m_d \log \left(\prod_{p=1}^{\infty} (1 - Q^{pd}) \right) - \frac{1}{12} \sum_{d=1}^{\infty} n_d \log(1 - Q^d). \quad (5.5)$$

We present the number of rational and elliptic curves for each case in Tables 3 and 4. Table 3 is based on the results presented in [10, p. 16]. The results for m_d , $d \leq 3$, agree with the results presented in [1, Table 2].

Table 3. Rational curves on Calabi–Yau weighted projective space.

d	$P(1, 1, 1, 1, 2)$, $k = 6$	$P(1, 1, 1, 1, 4)$, $k = 8$	$P(1, 1, 1, 2, 5)$, $k = 10$
1	7884	29504	231200
2	6028452	128834912	12215785600
3	11900417220	1423720546880	1700894366474400
4	34600752005688	23193056024793312	350154658851324656000
5	124595034333130080	467876474625249316800	89338191421813572850115680

5.3 Examples of complete intersections in projective space

In this subsection, we present the results for complete intersections in projective space CP^{N-1} , denoted as $P(1, 1, \dots, 1 \mid k_1, k_2, \dots, k_m)$. Since $a_1 = a_2 = \dots = a_N = 1$, we abbreviate $P(1, 1, \dots, 1 \mid k_1, k_2, \dots, k_m)$ as $(k_1, k_2, \dots, k_m)_N$. We now introduce four examples: $(2, 2)_5$, $(2, 2, 2)_6$, $(2, 2, 2)_7$, and $(2, 2, 3)_7$.

First, we present the results for $(2, 2)_5$. The mirror maps are given by

$$t^0 = x^0 + 4q + 56q^2 x^2 + 1696q^3 (x^2)^2 + \dots,$$

Table 4. Elliptic curves on Calabi–Yau weighted projective space.

d	$P(1, 1, 1, 1, 2), k = 6$	$P(1, 1, 1, 1, 4), k = 8$	$P(1, 1, 1, 2, 5), k = 10$
1	0	0	280
2	7884	41312	207680680
3	145114704	21464350592	161279120326560
4	1773044315001	1805292092664544	103038403740690105440
5	17144900584158168	101424054914016355712	59221844124053623534386928

$$\begin{aligned}
 t^1 &= x^1 + 12qx^2 + 272q^2(x^2)^2 + 10432q^3(x^2)^3 + \dots, \\
 t^2 &= x^2 + 12q(x^2)^2 + \frac{1328}{3}q^2(x^2)^3 + \frac{64064}{3}q^3(x^2)^4 + \dots.
 \end{aligned}$$

Their inversions are given as follows:

$$\begin{aligned}
 x^0 &= t^0 - 4Q - 8Q^2t^2 - 32Q^3(t^2)^2 + \dots, \\
 x^1 &= t^1 - 12Qt^2 + 16Q^2(t^2)^2 - 32Q^3(t^2)^3 + \dots, \\
 x^2 &= t^2 - 12Q(t^2)^2 - \frac{32Q^2(t^2)^3}{3} - \frac{2336Q^3(t^2)^4}{3} + \dots.
 \end{aligned}$$

The generating functions are given by

$$F_1^B = -\frac{x^1}{6} - 2qx^2 - \frac{136}{3}q^2(x^2)^2 - \frac{5216}{3}q^3(x^2)^3 + \dots, \quad F_1^A = -\frac{t^1}{6} + \dots.$$

We present the genus 1 Gromov–Witten invariants obtained from these results in Table 5, where w is the value of the corresponding elliptic virtual structure constant.

