On the Asymptotics of Orthogonal Polynomials on Multiple Intervals with Non-Analytic Weights

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Abstract. We consider the asymptotics of orthogonal polynomials for measures that are differentiable, but not necessarily analytic, multiplicative perturbations of Jacobi-like measures supported on disjoint intervals. We analyze the Fokas–Its–Kitaev Riemann–Hilbert problem using the Deift–Zhou method of nonlinear steepest descent and its $\overline{\partial}$ extension due to Miller and McLaughlin. Our results extend that of Yattselev in the case of Chebyshev-like measures with error bounds that give similar rates while allowing less regular perturbations. For the general Jacobi-like case, we present, what appears to be the first result for asymptotics when the perturbation of the measure is only assumed to be differentiable with bounded second derivative.

Key words: orthogonal polynomials; Riemann–Hilbert problems; steepest descent; dbar problems

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1 Introduction

Consider a Borel measure μ on $\mathbb{R}$ of the form

$$\mu(\mathrm{d}x) = \sum_{j=1}^{g+1} h_j(x)\mathbb{1}_{[a_j,b_j]}(x)(b_j-x)^{\alpha_j}(x-a_j)^{\beta_j}\,\mathrm{d}x + \sum_{j=1}^{P} r_j\delta_{c_j} = \rho(x)\mathrm{d}x + \sum_{j=1}^{P} r_j\delta_{c_j}, \quad (1.1)$$

where $a_j < b_j < a_{j+1}$ and $\{c_1,\ldots,c_P\}$ is a subset of the complement of $\bigcup_j[a_j,b_j]$. We assume that $\alpha_j > -1$, $\beta_j > -1$, $r_j > 0$ for each j and $h_j(x) > 0$ on $[a_j,b_j]$. We consider the problem of determining the strong asymptotics of the (monic) orthogonal polynomials, denoted by $\pi_n(;\mu)$, with respect to μ as $n \to \infty$ when h_j is not assumed to be analytic. Polynomials orthogonal over disjoint intervals were originally considered by Akhiezer [1] and Akhiezer and Tomchuck [2] for a special class of weight functions. These authors described the orthogonal polynomials and their associated weighted Cauchy integrals, two of the main objects of study here. See [24] for a method to study a different special class using polynomial mappings.

We are interested in obtaining asymptotic formulae for many reasons. One is in the use of the asymptotics in approximation theory. Specifically, the approximation of functions across multiple disjoint intervals has been found to be fruitful in computing matrix functions [7]. Within random matrix theory there are other applications. First, the local asymptotics near the edge(s) and bulk(s) imply universality for certain invariant ensembles [12, 28] and bounds on the growth of the polynomials can imply a certain stability of the recurrence coefficients that is useful in statistical estimation problems [17, 19].

The so-called strong asymptotics of orthogonal polynomials was developed in the book of Szegő [34] for the classical families of orthogonal polynomials where one has a related differential equation and integral formula. The asymptotics are termed "strong" because they provide precise, pointwise, leading-order behavior. Two slightly more modern references are [29, 38].

The majority of these results focus on orthogonal polynomials on a single finite interval, on the entire real axis or on a semi-axis.

The Riemann–Hilbert approach to determining strong asymptotics was pioneered in [12, 15, 26] for weights on the real axis and in [28] for the interval $[-1, 1]$. Crucially, these works used that weight function ρ in $\mu(\mathrm{d}x) = \rho(x)\,\mathrm{d}x$ had an appropriate analytic continuation to a neighborhood of the support of μ. It is worth noting here that the approach in [29], for example, does not require analyticity. For a special choice of weight function, explicit formulae exist in terms of Riemann theta functions [11].

The Riemann–Hilbert approach was extended to non-analytic weights on the unit circle and the real line in [30, 31]. Then, in [8], the approach was used to obtain asymptotics for orthogonal polynomials on $[-1, 1]$ for smooth, but not analytic, perturbations of the Jacobi weight $(1 - x)^\beta (1 + x)^\alpha$. Continuing, in [41] the asymptotics were refined in the case where $\alpha, \beta \in \{-1/2, 1/2\}$. This refinement is important because if $\rho(x) = h(x)(1-x)^\beta(1+x)^\alpha$, $\alpha, \beta \in \{-1/2, 1/2\}$ and h is analytic then one knows the error terms in the asymptotic expansions are exponentially small with respect to the degree of the polynomial [27]. The work of [41] uses a Riemann–Hilbert analysis to show that as the smoothness of h increases, the order of the error term decreases accordingly. As noted in [41], this can also be seen to be a consequence of theorems concerning orthogonal polynomials on the unit circle, using the canonical (Joukowski) mapping of the unit circle to the unit interval $[-1, 1]$.

For orthogonal polynomials for weight functions supported on multiple disjoint intervals of the form (1.1) for $\alpha_j, \beta_j \in \{-1/2, 1/2\}$, i.e., the Chebyshev-like case, the relevant works are first [40] and then [17], where the quantities in [40] were made more explicit and used for perturbation theory. We are unaware of results for multiple intervals for general α_j, β_j, i.e., the general Jacobi-like case, even for analytic perturbations. Thus, specifically, in this work we are interested in extending these results to the case where (i) h_j in (1.1) is not analytic and (ii) considering general $\alpha_j, \beta_j > -1$. The goal is not specifically to obtain new formulae for the asymptotics in such situations, since the formulae for analytic perturbations h_j broadly hold for non-analytic perturbations. Rather, one is interested in the size of the error terms in the asymptotics. For this reason, we largely leave our main results, Theorems 2.6 and 3.1, in an abstract form. If more specifics about the details of the asymptotics is desired, we refer the reader to [17] for asymptotics away from the edges and [28] for local behavior near the edge.

The notation $C^k(I)$ is used to denote the space of k times continuously differentiable functions f on a set I with norm $\|f\|_{C^k(I)} := \max_{0 \le \ell \le k} \max_{x \in I} |f^{(\ell)}(x)|$. And let $C^{k,\alpha}(I)$ denote the space of k times continuously differentiable functions f on a set I such that $f^{(k)}$ is α-Hölder continuous, with norm

$$\|f\|_{C^{k,\alpha}(I)} := \max_{0 \le \ell \le k} \sup_{x \in I} |f^{(\ell)}(x)| + \sup_{y \ne x} \frac{|f^{(k)}(x) - f^{(k)}(y)|}{|x - y|^\alpha}.$$

For general α_j, β_j, we assume that $h_j \in C^{k,1}([a_j, b_j])$, $k \ge 1$ and $k > 2\max_j \max\{|\alpha_j|, |\beta_j|\} - \frac{1}{2}$ (see Theorem 2.6). We also utilize the Sobolev spaces $W^{k,p}([a_j, b_j])$ consisting of functions that have k weak derivatives that are all in $L^p([a_j, b_j])$ with respect to standard Lebesgue measure. In the special case, where $\alpha_j, \beta_j \in \{1/2, -1/2\}$, we allow $h_j \in W^{k,q}([a_j, b_j])$, $k \ge 1$, $q > 4$. Below, we reserve the index k to refer to this assumed smoothness class.

As noted above, [41] considered the case of a single interval with $\alpha_1, \beta_1 \in \{1/2, -1/2\}$. It was shown that the error term is, in particular, $\mathrm{O}\left(\frac{\log n}{n^{k+\alpha}}\right)$ if $h_1 \in C^{k,\alpha}([a_1, b_1])$ and $k \ge 3$. Here we obtain similar error bounds[1] while allowing $g > 0$, $P > 0$, and only requiring $k \ge 1$, see Theorem 3.1. There is a strong reason to believe this is still less than optimal, see Figures 4

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[1] In a previous version of this manuscript, an error bound without the $\log n$ factor was stated. While we believe this bound to still be valid, there was a flaw in the argument.

and 5. These plots seem to indicate that this bound is suboptimal by a factor of $1/n$ in the situation where the density has, in a sense, a single point of non-analyticity.

In the same vein as [41], the method described in this paper works by constructing close-to-analytic extensions of each function h_j using polynomial approximation. Then, again borrowing directly from [41], we reuse an argument based on the Bernstein–Walsh inequality. Here it requires a bit more care in its implementation using a comparison, see Proposition A.1.

For the case of general α_j, β_j, our approach differs moderately from the most closely related work in [8]. In [8], the authors construct an approximately analytic extension of a Szegő-like function that appears in the exponent. This has the benefit that one can directly use the exact classical parametrices to solve local problems. It appears to have the drawback that as the assumed smoothness is increased, one does not recover the order of the error term that is found in the case of analytic h_j. The method proposed here directly constructs a nearly analytic extension of h_j and one obtains error terms on the order of $\mathrm{O}\left(n^{-1+\epsilon}\right)$ for any $\epsilon > 0$ by supposing that h_j is smooth enough. The drawback of our approach is that there is a small interval on the real axis where we do not obtain uniform estimates, determined by the δ parameter in Theorem 2.6. Importantly, $\delta \gg 1/n^2$ is always possible and the so-called Bessel asymptotics for the orthogonal polynomials near a_j, b_j can be determined using the method here. If uniform asymptotics are desired using this method, we conjecture that a Sobolev improvement of our L^2 bounds is possible, with possibly greater smoothness assumptions on h_j.

The outline of the paper is as follows. In the remainder of this section, we introduce the Fokas–Its–Kitaev Riemann–Hilbert problem and fix more notation. In Section 2, we present the $\overline{\partial}$ deformations for the general Jacobi-type case $\alpha_j, \beta_j > -1$. This leads to Theorem 2.6. In Section 3, we first present an improvement for the Chebyshev-like case $\alpha_j, \beta_j \in \{-1/2, 1/2\}$ in Theorem 3.1 and then discuss the addition of a finite number of point masses. Section 4 discusses the implied asymptotics for recurrence coefficients and uses this to present numerical experiments demonstrating that Theorem 3.1 is likely suboptimal. We also include five appendices that include technical developments.

On a notational note, with possible subscripts, C, C', C'', c, c', D, D' will be used to denote generic constants that may vary from line to line. Capital bold symbols $\mathbf{A}, \mathbf{B}, \ldots$ will be used to denote matrices and matrix-valued functions. Lower-case bold characters are used to denote vectors. For concreteness, $\|\mathbf{A}\|$ denotes the Frobenius (Hilbert–Schmidt) matrix norm.

1.1 The asymptotics of the recurrence coefficients

We briefly review the implications of the asymptotic estimates obtained here by reviewing the formulae given in [18]. For error terms E_1, E_2, E_3, the recurrence coefficients for the orthonormal polynomials with respect to a measure μ (1.1) satisfy

$$b_n(\mu)^2 = \frac{1}{\mathfrak{c}^2} \frac{\dfrac{\Theta_2(\infty; \mathbf{d}_2; (n+1-P)\mathbf{\Delta} + \boldsymbol{\zeta})}{\Theta_1(\infty; \mathbf{d}_2; (n+1-P)\mathbf{\Delta} + \boldsymbol{\zeta})} + E_1}{\dfrac{\Theta_2(\infty; \mathbf{d}_2; (n-P)\mathbf{\Delta} + \boldsymbol{\zeta})}{\Theta_1(\infty; \mathbf{d}_2; (n-P)\mathbf{\Delta} + \boldsymbol{\zeta})} + E_2},$$

$$a_n(\mu) = \frac{\Theta_1^{(1)}(\mathbf{d}_2; (n-P)\mathbf{\Delta} + \boldsymbol{\zeta})}{\Theta_1(\infty; \mathbf{d}_2; (n-P)\mathbf{\Delta} + \boldsymbol{\zeta})} - \frac{\Theta_1^{(1)}(\mathbf{d}_2; (n+1-P)\mathbf{\Delta} + \boldsymbol{\zeta})}{\Theta_1(\infty; \mathbf{d}_2; (n+1-P)\mathbf{\Delta} + \boldsymbol{\zeta})} + \mathfrak{g}_1 + E_3.$$

For the definition of Θ_j see (D.1), and the superscript $^{(1)}$ denotes the $\mathrm{O}(1/z)$ term in the expansion of the function at ∞. The vectors $\mathbf{d}_j$, $\mathbf{\Delta}$, and $\boldsymbol{\zeta}$ are determined in (2.3) and the constant $\mathfrak{g}_1$ is the $\mathrm{O}(1/z)$ term in the expansion of $\mathfrak{g}$ at ∞ (see (A.2)). The size of the error terms E_1, E_2, E_3 is determined by the size of the error terms, with respect to n, obtained in Theorem 2.6 or Theorem 3.1.

1.2   The Fokas–Its–Kitaev Riemann–Hilbert problem

In [22], the authors found a characterization of orthogonal polynomials in terms of a matrix Riemann–Hilbert (RH) problem. We now review such a formulation. Define the Cauchy transforms of the monic orthogonal polynomials[2]

$$c_n(z;\mu) = \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{\pi_n(\lambda;\mu)}{\lambda - z} \mu(d\lambda),$$

and the matrix-valued function

$$\mathbf{Y}_n(z;\mu) = \begin{bmatrix} \pi_n(z;\mu) & c_n(z;\mu) \\ \gamma_{n-1}(\mu)\pi_{n-1}(z;\mu) & \gamma_{n-1}(\mu)c_{n-1}(z;\mu) \end{bmatrix}, \qquad z \notin \mathrm{supp}(\mu),$$

where we use the notation $\gamma_n(\mu) = -2\pi i \|\pi_n(\cdot;\mu)\|_{L^2(\mu)}^{-2}$. It then follows that (see [22] or [27])

$$\mathbf{Y}_n^+(z;\mu) = \mathbf{Y}_n^-(z;\mu) \begin{bmatrix} 1 & \rho(z) \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{Y}_n^{\pm}(z;\mu) := \lim_{\epsilon \to 0^+} \mathbf{Y}_n(z \pm i\epsilon;\mu), \tag{1.2}$$

at all points $z \in \mathbb{R}$ where μ has a continuous density ρ. From the discrete contributions to μ,

$$\mathrm{Res}_{z=c_j} \mathbf{Y}_n(z;\mu) = \lim_{z \to c_j} \mathbf{Y}_n(z;\mu) \begin{bmatrix} 0 & \frac{r_j}{2\pi i} \\ 0 & 0 \end{bmatrix}, \qquad j = 1, 2, \dots, P. \tag{1.3}$$

We will initially suppose that $P = 0$ and discuss the requisite modifications later.

Additionally,

$$\mathbf{Y}_n(z;\mu) \begin{bmatrix} z^{-n} & 0 \\ 0 & z^n \end{bmatrix} = \mathbf{I} + \mathrm{O}(1/z), \qquad z \to \infty. \tag{1.4}$$

We need to impose singularity conditions at the endpoints, following [27], we require the entry-wise asymptotics

$$\mathbf{Y}_n(z;\mu) = \begin{cases} \mathrm{O} \begin{bmatrix} 1 & 1 + |z - b_j|^{\alpha_j} \\ 1 & 1 + |z - b_j|^{\alpha_j} \end{bmatrix}, & \alpha_j \neq 0, \\ \mathrm{O} \begin{bmatrix} 1 & \log|z - b_j| \\ 1 & \log|z - b_j| \end{bmatrix}, & \alpha_j = 0, \end{cases} \qquad z \to b_j,$$

$$\mathbf{Y}_n(z;\mu) = \begin{cases} \mathrm{O} \begin{bmatrix} 1 & 1 + |z - a_j|^{\beta_j} \\ 1 & 1 + |z - a_j|^{\beta_j} \end{bmatrix}, & \beta_j \neq 0, \\ \mathrm{O} \begin{bmatrix} 1 & \log|z - a_j| \\ 1 & \log|z - a_j| \end{bmatrix}, & \beta_j = 0, \end{cases} \qquad z \to a_j. \tag{1.5}$$

We make an important note here. First, it follows that $\det \mathbf{Y}_n(z;\mu) = 1$. Second, if we construct another function, $\tilde{\mathbf{Y}}_n(z;\mu)$ that also satisfies (1.2), (1.3), (1.4) and (1.5) then the ratio $\tilde{\mathbf{Y}}_n(z;\mu)\mathbf{Y}_n(z;\mu)^{-1}$ will be analytic except for possibly a_j, b_j where isolated singularities could persist. Indeed, we check, as $z \to b_j$, if $\alpha_j \neq 0$, for example,

$$\tilde{\mathbf{Y}}_n(z;\mu)\mathbf{Y}_n(z;\mu)^{-1} = \mathrm{O} \begin{bmatrix} 1 & 1 + |z - b_j|^{\alpha_j} \\ 1 & 1 + |z - b_j|^{\alpha_j} \end{bmatrix} \begin{bmatrix} 1 + |z - b_j|^{\alpha_j} & 1 + |z - b_j|^{\alpha_j} \\ 1 & 1 \end{bmatrix}$$

$$= \mathrm{O} \begin{bmatrix} 1 + |z - b_j|^{\alpha_j} & 1 + |z - b_j|^{\alpha_j} \\ 1 + |z - b_j|^{\alpha_j} & 1 + |z - b_j|^{\alpha_j} \end{bmatrix}.$$

From the condition $\alpha_j > -1$, we see that this isolated singularity must be removable. Liouville's theorem gives $\tilde{\mathbf{Y}}_n(z;\mu)\mathbf{Y}_n(z;\mu)^{-1} = \mathbf{I}$.

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[2]The monic orthogonal polynomials are defined by the requirement that $\pi_n(x;\mu) = x^n(1 + o(1))$, $n \to \infty$, and $\int \pi_n(x;\mu)\pi_m(x;\mu)\mu(dx) \propto \delta_{n,m}$.
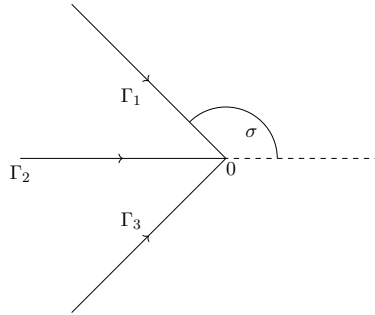
Remark 1.1. First, in all our calculations, $z = x + iy$ where $x, y \in \mathbb{R}$. We will treat a function $f(z)$ of a complex variable also a function two variables $f(x, y)$ and will abuse notation here. We typically write $f(x, y)$ when f may not be an analytic function of $z = x + iy$.

Remark 1.2. We typically use $f^{\pm}(z)$ to denote the boundary values of a function f from the left $(+)$ or right $(-)$ side of an oriented curve. In some cases, it may be convenient to use subscripts to denote the boundary values, e.g., $f'_{\pm}(z)$.

2   Dbar deformation

We begin the deformation procedure by first recalling the definition and properties of the so-called g-function, see (A.2). Historically speaking, this construction is found in [2]. This function is also related to the exterior Green's function with pole at ∞ [39]. Often this function is determined during the process of deformation, but we need it at the outset. Throughout what follows, we now suppose that $\epsilon' > \epsilon > 0$ are sufficiently small so that both $\varphi(z; a_j)$, $\varphi(z; b_j)$ from (A.3) are conformal on the balls $B_{\epsilon'}(a_j)$, $B_{\epsilon'}(b_j)$, respectively,[3] for all j. We assume that all the sets $B_{\epsilon'}(a_j)$, $B_{\epsilon'}(b_j)$, $j = 1, 2, \ldots, g + 1$ are at a distance at least, say, $10\epsilon'$ from one another. Next, define $\mathcal{A}_j = \varphi(B_{\epsilon}(a_j); a_j)$, $\mathcal{B}_j = \varphi(B_{\epsilon}(b_j); b_j)$. We also now suppose that $\epsilon > 0$ is sufficiently small so that these sets are convex.