Table 5. $(2, 2)_5$.

d	N_d^0	N_d^1	w
1	16	0	-2
2	40	0	$-\frac{272}{3}$
3	256	0	-10432
4	3328	256	$-\frac{6007040}{3}$
5	69632	16384	$-\frac{1633808384}{3}$

Next, we present the results for $(2, 2, 2)_6$, which is a K3 surface. The mirror map is given by

$$\begin{aligned}
 t &= x + 24q + 564q^2 + 19904q^3 + \dots, \\
 x &= t - 24Q + 12Q^2 - 32Q^3 + \dots.
 \end{aligned}$$

All elliptic virtual structure constants up to degree 5 vanish. Therefore, the generating functions up to degree 5 are given by $F_1^B = 0$, $F_1^A = 0$. This result agrees with the discussions in [1], where the vanishing of genus 1 Gromov–Witten invariants for the K3 surface is suggested.

We then turn to the results for $(2, 2, 2)_7$, which is a Fano threefold. We present the results up to degree 3. The mirror maps are given by

$$\begin{aligned}
 t^0 &= x^0 + 8q + 368q^2x^2 + q^3(40320(x^2)^2 + 22656x^3) + \dots, \\
 t^1 &= x^1 + 32qx^2 + q^2(2560(x^2)^2 + 1936x^3) + q^3\left(\frac{1113344(x^2)^3}{3} + 561920x^2x^3\right) + \dots,
 \end{aligned}$$

$$\begin{aligned}
t^2 &= x^2 + q(40(x^2)^2 + 56x^3) + q^2 \left(\frac{16960(x^2)^3}{3} + 11936x^2x^3 \right) \\
&\quad + q^3(1073152(x^2)^4 + 3028224(x^2)^2x^3 + 714496(x^3)^2) + \dots, \\
t^3 &= x^3 + q \left(\frac{80(x^2)^3}{3} + 112x^2x^3 \right) + q^2(6400(x^2)^4 + 26560(x^2)^2x^3 + 9248(x^3)^2) \\
&\quad + q^3 \left(\frac{7816192(x^2)^5}{5} + \frac{21595136}{3}(x^2)^3x^3 + 5016576x^2(x^3)^2 \right) + \dots.
\end{aligned}$$

Their inversions are given as follows:

$$\begin{aligned}
x^0 &= t^0 - 8Q - 112Q^2t^2 + Q^3(-4096(t^2)^2 - 896t^3) + \dots, \\
x^1 &= t^1 - 32Qt^2 + Q^2(-256(t^2)^2 - 144t^3) + Q^3(-10240(t^2)^3 - 6656t^2t^3) + \dots, \\
x^2 &= t^2 + Q(-40(t^2)^2 - 56t^3) + Q^2(320(t^2)^3 + 608t^2t^3) \\
&\quad + Q^3(-512(t^2)^4 + 1664(t^2)^2t^3 + 3200(t^3)^2) + \dots, \\
x^3 &= t^3 + Q \left(-\frac{80(t^2)^3}{3} - 112t^2t^3 \right) + Q^2(640(t^2)^4 - 1472(t^2)^2t^3 - 2976(t^3)^2) \\
&\quad + Q^3 \left(\frac{145408(t^2)^5}{5} - \frac{396544(t^2)^3t^3}{3} - 251136t^2(t^3)^2 \right) + \dots.
\end{aligned}$$

The generating functions are given by

$$\begin{aligned}
F_1^B &= -x^1 - \frac{80}{3}qx^2 + q^2 \left(-2176(x^2)^2 - \frac{4912x^3}{3} \right) \\
&\quad + q^3 \left(-\frac{958720}{3}(x^2)^3 - 479488x^2x^3 \right) + \dots, \\
F_1^A &= -t^1 + \frac{16t^2Q}{3} + \left(-\frac{16384(t^2)^3}{9} - \frac{3328t^2t^3}{3} \right) Q^3 + \dots.
\end{aligned}$$

We present the results for genus 1 Gromov–Witten invariants in Table 6, where $N_{d,a,b}^0$ represents $\langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{0,d}$ and $N_{d,a,b}^1$ represents $\langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{1,d}$.