When h_j is assumed analytic, one arrives at a local RH problem that needs to be solved explicitly near a_j, b_j. After employing the local conformal mappings, one finds a model problem with piecewise constant jump matrices. The solution of this problem is built out of Bessel and Hankel functions [28]. As will be made clearer below, the jump contours for the model problem are given in Figure 1. The contours will then need to be mapped using the inverse mappings. So, define $\Sigma_{a_j, \ell} = \varphi^{-1}\big(\overline{\mathcal{A}_j} \cap \Gamma_\ell; a_j\big)$, $\Sigma_{b_j, \ell} = \varphi^{-1}\big(\overline{\mathcal{B}_j} \cap \Gamma_\ell; b_j\big)$, for $j = 1, 2, \ldots, g + 1$, $\ell = 1, 2, 3$.

Figure 1. Model contours. Here we use $\sigma = 2\pi/3$. The contours are symmetric about the real axis.

The contours in Figure 2 will become jump contours in a deformed hybrid RH-$\bar{\partial}$ problem.

For $z \in \mathbb{C}$, $z = re^{i\theta}, \theta \in [0, 2\pi)$, define $(z)^{\alpha}_{\rightarrow} := r^{\alpha}e^{i\alpha\theta}$. Similarly, for $z \in \mathbb{C}$, $z = re^{i\theta}$, $\theta \in [-\pi, \pi)$, define $(z)^{\alpha} := r^{\alpha}e^{i\alpha\theta}$, $\sqrt{z} = z^{1/2}$. Consider the function $w_j(x) = (b_j - x)^{\alpha_j}(x - a_j)^{\beta_j}$, $a_j \leq x \leq b_j$, and

$$\omega_j(z) = e^{-i\alpha_j \pi}(z - b_j)^{\alpha_j}_{\rightarrow}(z - a_j)^{\beta_j}, \qquad z \in \mathbb{C} \setminus ((-\infty, a_j] \cup [b_j, \infty)).$$

Then for $a_j < x < b_j$, $\lim_{\epsilon \to 0} \omega_j(x + i\epsilon) = e^{-i\alpha_j \pi}(x - b_j)^{\alpha_j}_{\rightarrow}(x - a_j)^{\beta_j} = w_j(x)$, and therefore it is the analytic continuation of $w_j(x)$ to the upper- and lower-half planes.

In Appendix B, we develop a $\sharp$ operator that extends a function off the real axis, while satisfying several convenient conditions. For $f_j(x) = 1/h_j(x)$, consider $f_j^{\sharp}(x, y)$ with $\kappa = k$, $\tau = \epsilon/2$. The choice of θ_1, θ_2 that appear in the definition of $f_j^{\sharp}$ will be discussed below.

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[3]Here we use the notation $B_{\epsilon}(c) = \{z \in \mathbb{C} \mid |z - c| < \epsilon\}$.
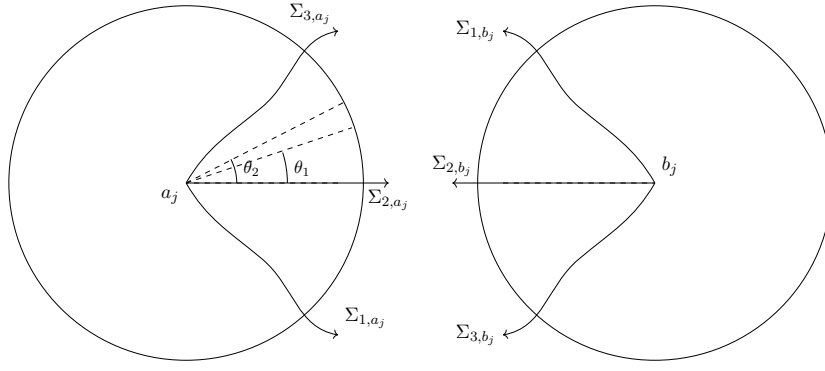
Figure 2. Mapping the contours Γ_j using the local conformal mappings, with valid choices for θ_1, θ_2 displayed.
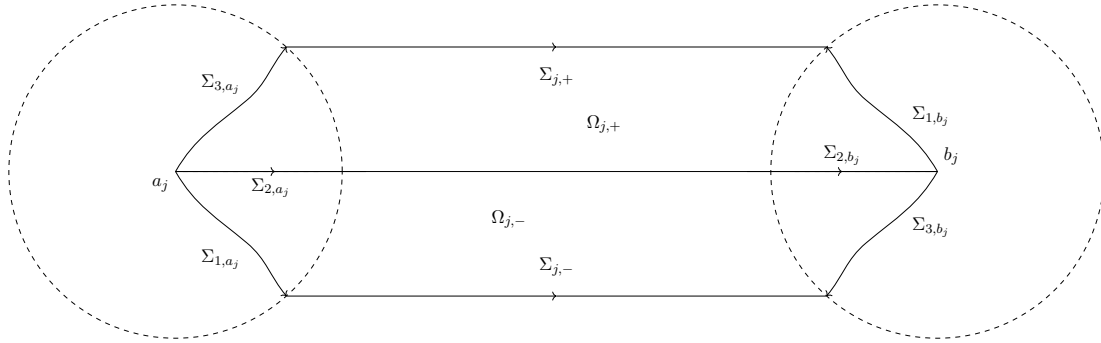
Figure 3. The addition of contours to form $\Sigma_{j,\pm}$ and enclose the regions $\Omega_{j,\pm}$. The definition of $\Sigma_{j,\pm}$ includes the mapped contours near a_j, b_j.

Before we state some properties of $f_j^\sharp$, we define a region where $f_j^\sharp$ will be used. The contours Σ_{1,a_j} and Σ_{3,a_j} intersect the circle $\{|z - a_j| = \epsilon\}$ at points off the real axis, and to the right of the line $a_j + \mathrm{i}\mathbb{R}$. The same is true of the intersection of the contours Σ_{1,b_j} and Σ_{3,b_j} and the circle $\{|z - b_j| = \epsilon\}$ to the left of $b_j + \mathrm{i}\mathbb{R}$. So, a straight line, lying within the lower-half plane, can be used to connect $\Sigma_{1,a_j} \cap \{|z - a_j| = \epsilon\}$ and $\Sigma_{3,b_j} \cap \{|z - b_j| = \epsilon\}$, giving a contour $\Sigma_{j,-}$, using left-to-right orientation. And a straight line, lying within the upper-half plane, can be used to connect $\Sigma_{3,a_j} \cap \{|z - a_j| = \epsilon\}$ and $\Sigma_{1,b_j} \cap \{|z - b_j| = \epsilon\}$, giving a contour $\Sigma_{j,+}$, using left-to-right orientation. This is depicted in Figure 3. The contours $\Sigma_{j,\pm}$ now enclose open regions $\Omega_{j,\pm}$, also depicted in Figure 3.

Define the $\overline{\partial}$ derivative $\overline{\partial} = \frac{1}{2}(\partial_x + \mathrm{i}\partial_y)$. We have the following properties of $f_j^\sharp$ (again, see Appendix B) so long as we choose θ_2 sufficiently small so that $\beta(x - a_j, y)$ and $\beta(x - b_j, y)$ in the definition of the $\sharp$ operator are identically equal to 1 on $\Sigma_{a_j,\ell}$, $\Sigma_{b_j,\ell}$, $\ell = 1, 3$. The extension $f_j^\sharp$ of f_j then satisfies:

(1) for $a_j \leq x \leq b_j$, $f_j^\sharp(x, 0) = f_j(x)$,

(2) for $c = a_j, b_j$, $|z - c| < \epsilon/2$, and $z \in \Sigma_{c,1} \cup \Sigma_{c,3}$, we have

$$f_j^\sharp(x, y) = f_j(z; c) = \sum_{\ell=0}^{k} \frac{f_j^{(\ell)}(c)}{\ell!}(z - c)^\ell,$$

(3) $f_j^\sharp$ extends to be continuous on $\overline{\Omega_{j,+} \cup \Omega_{j,-}}$, and

(4) for $(x, y) \in \Omega_{j,+} \cup \Omega_{j,-}$, $\left|\overline{\partial} f_j^\sharp(x, y)\right| \leq C\|f_j\|_{C^{k,1}([a_j,b_j])}|y|^k$, for some $C > 0$.

We further suppose that ϵ is sufficiently small so that $f_j(z; c)$ does not vanish for $|z - c| \leq \epsilon/2$.

2.1   The first $\overline{\partial}$ deformation of the FIK RH problem

To begin to deform the RH problem, we notice the factorization

$$\begin{bmatrix} 1 & \rho(x) \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1/\rho(x) & 1 \end{bmatrix} \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/\rho(x) & 1 \end{bmatrix}.$$

We will use this factorization locally near each interval $[a_j, b_j]$. Define

$$\mathbf{Z}_n(x, y) = \mathbf{Y}_n(z; \mu) \begin{cases} \begin{bmatrix} 1 & 0 \\ -f_j^\sharp(x, y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,+}, \\[2ex] \begin{bmatrix} 1 & 0 \\ f_j^\sharp(x, y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,-}, \\[2ex] \mathbf{I}, & \text{otherwise.} \end{cases}$$

We compute

$$\overline{\partial}\mathbf{Z}_n(x, y) = \begin{cases} \mathbf{Y}_n(z; \mu) \begin{bmatrix} 0 & 0 \\ -\overline{\partial} f_j^\sharp(x, y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,+}, \\[2ex] \mathbf{Y}_n(z; \mu) \begin{bmatrix} 0 & 0 \\ \overline{\partial} f_j^\sharp(x, y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,-}, \\[2ex] \mathbf{0}, & z \in \mathbb{C} \setminus \left(\overline{\Omega}_{j,+} \cup \overline{\Omega}_{j,-} \right). \end{cases}$$

This can actually be rewritten as

$$\overline{\partial}\mathbf{Z}_n(x, y) = \begin{cases} \mathbf{Z}_n(z; \mu) \begin{bmatrix} 0 & 0 \\ -\overline{\partial} f_j^\sharp(x, y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,+}, \\[2ex] \mathbf{Z}_n(z; \mu) \begin{bmatrix} 0 & 0 \\ \overline{\partial} f_j^\sharp(x, y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,-}, \\[2ex] \mathbf{0}, & z \in \mathbb{C} \setminus \left(\overline{\Omega}_{j,+} \cup \overline{\Omega}_{j,-} \right). \end{cases}$$

Setting $\Sigma_j = \Sigma_{j,+} \cup \Sigma_{j,-}$, as in Figure 3, we have the jump conditions

$$\mathbf{Z}_n^+(x, y) = \mathbf{Z}_n^-(x, y) \begin{cases} \begin{bmatrix} 1 & 0 \\ f_j^\sharp(x, y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Sigma_j, \\[2ex] \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix}, & z \in [a_j, b_j]. \end{cases}$$

Next, to handle the asymptotics at infinity, which we have not changed, we recall the definition of $\mathfrak{g}(z)$ in Appendix A and set $\mathfrak{c} = \lim_{z \to \infty} \frac{e^{\mathfrak{g}(z)}}{z}$. We define

$$\mathbf{S}_n(x, y) = \mathfrak{c}^{n\sigma_3} \mathbf{Z}_n(x, y) e^{-n\mathfrak{g}(z)\sigma_3}, \qquad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

We find that $\mathbf{S}_n(x, y)$ solves the normalized hybrid RH-$\overline{\partial}$ problem

$$\mathbf{S}_n^+(x, y) = \mathbf{S}_n^-(x, y) \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{-2n\mathfrak{g}(z)} f_j^\sharp(x, y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Sigma_j, \\[2ex] \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix} & z \in (a_j, b_j), \\[2ex] e^{-n\Delta_j\sigma_3}, & z \in (b_j, a_{j+1}), \end{cases}$$

$$\overline{\partial}\mathbf{S}_n(x,y) = \mathbf{S}_n(z) \begin{cases} \begin{bmatrix} 0 & 0 \\ -\mathrm{e}^{-2n\mathfrak{g}(z)}\overline{\partial}f_j^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,+}, \\[2em] \begin{bmatrix} 0 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}\overline{\partial}f_j^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,-}, \\[2em] \mathbf{0}, & z \in \mathbb{C} \setminus \left(\overline{\Omega}_{j,+} \cup \overline{\Omega}_{j,-}\right), \end{cases}$$

$$\mathbf{S}_n(x,y) = \mathbf{I} + \mathrm{O}(z^{-1}), \qquad z \to \infty.$$

We recall here that $\mathfrak{g}^+(x) + \mathfrak{g}^-(x) = 0$ for $x \in [a_j, b_j]$ and $\mathfrak{g}^+(x) - \mathfrak{g}_-(x) = \Delta_j$ for $x \in [b_j, a_{j+1}]$ where Δ_j is a constant.

2.2   The second $\overline{\partial}$ deformation of the FIK RH problem

While one can proceed to construct a parametrix for $\mathbf{S}_n(x,y)$ — an approximate solution as $n \to \infty$, there is a fundamental difficulty. The outer solution will be determined, as we discuss further below, by a function that depends on the whole of ρ — the Szegő function. The natural way to solve $\mathbf{S}_n(x,y)$ near $z = a_j, b_j$ is to use the Bessel parametrices that depend on $f_j(z; c)$, $c = a_j, b_j$, using its analyticity in a crucial way. The matching of this parametrix with the outer solution will not remove the jump on the real axis if h_j does not have an analytic extension. The discrepancy can be estimated but it is not the most convenient way to handle the issue.

Define (using the notation in (B.1))

$$d_{j,-1}(x;c) = \left[\sqrt{\frac{f_j(x;c)}{f_j(x)}} - 1\right](1 - b_\tau(x;c)), \qquad c = a_j, b_j,$$

$$d_{j,+}(x;c) = \left[\sqrt{\frac{f_j(x)}{f_j(x;c)}} - 1\right](1 - b_\tau(x;c)), \qquad c = a_j, b_j,$$

and set $d_{j,\pm}^{\sharp}(x,y;c)$ to be the extension, as in Appendix B, with $\tau = \delta < \epsilon/4$ and $\kappa = 0$ — $d_{j,\pm}^{\sharp}(x,y;c) = d_{j,\pm}(x;c)$ for $|y| < \epsilon/4$. More explicitly, for example,

$$d_{j,\pm}^{\sharp}(x,y;a_j) = (1 - b_\delta(|y|;0))\left[\tilde{b}(x)d_{j,\pm}(x;a_j) + (1 - b_\delta(x;a_j))(1 - \beta(x - a_j,y))d_{j,\pm}(x;a_j)\right. \\ \left. + \beta(x - a, y)(1 - b_\delta(x;a_j))d_{j,\pm}(a_j;a_j)\right].$$

We write

$$\left|\sqrt{\frac{f_j(x;c)}{f_j(x)}} - 1\right| = \left|\frac{\sqrt{f_j(x;c)} - \sqrt{f_j(x)}}{\sqrt{f_j(x)}}\right| \leq \frac{1}{2\sqrt{\xi}\sqrt{f_j(x)}}|f_j(x;c) - f_j(x)|,$$

for ξ between $f_j(x;c)$ and $f_j(x)$. Since both of these functions are bounded below on $[0, 2\delta]$, we find

$$\|d_{j,\pm}(\cdot;a_j)\|_{C^0([a_j,b_j])} \leq C\|f_j(\cdot;a_j) - f_j(\cdot)\|_{C^0([a_j,a_j+2\delta])},$$

$$\|d_{j,\pm}(\cdot;b_j)\|_{C^0([a_j,b_j])} \leq C\|f_j(\cdot;a_j) - f_j(\cdot)\|_{C^0([b_j-2\delta,b_j])}.$$

Similarly, we find

$$\|d_{j,\pm}(\cdot;a_j)\|_{C^1([a_j,b_j])} \leq C\|f_j(\cdot;a_j) - f_j(\cdot)\|_{C^{0,1}([a_j,a_j+2\delta])},$$

$$\|d_{j,\pm}(\cdot;b_j)\|_{C^1([a_j,b_j])} \leq C\|f_j(\cdot;b_j) - f_j(\cdot)\|_{C^{0,1}([b_j-2\delta,b_j])}.$$

We immediately have the following.

Lemma 2.1.

$$\left\|d_{j,\pm}^\sharp(\cdot,y;c)\right\|_{L^\infty([a_j,b_j])} \le C\delta^{k+1}, \qquad \left\|\overline\partial d_{j,\pm}^\sharp(\cdot,y;c)\right\|_{L^\infty([a_j,b_j])} \le C\delta^k.$$

Next, define

$$\mathbf{T}_n(x,y) = \mathbf{S}_n(x,y)\begin{cases}\begin{bmatrix} d_{j,-}^\sharp(x,y;c)+1 & 0 \\ 0 & d_{j,+}^\sharp(x,y;c)+1 \end{bmatrix}, & z \in \Omega_{j,\pm},\ |z-c|<\epsilon, \\ & c = a_j, b_j, \\ \mathbf{I}, & \text{otherwise.} \end{cases}$$

We find that $\mathbf{T}_n(x,y)$ solves the hybrid RH-$\overline\partial$ problem for $c = a_j, b_j$

$$\mathbf{T}_n^+(x,y) = \mathbf{T}_n^-(x,y)\begin{cases}\begin{bmatrix} 1 & 0 \\ e^{-2ng(z)}f_j^\sharp(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Sigma_j, \\[4pt] \begin{bmatrix} 0 & w_j(x)/f_j(x;c) \\ -f_j(x;c)/w_j(x) & 0 \end{bmatrix}, & \begin{matrix}|z-c|<\delta, \\ z \in (a_j,b_j),\end{matrix} \\[4pt] \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix}, & \begin{matrix}|z-c|>2\delta, \\ z \in (a_j,b_j),\end{matrix} \\[4pt] \begin{bmatrix} 0 & \rho(x)\dfrac{d_{j,-}^\sharp(x,y;c)+1}{d_{j,+}^\sharp(x,y;c)+1} \\ -\dfrac{d_{j,+}^\sharp(x,y;c)+1}{d_{j,-}^\sharp(x,y;c)+1}\rho(x)^{-1} & 0 \end{bmatrix}, & \begin{matrix}\delta \le |z-c| \le 2\delta, \\ z \in (a_j,b_j),\end{matrix} \\[4pt] e^{-\Delta_j\boldsymbol\sigma_3}, & z \in (b_j, a_{j+1}), \end{cases}$$

$$\overline\partial\mathbf{T}_n(x,y) = \mathbf{T}_n(x,y)\mathbf{W}_n(x,y),$$

$$\mathbf{W}_n(x,y) = \begin{cases}\begin{bmatrix} \dfrac{\overline\partial d_{j,+}^\sharp(x,y;c)}{d_{j,+}^\sharp(x,y;c)+1} & 0 \\ -e^{-2ng(z)}\overline\partial f_j^\sharp(x,y)/\omega_j(z)\dfrac{d_{j,+}^\sharp(x,y;c)+1}{d_{j,-}^\sharp(x,y;c)+1} & \dfrac{\overline\partial d_{j,-}^\sharp(x,y;c)}{d_{j,-}^\sharp(x,y;c)+1} \end{bmatrix}, & \begin{matrix}z \in \Omega_{j,+}, \\ |z-c|<\epsilon,\end{matrix} \\[4pt] \begin{bmatrix} \dfrac{\overline\partial d_{j,+}^\sharp(x,y;c)}{d_{j,+}^\sharp(x,y;c)+1} & 0 \\ e^{-2ng(z)}\overline\partial f_j^\sharp(x,y)/\omega_j(z)\dfrac{d_{j,+}^\sharp(x,y;c)+1}{d_{j,-}^\sharp(x,y;c)+1} & \dfrac{\overline\partial d_{j,-}^\sharp(x,y;c)}{d_{j,-}^\sharp(x,y;c)+1} \end{bmatrix}, & \begin{matrix}z \in \Omega_{j,-}, \\ |z-c|<\epsilon,\end{matrix} \\[4pt] \begin{bmatrix} 0 & 0 \\ -e^{-2ng(z)}\overline\partial f_j^\sharp(x,y)/\omega_j(z) & 0 \end{bmatrix}, & \begin{matrix}z \in \Omega_{j,+}, \\ |z-a_j|>\epsilon, \\ |z-b_j|>\epsilon,\end{matrix} \\[4pt] \begin{bmatrix} 0 & 0 \\ e^{-2ng(z)}\overline\partial f_j^\sharp(x,y)/\omega_j(z) & 0 \end{bmatrix}, & \begin{matrix}z \in \Omega_{j,-}, \\ |z-a_j|>\epsilon, \\ |z-b_j|>\epsilon,\end{matrix} \\[4pt] \mathbf{0}, & z \in \mathbb{C}\backslash(\overline\Omega_{j,+}\cup\overline\Omega_{j,-}), \end{cases}$$

$$\mathbf{T}_n(x,y) = \mathbf{I} + O(z^{-1}), \qquad z \to \infty.$$

And motivated by this, we define $\rho(z;c) := \omega_j(z)/f_j(z;c)$,

$$\tilde\rho(x;\delta) := \begin{cases} \rho(x;c), & |x-c|<\delta,\ c = a_j, b_j, \\ \rho(x), & |x-c|>2\delta,\ c = a_1, b_1, \ldots, a_{g+1}, b_{g+1}, \\ \rho(x)\dfrac{d_{j,-}^\sharp(x,y;c)+1}{d_{j,+}^\sharp(x,y;c)+1}, & \delta \le |x-c| \le 2\delta,\quad c = a_j, b_j. \end{cases}$$