Lastly, we present the result for $(2, 2, 3)_7$, which is a Calabi–Yau threefold. The mirror map and its inversion are given by

$$\begin{aligned}
t &= x + 108q + 14202q^2 + 2974032q^3 + \dots, \\
x &= t - 108Q - 2538Q^2 - 262152Q^3 + \dots.
\end{aligned}$$

The generating functions are given as follows:

$$\begin{aligned}
F_1^B &= -\frac{5x}{2} - 210q - 27126q^2 - 5688480q^3 + \dots, \\
F_1^A &= -\frac{5t}{2} + 60Q + 1899Q^2 + 134376Q^3 + \dots.
\end{aligned}$$

From this F_1^A , we obtain the number of rational curves n_d and the number of elliptic curves m_d by using equations (5.4) and (5.5). The results are presented in Table 7.

Table 6. $(2, 2, 2)_7$.

d	(a, b)	$N_{d,a,b}^0$	$N_{d,a,b}^1$	$\frac{d-2}{24}N_{d,a,b}^0 + N_{d,a,b}^1$	$w_{a,b}$
1	(1,0)	128	$\frac{16}{3}$	0	$-\frac{80}{3}$
2	(0,1)	608	0	0	$-\frac{4912}{3}$
2	(2,0)	3200	0	0	-4352
3	(1,1)	26624	$-\frac{3328}{3}$	0	-479488
3	(3,0)	262144	$-\frac{32768}{3}$	0	-1917440
4	(0,2)	242176	$-\frac{57856}{3}$	896	$-\frac{150728192}{3}$
4	(2,1)	2914304	$-\frac{696320}{3}$	10752	$-\frac{808314880}{3}$
4	(4,0)	41943040	$-\frac{10141696}{3}$	114688	$-\frac{4322443264}{3}$
5	(1,2)	33062912	-3444736	688128	$-\frac{103377682432}{3}$
5	(3,1)	549453824	-57344000	11337728	$-\frac{693950070784}{3}$
5	(5,0)	10401873920	-1118306304	181927936	$-\frac{4649625714688}{3}$

Table 7. n_d and m_d in degrees $(2, 2, 3)_7$.

d	n_d	m_d
1	720	0
2	22428	0
3	1611504	64
4	168199200	265113
5	21676931712	198087264

5.4 Examples of complete intersections in weighted projective space

In this subsection, we present the results for complete intersections in weighted projective space. Specifically, we compute the two examples: $P(1, 1, 1, 1, 2 \mid 2, 2)$ and $P(1, 1, 1, 1, 2 \mid 2, 2)$. The results for $P(1, 1, 1, 1, 2 \mid 2, 2)$ are summarized in Table 8, where $n_d = \langle (\mathcal{O}_{h^2})^{2d-1} \rangle_{0,d}$ and $m_d = \langle (\mathcal{O}_{h^2})^{2d} \rangle_{1,d}$. Since the dimension of $P(1, 1, 1, 1, 2 \mid 2, 2)$ is two, the mirror maps are given by

$$\begin{aligned} t^0 &= x^0 + 2qx^2 + 10q^2(x^2)^3 + \frac{320}{3}q^3(x^2)^5 + \dots, \\ t^1 &= x^1 + 3q(x^2)^2 + \frac{131}{6}q^2(x^2)^4 + \frac{12329}{45}q^3(x^2)^6 + \dots, \\ t^2 &= x^2 + 2q(x^2)^3 + \frac{313}{15}q^2(x^2)^5 + \frac{10764}{35}q^3(x^2)^7 + \dots, \end{aligned}$$

and their inversions are

$$\begin{aligned} x^0 &= t^0 - 2Qt^2 - \frac{4Q^3(t^2)^5}{15} + \dots, \\ x^1 &= t^1 - 3Q(t^2)^2 - \frac{5Q^2(t^2)^4}{6} - \frac{91Q^3(t^2)^6}{9} + \dots, \\ x^2 &= t^2 - 2Q(t^2)^3 - \frac{43Q^2(t^2)^5}{15} - \frac{500Q^3(t^2)^7}{21} + \dots. \end{aligned}$$

The generating functions are given by

$$F_1^B = -\frac{x^1}{6} - \frac{1}{2}q(x^2)^2 - \frac{131}{36}q^2(x^2)^4 - \frac{12329}{270}q^3(x^2)^6 + \dots, \quad F_1^A = -\frac{t^1}{6} + \dots.$$

Since $P(1, 1, 1, 1, 2 \mid 2, 2)$ is biholomorphically equivalent to a degree 2 hypersurface in CP^3 , the result agrees with the corresponding result in [7].

Table 8. $P(1, 1, 1, 1, 2 \mid 2, 2)$.

d	n_d	m_d
1	4	0
2	8	0
3	64	0
4	1792	256
5	99328	40960

Then, we present the results for $P(1, 1, 1, 1, 1, 2 \mid 2, 2)$, which is a Fano threefold. The results for the Gromov–Witten invariants of $P(1, 1, 1, 1, 1, 2 \mid 2, 2)$ are presented in Table 9. In this table, $N_{d,a,b}^0 = \langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{0,d}$ and $m_d = \langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{1,d}$, respectively. The selection rules are given by

$$\begin{aligned} \langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{0,d} \neq 0 &\implies 3d = a + 2b, \\ \langle (\mathcal{O}_{h^2})^a (\mathcal{O}_{h^3})^b \rangle_{1,d} \neq 0 &\implies 3d = a + 2b. \end{aligned}$$

The mirror maps are given by

$$\begin{aligned} t^0 &= x^0 + q((x^2)^2 + 2x^3) + q^2 \left(\frac{58(x^2)^5}{15} + \frac{98}{3}(x^2)^3 x^3 + 42x^2(x^3)^2 \right) \\ &\quad + q^3 \left(\frac{4201(x^2)^8}{126} + \frac{7546}{15}(x^2)^6(x^3) + \frac{6110}{3}(x^2)^4(x^3)^2 \right. \\ &\quad \left. + \frac{6634}{3}(x^2)^2(x^3)^3 + 301(x^3)^4 \right) + \dots, \\ t^1 &= x^1 + q \left(\frac{4(x^2)^3}{3} + 6x^2 x^3 \right) \\ &\quad + q^2 \left(\frac{326(x^2)^6}{45} + \frac{244}{3}(x^2)^4 x^3 + 182(x^2)^2(x^3)^2 + \frac{134(x^3)^3}{3} \right) \\ &\quad + q^3 \left(\frac{13499(x^2)^9}{189} + \frac{26812}{21}(x^2)^7 x^3 + \frac{33252}{5}(x^2)^5(x^3)^2 + \frac{98872}{9}(x^2)^3(x^3)^3 \right. \\ &\quad \left. + \frac{12196}{3}x^2(x^3)^4 \right) + \dots, \\ t^2 &= x^2 + q \left(\frac{5(x^2)^4}{6} + 7(x^2)^2 x^3 + 5(x^3)^2 \right) \\ &\quad + q^2 \left(\frac{94(x^2)^7}{15} + \frac{1393}{15}(x^2)^5 x^3 + \frac{986}{3}(x^2)^3(x^3)^2 + 234x^2(x^3)^3 \right) \\ &\quad + q^3 \left(\frac{666611(x^2)^{10}}{9450} + \frac{21069}{14}(x^2)^8 x^3 + \frac{450524}{45}(x^2)^6(x^3)^2 + \frac{214577}{9}(x^2)^4(x^3)^3 \right. \\ &\quad \left. + 17085(x^2)^2(x^3)^4 + \frac{24577(x^3)^5}{15} \right) + \dots, \end{aligned}$$