2.3 Construction of the global parametrix

The value of the so-called Szegő function $G(z)$ is that it allows one to replace the jumps (a_j, b_j) with something simpler at the cost of adding to the jumps on (b_j, a_{j+1}). Define[4]

$$G(z) = -\frac{R(z)}{2\pi i}\left[\sum_{j=1}^{g+1}\int_{a_j}^{b_j}\frac{\log\rho(x)}{x-z}\frac{dx}{R_+(x)} + \sum_{j=1}^{g}\int_{b_j}^{a_{j+1}}\frac{\zeta_j}{x-z}\frac{dx}{R(x)}\right],$$

where the constants ζ_j are yet to be determined. Note that

$$G^+(z) + G^-(z) = -\log\rho(z), \qquad z \in (a_j, b_j),$$
$$G^+(z) - G^-(z) = -\zeta_j, \qquad z \in (b_j, a_{j+1}).$$

Since $R(z) = O(z^{g+1})$, we see that $G(z) = O(z^g)$. To avoid unbounded behavior of G at infinity, we choose $\boldsymbol{\zeta} = (\zeta_j)_{j=1}^g$ so that as $z \to \infty$ $G(z) = O(1)$. Indeed, we find a linear system of equations

$$m_\ell = -\sum_{j=1}^{g}\int_{a_j}^{b_j}\log\rho(x)x^{\ell-1}\frac{dx}{R_+(x)} - \sum_{j=1}^{g}\int_{b_j}^{a_{j+1}}\zeta_j x^{\ell-1}\frac{dx}{R(x)} = 0, \qquad \ell = 1, 2, \ldots, g.$$

This system of equations is uniquely solvable for $\boldsymbol{\zeta}$ using the fact that the normalized differentials exist, and involves the same coefficient matrix that is used to determine the polynomials Q_g in Appendix D.

As a first approximation, we consider the matrix $\tilde{\mathbf{T}}_n(z)$ that is obtained from $\mathbf{T}_n(x, y)$ by dropping the $\bar{\partial}$ conditions, and just using ρ on the real axis

$$\tilde{\mathbf{T}}_n^+(z) = \tilde{\mathbf{T}}_n^-(z)\begin{cases}\begin{bmatrix}1 & 0\\ e^{-2ng(z)}f_j^\sharp(x,y)/\omega_j(z) & 1\end{bmatrix}, & z \in \Sigma_j,\\[1.5em]\begin{bmatrix}0 & \rho(x)\\ -1/\rho(x) & 0\end{bmatrix}, & z \in (a_j, b_j),\\[1.5em]e^{-\Delta_j\boldsymbol{\sigma}_3}, & z \in (b_j, a_{j+1}),\end{cases}$$
$$\tilde{\mathbf{T}}_n(z) = \mathbf{I} + O(z^{-1}), \qquad z \to \infty.$$

Then consider $\mathbf{U}_n(x, y; \mu) = e^{\boldsymbol{\sigma}_3 G(\infty)}\tilde{\mathbf{T}}_n(z; \mu)e^{-\boldsymbol{\sigma}_3 G(z)}$. We check the jumps of $\mathbf{U}_n$

$$\mathbf{U}_n^+(z; \mu) = \begin{cases}\mathbf{U}_n^-(z; \mu)\begin{bmatrix}1 & 0\\ e^{-2(ng(z)-G(z))}f_j^\sharp(x,y)/\omega_j(z) & 1\end{bmatrix}, & z \in \Sigma_j \setminus \mathbb{R},\\[1.5em]\mathbf{U}_n^-(z; \mu)\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, & z \in (a_j, b_j),\\[1.5em]\mathbf{U}_n^-(z; \mu))\begin{bmatrix}e^{-n\Delta_j-\zeta_j} & 0\\ 0 & e^{n\Delta_j+\zeta_j}\end{bmatrix}, & z \in (b_j, a_{j+1}),\end{cases}$$

and asymptotics $\mathbf{U}_n(z) = \mathbf{I} + O(z^{-1})$, $z \to \infty$.

Due to the exponential decay that $e^{-ng(z)}$ will induce, see Appendix D, we expect the dominant contribution to the solution of this RH problem to come from the jump conditions on

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[4]See Appendix D for the definition of $R(z)$.

the real axis, at least away from the endpoints a_j, b_j. From Appendix D, (D.2) specifically, we expect $\mathbf{U}_n(x,y) \approx \mathbf{L}(z; n\boldsymbol{\Delta} + \boldsymbol{\zeta})$. And so, the global parametrix to $\mathbf{T}_n$ is

$$\mathbf{G}_n(z) = \mathrm{e}^{-\boldsymbol{\sigma}_3 G(\infty)} \mathbf{L}(z; n\boldsymbol{\Delta} + \boldsymbol{\zeta}) \mathrm{e}^{\boldsymbol{\sigma}_3 G(z)}.$$

While this will indeed give the asymptotics of the orthogonal polynomials away from the end-points, to complete the analysis, we need a modified version of this function. Define

$$\tilde{G}(z) = -\frac{R(z)}{2\pi\mathrm{i}} \left[\sum_{j=1}^{g+1} \int_{a_j}^{b_j} \frac{\log \tilde{\rho}(x;\delta)}{x-z} \frac{\mathrm{d}x}{R_+(x)} + \sum_{j=1}^{g} \int_{b_j}^{a_{j+1}} \frac{\zeta_j}{x-z} \frac{\mathrm{d}x}{R(x)} \right],$$

$$\tilde{\mathbf{G}}_n(z) = \tilde{\mathbf{G}}_n(z; \delta) = \mathrm{e}^{-\boldsymbol{\sigma}_3 G(\infty)} \mathbf{L}(z; n\boldsymbol{\Delta} + \boldsymbol{\zeta}) \mathrm{e}^{\boldsymbol{\sigma}_3 \tilde{G}(z)}.$$

Note that the constants ζ_j are the same as those in the definition of $G(z)$. This is so that the ratio $G(z)/\tilde{G}(z)$ will be analytic across the gaps (a_{j+1}, b_j).

For a given $z \in \mathbb{C} \setminus [a,b]$ let $\eta(z)$ be the closest point in $[a,b]$ to z. Set $S(z) = \sqrt{z-a}\sqrt{z-b}$. Then

$$S(z) \int_a^b \frac{f(x')}{x'-z} \frac{\mathrm{d}x'}{S_+(x')} = f(\eta(z)) \int_a^b \frac{1}{x'-z} \frac{S(z)}{S_+(x')} \mathrm{d}x' + \int_a^b \frac{f(x') - f(\eta(z))}{x'-z} \frac{S(z)}{S_+(x')} \mathrm{d}x'.$$

The first integral can be computed explicitly and seen to be bounded on all of $\mathbb{C}$. If L is the Lipschitz constant for f on $[a,b]$ then because $|x' - \eta(z)| \leq |x' - z|$ the last integral is bounded by $L \int_a^b \frac{|S(z)|}{|S_+(x')|} \mathrm{d}x'$, which is bounded on bounded subsets of $\mathbb{C}$. Then using Lemma 2.1

$$|G(z) - \tilde{G}(z)| \leq C \sum_{j=1}^{g+1} \|\rho/w_j - \tilde{\rho}(\cdot, \delta)/w_j\|_{C^{0,1}([a_j,b_j])} \leq C'\delta^k, \tag{2.1}$$

where the constants depend on the subset.

2.4 Using local solutions

In this subsection, we make heavy use of the definitions and jump conditions established in Appendix C. Consider the function

$$\mathbf{R}_n(x,t) = \mathbf{T}_n(x,y) \left(\tilde{\mathbf{G}}_n(z) \mathbf{G}_n(z)^{-1} \mathbf{P}_n(z; b_j) \right)^{-1}.$$

We claim that for $|z - b_j| < \epsilon$, that the function $\mathbf{R}_n$ is continuous for $z \notin (b_j - \epsilon, b_j)$, and for $|z - b_j| < \delta$. Indeed, $\mathbf{T}_n(x,y)$ and $\mathbf{P}_n(z; b_j)$ have the same jump condition for $z \notin (b_j - \epsilon, b_j)$ and $\tilde{\mathbf{G}}_n(z) \mathbf{G}_n(z)^{-1}$ is analytic for $z \notin (b_j - \epsilon, b_j)$.

We then recall that for $b_j - \epsilon < z < b_j$ we have

$$\mathbf{T}_n^+(x,y) = \mathbf{T}_n^-(x,y) \begin{bmatrix} 0 & \tilde{\rho}(z; b_j) \\ -1/\tilde{\rho}(z; b_j) & 0 \end{bmatrix}.$$

Using that for $b_j - \epsilon < z < b_j$

$$\mathbf{A}_n^+(z; b_j) = \mathbf{A}_n^-(z; b_j) \Psi^+(z)^{-\boldsymbol{\sigma}_3} \mathbf{E} \left(\frac{\rho(z)}{\rho(z; b_j)} \right)^{\boldsymbol{\sigma}_3} \mathbf{E}^{-1} \Psi^+(z)^{\boldsymbol{\sigma}_3},$$

and

$$\mathbf{Q}_n^+(z; b_j) = \mathbf{Q}_n^-(z; b_j) \begin{bmatrix} 0 & \rho(z; b_j) \\ -1/\rho(z; b_j) & 0 \end{bmatrix}, \qquad b_j - \epsilon < z < b_j,$$

we compute, again for $b_j - \epsilon < z < b_j$, that

$$\mathbf{R}_n^+(z) = \mathbf{R}_n^-(z)\tilde{\mathbf{G}}_n^-(z)\mathbf{G}_n^-(z)^{-1}\mathbf{A}_n^-(z;b_j)\mathbf{Q}_n^-(z;b_j)$$
$$\times \left(\frac{\tilde{\rho}(z;\delta)}{\rho(z;b_j)}\right)^{\boldsymbol{\sigma}_3}\mathbf{Q}_n^-(z;b_j)^{-1}\Psi^+(z)^{-\boldsymbol{\sigma}_3}\mathbf{E}\left(\frac{\tilde{\rho}(z;\delta)}{\rho(z;b_j)}\right)^{-\boldsymbol{\sigma}_3}$$
$$\times \mathbf{E}^{-1}\Psi^+(z)^{\boldsymbol{\sigma}_3}\mathbf{A}_n^-(z;b_j)^{-1}\mathbf{G}_n^-(z)\tilde{\mathbf{G}}_n^-(z)^{-1}.$$

For $b_j - \delta < z < b_j$, we have that $\tilde{\rho}(z;\delta) = \rho(z;b_j)$ and the claims about $\mathbf{R}_n$ follow.

Define

$$\mathbf{B}_1(z) = \tilde{\mathbf{G}}_n^-(z)\mathbf{G}_n^-(z)^{-1}\mathbf{A}_n^-(z;b_j)\mathbf{Q}_n^-(z;b_j),$$
$$\mathbf{B}_2(z) = \mathbf{Q}_n^-(z;b_j)^{-1}\Psi^+(z)^{-\boldsymbol{\sigma}_3}\mathbf{E},$$
$$\mathbf{B}_3(z) = \mathbf{E}^{-1}\Psi^+(z)^{\boldsymbol{\sigma}_3}\mathbf{A}_n^-(z;b_j)^{-1}\mathbf{G}_n^-(z)\tilde{\mathbf{G}}_n^-(z)^{-1}.$$

The following is a direct consequence of Lemma C.5.

Lemma 2.2. *Let $b_j - \epsilon < z < b_j$, and*

$$\frac{\tilde{\rho}(z;\delta)}{\rho(z;b_j)} = 1 + d_1(z), \qquad \frac{\rho(z;b_j)}{\tilde{\rho}(z;\delta)} = 1 + d_2(z), \qquad e(z) = \max\{|d_1(z)|, |d_2(z)|\},$$

and define

$$\varrho_n(z;b_j) = \left\|\mathbf{I} - \mathbf{B}_1(z)\left(\frac{\tilde{\rho}(z;\delta)}{\rho(z;b_j)}\right)^{\boldsymbol{\sigma}_3}\mathbf{B}_2(z)\left(\frac{\tilde{\rho}(z;\delta)}{\rho(z;b_j)}\right)^{-\boldsymbol{\sigma}_3}\mathbf{B}_3(z)\right\|.$$

As $n^2(b_j - z) \to \infty$, $\varrho_n(z;b_j) = \mathrm{O}\left(n^{-1}|z - b_j|^{-1}e(z)^2\right)$.

2.5 Global approximation

We have already defined $\mathbf{R}_n(z)$ for $|z - b_j| < \epsilon$. To complete the definition, we set

$$\mathbf{R}_n(x,y) = \mathbf{T}_n(x,y)\begin{cases} \left(\tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1}\mathbf{P}_n(z;b_j)\right)^{-1}, & |z - b_j| < \epsilon, \\ \left(\tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1}\mathbf{P}_n(z;a_j)\right)^{-1}, & |z - a_j| < \epsilon, \\ \mathbf{G}_n(z)^{-1}, & \text{otherwise.} \end{cases}$$

We now make the restriction that $n^{-2} \ll \delta \ll \epsilon$ to find that, for some $c > 0$, $\mathbf{R}_n(x,y)$ should solve the hybrid $\bar{\partial}$-RH problem

$$\mathbf{R}_n^+(x,y) = \mathbf{R}_n^-(x,y)\begin{cases} \mathbf{I} + \mathrm{O}(\mathrm{e}^{-cn}), & z \in \Sigma_{j,+} \cup \Sigma_{j,-}, \ |z - b_j| > \epsilon, \ |z - a_j| > \epsilon, \\ \mathbf{I} + \mathrm{O}(n^{-1} + \delta^k), & |z - c| = \epsilon, \ c = a_j, b_j, \\ \mathbf{I} + \mathrm{O}(n^{-1}\delta^{-1}), & z \in (b_j - \epsilon, b_j - \delta) \cup (a_j + \delta, a_j + \epsilon), \end{cases}$$
$$\bar{\partial}\mathbf{R}_n(x,y) = \mathbf{R}_n(x,y)\mathbf{X}_n(x,y),$$
$$\mathbf{X}_n(x,y) = \begin{cases} \mathbf{G}_n(z)\mathbf{W}_n(x,y)\mathbf{G}_n(z)^{-1}z \in \Omega_{j,\pm} \cap \left(\overline{B_\epsilon(a_j) \cup B_\epsilon(b_j)}\right)^c, \\ \tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1}\mathbf{P}_n(z;c)\mathbf{W}_n(x,y)\mathbf{P}_n(z;c)^{-1}\mathbf{G}_n(z)\tilde{\mathbf{G}}_n(z)^{-1} \\ \qquad \times z \in \Omega_{j,\pm} \cap B_\epsilon(c), \qquad c = a_j, b_j, \\ \mathbf{0}, \qquad z \in \mathbb{C} \setminus \left(\overline{\Omega}_{j,+} \cup \overline{\Omega}_{j,-}\right), \end{cases}$$
$$\mathbf{R}_n(x,y) = \mathbf{I} + \mathrm{O}\left(z^{-1}\right), \qquad z \to \infty,$$

where the jump condition error terms are uniform in z.

The jump condition on $(b_j - \epsilon, b_j - \delta)$, for example, comes from Lemma 2.2, and the jump condition on $|z - c| = \epsilon$ follows from (2.1) and the fact that $\mathbf{G}_n^{-1}(z)\mathbf{P}_n(z; c) = \mathbf{I} + \mathrm{O}(n^{-1})$ for $|z - c| = \epsilon$, see Appendix C.

To solve this problem, asymptotically, we first consider the problem with the RH component removed. We seek $\mathbf{C}_n(x, y)$ that is continuous on $\mathbb{C}$ and satisfies

$$\overline{\partial}\mathbf{C}_n(x, y) = \mathbf{C}_n(z)\mathbf{X}_n(x, y), \qquad z \in \Omega_{j,+} \cup \Omega_{j,-},$$
$$\mathbf{C}_n(x, y) = \mathbf{I} + \mathrm{O}(z^{-1}), \qquad z \to \infty,$$

where derivatives are understood to hold in a distributional sense. We set $\Omega = \bigcup_{o=\pm}\bigcup_j \Omega_{j,o}$.