$$\begin{aligned}
 t^3 &= x^3 + q \left(\frac{(x^2)^5}{3} + \frac{14}{3}(x^2)^3 x^3 + 10x^2(x^3)^2 \right) \\
 &+ q^2 \left(\frac{629(x^2)^8}{180} + \frac{1027}{15}(x^2)^6 x^3 + \frac{1088}{3}(x^2)^4 (x^3)^2 + 518(x^2)^2 (x^3)^3 + \frac{280(x^3)^4}{3} \right) \\
 &+ q^3 \left(\frac{2362258(x^2)^{11}}{51975} + \frac{41093}{35}(x^2)^9 x^3 + \frac{3145468}{315}(x^2)^7 (x^3)^2 + \frac{1495106}{45}(x^2)^5 (x^3)^3 \right. \\
 &\left. + 39738(x^2)^3 (x^3)^4 + \frac{57354}{5}x^2(x^3)^5 \right) + \dots,
 \end{aligned}$$

and their inversions are given as follows:

$$\begin{aligned}
 x^0 &= t^0 + Q \left(-(t^2)^2 - 2t^3 \right) + Q^2 \left(-\frac{(t^2)^5}{5} - \frac{2(t^2)^3 t^3}{3} \right) \\
 &+ Q^3 \left(-\frac{367(t^2)^8}{1260} - \frac{31(t^2)^6 t^3}{15} - \frac{13(t^2)^4 (t^3)^2}{3} - \frac{8(t^2)^2 (t^3)^3}{3} \right) + \dots, \\
 x^1 &= t^1 + Q \left(-\frac{4(t^2)^3}{3} - 6t^2 t^3 \right) + Q^2 \left(-\frac{2(t^2)^6}{15} - \frac{13(t^2)^4 t^3}{3} - 24(t^2)^2 (t^3)^2 - \frac{44(t^3)^3}{3} \right) \\
 &+ Q^3 \left(-\frac{989(t^2)^9}{1890} - \frac{778(t^2)^7 t^3}{35} - \frac{3338(t^2)^5 (t^3)^2}{15} - \frac{5968(t^2)^3 (t^3)^3}{9} \right. \\
 &\left. - \frac{1192t^2 (t^3)^4}{3} \right) + \dots, \\
 x^2 &= t^2 + Q \left(-\frac{5(t^2)^4}{6} - 7(t^2)^2 t^3 - 5(t^3)^2 \right) \\
 &+ Q^2 \left(-\frac{2(t^2)^7}{45} - \frac{113(t^2)^5 t^3}{15} - \frac{146(t^2)^3 (t^3)^2}{3} - 34t^2 (t^3)^3 \right) \\
 &+ Q^3 \left(-\frac{2497(t^2)^{10}}{3150} - \frac{12365(t^2)^8 t^3}{252} - 547(t^2)^6 (t^3)^2 - 1822(t^2)^4 (t^3)^3 \right. \\
 &\left. - 1432(t^2)^2 (t^3)^4 - \frac{684(t^3)^5}{5} \right) + \dots, \\
 x^3 &= t^3 + Q \left(-\frac{(t^2)^5}{3} - \frac{14(t^2)^3 t^3}{3} - 10t^2 (t^3)^2 \right) \\
 &+ Q^2 \left(-\frac{19(t^2)^8}{180} - \frac{127(t^2)^6 t^3}{15} - \frac{200(t^2)^4 (t^3)^2}{3} - 118(t^2)^2 (t^3)^3 - \frac{130(t^3)^4}{3} \right) \\
 &+ Q^3 \left(-\frac{48793(t^2)^{11}}{51975} - \frac{22417(t^2)^9 t^3}{378} - \frac{236618(t^2)^7 (t^3)^2}{315} - \frac{144296(t^2)^5 (t^3)^3}{45} \right. \\
 &\left. - \frac{41120(t^2)^3 (t^3)^4}{9} - \frac{26012t^2 (t^3)^5}{15} \right) + \dots.
 \end{aligned}$$