Lemma 2.3. *For $p > 2$, $k \geq 0$, suppose*

$$2p(2|\alpha_j| - k) + p - 4 < 0, \qquad 2p(2|\beta_j| - k) + p - 4 < 0,$$

for all j. Then

$$\|\mathbf{X}_n\|_{L^p(\Omega)} = \mathrm{O}\Big((1 + \mathbb{1}_{\eta=0}\log n)n^{2(\eta-k)+1-\frac{4}{p}} + n^{-\frac{((3+k)p-1)((2k-1)p+4)}{(2k+5)p^2}} + \delta^k\Big),$$
$$\eta = 2\max_j\max\{|\alpha_j|, |\beta_j|\}.$$

Proof. We define three regions

$$\Omega_{b_j,\mathrm{I}} = (\Omega_{j,+} \cup \Omega_{j,-}) \cap \big\{z \mid n^2|z - b_j| \leq c\big\},$$
$$\Omega_{b_j,\mathrm{II}} = (\Omega_{j,+} \cup \Omega_{j,-}) \cap \big\{z \mid c < n^2|z - b_j| < N\big\},$$
$$\Omega_{b_j,\mathrm{III}} = (\Omega_{j,+} \cup \Omega_{j,-}) \cap \big\{z \mid N \leq n^2|z - b_j| < n^2\epsilon\big\},$$

where c is sufficiently small, but fixed, such that the estimates in Lemma C.1 hold, and $N \ll n^2$. In $\Omega_{b_j,\mathrm{I}}$ and $\Omega_{b_j,\mathrm{II}}$ we simply estimate $|e^{-\mathfrak{g}(z)}| \leq 1$. Then for $\alpha_j \neq 0$, $z \in \Omega_{b_j,\mathrm{I}}$ we have for $D_\mathrm{I} > 0$, using Lemma C.1,

$$\|\mathbf{X}_n(x, y)\| \leq D_\mathrm{I}\big(n^{2|\alpha_j|+1}|z - b_j|^{-|\alpha_j|}|y|^k + \delta^k\big) \leq D_\mathrm{I}\big(n^{2|\alpha_j|+1}|z - b_j|^{-|\alpha_j|+k} + \delta^k\big),$$

where we used that

$$W(z; b_j)^{-\sigma_3}\begin{bmatrix} 0 & 0 \\ 1/\omega_j(z) & 0 \end{bmatrix}W(z; b_j)^{\sigma_3} = \begin{bmatrix} 0 & 0 \\ W(z; b_j)^2/\omega_j(z) & 0 \end{bmatrix}$$

is bounded. Then rescaling and using polar coordinates[5]

$$\int_{\Omega_{b_j,\mathrm{I}}} \|\mathbf{X}_n(x, y)\|^p\, \mathrm{d}x\, \mathrm{d}y \leq D_\mathrm{I}^p n^{2p(2|\alpha_j|-k)+p-4}\int_0^{2\pi}\int_0^c r^{-p|\alpha_j|+pk+1}\, \mathrm{d}r\, \mathrm{d}\theta + \mathrm{O}(\delta^{kp}).$$

From this, we find the conditions $p(k - |\alpha_j|) + 1 > -1$, $2p(2|\alpha_j| - k) + p - 4 < 0$, where the first is required for integrability and the second is needed to obtain a meaningful estimate. The second is more restrictive. Now, if $\alpha_j = 0$, we have

$$\|\mathbf{X}_n(x, y)\| \leq D_\mathrm{I}\big(n|z - b_j|^k|\log n^2|z - b_j||^2 + \delta^k\big),$$

giving

$$\int_{\Omega_{b_j,\mathrm{I}}} \|\mathbf{X}_n(x, y)\|^p\, \mathrm{d}x\, \mathrm{d}y \leq D_\mathrm{I}^p n^{-2pk+p-4}(\log n)^2 + \mathrm{O}(\delta^{kp}), \qquad -2pk + p - 4 < 0.$$

---

[5]For convenience, we will routinely use the fact that for $a, b \geq 0$, $p > 0$ that $(a + b)^p \leq C_p(a^p + b^p)$.

Then, again, rescaling and using polar coordinates and Lemma C.4,

$$\int_{\Omega_{b_j,\mathrm{II}}} \|\mathbf{X}_n(x,y)\|^p \, \mathrm{d}x \, \mathrm{d}y \leq D_{\mathrm{II}}^p \left[N^{3p/2} n^{-4-2kp} \int_0^N r^{kp+1} \, \mathrm{d}r \right] + \mathrm{O}(\delta^{kp})$$

$$= \frac{D_{\mathrm{II}}^p}{kp+2} N^{3p/2+kp+2} n^{-4-2kp} + \mathrm{O}(\delta^{kp}).$$

Now, in $\Omega_{b_j,\mathrm{III}}$, the change of variables will not be sufficient. So, we need an estimate on $\mathfrak{g}(z)$. For any $\epsilon > 0$, sufficiently small, there is $c' > 0$, such that for $|y| \leq \epsilon$ we have

$$\operatorname{Re} \mathfrak{g}(z) \leq -c'|y|, \tag{2.2}$$

which, of course, implies $\left| e^{\mathfrak{g}(z)} \right| \leq e^{-c'|y|}$. For $z \in \Omega_{b_j,\mathrm{III}}$,

$$\|\mathbf{X}_n(x,y)\|^p \leq D_{\mathrm{III}}^p \big(|y|^{pk} |x - b_j|^{-p/2} e^{-c'pn|y|} + \delta^{kp} \big).$$

And then because $\Omega_{b_j,\mathrm{III}} \subset \big\{ z \mid b_j + m^{-1} N^2/n^2 \leq x \leq b_j + \epsilon, \, |y| \leq \epsilon \big\}$ for some $m > 0$, we have, using Lemma C.5,

$$\int_{\Omega_{b_j,\mathrm{III}}} \|\mathbf{X}_n(x,y)\|^p \, \mathrm{d}x \, \mathrm{d}y \leq 2 D_{\mathrm{III}}^p \int_{b_j+m^{-1}N^2/n^2}^{b_j+\epsilon} \int_0^\epsilon y^{pk} e^{-c'pny} |x - b_j|^{-p/2} \, \mathrm{d}x \, \mathrm{d}y + O(\delta^{kp}).$$

So, we set $x' = x - b_j$, $y' = c'ny$ giving

$$\int_{\Omega_{b_j,\mathrm{III}}} \|\mathbf{X}_n(x,y)\|^p \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq 2 D_{\mathrm{III}}^p (c'pn)^{-pk-1} \left[\int_0^\infty y^{pk} e^{-y} \, \mathrm{d}y \right] \frac{1}{1-p/2} \left. x^{1-p/2} \right|_{m^{-1}N^2/n^2}^{\epsilon} + O(\delta^{kp}).$$

We are left with, by possibly increasing D_{III},

$$\int_{\Omega_{b_j,\mathrm{III}}} \|\mathbf{X}_n(x,y)\|^p \, \mathrm{d}x \, \mathrm{d}y \leq D_{\mathrm{III}}^p n^{p(1-k)-3} N^{2-p} + O(\delta^{kp}).$$

To set N, we set

$$n^{p(1-k)-3} N^{2-p} = N^{3p/2+kp+2} n^{-4-2kp}, \qquad N^{3p/2+(k+1)p} = n^{1+(k+1)p},$$

$$N = n^{\frac{1+(k+1)p}{3p/2+(k+1)p}}.$$

We note that this is a valid choice for N because, for $p > 2$,

$$\frac{1+(k+1)p}{3p/2+(k+1)p} < 1.$$

Then for $\Sigma = \Omega \setminus \bigcup_j (B_\epsilon(a_j) \cup B_\epsilon(b_j))$, we use (2.2), the formula for $\mathbf{W}_n$ and the boundedness of $\mathbf{G}_n$ to conclude

$$\int_\Sigma \|\mathbf{X}_n(x,y)\|^p \, \mathrm{d}x \, \mathrm{d}y \leq D \int_0^\epsilon y^{pk} e^{-npc'y} \, \mathrm{d}y \leq D' n^{-pk-1}. \qquad \blacksquare$$

Proposition 2.4. *Suppose $p > 2$ and $p(k - 2|\alpha_j|) + 2 > 0$, $p(k - 2|\beta_j|) + 2 > 0$ for all j. Suppose also that $\mathbf{V}_n(x,y) \in L^\infty(\mathbb{C})$ satisfies*

$$\mathbf{V}_n(x,y) - \frac{1}{\pi} \int_\Omega \frac{\mathbf{V}_n(x',y') \mathbf{X}_n(x',y')}{z' - z} \, \mathrm{d}A(z') = \frac{1}{\pi} \int_\Omega \frac{\mathbf{X}_n(x',y')}{z' - z} \, \mathrm{d}A(z'). \tag{2.3}$$

Then $\mathbf{V}_n(x,y) + \mathbf{I}$ is a solution of the $\overline{\partial}$ problem solved by $\mathbf{C}_n(x,y)$. Lastly, $\mathbf{V}_n$ is α-Hölder continuous for some $0 < \alpha < 1$ and $\mathbf{V}_n|_\Omega$ is differentiable almost everywhere.

Proof. We use [3, Theorem 4.3.10] which states that, in a distributional sense, the integral operator

$$\mathcal{K}_\Omega u(x,y) = \frac{1}{\pi} \int_\Omega \frac{u(x',y')}{z'-z} \, \mathrm{d}A(z')$$

is inverse to $\overline{\partial}$, provided $u \in L^2(\mathbb{C})$. The condition imposed on k implies that $\mathbf{X}_n \in L^2(\mathbb{C})$. So, if $\mathbf{V}_n \in L^\infty(\mathbb{C})$, $\mathbf{V}_n \mathbf{X}_n \in L^2(\mathbb{C})$ so that $\overline{\partial} \mathbf{V}_n = \mathbf{V}_n \mathbf{X}_n + \mathbf{X}_n$, $\overline{\partial}(\mathbf{V}_n + \mathbf{I}) = (\mathbf{V}_n + \mathbf{I})\mathbf{X}_n$. Next, from [3, Theorem 4.3.13], we have that $\mathcal{K}_\Omega$ maps $L^p(\mathbb{C})$ into the space of α-Hölder continuous functions for $\alpha = 1 - 2/p$, $p > 2$. And we again use the relation $\mathbf{V}_n = \mathcal{K}_\Omega(\mathbf{V}_n \mathbf{X}_n) + \mathcal{K}_\Omega \mathbf{X}_n$, to get the desired conclusion because $\mathbf{X}_n \in L^p(\Omega)$. The claim about $\mathbf{V}_n|_\Omega$ follows from the elliptic regularity theorem, see [23, Theorem 9.26], since $\mathbf{V}_n|_\Omega \in L^\infty(\Omega)$. ∎

We also see that for any $p > 2 > q, 1/p + 1/q = 1$, and $\mathbf{U} \in L^\infty(\mathbb{C})$,

$$\|\mathbf{K}_n\|_{L^\infty(\mathbb{C})} \leq C\|\mathbf{U}\|_{L^\infty(\mathbb{C})}\|\mathbf{X}_n\|_{L^p(\Omega)} \left(\sup_{z \in \Omega} \int_\Omega \frac{\mathrm{d}A(z')}{\pi|z'-z|^q}\right)^{1/q},$$

$$\mathbf{K}_n(x,y) = \frac{1}{\pi} \int_\Omega \frac{\mathbf{U}(x',y')\mathbf{X}_n(x',y')}{z'-z} \, \mathrm{d}A(z').$$

We arrive at our main theorem concerning the existence of a solution $\mathbf{C}_n(x,y)$.

Theorem 2.5. *Fix $k \geq 1$ and set $\eta = 2\max_j \max\{|\alpha_j|, |\beta_j|\}$. Suppose $2(\eta - k) - 1 < 0$. Then for every $\gamma > 0$, and n sufficiently large, there is a unique $L^\infty(\mathbb{C})$ solution of (2.3) that satisfies*

$$\|\mathbf{V}_n\|_{L^\infty(\mathbb{C})} = \mathrm{O}\big(n^{2(\eta-k)-1+\gamma} + \delta^k\big),$$

which is also Hölder continuous on $\mathbb{C}$. Furthermore, $\mathbf{V}_n + \mathbf{I}$ solves the $\overline{\partial}$ problem for $\mathbf{C}_n$ satisfying the pointwise estimate

$$|\mathbf{V}_n(x,y)| = \mathrm{O}\left(\frac{\|\mathbf{V}_n\|_{L^\infty(\mathbb{C})}}{1+|z|}\right), \qquad z \in \mathbb{C}.$$

Proof. By choosing $p > 2$ in Lemma 2.3 sufficiently close to 2, the theorem follows by Proposition 2.4. ∎

This implies that $\|\mathbf{C}_n(x,y)\|$ and $\|\mathbf{C}_n(x,y)^{-1}\|$ are uniformly bounded for n sufficiently large. So, set $\tilde{\mathbf{R}}_n(x,y) = \mathbf{R}_n(x,y)\mathbf{C}_n(x,y)^{-1}$. It then follows that, for some $c > 0$, $\tilde{\mathbf{R}}_n(x,y)$ should solve the RH problem

$$\tilde{\mathbf{R}}_n^+(z) = \tilde{\mathbf{R}}_n^-(z) \begin{cases} \mathbf{I} + \mathrm{O}(e^{-cn}), & z \in \Sigma_{j,+} \cup \Sigma_{j,-}, \ |z-b_j| > \epsilon, \ |z-a_j| > \epsilon, \\ \mathbf{I} + \mathrm{O}\big(n^{-1} + \delta^k\big), & |z-c| = \epsilon, \ c = a_j, b_j, \\ \mathbf{I} + \mathrm{O}\big(n^{-1}\delta^{-1}\big), & z \in (b_j - \epsilon, b_j - \delta) \cup (a_j + \delta, a_j + \epsilon), \end{cases}$$

$$\tilde{\mathbf{R}}_n(z) = \mathbf{I} + \mathrm{O}\big(z^{-1}\big), \qquad z \to \infty,$$

where the jump condition error terms are uniform in z. Standard theory, see [12], for example, gives estimates on $\tilde{\mathbf{R}}_n$. We find the following.

Theorem 2.6. *Suppose $h_j \in C^{k,1}([a_j, b_j])$ for each j. For $k \geq 1$ such that $2(\eta - k) < 1$, $\eta = 2\max_j \max\{|\alpha_j|, |\beta_j|\}$ and any fixed $\gamma > 0$, sufficiently small, set*

$$\mathbf{D}_j^\flat(z; c) = \begin{bmatrix} d_{j,+}^\sharp(x,y;c) + 1 & 0 \\ 0 & d_{j,-}^\sharp(x,y;c) + 1 \end{bmatrix},$$

$$\mathbf{D}^\sharp(x,y) = \begin{cases} \mathbf{D}_j^\sharp(z;a_j)^{-1}\mathbf{D}_j^\sharp(z;b_j)^{-1}, & z \in \Omega_{j,\pm}, \\ \mathbf{I}, & \text{otherwise}, \end{cases}$$

$$\mathbf{F}^\sharp(x,y) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \pm f_j^\sharp(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,\pm}, \\ \mathbf{I}, & \text{otherwise}. \end{cases}$$

Then for $n^{-1} \ll \delta \ll 1$,

$$\mathbf{Y}_n(z;\mu) = \mathfrak{c}^{-n\boldsymbol{\sigma}_3}\mathbf{T}_n(x,y)\mathbf{D}^\sharp(x,y)\mathrm{e}^{n\mathfrak{g}(z)\boldsymbol{\sigma}_3}\mathbf{F}^\sharp(x,y),$$

$$\mathbf{T}_n(x,y) = \tilde{\mathbf{R}}_n(z)(\mathbf{V}_n(x,y) + \mathbf{I}) \begin{cases} \tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1}\mathbf{P}_n(z;b_j), & |z - b_j| < \epsilon, \\ \tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1}\mathbf{P}_n(z;a_j), & |z - a_j| < \epsilon, \\ \mathbf{G}_n(z), & \text{otherwise}, \end{cases}$$

where

$$\tilde{\mathbf{R}}_n(z) = \mathbf{I} + \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\delta} \frac{\mathbf{O}_n(z')}{z' - z}\, \mathrm{d}z', \qquad \|\mathbf{O}_n\|_{L^2(\Gamma_\delta)} = \mathrm{O}\big(n^{-1} + n^{-1}\delta^{-1} + \delta^k\big),$$

$$\mathbf{V}_n(x,y) = \mathrm{O}\left(\frac{n^{-k+\gamma} + n^{2(\eta-k)-1+\gamma} + \delta^k}{1 + |z|}\right),$$

and therefore

$$\tilde{\mathbf{R}}_n(z)(\mathbf{V}_n(x,y) + \mathbf{I}) = \mathbf{I} + \mathrm{O}\left(\frac{n^{-k+\gamma} + n^{2(\eta-k)-1+\gamma} + n^{-1}\delta^{-1} + \delta^k}{1 + |z|}\right),$$

$$\tilde{\mathbf{G}}_n(z)\mathbf{G}_n(z)^{-1} = \mathbf{I} + \mathrm{O}\big(\delta^k\big), \qquad \mathbf{D}^\sharp(x,y) = \mathbf{I} + \mathrm{O}\big(\delta^k\big),$$

uniformly for z in sets bounded away from Γ_δ, where

$$\Gamma_\delta = \bigcup_j \left(\{z \mid |z - a_j| = \epsilon\} \cup \{z \mid |z - b_j| = \epsilon\} \cup (b_j - \epsilon, b_j - \delta) \cup (a_j + \delta, a_j + \epsilon)\right).$$

Proof. Let $\tilde{\mathbf{Y}}_n(z;\mu) = \mathfrak{c}^{-n\boldsymbol{\sigma}_3}\mathbf{T}_n(x,y)\mathbf{D}^\sharp(x,y)\mathrm{e}^{n\mathfrak{g}(z)\boldsymbol{\sigma}_3}\mathbf{F}^\sharp(x,y)$. All the estimates have been previously discussed and the remaining issue is the equality — that $\tilde{\mathbf{Y}}_n(z;\mu) = \mathbf{Y}_n(z;\mu)$. It can be shown that the jump conditions for $\tilde{\mathbf{R}}_n(z)$ satisfy the so-called product condition [37, Definition 2.55] and derivatives of the jump matrix are essentially bounded. Thus $\mathbf{O}_n \in H_z^1(\Gamma_\delta)$, [37, Definition 2.48], an appropriate Sobolev space to ensure that $\tilde{\mathbf{R}}_n(z)$, as defined, is uniformly α-Hölder continuous and takes α-Hölder continuous boundary values for some $\alpha > 0$. And $\tilde{\mathbf{Y}}_n(z;\mu)$ satisfies the jump conditions set for the FIK RH problem, in a continuous sense (away from a_j, b_j).

Next, we need to verify analyticity. But from classical elliptic regularity, [23, Theorem 9.26], we have that $\tilde{\mathbf{Y}}_n(\cdot;\mu) \in C^\infty(\Omega')$ for every open set Ω' that does not intersect any one of the contours used in the deformation, i.e., for

$$\Omega' \subset \mathbb{C} \setminus \Sigma, \qquad \Sigma := \bigcup_j (\Sigma_{j,-} \cup \Sigma_{j,+} \cup [a_j, b_j] \cup \{z \mid |z - a_j| = \epsilon\} \cup \{z \mid |z - b_j| = \epsilon\}).$$

Furthermore, $\overline{\partial}\tilde{\mathbf{Y}}_n = 0$ in such a set Ω', implying analyticity.

Lastly, from the singularity structure of $\mathbf{P}_{\text{Bessel}}$ it follows that $\tilde{\mathbf{Y}}_n$ has the same singularity orders as $\mathbf{Y}_n$ and therefore $\tilde{\mathbf{Y}}_n = \mathbf{Y}_n$. ∎

Effective use of this theorem then requires choosing δ. If asymptotics are desired away from the endpoints a_j, b_j, one can choose k so that $n^{-1}\delta^{-1} = \delta^k$, $\delta = n^{-1/(k+1)}$. For $2(\eta - k) \leq 0$, this will give errors of $\mathrm{O}\big(n^{-k/(k+1)}\big)$. On the other hand, if one wants asymptotics near $z = a_j$, for example, care has to be taken in estimating $\tilde{\mathbf{R}}_n(z)$. We write Γ_δ as a disjoint union

$$\Gamma_\delta = (a_j + \delta, a_j + \epsilon) \cup \Gamma_\delta^o,$$

and find the entry-wise estimate for $z \in B_{\delta'}(z_j)$, $\delta' \ll \delta$,

$$\left|\big(\tilde{\mathbf{R}}_n(z) - \mathbf{I}\big)_{ij}\right| \leq \|\mathbf{O}_n\|_{L^2(\Gamma_\delta)} \left(\int_{a_j+\delta}^{a_j+\epsilon} \frac{\mathrm{d}x}{|x-z|^2}\right)^{1/2} + \mathrm{O}(\|\mathbf{O}_n\|_{L^2(\Gamma_\delta)})$$
$$= \mathrm{O}\big(\delta^{-1/2}\|\mathbf{O}_n\|_{L^2(\Gamma_\delta)}\big).$$

So, if $n^{-1}\delta^{-3/2} \ll 1$, or $\delta = n^{-2/3+\gamma}$, and $2(\eta - k) + 1/3 < 0$, we obtain a valid error term giving uniform asymptotics on scaled neighborhoods of the endpoints a_j, b_j. Furthermore, by taking k sufficiently large, $\gamma < 2/3$ here can be chosen arbitrarily close to $2/3$.

3 Extensions and improvements

In this section, we discuss two topics. The first is the improvement in the estimates for the $\alpha_j, \beta_j \in \{1/2, -1/2\}$, the perturbed Chebyshev-like case. The second is the addition of point masses to the measure μ, i.e., $P > 0$.

3.1 Chebyshev-like polynomials on multiple intervals

So, suppose $\alpha_j, \beta_j \in \{1/2, -1/2\}$. In the case where $f_j = 1/h_j$ is analytic in a neighborhood of $[a_j, b_j]$ for every j, local parametrices are not needed [27] (see also [17]). To treat the case of non-analytic f_j using this fact, one needs to find an extension of f_j to an entire neighborhood of $[a_j, b_j]$ to appropriately pose a hybrid $\bar{\partial}$-RH problem. This was accomplished in [41]. While we follow this idea, we use a slightly different method of extension, since we have already developed our extension operator $\sharp$. First, we recall that for $\epsilon > 0$, $f_j \in W^{k,p}([a_j, b_j])$ has a $W^{k,p}([a_j - 2\epsilon, b_j + 2\epsilon])$ extension. Similarly, $f_j \in C^{k,\alpha}([a_j, b_j])$ has a $C^{k,\alpha}([a_j - 2\epsilon, b_j + 2\epsilon])$ extension. This can be constructed simply by polynomial interpolation of the function and its first k derivatives at the endpoints — the Taylor polynomial. For what follows, we will assume $f_j \in C^{k,\alpha}([a_j, b_j])$, $k \geq 1$ and any $\alpha > 0$ or $f_j \in W^{k,q}([a_j, b_j])$, $k \geq 1$ and $q > 4$. Then, we find a sequence of polynomials $(p'_{m,j})_{m\geq 1}$ of degree m such that

$$E_{j,m,q} := \|f'_j - p'_{m,j}\|_{L^q([a_j - 2\epsilon, b_j + 2\epsilon])} \to 0, \qquad q > 6.$$

From Jackson's theorem, see [4, Theorem 3.7.2], we can select the sequence such that $E_{j,m,\infty} = \mathrm{O}\big(m^{-k+1-\alpha}\big)$ if $f \in C^{k,\alpha}([a_j, b_j])$, but more refined estimates are possible for $W^{k,q}([a_j, b_j])$ [10, 20]. Specifically, if $f_j \in W^{k,p}([a_j, b_j])$, we can select the sequence such that [10, (5.4.16)]

$$E_{j,m,q} \leq Cm^{-k+1} \sum_{\ell=\min\{k-1,m+1\}}^{k-1} \|f_j^{(\ell+1)}\|_{L^q([a_j - 2\epsilon, b_j + 2\epsilon])}$$
$$\leq C'm^{-k+1}\|f_j\|_{W^{k,p}([a_j - 2\epsilon, b_j + 2\epsilon])}.$$

Set, for example, $p_{m,j} = f_j(a_j) + \int_{a_j}^x p'_{m,j}(x')\,\mathrm{d}x'$. Then, define $r_{m,j} = f_j - p_{m,j}$, and its extension $r^\sharp_{m,j}(x, y)$ with $\kappa = 0$ and $\tau = \epsilon/2$. Then the extension of f_j is given by

$$f_j^\ddagger(x, y) = r^\sharp_{m,j}(x, y) + p_{m,j}(z).$$

Note that for $z \in \Omega_j' := \{z \mid \epsilon/2 - a_j < x < b_j + \epsilon/2, \, 0 < |y| < \epsilon/2\}$, we simply have

$$f_j^{\ddagger}(x,y) = r_{m,j}(x) + p_{m,j}(z).$$

In our deformations, we reuse the analogous notation from the previous sections and trust it will not cause too much confusion. Define $\Omega_{j,\pm}' := \Omega_j' \cap \mathbb{C}^{\pm}$ and

$$\mathbf{Z}_n(x,y) = \mathbf{Y}_n(z;\mu) \begin{cases} \begin{bmatrix} 1 & 0 \\ -f_j^{\ddagger}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,+}', \\[2ex] \begin{bmatrix} 1 & 0 \\ f_j^{\ddagger}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,-}', \\[2ex] \mathbf{I}, & \text{otherwise.} \end{cases}$$

So that

$$\overline{\partial}\mathbf{Z}_n(x,y) = \begin{cases} \mathbf{Z}_n(z;\mu) \begin{bmatrix} 0 & 0 \\ -\overline{\partial}r_{m,j}^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,+}', \\[2ex] \mathbf{Z}_n(z;\mu) \begin{bmatrix} 0 & 0 \\ \overline{\partial}r_{m,j}^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,-}', \\[2ex] \mathbf{0}, & z \in \mathbb{C} \setminus \overline{\Omega}_j'. \end{cases}$$

Set

$$\Sigma_{j,+}' = \{z \mid x = a_j - \epsilon/2, \, 0 \le y \le \epsilon/2\}$$
$$= \cup\{z \mid y = \epsilon/2, \, a_j - \epsilon/2 \le x \le b_j + \epsilon/2\} \cup \{z \mid x = b_j + \epsilon/2, \, 0 \le y \le \epsilon/2\},$$

i.e., the "top" boundary of the box Ω_j'. We give this negative orientation (as the boundary of Ω_j'). And set $\Sigma_{j,-}' = \{z \mid \bar{z} \in \Sigma_{j,+}'\}$, inheriting orientation.

It follows that the matrix used in the definition of $\mathbf{Z}_n$ is continuous across the real axis for $a_j - \epsilon/2 < x < a_j$ and $b_j < x < b_j + \epsilon/2$ [27] giving the resulting jump conditions

$$\mathbf{Z}_n^+(x,y) = \mathbf{Z}_n^-(x,y) \begin{cases} \begin{bmatrix} 1 & 0 \\ f_j^{\ddagger}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Sigma_{j,-}' \cup \Sigma_{j,+}', \\[2ex] \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix}, & z \in [a_j, b_j]. \end{cases}$$

As before, define $\mathbf{T}_n(x,y) = \mathfrak{c}^{n\boldsymbol{\sigma}_3}\mathbf{Z}_n(x,y)\mathrm{e}^{-n\mathfrak{g}(z)\boldsymbol{\sigma}_3}$. We find that $\mathbf{T}_n(x,y)$ solves the normalized hybrid RH-$\overline{\partial}$ problem

$$\mathbf{T}_n^+(x,y) = \mathbf{T}_n^-(x,y) \begin{cases} \begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}f_j^{\ddagger}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Sigma_{j,-}' \cup \Sigma_{j,+}', \\[2ex] \begin{bmatrix} 0 & \rho(x) \\ -1/\rho(x) & 0 \end{bmatrix}, & z \in (a_j, b_j), \\[2ex] \mathrm{e}^{-\Delta_j\boldsymbol{\sigma}_3}, & z \in (b_j, a_{j+1}), \end{cases}$$

$$\overline{\partial}\mathbf{T}_n(x,y) = \mathbf{T}_n(z) \begin{cases} \begin{bmatrix} 0 & 0 \\ -\mathrm{e}^{-2n\mathfrak{g}(z)}\overline{\partial}r_{m,j}^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,+}', \\[2ex] \begin{bmatrix} 0 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}\overline{\partial}r_{m,j}^{\sharp}(x,y)/\omega_j(z) & 0 \end{bmatrix}, & z \in \Omega_{j,-}', \\[2ex] \mathbf{0}, & z \in \mathbb{C} \setminus \overline{\Omega}_j', \end{cases}$$

$$\mathbf{T}_n(x,y) = \mathbf{I} + \mathrm{O}\big(z^{-1}\big), \qquad z \to \infty.$$

With the notation for $\mathbf{G}_n(z)$ being precisely the same, we set $\mathbf{R}_n(x,y) = \mathbf{T}_n(x,y)\mathbf{G}_n(z)^{-1}$, and find that $\mathbf{R}_n(x,y)$ solves

$$\mathbf{R}_n^+(x,y) = \mathbf{R}_n^-(x,y)\mathbf{G}_n(z)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}f_j^\ddagger(x,y)/\omega_j(z) & 1 \end{bmatrix}\mathbf{G}_n(z)^{-1}, \qquad z \in \Sigma'_{j,-} \cup \Sigma'_{j,+},$$

$$\overline{\partial}\mathbf{R}_n(x,y) = \mathbf{R}_n(z)\begin{cases} \mathbf{G}_n(z)\mathrm{e}^{-\boldsymbol{\sigma}_3 G(z)}\begin{bmatrix} 0 & 0 \\ -\mathrm{e}^{-2n\mathfrak{g}(z)-2G(z)}\overline{\partial}r_{m,j}^\sharp(x,y)/\omega_j(z) & 0 \end{bmatrix} \\ \qquad \times \mathrm{e}^{\boldsymbol{\sigma}_3 G(z)}\mathbf{G}_n(z)^{-1}, \quad z \in \Omega'_{j,+}, \\ \mathbf{G}_n(z)\mathrm{e}^{-\boldsymbol{\sigma}_3 G(z)}\begin{bmatrix} 0 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)-2G(z)}\overline{\partial}r_{m,j}^\sharp(x,y)/\omega_j(z) & 0 \end{bmatrix} \\ \qquad \times \mathrm{e}^{\boldsymbol{\sigma}_3 G(z)}\mathbf{G}_n(z)^{-1}, \qquad z \in \Omega'_{j,-}, \\ \mathbf{0}, \qquad z \in \mathbb{C}\setminus\overline{\Omega'_j}, \end{cases}$$

$$\mathbf{R}_n(x,y) = \mathbf{I} + \mathrm{O}\big(z^{-1}\big), \qquad z \to \infty.$$

As above, we then seek $\mathbf{C}_n(x,y)$ that is continuous on $\mathbb{C}$ and satisfies

$$\overline{\partial}\mathbf{C}_n(x,y) = \mathbf{C}_n(z)\mathbf{X}_n(x,y), \qquad z \in \Omega'_j,$$
$$\mathbf{C}_n(x,y) = \mathbf{I} + \mathrm{O}\big(z^{-1}\big), \qquad z \to \infty,$$

where derivatives are understood to hold in an $L^2(\mathbb{C})$ distributional sense. As above, we seek a solution $\mathbf{C}_n = \mathbf{V}_n + \mathbf{I}$ where $\mathbf{V}_n$ satisfies

$$\mathbf{V}_n(x,y) - \frac{1}{\pi}\int_{\Omega'}\frac{\mathbf{V}_n(x',y')\mathbf{X}_n(x',y')}{z'-z}\,\mathrm{d}A(z') = \frac{1}{\pi}\int_{\Omega'}\frac{\mathbf{X}_n(x',y')}{z'-z}\,\mathrm{d}A(z'), \qquad \Omega' = \bigcup_j \Omega'_j.$$

Define the operator $\mathcal{L}_n\mathbf{U} := \mathcal{K}_{\Omega'}(\mathbf{U}\mathbf{X}_n)$, and the companion operator

$$\underline{\mathcal{K}}_{\Omega'}u(z) = \frac{1}{\pi}\int_{\Omega'}\frac{u(x',y')}{|z'-z|}\,\mathrm{d}A(z').$$

Below, we will establish conditions under which $\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})} \to 0$. This will then imply, via a standard Neumann series argument, that $\mathbf{C}_n(x,y)$ and $\mathbf{C}_n(x,y)^{-1}$ are uniformly bounded, for n sufficiently large, and tend uniformly to the identity matrix.

Next, we make a similar observation to what was made in [41]. The Bernstein–Walsh inequality, see [33, Theorem III.2.1], in its simplest form, states that if a polynomial $p(x)$, of degree n, satisfies $\max_{-1\le x\le 1}|p(x)| \le C$, then

$$|p(z)| \le C|\phi(z)|^n, \qquad z \in \mathbb{C}, \qquad \phi(z) = z + \sqrt{z-1}\sqrt{z+1}$$

Therefore, if a polynomial $p(x)$, of degree n, satisfies $\max_{a\le x\le b}|p(x)| \le C$, then

$$|p(z)| \le C|\phi(M(z))|^n, \qquad z \in \mathbb{C}, \qquad M(z) = M(z;a,b) = \frac{2}{b-a}\left(x - \frac{b+a}{2}\right). \qquad (3.1)$$

Thus, for the sequence of polynomials $p_{m,j}(x)$ that converge uniformly on $[a_j,b_j]$, we have $|p_{m,j}(x)| \le C$ for some constant, independent of m (and j). From this, and Proposition A.1, provided that ϵ is sufficiently small we have that

$$|p_{m,j}(z)|\big|\mathrm{e}^{-2n\mathfrak{g}(z)}\big| \le C|\phi(M(z;a_j,b_j))|^{-2cn+m}, \qquad (3.2)$$

for some $c > 0$. So, we choose, for example $m = \lfloor cn\rfloor$ so that (3.2) is $\mathrm{O}(\mathrm{e}^{-cn})$.

Now, as before, set $\tilde{\mathbf{R}}_n(x,y) = \mathbf{R}_n(x,y)\mathbf{C}_n(x,y)^{-1}$. It then follows that $\tilde{\mathbf{R}}_n(x,y)$ should solve the RH problem

$$\tilde{\mathbf{R}}_n^+(z) = \tilde{\mathbf{R}}_n^-(z)(\mathbf{I} + \mathrm{O}(\mathrm{e}^{-cn})), \qquad z \in \Sigma', \qquad \Sigma' = \bigcup_j (\Sigma'_{j,-} \cup \Sigma'_{j,+})$$

$$\tilde{\mathbf{R}}_n(z) = \mathbf{I} + \mathrm{O}(z^{-1}), \qquad z \to \infty,$$

where the jump condition error terms are uniform in z. Standard theory, see [12], for example, again gives estimates on $\tilde{\mathbf{R}}_n$, and we find the following.

Theorem 3.1. *For $k \geq 1$, $q > 6$ and[6] ϵ sufficiently small set*

$$\mathbf{F}^\natural(x,y) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \pm f_j^\ddagger(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega'_{j,\pm}, \\ \mathbf{I}, & \text{otherwise}, \end{cases}$$

and suppose $\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})} = o(1)$, $n \to \infty$. Then for n sufficiently large,

$$\mathbf{Y}_n(z;\mu) = \mathfrak{c}^{-n\boldsymbol{\sigma}_3}\mathbf{T}_n(x,y)\mathrm{e}^{n\mathfrak{g}(z)\boldsymbol{\sigma}_3}\mathbf{F}^\natural(x,y),$$
$$\mathbf{T}_n(x,y) = \tilde{\mathbf{R}}_n(z)(\mathbf{V}_n(x,y) + \mathbf{I})\mathbf{G}_n(z),$$

where

$$\tilde{\mathbf{R}}_n(z) = \mathbf{I} + \frac{1}{2\pi\mathrm{i}} \int_{\Sigma'} \frac{\mathbf{O}_n(z')}{z' - z}\,\mathrm{d}z', \qquad \|\mathbf{O}_n\|_{L^2(\Sigma')} = \mathrm{O}(\mathrm{e}^{-cn}),$$

and

$$\mathbf{V}_n(x,y) = \mathrm{O}\left(\frac{\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})}}{1 + |z|}\right), \qquad z \in \mathbb{C}, \qquad m = \lfloor cn \rfloor.$$

Proof. The only remaining detail to spell out is the estimate on $\mathbf{V}_n$. First, it follows that

$$\|\mathcal{L}_n\|_{L^\infty(\Omega')} \leq \|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})}.$$

This implies that the integral equation $\mathbf{V}_n - \mathcal{L}_n\mathbf{V}_n = \mathcal{K}\mathbf{X}_n$ is near-identity. For n sufficiently large, it has a unique solution, tending to zero as $n \to \infty$, and

$$\|\mathbf{V}_n\| \leq \|\mathcal{L}_n\mathbf{V}_n\| + \|\underline{\mathcal{K}}_{\Omega'}\mathbf{X}_n\| \leq (1 + \|\mathbf{V}_n\|_{L^\infty(\Omega')})\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})}. \qquad \blacksquare$$

To finalize the preceding theorem, we have the following estimates.

Lemma 3.2 (norm bounds). *For $p > 2$, $q > 6$, and $1/p = 1/q + 1/r$, and $m = \lfloor cn \rfloor$,*

$$\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})} = \mathrm{O}\left(n^{-1/r} \max_j E_{j,m,q}\right).$$

Furthermore,

$$\|\underline{\mathcal{K}}_{\Omega'}(\|\mathbf{X}_n\|)\|_{L^\infty(\mathbb{C})} = \mathrm{O}\left(n^{-1} \log n \max_j E_{j,m,\infty}\right).$$

---

[6]Recall that ϵ appears in the definition of $\Sigma'_{j,\pm}$.

Proof. Importantly, we note that by Lemma C.3, $e^{-2G(z)}/\omega_j(z)$ is bounded in a neighborhood of $[a_j, b_j]$. From the (at most) quarter-root singularities of $\mathbf{G}_n(z)e^{-\boldsymbol{\sigma}_3 G(z)}$, we find for $z \in \Omega'_j$ that

$$\mathbf{X}_n(x,y) = \mathbf{G}_n(z)e^{-\boldsymbol{\sigma}_3 G(z)} \begin{bmatrix} 0 & 0 \\ -e^{-2n\mathfrak{g}(z)-2G(z)}\overline{\partial}r^{\sharp}_{m,j}(x,y)/\omega_j(z) & 0 \end{bmatrix} e^{\boldsymbol{\sigma}_3 G(z)}\mathbf{G}_n(z)^{-1}$$

satisfies

$$\|\mathbf{X}_n(x,y)\| \leq C\big|\overline{\partial}r^{\sharp}_{m,j}(x,y)\big||z-a_j|^{-1/2}|z-b_j|^{-1/2}e^{-2n\,\mathrm{Re}\,\mathfrak{g}(z)}.$$

With the aim of using (E.1), we have

$$\|\mathbf{X}_n\|_{L^p(\Omega'_j)} \leq C\big\|\overline{\partial}r^{\sharp}_{m,j}\big\|_{L^q(\Omega'_j)}\|h_{j,n}\|_{L^r(\Omega'_j)},$$

$$h_{j,n}(z) := |z-a_j|^{-1/2}|z-b_j|^{-1/2}e^{-2n\,\mathrm{Re}\,\mathfrak{g}(z)},$$

for $1 \leq p \leq 4$, and $1/p = 1/q + 1/r$. Since we will need $p > 2$ below, and using (E.1) requires $\gamma < 3/2$, we choose $r < 3$, and $q > 6$, giving $\|\mathbf{X}_n\|_{L^p(\Omega'_j)} = O\big(n^{-1/r}E_{j,m,q}\big)$. The first claim follows from the fact that $\|\underline{\mathcal{K}}_{\Omega'}u\|_{L^\infty(\mathbb{C})} \leq C_p\|u\|_{L^p(\Omega')}$ for some $C_p > 0$, provided $p > 2$.