The generating functions are the following:

$$F_1^B = -\frac{x^1}{3} + q \left(-\frac{1}{2}(x^2)^3 - \frac{13}{6}x^2 x^3 \right)$$

$$\begin{aligned}
& + q^2 \left(-\frac{1489}{540}(x^2)^6 - \frac{1087}{36}(x^2)^4 x^3 - 66(x^2)^2 (x^3)^2 - \frac{287(x^3)^3}{18} \right) \\
& + q^3 \left(-\frac{461369(x^2)^9}{17010} - \frac{450599}{945}(x^2)^7 x^3 - \frac{109972}{45}(x^2)^5 (x^3)^2 - \frac{107299}{27}(x^2)^3 (x^3)^3 \right. \\
& \quad \left. - \frac{26125}{18}x^2(x^3)^4 \right) + \dots, \\
F_1^A = & -\frac{t^1}{3} + Q \left(-\frac{(t^2)^3}{18} - \frac{t^2 t^3}{6} \right) + Q^2 \left(-\frac{2(t^2)^6}{27} - \frac{4(t^2)^4 t^3}{9} - \frac{2(t^2)^2 (t^3)^2}{3} - \frac{2(t^3)^3}{9} \right) \\
& + Q^3 \left(-\frac{121(t^2)^9}{1215} - \frac{118(t^2)^7 t^3}{135} - \frac{112(t^2)^5 (t^3)^2}{45} - \frac{70(t^2)^3 (t^3)^3}{27} - \frac{7t^2 (t^3)^4}{9} \right) + \dots.
\end{aligned}$$

Since $P(1, 1, 1, 1, 1, 2 \mid 2, 2)$ is deformation equivalent to a degree 2 hypersurface in CP^4 , the result agrees with the corresponding result in [7].

Table 9. $P(1, 1, 1, 1, 1, 2 \mid 2, 2)$.

d	(a, b)	$N_{d,a,b}^0$	$N_{d,a,b}^1$	$\frac{3d-2}{24}N_{d,a,b}^0 + N_{d,a,b}^1$	$w_{a,b}$
1	(1,1)	4	$-\frac{1}{6}$	0	$-\frac{13}{6}$
1	(3,0)	8	$-\frac{1}{3}$	0	-3
2	(0,3)	8	$-\frac{4}{3}$	0	$-\frac{287}{3}$
2	(2,2)	16	$-\frac{8}{3}$	0	-264
2	(4,1)	64	$-\frac{32}{3}$	0	$-\frac{2174}{3}$
2	(6,0)	320	$-\frac{160}{3}$	0	$-\frac{5956}{3}$
3	(1,4)	64	$-\frac{56}{3}$	0	$-\frac{104500}{3}$
3	(3,3)	320	$-\frac{280}{3}$	0	$-\frac{429196}{3}$
3	(5,2)	2048	$-\frac{1792}{3}$	0	$-\frac{1759552}{3}$
3	(7,1)	15104	$-\frac{13216}{3}$	0	$-\frac{7209584}{3}$
3	(9,0)	123904	$-\frac{108416}{3}$	0	$-\frac{29527616}{3}$
4	(0,6)	384	-160	0	$-\frac{18667312}{3}$
4	(2,5)	2560	$-\frac{3200}{3}$	0	$-\frac{101879272}{3}$
4	(4,4)	18944	$-\frac{22912}{3}$	256	$-\frac{555449168}{3}$
4	(6,3)	163840	$-\frac{194048}{3}$	3584	$-\frac{3026251616}{3}$
4	(8,2)	1583104	$-\frac{1849856}{3}$	43008	$-\frac{16485590720}{3}$
4	(10,1)	16687104	-6440960	512000	$-\frac{89806527616}{3}$
4	(12,0)	189358080	-72652800	6246400	-163085218816
5	(1,7)	27136	$-\frac{41792}{3}$	768	$-\frac{28726121392}{3}$
5	(3,6)	229376	$-\frac{331264}{3}$	13824	$-\frac{195282001984}{3}$
5	(5,5)	2232320	$-\frac{3049984}{3}$	192512	$-\frac{1326874482304}{3}$
5	(7,4)	24391680	-10660352	2551808	$-\frac{9013280450048}{3}$
5	(9,3)	291545088	-123583488	34336768	$-\frac{61226330115584}{3}$
5	(11,2)	3750199296	-1553444864	477913088	-138652119786496
5	(13,1)	51384877056	-20917362688	6916112384	$-\frac{2826429058966016}{3}$
5	(15,0)	744875950080	-299359264768	104115208192	$-\frac{19209989184830464}{3}$