To establish the second claim, we need to establish the estimate $\underline{\mathcal{K}}_{\Omega'}h_{j,n}(z) = O\big(n^{-1}\log n\big)$. This essentially follows from the proof of [41, Lemma 4] once we note that, after the change of variable $z' = 1/2(\tau + 1/\tau)$, there is a helpful factor of $\big|\tau^2 - 1\big|$ that can be used to cancel the singularity in

$$\big|z^2 - 1\big|^{-1/2} = \frac{2\tau}{\big|\tau^2 - 1\big|}. \qquad \blacksquare$$

3.2 The addition of a finite number of point masses

The addition of point masses follows a simple procedure, see, for example, [17]. Diagonal and triangular rational modifications are made to the original RH problem, with the second applying only in a neighborhood of each of the point masses, converting a residue condition (1.3) into a rational jump condition. The effect of this is a rational modification of the densities on $[a_j, b_j]$ for every j, a change in the asymptotics at infinity and the introduction of a jump condition on a small curve encircling the point masses. Once the g-function is introduced, this jump condition becomes exponentially close to the identity. We sketch this procedure here but refer the reader to [17] for more detail.

Define

$$\hat{\mathbf{Y}}_n(z) = \mathbf{Y}_n(z;\mu)\begin{bmatrix} \prod_{j=1}^P(z-c_j)^{-1} & 0 \\ 0 & \prod_{j=1}^P(z-c_j) \end{bmatrix}.$$

Then let Ξ_j be a small neighborhood of c_j. Define

$$\check{\mathbf{Y}}_n(z) = \hat{\mathbf{Y}}_n(z)\begin{cases} \begin{bmatrix} 1 & 0 \\ -\frac{\tilde{r}_j}{z-c_j} & 1 \end{bmatrix}, & z \in \Xi_j, \\ \mathbf{I}, & \text{otherwise,} \end{cases}$$

where $\tilde{r}_j$ is defined as

$$\tilde{r}_j := \frac{2\pi i}{r_j}\prod_{k\neq j}(c_j - c_k)^{-2}.$$

The RH problem satisfied by $\check{\mathbf{Y}}_n$ is found by modifying the RH problem satisfied by $\mathbf{Y}_n$ in three ways:

- Replace $\rho(x)$ with

$$\rho(x)\prod_{j=1}^{P}(x-c_j)^2. \tag{3.3}$$

- Include the jump

$$\check{\mathbf{Y}}_n^+(z) = \check{\mathbf{Y}}_n^-(z)\begin{bmatrix} 1 & 0 \\ -\frac{\tilde{r}_j}{z-c_j} & 1 \end{bmatrix}$$

on the positively-oriented boundary of Ξ_j.

- Replace the asymptotics at infinity with $\check{\mathbf{Y}}_n(z)z^{-(n-P)\sigma_3} = \mathbf{I} + \mathrm{O}(z^{-1})$, $z \to \infty$.

Then Theorem 2.6 holds if one makes the new definition

$$\mathbf{F}^{\natural}(x,y) = \mathbf{H}^{\natural}(x,y)\begin{bmatrix} \prod_{j=1}^{P}(z-c_j) & 0 \\ 0 & \prod_{j=1}^{P}(z-c_j)^{-1} \end{bmatrix},$$

$$\mathbf{H}^{\natural}(x,y) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \pm f_j^{\natural}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega_{j,\pm}, \\ \begin{bmatrix} 1 & 0 \\ \frac{\tilde{r}_j}{z-c_j} & 1 \end{bmatrix}, & z \in \Xi_j, \\ \mathbf{I}, & \text{otherwise,} \end{cases}$$

ρ is replaced with (3.3) and n is replaced with $n-P$. Similarly, Theorem 3.1 holds if one makes the new definition

$$\mathbf{F}^{\sharp}(x,y) = \mathbf{H}^{\natural}(x,y)\begin{bmatrix} \prod_{j=1}^{P}(z-c_j) & 0 \\ 0 & \prod_{j=1}^{P}(z-c_j)^{-1} \end{bmatrix},$$

$$\mathbf{H}^{\natural}(x,y) = \begin{cases} \begin{bmatrix} 1 & 0 \\ \pm f_j^{\ddagger}(x,y)/\omega_j(z) & 1 \end{bmatrix}, & z \in \Omega'_{j,\pm}, \\ \begin{bmatrix} 1 & 0 \\ \frac{\tilde{r}_j}{z-c_j} & 1 \end{bmatrix}, & z \in \Xi_j, \\ \mathbf{I}, & \text{otherwise,} \end{cases}$$

ρ is replaced with (3.3) and n is replaced with $n-P$.

4 Asymptotics of recurrence coefficients and estimating optimal errors

While asymptotics of the polynomials themselves are directly available using the above calculations, we are primarily interested in the asymptotics of the associated recurrence coefficients. The recurrence coefficients $(a_n(\mu))_{n\geq 0}$ and $(b_n(\mu))_{n\geq 0}$, $b_n(\mu) > 0$, are the coefficients in the relation

$$p_{-1}(x;\mu) := 0, \qquad p_0(x;\mu) = 1,$$
$$xp_n(x;\mu) = b_n(\mu)p_{n+1}(x;\mu) + a_n(\mu)p_n(x;\mu) + b_{n-1}(\mu)p_{n-1}(x;\mu).$$

If we write

$$\mathbf{Y}_n(z;\mu) = \big(\mathbf{I} + \mathbf{Y}_n^{(1)}(\mu)z^{-1} + \mathrm{O}(z^{-2})\big)z^{\boldsymbol{\sigma}_3 n}, \qquad \mathbf{Y}_n^{(1)}(\mu) = \begin{bmatrix} y_{11}(n;\mu) & y_{12}(n;\mu) \\ y_{21}(n;\mu) & y_{22}(n;\mu) \end{bmatrix},$$

we have

$$p_n(z;\mu) = \ell_n(\mu)\pi_n(z;\mu) = \ell_n(\mu)\left(z^n + y_{11}(n;\mu)z^{n-1} + \cdots\right),$$

$$c_n(z;\mu) = -\frac{1}{2\pi\mathrm{i}}\ell_n(\mu)^2 z^n + \cdots = y_{12}(n;\mu)z^n + \cdots.$$

Equating coefficients in the recurrence, gives

$$\ell_n(\mu) = b_n(\mu)\ell_{n+1}(\mu), \qquad \ell_n(\mu)y_{11}(n;\mu) = b_n(\mu)\ell_{n+1}(\mu)y_{11}(n+1;\mu) + a_n(\mu)\ell_n(\mu).$$

And therefore

$$b_n(\mu) = \frac{\ell_{n+1}(\mu)}{\ell_n(\mu)}, \qquad b_n(\mu)^2 = \frac{y_{12}(n+1;\mu)}{y_{12}(n;\mu)}, \qquad a_n(\mu) = y_{11}(n;\mu) - y_{11}(n+1;\mu).$$

In Theorem 2.6 or Theorem 3.1, we have as $z \to \infty$

$$\mathbf{T}_n(x,y) = \big(\mathbf{I} + \mathbf{T}_n^{(1)}z^{-1} + \mathrm{O}(z^{-2})\big)\mathbf{G}_n(z), \qquad \mathbf{G}_n(z) = \mathbf{I} + \mathbf{G}_n^{(1)}z^{-1} + \mathrm{O}(z^{-2})$$

for $\mathbf{T}_n^{(1)} \to 0$. Then using that

$$\mathfrak{g}(z) = \log\mathfrak{c} + \log z + \mathfrak{g}_1 z^{-1} + \mathrm{O}(z^{-2}),$$

we have

$$\mathrm{e}^{n\mathfrak{g}(z)\boldsymbol{\sigma}_3} = \mathfrak{c}^{n\boldsymbol{\sigma}_3}z^{n\boldsymbol{\sigma}_3}\big(\mathbf{I} + n\mathfrak{g}_1\boldsymbol{\sigma}_3 z^{-1} + \mathrm{O}(z^{-2})\big),$$

$$\mathbf{Y}_n(z;\mu) = \mathfrak{c}^{-n\boldsymbol{\sigma}_3}\big(\mathbf{I} + \big[\mathbf{G}_n^{(1)} + \mathbf{T}_n^{(1)}\big]z^{-1} + \mathrm{O}(z^{-2})\big)\big(\mathbf{I} + n\mathfrak{g}_1\boldsymbol{\sigma}_3 z^{-1} + \mathrm{O}(z^{-2})\big)\mathfrak{c}^{n\boldsymbol{\sigma}_3}z^{n\boldsymbol{\sigma}_3}$$

$$= \big(\mathbf{I} + \big[\mathfrak{c}^{-n\boldsymbol{\sigma}_3}\big[\mathbf{G}_n^{(1)} + \mathbf{T}_n^{(1)}\big]\mathfrak{c}^{n\boldsymbol{\sigma}_3} + n\mathfrak{g}_1\sigma_3\big]z^{-1} + \mathrm{O}(z^{-2})\big)z^{n\boldsymbol{\sigma}_3}.$$

This gives the expansion

$$\mathbf{Y}_n^{(1)}(\mu) = \mathfrak{c}^{-n\boldsymbol{\sigma}_3}\big[\mathbf{G}_n^{(1)} + \mathbf{T}_n^{(1)}\big]\mathfrak{c}^{n\boldsymbol{\sigma}_3} + n\mathfrak{g}_1\sigma_3.$$

If we set

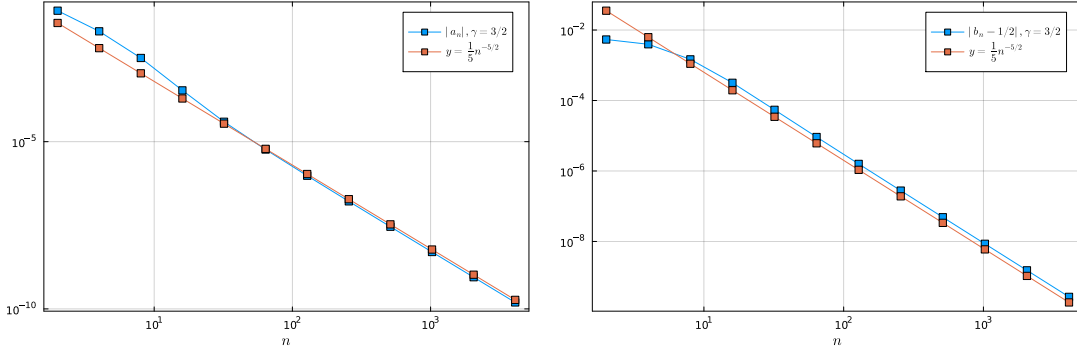$$\mathbf{G}_n^{(1)} = \begin{bmatrix} g_{11}(n;\mu) & g_{12}(n;\mu) \\ g_{21}(n;\mu) & g_{22}(n;\mu) \end{bmatrix},$$

we have

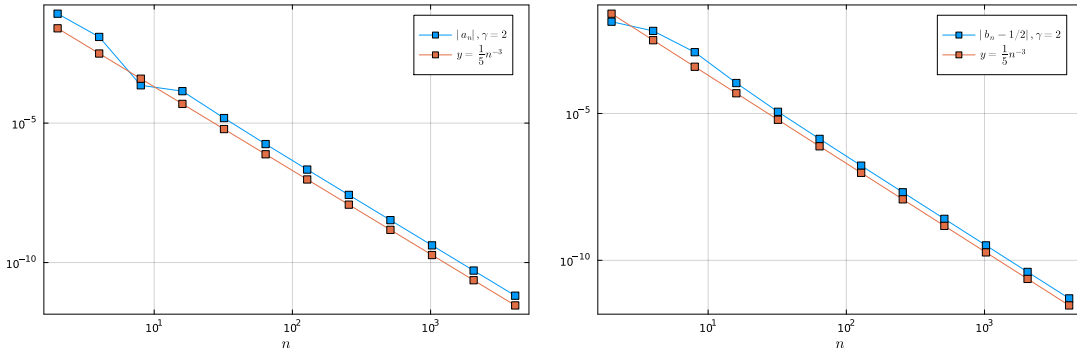$$a_n(\mu) = g_{11}(n;\mu) - g_{11}(n+1;\mu) - \mathfrak{g}_1 + \mathrm{O}\big(\mathbf{T}_n^{(1)}\big),$$

$$b_n(\mu)^2 = \frac{g_{12}(n+1;\mu) + \mathrm{O}\big(\mathbf{T}_n^{(1)}\big)}{g_{12}(n;\mu) + \mathrm{O}\big(\mathbf{T}_n^{(1)}\big)}.$$

To compare our error estimates to the true errors, we consider a single interval $[a_1, b_1] = [-1, 1]$, $g = 0$. We use $\alpha_1 = \beta_1 = 1/2$,

$$h_1(x) = \begin{cases} 1, & -1 \le x \le 0, \\ 1 + x^\gamma(1+x)^{-1/2}, & 0 \le x \le 1. \end{cases}$$

Figure 4. A comparison of the recurrence coefficients when $\gamma = 3/2$. The observed discrepancy between the recurrence coefficients and their known asymptotics appears to be $\mathrm{O}\!\left(n^{-\gamma-1}\right)$.

Figure 5. A comparison of the recurrence coefficients when $\gamma = 2$. The observed discrepancy between the recurrence coefficients and their known asymptotics appears to be $\mathrm{O}\!\left(n^{-\gamma-1}\right)$.

Then, if one uses the Gauss–Jacobi quadrature rules (x_j, w_j), (y_j, v_j) such that

$$\int_{-1}^{1} p(x)\sqrt{1-x^2}\,\mathrm{d}x = \sum_{j=1}^{n} p(x_j)w_j,$$

$$\int_{0}^{1} p(x)x^{\gamma}\sqrt{1-x}\,\mathrm{d}x = \sum_{j=1}^{n} p(y_j)v_j,$$

for all polynomials p of degree less than $2n$, the inner product can be approximated well because

$$\int_{-1}^{1} f(x)\overline{g(x)}\sqrt{1-x^2}h_1(x)\,\mathrm{d}x = \sum_{j=1}^{n} f(x_j)\overline{g(x_j)}w_j + \sum_{j=1}^{n} f(y_j)\overline{g(x_j)}v_j,$$

when f and g are polynomials of degree less than or equal to $n-1$. This implies the recurrence coefficients can be generated exactly (up to roundoff). The ideal algorithm is the RKPW method as outlined in [25] which has $O\!\left(n^2\right)$ complexity. In Figures 4 and 5, we provide evidence that error term in the asymptotics of the recurrence coefficients is $O\!\left(n^{-\gamma-1}\right)$ whereas our method, and that of Yattselev [41], using $q = \infty$ predicts $O\!\left(n^{-\gamma}\log n\right)$. We do note that, while potentially indicative, this density is piecewise analytic. A more exotic density might be needed to see the rate predicted by Yattselev, but running computations in such a case will likely be its own entirely separate challenge.

5 Conclusions and outlook

The main open problem here is to obtain error bounds that reflect the true rates demonstrated in Figures 4 and 5. It is also unknown how close to optimal the rate we have obtained for the general Jacobi-type case is.

It is also of interest to see if these deformations, at least in some way, could be helpful in computing the recurrence coefficients as was done in [6, 35, 36], potentially enabling computations with optimal complexity. We also anticipate using the asymptotics discussed here in perturbation theory à la [17] for random matrices that have anomalous edge/bulk behavior.

A The g-function

In this appendix, we define what is commonly called the external Green's function with pole at infinity [39], or rather the analytic function whose real part is this Green's function, and determine some essential properties. This construction can be also be found in [2]. Specifically, we find a function $\mathfrak{g}$ that satisfies

(a) $\mathfrak{g}'(z) = 1/z + \mathrm{O}\big(1/z^2\big)$ as $z \to \infty$,

(b) $\mathfrak{g}'_+(z), \mathfrak{g}'_-(z) \in i\mathbb{R}$ on $[a_j, b_j]$,

(c) $\int_{b_j}^{a_{j+1}} \mathfrak{g}'(z)\,\mathrm{d}z = 0$, $j = 1, 2, \ldots, g$.

The following construction can be found in [17]. Set[7]

$$\mathfrak{g}'(z) = \frac{Q_g(z)}{R(z)}, \qquad \text{where} \quad R(z)^2 = \prod_{j=1}^{g+1}(z - a_j)(z - b_j), \tag{A.1}$$

where Q_g is a monic polynomial of degree g. The coefficients for Q_g, $Q_g(z) = \sum_\ell \alpha_\ell z^\ell$, are found by solving the linear system, which enforces (c)

$$\int_{b_j}^{a_{j+1}} \sum_{\ell=0}^{g-1} \alpha_k \frac{z^\ell}{R(z)}\mathrm{d}z = -\int_{b_j}^{a_{j+1}} \frac{z^g}{R(z)}\mathrm{d}z, \qquad j = 1, 2, \ldots, g.$$

This system is uniquely solvable due to abstract theory, see Appendix D.

Then set

$$\mathfrak{g}(z) = \int_{a_1}^z \mathfrak{g}'(z')\,\mathrm{d}z', \qquad z \notin [a_1, \infty), \tag{A.2}$$

where the path of integration is taken to be a straight line.

The following properties then hold:

(i) $\alpha_\ell \in \mathbb{R}$ for every ℓ,

(ii) $\mathfrak{g}^+(z) + \mathfrak{g}^-(z) = 0$, $z \in (a_j, b_j)$,

(iii) $\mathfrak{g}^+(z) - \mathfrak{g}^-(z) =: \Delta_j \in i\mathbb{R}$ is constant on $[b_j, a_{j+1}]$, and

(iv) $\operatorname{Re}\mathfrak{g}(z)$ is strictly positive on any closed subset of $\mathbb{R} \setminus \bigcup_j [a_j, b_j]$.

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[7]See Appendix D for the definition of $R(z)$.

We will need to characterize the last property more quantitatively. Consider the function

$$\phi(z) = z + \sqrt{z-1}\sqrt{z+1}.$$

It follows that $\phi(z)$ is a conformal mapping from $\mathbb{C} \setminus [-1, 1]$ to the exterior of the unit disk. It can also be shown that

$$\log |\phi(z)| = \operatorname{Re} \int_{-1}^{z} \frac{\mathrm{d}z'}{\sqrt{z'-1}\sqrt{z'+1}}.$$

Then for $a < b$, set $M(z; a, b) = \frac{2}{b-a}\left(z - \frac{b+a}{2}\right)$ and we establish the following in Appendix C.5.