A Appendix: Proof of Proposition 5.1

In this appendix, we prove Proposition 5.1. The proof fundamentally follows the same approach as the corresponding proposition in [7]; specifically, we only need to show that the residue of f_Γ for type (iii) graphs at $w = z_0$ vanishes. If $a_i = 1$, then $q(x, y) = \prod_{j=1}^{a_i-1} (jx + (a_i - j)y) := 1$.

We assume

$$P(a_1, a_2, \dots, a_N \mid k_1, k_2, \dots, k_m) = P(1, 1, \dots, 1, a_{r+1}, a_{r+2}, \dots, a_N \mid k_1, k_2, \dots, k_m),$$

which implies $a_1 = \dots = a_r = 1$. We also assume that $a_i > 1$ for $i \in \{r+1, r+2, \dots, N\}$. From the Calabi–Yau condition, we have $r + \sum_{i=r+1}^N a_i = \sum_{j=1}^m k_m$. In the Calabi–Yau case, the condition $n_a = 0$, $a = 2, 3, \dots, N - m - 1$, holds. Hence, the corresponding integrand is given by

$$\begin{aligned} f_\Gamma &= \frac{\text{Sym}(\sigma)}{24(\prod_{i=1}^N a_i)^d (z_0)^{N(f-1)}} \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i} \frac{1}{(z_{i,j})^N} \right) \right) \left(-\frac{N-m}{N} \frac{1}{w^N} - \frac{N+m}{N} \frac{1}{(z_0)^N} \right) \\ &\times \frac{1}{(w-z_0)^2 q(w, z_0) (q(z_0, z_0))^{f-1}} \left(\prod_{p=1}^m \frac{1}{(k_p z_0)^{l-1}} \frac{e_{k_p}(w, z_0) (e_{k_p}(z_0, z_0))^{f-1}}{k_p w (k_p z_0)^f} \right) \\ &\times \left(\prod_{i=1}^l \frac{\prod_{p=1}^m e_{k_p}(z_0, z_{i,1})}{q(z_0, z_{i,1}) (z_{i,1} - z_0)} \right) \\ &\times \left(\prod_{i=1}^l \left(\prod_{j=1}^{d_i-1} \frac{\prod_{p=1}^m e_{k_p}(z_{i,j}, z_{i,j+1})}{q(z_{i,j}, z_{i,j+1}) (2z_{i,j} - z_{i,j-1} - z_{i,j+1}) (\prod_{p=1}^m k_p z_{i,j})} \right) \right) \\ &= \left(-\frac{N-m}{N} \frac{1}{w^N} - \frac{N+m}{N} \frac{1}{(z_0)^N} \right) \frac{1}{(w-z_0)^2 q(w, z_0)} \left(\prod_{l=1}^m \frac{e_{k_l}(w, z_0)}{k_l w} \right) g(z). \end{aligned}$$

Here, $g(z)$ in the last line represents the factor that does not contain w .