Proposition A.1. *For every $\epsilon > 0$ sufficiently small, there exist $\gamma, \gamma' > 0$ such that if $a_j - \epsilon \leq x \leq b_j + \epsilon$ and $|y| \leq \epsilon$, then*

$$\mathrm{e}^{\operatorname{Re}\mathfrak{g}(z)} \geq |\phi(M(z; a_j, b_j))|^{\gamma} \qquad and \qquad \mathrm{e}^{\operatorname{Re}\mathfrak{g}(z)} \leq |\phi(M(z; a_j, b_j))|^{\gamma'}.$$

Define the vector $\boldsymbol{\Delta} = (\Delta_j)_{j=1}^{g}$. We now define conformal maps in the neighborhood of each endpoints a_j, b_j. Specifically, define for $j = 1, 2, \ldots, g$,

$$\varphi(z; a_j) = \left(\int_{a_j}^{z} \mathfrak{g}'(z')\,\mathrm{d}z'\right)^2, \qquad \varphi(z; b_j) = \left(\int_{b_j}^{z} \mathfrak{g}'(z')\,\mathrm{d}z'\right)^2. \tag{A.3}$$

The following is immediate.

Lemma A.2. *For each $j = 1, 2, \ldots, g+1$, $\varphi(z; a_j)$ (resp. $\varphi(z; b_j)$) is a conformal mapping from a neighborhood of a_j (resp. b_j) to a neighborhood of the origin. For ϵ sufficiently small, and $0 < s < \epsilon$, $\varphi(a_j + s; a_j) < 0$ and $\varphi(b_j - s; b_j) < 0$.*

B   Extension operator

In this appendix, we define an extension operator $\sharp$ that will take a sufficiently smooth function defined on the interval $[a, b]$ and extend it to the complex plane suitably. Let $\hat{b}(x)$ be a smooth function satisfying

$$\hat{b}(x) = \begin{cases} 0, & x < 1, \\ 1, & x > 2, \end{cases}$$

that is monotonic on the interval $[1, 2]$. Set

$$b_\tau(x; c) = \hat{b}(|x - c|/\tau), \qquad 0 < \tau < (b-a)/8. \tag{B.1}$$

Then consider $\tilde{b}(x) = b_\tau(x; a)b_\tau(x; b)$. We use b and $\tilde{b}$ to define an extension of a function $m \colon [a, b] \to \mathbb{R}$. We assume m is κ times differentiable with its $(\kappa + 1)$-th derivative existing almost everywhere. Define the Taylor extension

$$m(x, y) = \sum_{\ell=0}^{\kappa} \frac{m^{(\ell)}(x)}{\ell!}(\mathrm{i}y)^\ell, \qquad x \in [a, b],$$

and the Taylor approximations

$$m(z; c) = \sum_{\ell=0}^{\kappa} \frac{m^{(\ell)}(c)}{\ell!}(z - c)^\ell, \qquad c = a, b.$$

For $0 < \theta_1 < \theta_2 < \pi/2$, construct an infinitely smooth function that for $\theta \in [-\pi/2, \pi/2]$ satisfies

$$\tilde{\beta}(\theta) = \begin{cases} 0, & -\theta_1 \le \theta \le \theta_1, \\ 1, & |\theta| > \theta_2. \end{cases}$$

Then extend $\tilde{\beta}(\theta)$ to the entirety of interval $[-\pi, \pi]$ using an even reflection about $\theta = \pi/2$. Then, one obtains a function on the unit circle in the obvious way $\hat{\beta}(z) = \tilde{\beta}(\theta)$, $z = e^{i\theta}$, which then extends to a function on $\mathbb{C} \setminus \{0\}$ $\beta(x, y) = \hat{\beta}(z/|z|)$. The extension is then defined as[8]

$$\begin{aligned} m^\sharp(x, y) = (1 - b_\tau(|y|; 0))[\tilde{b}(x)m(x, y) \\ + (1 - b_\tau(x; a))(1 - \beta(x - a, y))m(x, y) + \beta(x - a, y)(1 - b_\tau(x; a))m(z; a) \\ + (1 - b_\tau(x; b))(1 - \beta(x - b, y))m(x, y) + \beta(x - b, y)(1 - b_\tau(x; b))m(z; b)]. \end{aligned}$$

We now verify the following properties of $m^\sharp$:

(1) $m^\sharp(x, y)$ extends to be continuous for $a \le x \le b$, $y \in \mathbb{R}$,

(2) for $a \le x \le b$, $m^\sharp(x, 0) = m(x)$,

(3) for $c = a, b$, $|z - c| < \tau$,

$$\begin{aligned} m^\sharp(x, y) = m(z; a), && \theta_2 < |\arg(z - a)| < \pi/2, \\ m^\sharp(x, y) = m(z; b), && \theta_2 < |\arg(b - z)| < \pi/2, \end{aligned}$$

(4) for $a \le x \le b$ and $|y| \le \tau$, $\|\bar{\partial} m^\sharp(\cdot, y)\|_{L^\infty([a,b])} \le C_\kappa \|m\|_{C^{\kappa,1}([a,b])}|y|^\kappa$, for some $C_\kappa > 0$.

B.1 Verification of the properties

The following lemma will assist in the verification above properties.

Lemma B.1. *Suppose $f \in C^{\kappa,1}(I)$, where I is an open interval containing $x = 0$. Then for $x \in I$,*

$$\left| \sum_{j=0}^{\kappa} \frac{f^{(j)}(x)}{j!}(iy)^j - \sum_{j=0}^{\kappa} \frac{f^{(j)}(0)}{j!}(x + iy)^j \right| \le C\|f\|_{C^{\kappa,1}(I)} \sum_{j=1}^{\kappa+1} |x|^j |y|^{\kappa+1-j}$$

for a constant C that only depends on I and κ.

Proof. We use Taylor's theorem with remainder to write

$$f^{(j)}(x) = \sum_{\ell=0}^{\kappa-j} \frac{f^{(j+\ell)}(0)}{\ell!} x^\ell + R_j(x),$$

where $|R_j(x)| \le M_j \sup_{\xi \in I} |f^{(\kappa+1)}(\xi)||x|^{\kappa-j+1}$. Then, the binomial theorem gives

$$\sum_{j=0}^{\kappa} \frac{f^{(j)}(x)}{j!}(iy)^j = \sum_{j=0}^{\kappa} \frac{f^{(j)}(0)}{j!}(x + iy)^j + \sum_{j=0}^{\kappa} \frac{R_j(x)(iy)^j}{j!},$$

and the claim follows. $\blacksquare$

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[8]Here $m(x, y)$ is only defined for $x \in [a, b]$, but because the $\tilde{b}(x) = 0$ for $|x - a| < \tau, |x - b| < \tau$, this naturally extends to $|z - c| < \tau$, $c = a, b$.

B.1.1 Property (1)

The only points where continuity does not follow immediately is $z = a, b$. So, for $|z - a| < \tau$ we have

$$m^\sharp(x, y) = (1 - \beta(x - a, y))m(x, y) + \beta(x - a, y)m(z; a).$$

We then write, using Lemma B.1,

$$m^\sharp(x, y) = m(z; a) + (1 - \beta(x - a, y))P_{k+1}(x - a, y),$$

where $P_{k+1}(x, y)$ is bounded by a homogeneous polynomial of degree $k + 1$ in x and y. With the definition $m^\sharp(a, 0) := m(a; a)$, continuity follows. Similar considerations hold near $z = b$.

B.1.2 Property (2)

For $a + 2\tau \leq x \leq b - 2\tau$, we have $b_\tau(x; a) = b_\tau(x; b) = 1$ giving $m^\sharp(x, 0) = m(x, 0) = m(x)$. Then for $a \leq x \leq a + 2\tau$, we have $b_\tau(x; b) = 1$ and

$$m^\sharp(x, 0) = b_\tau(x; a)m(x, 0) + (1 - b_\tau(x; a))m(x, 0).$$

A similar calculation follows near $x = b$.

B.1.3 Property (3)

By definition, for $|z - a| < \tau$ and $\theta_2 < |\arg(z - a)| < \pi/2$, we find that $m^\sharp(x, y) = m(z; a)$. The same calculation follows near $x = b$.

B.1.4 Property (4)

We first note that, in general, for almost every x

$$2\bar{\partial}m(x, y) = \sum_{\ell=0}^{\kappa} \frac{m^{(\ell+1)}(x)}{\ell!}(iy)^\ell - \sum_{\ell=1}^{\kappa} \frac{m^{(\ell)}(x)}{(\ell-1)!}(iy)^{\ell-1} = \frac{m^{(\kappa+1)}(x)}{\kappa!}(iy)^\kappa.$$

So, we immediately have the property we desire for z such that $a + 2\tau \leq x \leq b - 2\tau$, when $\tilde{b}(x) = 1$. Then for $|z - a| \leq 2\tau$ and $-\theta_1 < \arg(z - a) < \theta_1$

$$m^\sharp(x, y) = m(x, y)(1 - \tilde{b}(x)) + m(x, y)\tilde{b}(x).$$

We again immediately have the desired property.

Now, consider $0 < |z - a| \leq \tau$ and $|\arg(z - a)| \geq \theta_1$. Then $\tilde{b}(x) = 0$ and we have

$$m^\sharp(x, y) = (1 - \beta(x - a, y))m(x, y) + \beta(x - a, y)m(z; a).$$

Since $\bar{\partial}m(z; a) = 0$, we find

$$2\bar{\partial}m^\sharp(x, y) = (2 - 2\beta(x - a, y))\bar{\partial}m(x, y) + 2\bar{\partial}\beta(x - a, y)\left[m(z; a) - m(x, y)\right].$$

The first term admits a bound of the desired type using previous considerations. We use Lemma B.1 to bound

$$|m(z; a) - m(x, y)| \leq C\|m\|_{C^{\kappa,1}([a,b])} \sum_{\ell=1}^{k+2} |x - a|^\ell |y|^{k-\ell+1}.$$

And because $|\arg(z-a)| \geq \theta_1$, we find

$$|m(z;a) - m(x,y)| \leq C' \|m\|_{C^{\kappa,1}([a,b])} |y|^{k+1}.$$

It remains to bound $\bar{\partial}\beta$.

Then we compute

$$\partial_x \beta(x,y) = \partial_x \hat{\beta}\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right) = -\hat{\beta}'(z/|z|)\frac{iy}{\bar{z}|z|},$$

$$\partial_y \beta(x,y) = \partial_y \hat{\beta}\left(\frac{x+iy}{\sqrt{x^2+y^2}}\right) = \hat{\beta}'(z/|z|)\frac{ix}{\bar{z}|z|}.$$

For some $A > 1$, $A^{-1}|y| \leq |x| \leq A|y|$ whenever $|\arg(z-a)| \geq \theta_1$. This then implies that

$$A^{-2}y^2 \leq x^2 \leq A^2 y^2, \qquad (A^{-2}+1)y^2 \leq x^2 + y^2 \leq (A^2+1)y^2.$$

So, within this sector, we have $\left|\frac{ix}{\bar{z}|z|}\right| \leq C|y|^{-1}$, $\left|\frac{iy}{\bar{z}|z|}\right| \leq C|y|^{-1}$, for a constant $C > 0$. Therefore, for a new constant C', we have

$$|\partial_x \beta(x,y)| \leq C'|y|^{-1}, \qquad |\partial_y \beta(x,y)| \leq C'|y|^{-1}.$$

The remaining case of $\tau \leq |z-a| \leq 2\tau$ and $|\arg(z-a)| \geq \theta_1$ can be established analogously by using $\tilde{b}(x) = b_\tau(x;a)$ and therefore

$$m^\sharp(x,y) = m(x,y) + (1 - b_\tau(x;a))\beta(x-a,y)(m(z;a) - m(x,y)).$$

This establishes property (4), with a τ-independent constant for $|y| \leq \tau$.

C   Construction and estimation of local parametrices

We first define the classical Bessel parametrix ($\sigma = 2\pi/3$)

$$\mathbf{P}_{\text{Bessel}}(\xi;\alpha) = \begin{cases} \begin{bmatrix} I_\alpha\left(2\xi^{1/2}\right) & \frac{i}{\pi} K_\alpha\left(2\xi^{1/2}\right) \\ 2\pi i \xi^{1/2} I_\alpha'\left(2\xi^{1/2}\right) & -2\xi^{1/2} K_\alpha'\left(2\xi^{1/2}\right) \end{bmatrix}, & \arg\xi \in (-\sigma,\sigma), \\[12pt] \begin{bmatrix} \frac{1}{2} H_\alpha^{(1)}\left(2(-\xi)^{1/2}\right) & \frac{1}{2} H_\alpha^{(2)}\left(2(-\xi)^{1/2}\right) \\ \pi\xi^{1/2} H_\alpha^{(1)'}\left(2(-\xi)^{1/2}\right) & \pi\xi^{1/2} H_\alpha^{(2)'}\left(2(-\xi)^{1/2}\right) \end{bmatrix} e^{\frac{1}{2}\alpha\pi i\boldsymbol{\sigma}_3}, & \arg\xi(\sigma,\pi), \\[12pt] \begin{bmatrix} \frac{1}{2} H_\alpha^{(2)}\left(2(-\xi)^{1/2}\right) & -\frac{1}{2} H_\alpha^{(1)}\left(2(-\xi)^{1/2}\right) \\ -\pi\xi^{1/2} H_\alpha^{(2)'}\left(2(-\xi)^{1/2}\right) & \pi\xi^{1/2} H_\alpha^{(1)'}\left(2(-\xi)^{1/2}\right) \end{bmatrix} \\ \qquad \times e^{-\frac{1}{2}\alpha\pi i\boldsymbol{\sigma}_3}, \qquad \arg\xi \in (-\pi,-\sigma). \end{cases}$$

Here I_α, K_α, $H_\alpha^{(2)}$ and $H_\alpha^{(1)}$ are the modified Bessel and Hankel functions [32]. From [28], we have the following.

The function $\mathbf{P}_{\text{Bessel}}$ satisfies the following jump conditions

$$\mathbf{P}_{\text{Bessel}}^+(\xi;\alpha) = \mathbf{P}_{\text{Bessel}}^-(\xi;\alpha)\mathbf{J}_{\text{Bessel}}(\xi;\alpha), \qquad \xi \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3,$$

$$\mathbf{J}_{\text{Bessel}}(\xi;\alpha) = \begin{cases} \begin{bmatrix} 1 & 0 \\ e^{\alpha\pi i} & 1 \end{bmatrix}, & \xi \in \Gamma_1, \\[12pt] \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & \xi \in \Gamma_2, \\[12pt] \begin{bmatrix} 1 & 0 \\ e^{-\alpha\pi i} & 1 \end{bmatrix}, & \xi \in \Gamma_3, \end{cases}$$

where contours are oriented in the direction of decreasing modulus. The asymptotics at the origin are given by the following three cases. If $\alpha > 0$,

$$\mathbf{P}_{\text{Bessel}}(\xi;\alpha) = \begin{cases} O\begin{bmatrix} |\xi|^{\alpha/2} & |\xi|^{-\alpha/2} \\ |\xi|^{\alpha/2} & |\xi|^{-\alpha/2} \end{bmatrix}, & |\arg\xi| < 2\pi/3, \\ & \text{as } \xi \to 0. \\ O\begin{bmatrix} |\xi|^{-\alpha/2} & |\xi|^{-\alpha/2} \\ |\xi|^{-\alpha/2} & |\xi|^{-\alpha/2} \end{bmatrix}, & 2\pi/3 < |\arg\xi| < \pi, \end{cases}$$

If $\alpha = 0$, then

$$\mathbf{P}_{\text{Bessel}}(\xi;\alpha) = O\begin{bmatrix} \log|\xi| & \log|\xi| \\ \log|\xi| & \log|\xi| \end{bmatrix}, \qquad \text{as } \xi \to 0.$$

And if $\alpha < 0$,

$$\mathbf{P}_{\text{Bessel}}(\xi;\alpha) = O\begin{bmatrix} |\xi|^{\alpha/2} & |\xi|^{\alpha/2} \\ |\xi|^{\alpha/2} & |\xi|^{\alpha/2} \end{bmatrix}, \qquad \text{as } \xi \to 0.$$

We also remark that $\det \mathbf{P}_{\text{Bessel}} = 1$ (see [32, 10.28.2]) so that asymptotics for the inverse of $\mathbf{P}_{\text{Bessel}}$ can be easily inferred. We also have

$$\mathbf{P}_{\text{Bessel}}(n^2\xi;\alpha) = \begin{bmatrix} \frac{1}{2}\left(\frac{1}{\pi n}\right)^{1/2}\xi^{-1/4}e^{2n\xi^{1/2}} & \frac{i}{2}\left(\frac{1}{\pi n}\right)^{1/2}\xi^{-1/4}e^{-2n\xi^{1/2}} \\ i(\pi n)^{1/2}\xi^{1/4}e^{2n\xi^{1/2}} & (\pi n)^{1/2}\xi^{1/4}e^{-2n\xi^{1/2}} \end{bmatrix}$$
$$\times \left(\mathbf{I} + O\left(n^{-1}|\xi|^{-1/2}\right)\right), \qquad n \to \infty.$$

We rewrite this in a more convenient form

$$\mathbf{P}_{\text{Bessel}}(n^2\xi;\alpha) = (\pi n)^{-\frac{1}{2}\sigma_3}\xi^{-\frac{1}{4}\sigma_3}\mathbf{E}_{\text{Bessel}}(n^2\xi)e^{2n\xi^{1/2}\sigma_3},$$

$$\mathbf{E}_{\text{Bessel}}(n^2\xi;\alpha) = \mathbf{E} + O\left(n^{-1}|\xi|^{-1/2}\right), \qquad \mathbf{E} = \begin{bmatrix} \frac{1}{2} & \frac{i}{2} \\ i & 1 \end{bmatrix}. \tag{C.1}$$

These asymptotics apply for all ξ with $|\arg\xi| < \pi$. Furthermore, the asymptotics remain valid up to the boundary, $\arg\xi = \pm\pi$.

Now set, following [27],

$$W(z;a_j) = e^{-i\pi(\alpha_j+\beta_j)/2}(z-b_j)_{\to}^{\alpha_j/2}(z-a_j)_{\to}^{\beta_j/2}f(z;a_j)^{-1/2},$$
$$W(z;b_j) = (z-b_j)^{\alpha_j/2}(z-a_j)^{\beta_j/2}f(z;b_j)^{-1/2}.$$

The last factor in each line can be seen to be analytic in a neighborhood of $z = a_j, b_j$, respectively, with the choice of the principal branch. For $c = a_j, b_j$, set $\rho(z;c) = \omega_j(z)/f(z;c)$, and we have for

$$W^{\pm}(z;b_j) = e^{\pm i\pi\alpha_j/2}\rho(z;b_j)^{1/2}, \qquad b_j - \epsilon < z < b_j,$$
$$W^{\pm}(z;a_j) = e^{\mp i\pi\beta_j/2}\rho(z;a_j)^{1/2}, \qquad a_j < z < a_j + \epsilon, \tag{C.2}$$

so that for appropriate choices of z

$$W^+(z;b_j)W^-(z;b_j) = \rho(z;b_j), \qquad W^+(z;a_j)W^-(z;a_j) = \rho(z;a_j).$$

Then for $c = a_j, b_j$, $\gamma = \beta_j, \alpha_j$, respectively, set

$$\mathbf{P}_n(z; c) = \mathbf{A}_n(z; c) \mathbf{Q}_n(z; c), \qquad \mathbf{Q}_n(z; c) = \mathbf{P}_{\text{Bessel}} \left(n^2 \frac{\varphi(z; c)}{4}; \gamma \right) W(z; c)^{-\boldsymbol{\sigma}_3} e^{-n\mathfrak{g}(z)\boldsymbol{\sigma}_3},$$

where $\mathbf{A}_n(z; c)$ is to be determined.