We note the following equalities:

$$\begin{aligned} e_k(z_0, z_0) &= (kz_0)^{k+1}, \quad \left. \frac{d}{dw} e_k(w, z_0) \right|_{w=z_0} = \frac{k(k+1)}{2} (kz_0)^k, \\ q(z_0, z_0) &= \prod_{i=r+1}^N (a_i z_0)^{a_i-1}, \\ \left. \frac{d}{dw} q(w, z_0) \right|_{w=z_0} &= \sum_{i=r+1}^N \left(\prod_{\substack{l=r+1 \\ l \neq i}}^N (a_l z_0)^{a_l-1} \right) (a_i z_0)^{a_i-2} \frac{a_i(a_i-1)}{2}. \end{aligned}$$

Using these equalities, the residue of f_Γ at $w = z_0$ is computed as follows:

$$\begin{aligned} \lim_{w \rightarrow z_0} \frac{d}{dw} (f_\Gamma) &= \frac{d}{dw} \left[\left(-\frac{N-m}{N} \frac{1}{w^N} - \frac{N+m}{N} \frac{1}{(z_0)^N} \right) \frac{1}{q(w, z_0)} \left(\prod_{l=1}^m \frac{e_{k_l}(w, z_0)}{k_l w} \right) \right]_{w=z_0} g(z) \\ &= g(z) \left[\left(\frac{N-m}{(z_0)^{N+1}} \right) \frac{1}{q(z_0, z_0)} \left(\prod_{l=1}^m \frac{e_{k_l}(z_0, z_0)}{k_l z_0} \right) \right. \\ &\quad + \frac{2}{(z_0)^N} \left[\frac{d}{dw} q(w, z_0) \right]_{w=z_0} \left(\prod_{l=1}^m \frac{e_{k_l}(z_0, z_0)}{k_l z_0} \right) \\ &\quad \left. - \frac{2}{(z_0)^N q(z_0, z_0)} \sum_{i=1}^m \left(\prod_{\substack{l=1 \\ l \neq i}}^m \frac{e_{k_l}(z_0, z_0)}{k_l z_0} \right) \left(\frac{k_i(k_i+1)}{2k_i z_0} (k_i z_0)^{k_i} - \frac{(k_i z_0)^{k_i+1}}{k_i (z_0)^2} \right) \right] \end{aligned}$$

$$\begin{aligned}
&= g(z) \left[\frac{N-m}{(z_0)^{N+1}} \frac{1}{\prod_{i=r+1}^N (a_i z_0)^{a_i-1}} \left(\prod_{l=1}^m (k_l z_0)^{k_l} \right) \right. \\
&\quad + \frac{2}{(z_0)^N \left(\prod_{i=r+1}^N (a_i z_0)^{a_i-1} \right)^2} \left(\prod_{l=1}^m (k_l z_0)^{k_l} \right) \\
&\quad \times \sum_{i=r+1}^N \left(\prod_{\substack{l=r+1 \\ l \neq i}}^N (a_l z_0)^{a_l-1} \right) (a_i z_0)^{a_i-2} \frac{a_i(a_i-1)}{2} \\
&\quad \left. - \frac{2 \prod_{l=1}^m (k_l z_0)^{k_l}}{(z_0)^N \prod_{i=r+1}^N (a_i z_0)^{a_i-1}} \sum_{i=1}^m \frac{k_i-1}{2z_0} \right] \\
&= g(z) \frac{\prod_{l=1}^m (k_l z_0)^{k_l}}{(z_0)^N \prod_{i=r+1}^N (a_i z_0)^{a_i-1}} \left[\frac{N-m}{z_0} + \sum_{i=r+1}^N \frac{a_i-1}{z_0} - \sum_{i=1}^m \frac{k_i-1}{z_0} \right] \\
&= g(z) \frac{\prod_{l=1}^m (k_l z_0)^{k_l}}{(z_0)^{N+1} \prod_{i=r+1}^N (a_i z_0)^{a_i-1}} \left[N-m + \sum_{i=r+1}^N a_i - (N-r) - \sum_{i=1}^m k_i + m \right] = 0.
\end{aligned}$$

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