We recall that

$$\varphi(z; c) = \left(\int_c^z \mathfrak{g}'(z') \, dz' \right)^2.$$

And for $\text{Im } z > 0$,

$$\mathfrak{g}(z) = \int_{a_1}^{b_j} \mathfrak{g}'_+(z') \, dz' + \int_{b_j}^z \mathfrak{g}'(z') \, dz' = \sum_{\ell=1}^j \int_{a_\ell}^{b_\ell} \mathfrak{g}'_+(z') \, dz' + \int_{b_j}^z \mathfrak{g}'(z') \, dz'$$

$$= \frac{\Delta_j}{2} + \int_{b_j}^z \mathfrak{g}'_+(z') \, dz'.$$

Similarly, for $\text{Im } z < 0$, we have

$$\mathfrak{g}(z) = -\frac{\Delta_j}{2} + \int_{b_j}^z \mathfrak{g}'(z) \, dz'.$$

Next, using the principal branch we consider $\varphi(z; b_j)^{1/2}$. As $\varphi(z; b_j)$ is injective for $|z - b_j| < \epsilon$, we conclude that $\varphi(z; b_j)^{1/2}$ can only fail to be analytic for $z \leq b_j$. Because $\varphi(z; b_j)$ is positive for $z > b_j$ the same is true of $\varphi(z; b_j)^{1/2}$. And then we consider $\int_{b_j}^z \mathfrak{g}'(z') \, dz'$, $z > b_j$. It can be shown that the monic polynomial $Q_g(z)$ in the (A.1) must have a root in each interval (b_j, a_{j+1}). From this, it follows that $\mathfrak{g}'(b_j + \epsilon') > 0$ for $\epsilon' > 0$ sufficiently small. Then

$$\varphi(z; b_j)^{1/2} = \int_{b_j}^z \mathfrak{g}'(z') \, dz'.$$

And therefore

$$\varphi(z; b_j)^{1/2} = \begin{cases} \mathfrak{g}(z) - \frac{\Delta_j}{2}, & |z - b_j| < \epsilon, \quad \text{Im } z > 0, \\ \mathfrak{g}(z) + \frac{\Delta_j}{2}, & |z - b_j| < \epsilon, \quad \text{Im } z < 0. \end{cases}$$

We repeat these calculations for $\varphi(z; a_j)$ and find that with the convention that $\Delta_0 = 0 = \Delta_{g+1}$

$$\varphi(z; a_j)^{1/2} = \begin{cases} \mathfrak{g}(z) - \frac{\Delta_{j-1}}{2}, & |z - a_j| < \epsilon, \quad \text{Im } z > 0, \\ \mathfrak{g}(z) + \frac{\Delta_{j-1}}{2}, & |z - a_j| < \epsilon, \quad \text{Im } z < 0. \end{cases}$$

For $|z - a_j| = \epsilon$, we have $\text{Im } z \neq 0$,

$$\mathbf{P}_n(z; a_j) = \mathbf{A}_n(z; a_j) \left(\frac{2}{\pi n} \right)^{\frac{1}{2}\boldsymbol{\sigma}_3} \varphi(z; a_j)^{-\frac{1}{4}\boldsymbol{\sigma}_3} \mathbf{E}_{\text{Bessel}} \left(n^2 \frac{\varphi(z; a_j)}{4}; \beta_j \right)$$

$$\times e^{-(\text{sign Im } z)\frac{n}{2}\Delta_{j-1}\boldsymbol{\sigma}_3} W(z; a_j)^{-\boldsymbol{\sigma}_3}.$$

And for $|z - b_j| = \epsilon$, we have $\text{Im } z \neq 0$,

$$\mathbf{P}_n(z; b_j) = \mathbf{A}_n(z; b_j) \left(\frac{2}{\pi n} \right)^{\frac{1}{2}\boldsymbol{\sigma}_3} \varphi(z; b_j)^{-\frac{1}{4}\boldsymbol{\sigma}_3} \mathbf{E}_{\text{Bessel}} \left(n^2 \frac{\varphi(z; b_j)}{4}; \alpha_j \right)$$

$$\times e^{-(\text{sign Im } z)\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3} W(z; b_j)^{-\boldsymbol{\sigma}_3}.$$

We set

$$\mathbf{A}_n(z;a_j) = \mathbf{G}_n(z) W(z;a_j)^{\boldsymbol{\sigma}_3} \mathrm{e}^{(\mathrm{sign}\,\mathrm{Im}\,z)\frac{n}{2}\Delta_{j-1}\boldsymbol{\sigma}_3} \mathbf{E}^{-1}\left(\frac{2}{\pi n}\right)^{-\frac{1}{2}\boldsymbol{\sigma}_3} \varphi(z;a_j)^{\frac{1}{4}\boldsymbol{\sigma}_3},$$

$$\mathbf{A}_n(z;b_j) = \mathbf{G}_n(z) W(z;b_j)^{\boldsymbol{\sigma}_3} \mathrm{e}^{(\mathrm{sign}\,\mathrm{Im}\,z)\frac{n}{2}\Delta_{j}\boldsymbol{\sigma}_3} \mathbf{E}^{-1}\left(\frac{2}{\pi n}\right)^{-\frac{1}{2}\boldsymbol{\sigma}_3} \varphi(z;b_j)^{\frac{1}{4}\boldsymbol{\sigma}_3}.$$

In the following appendices, we discuss the behavior of the parametrix near b_j leaving the calculations near a_j to the reader.

C.1 The jumps of $\mathbf{Q}_n$ near b_j

We then compute, for example, for $b_j - \epsilon < z < b_j$

$$\mathbf{Q}_n^+(z;b_j) = \left[\lim_{\epsilon\downarrow 0} \mathbf{P}_{\mathrm{Bessel}}\left(n^2\frac{\varphi(z-\mathrm{i}\epsilon;b_j)}{4};\alpha_j\right)\right]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rho(z;b_j)^{-\boldsymbol{\sigma}_3} W_-(z;b_j)^{\boldsymbol{\sigma}_3}\mathrm{e}^{n\mathfrak{g}^-(z)\boldsymbol{\sigma}_3}$$

$$= \mathbf{Q}_n^-(z;b_j)\begin{bmatrix} 0 & \rho(z;b_j) \\ -1/\rho(z;b_j) & 0 \end{bmatrix}.$$

Then for $b_j < z < b_j + \epsilon$,

$$\mathbf{Q}_n^+(z;b_j) = \mathbf{P}_{\mathrm{Bessel}}\left(n^2\frac{\varphi(z;b_j)}{4};\alpha_j\right) W_+(z;b_j)^{-\boldsymbol{\sigma}_3}\mathrm{e}^{-n\mathfrak{g}^+(z)\boldsymbol{\sigma}_3}$$

$$= \mathbf{P}_{\mathrm{Bessel}}\left(n^2\frac{\varphi(z;b_j)}{4};\alpha_j\right) W_-(z;b_j)^{-\boldsymbol{\sigma}_3}\mathrm{e}^{-n\mathfrak{g}^-(z)\boldsymbol{\sigma}_3}\mathrm{e}^{-n\Delta_j\boldsymbol{\sigma}_3}$$

$$= \mathbf{Q}_n^-(z;b_j)\mathrm{e}^{-n\Delta_j\boldsymbol{\sigma}_3}.$$

For $z \in \Sigma_{b_j,1}$,

$$\mathbf{Q}_n^+(z;b_j) = \mathbf{Q}_n^-(z;b_j)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{\mathrm{i}\alpha_j\pi}W(z;b_j)^{-2}\mathrm{e}^{-2n\mathfrak{g}(z)} & 1 \end{bmatrix}.$$

From (C.2), it follows that $W^+(z;b_j)^2 = \mathrm{e}^{\mathrm{i}\alpha_j\pi}\rho(z;b_j)$, so that $\mathrm{e}^{\mathrm{i}\alpha_j\pi}W^+(z;b_j)^{-2}$ can be seen to be the analytic continuation of $1/\rho(z;b_j)$ to the upper-half plane

$$\mathbf{Q}_n^+(z;b_j) = \mathbf{Q}_n^-(z;b_j)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}f_j^{\sharp}(x,y)/\omega_j(z) & 1 \end{bmatrix}, \qquad z \in \Sigma_{b_j,1}.$$

Then because $\mathrm{e}^{-\mathrm{i}\alpha_j\pi}W^-(z;b_j)^{-2}$ can be seen to be the analytic continuation of $1/\rho(z;b_j)$ to the lower-half plane

$$\mathbf{Q}_n^+(z;b_j) = \mathbf{Q}_n^-(z;b_j)\begin{bmatrix} 1 & 0 \\ \mathrm{e}^{-2n\mathfrak{g}(z)}f_j^{\sharp}(x,y)/\omega_j(z) & 1 \end{bmatrix}, \qquad z \in \Sigma_{b_j,3}.$$

C.2 The jumps of vec A_n near b_j

We then need to analyze the analyticity and jump behavior of $\mathbf{A}_n$. We note that it is immediate that $\mathbf{A}_n$ is analytic off the real axis. For $b_j < z < b_j + \epsilon$,

$$\mathbf{A}_n^+(z;b_j) = \mathbf{G}_n^+(z) W(z;b_j)^{\boldsymbol{\sigma}_3} \mathrm{e}^{\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3} \mathbf{E}^{-1}\left(\frac{2}{\pi n}\right)^{-\frac{1}{2}\boldsymbol{\sigma}_3}\varphi(z;b_j)^{\frac{1}{4}\boldsymbol{\sigma}_3}$$

$$= \mathbf{G}_n^-(z) W(z;b_j)^{\boldsymbol{\sigma}_3} \mathrm{e}^{-\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3} \mathbf{E}^{-1}\left(\frac{2}{\pi n}\right)^{-\frac{1}{2}\boldsymbol{\sigma}_3}\varphi(z;b_j)^{\frac{1}{4}\boldsymbol{\sigma}_3} = \mathbf{A}_n^-(z;b_j).$$

Then for $b_j - \epsilon < z < b_j$ since $\varphi(z; b_j)$ is conformal and real-valued on the real axis,

$$\lim_{\epsilon \downarrow 0} \varphi(z \pm i\epsilon; b_j)^{1/4} = e^{\pm i\pi/4} |\varphi(z; b_j)|^{1/4}.$$

Thus

$$\lim_{\epsilon \downarrow 0} \varphi(z + i\epsilon; b_j)^{1/4} =: \varphi(z; b_j)_+^{1/4} = i\varphi(z; b_j)_-^{1/4} := \lim_{\epsilon \downarrow 0} \varphi(z - i\epsilon; b_j)^{1/4}.$$

Then we compute

$$\mathbf{A}_n^+(z; b_j) = \mathbf{G}_n^-(z) \begin{bmatrix} 0 & \rho(z) \\ -1/\rho(z) & 0 \end{bmatrix} \rho(z; b_j)^{\sigma_3} W^-(z; b_j)^{-\sigma_3} e^{\frac{n}{2} \Delta_j \sigma_3}$$

$$\times \mathbf{E}^{-1} \left(\frac{2}{\pi n} \right)^{-\frac{1}{2}\sigma_3} i^{\sigma_3} \left(\varphi(z; b_j)_-^{\frac{1}{4}} \right)^{\sigma_3}.$$

To simplify this expression, we first note that for $y \neq 0$,

$$\begin{bmatrix} 0 & y \\ -1/y & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y^{-\sigma_3},$$

and

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} y^{\sigma_3} \mathbf{E}^{-1} \left(\frac{i}{\sqrt{2}} \right)^{\sigma_3} = y^{-\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\frac{i}{2} \\ -i & \frac{1}{2} \end{bmatrix} \left(\frac{i}{\sqrt{2}} \right)^{\sigma_3} = y^{-\sigma_3} \mathbf{E}^{-1} 2^{-\sigma_3/2},$$

and therefore

$$\mathbf{A}_n^+(z; b_j) = \mathbf{G}_n^-(z) \left(\frac{\rho(z)}{\rho(z; b_j)} \right)^{\sigma_3} W^-(z; b_j)^{\sigma_3} e^{-\frac{n}{2} \Delta_j \sigma_3} \mathbf{E}^{-1} \left(\frac{2}{\pi n} \right)^{-\frac{1}{2}\sigma_3} \left(\varphi(z; b_j)_-^{\frac{1}{4}} \right)^{\sigma_3}$$

$$= \mathbf{A}_n^-(z; b_j) \Psi^+(z)^{-\sigma_3} \mathbf{E} \left(\frac{\rho(z)}{\rho(z; b_j)} \right)^{\sigma_3} \mathbf{E}^{-1} \Psi^+(z)^{\sigma_3},$$

$$\Psi(z) = \left(\frac{2}{\pi n} \right)^{-\frac{1}{2}\sigma_3} \varphi(z; b_j)^{\frac{1}{4}\sigma_3}.$$

C.3 The jumps of vec P_n near b_j

For all but one contour near b_j, the jump of $\mathbf{P}_n$ is precisely the same as that of $\mathbf{Q}_n$ (and therefore the same as $\mathbf{S}_n$) because $\mathbf{A}_n$ is analytic. So, it remains to consider $b_j - \epsilon < z < b_j$

$$\mathbf{P}_n^+(z; b_j) = \mathbf{A}_n^+(z; b_j) \mathbf{Q}_n^+(z; b_j) = \mathbf{A}_n^+(z; b_j) \mathbf{Q}_n^-(z; b_j) \begin{bmatrix} 0 & \rho(z; b_j) \\ -1/\rho(z; b_j) & 0 \end{bmatrix}$$

$$= \mathbf{A}_n^-(z; b_j) \Psi^+(z)^{-\sigma_3} \mathbf{E} \left(\frac{\rho(z)}{\rho(z; b_j)} \right)^{\sigma_3} \mathbf{E}^{-1} \Psi^+(z)^{\sigma_3} \mathbf{Q}_n^-(z; b_j) \rho(z; b_j)^{\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \mathbf{P}_n^-(z; b_j) \mathbf{Q}_n^-(z; b_j)^{-1} \Psi^+(z)^{-\sigma_3} \mathbf{E} \left(\frac{\rho(z)}{\rho(z; b_j)} \right)^{\sigma_3}$$

$$\times \mathbf{E}^{-1} \Psi^+(z)^{\sigma_3} \mathbf{Q}_n^-(z; b_j) \rho(z; b_j)^{\sigma_3} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

C.4 Local estimates

We have the following estimates which we state in a series of lemmas.

Lemma C.1. *Suppose $|z - b_j| < \epsilon$, $z \in \Omega_{j,+} \cup \Omega_{j,-}$. If $|\alpha_j| > 0$, then*

$$\|\mathbf{Q}_n(z; b_j) W(z; b_j)^{\boldsymbol{\sigma}_3}\| = \mathrm{O}\big(n^{|\alpha_j|}|z - b_j|^{-|\alpha_j|/2}\big), \qquad n^2(z - b_j) \to 0.$$

If $\alpha_j = 0$, then

$$\|\mathbf{Q}_n(z; b_j) W(z; b_j)^{\boldsymbol{\sigma}_3}\| = \mathrm{O}(\log n|z - b_j|), \qquad n^2(z - b_j) \to 0.$$

Also, $\mathrm{Re}\,\mathfrak{g}(z) = \mathrm{O}\big(|z - b_j|^{1/2}\big)$, $z \to b_j$.

Lemma C.2. *The function*

$$\mathbf{A}_n(z; c) \left(\frac{2}{\pi n}\right)^{\frac{1}{2}\boldsymbol{\sigma}_3}$$

is uniformly bounded in both z and n for $|z - c| < \epsilon$, $c = a_j, b_j$. The same holds for its inverse.

To prove this lemma we use the following lemma[9] of [17, Lemma A.2] which shows that $\mathrm{e}^{G(z)\boldsymbol{\sigma}_3} W(z; b_j)^{\boldsymbol{\sigma}_3}$ is bounded in a neighborhood of $z = b_j$.

Lemma C.3. *If $k \geq 0$, for some $\epsilon > 0$, and for every $j = 1, 2, \ldots, g + 1$, we have*

$$G(z) = -\frac{\alpha_j}{2} \log(z - b_j) - \frac{\beta_j}{2} \log(a_j - z) + R_j(z), \qquad \mathrm{dist}(z, [a_j, b_j]) \leq \epsilon,$$

where $R_j(z)$ is a uniformly bounded function for $\mathrm{dist}(z, [a_j, b_j]) \leq \epsilon$.

Proof of Lemma C.2. By Lemma C.3, it follows that $\mathrm{e}^{G(z)\boldsymbol{\sigma}_3} W(z; b_j)^{\boldsymbol{\sigma}_3}$ is bounded in a neighborhood of $z = b_j$. And then one can show, as is classically used the derivation of the local solutions [28], that if $\rho(z)$ is replaced with $\rho(z; b_j)$ in the definition of $G(z)$ for, say, $b_j - 2\epsilon < z < b_j$, then $\mathbf{A}_n(z; b_j)$ is analytic for $|z - b_j| < \epsilon$ as the singular factors produce weaker-than-pole singularities. Thus, the actual $\mathbf{A}_n(z; b_j)$ is a bounded perturbation of this analytic approximation. The theta function terms that contribute to $\mathbf{Q}_n$ are uniformly bounded, and the claim follows. ∎

Lemma C.4. *Suppose $|z - b_j| < \epsilon$, $z \in \Omega_{j,+} \cup \Omega_{j,-}$. For any $c < C = C(n)$, $C \ll n^2$, for sufficiently large n, there exists a constant D such that if $c \leq \big|n^2(b_j - z)\big| \leq C$, then*

$$\|\Psi(z)^{\boldsymbol{\sigma}_3}\| \leq DC^{1/4}, \qquad \|\mathbf{A}_n(z; b_j)\| \leq DC^{1/4}, \qquad \|\mathbf{Q}_n(z; b_j) W(z; b_j)^{\boldsymbol{\sigma}_3}\| \leq DC^{1/2}.$$

Proof. We have for $|z - b_j| \leq \epsilon$ and some $d \in \mathbb{R}$, $D' > 0$,

$$|\varphi(z; b_j) - d(z - b_j)| \leq D'|z - b_j|^2,$$
$$|n^2\varphi(z; b_j)| - \big|dn^2(z - b_j)\big| \leq (n^4 D'|z - b_j|^2)n^{-2} \leq D'C^2/n^2.$$

So $|d|/C - D'C^2/n^2 \leq |n^2\varphi(z; b_j)| \leq |d|C + D'C^2/n^2$, and the bound on $\Psi(z)^{\boldsymbol{\sigma}_3}$ follows for n sufficiently large. It follows from previous arguments that $\mathbf{A}_n(z; b_j)\Psi(z)^{-\boldsymbol{\sigma}_3}$ is bounded for $|z - b_j| < \epsilon$ as $n \to \infty$. The claim then follows for $\mathbf{A}_n(z; b_j)$. Then, for $\mathbf{Q}_n(z; b_j) W(z; b_j)^{\boldsymbol{\sigma}_3}$ factors of $W(z; b_j)^{\boldsymbol{\sigma}_3}$ cancel. Then if $C = \mathrm{O}(1)$, we have from Lemma C.1 $\mathrm{e}^{n\mathfrak{g}(z)} = \mathrm{O}(1)$, and $\mathbf{P}_{\mathrm{Bessel}}\big(n^2\frac{\varphi(z; b_j)}{4}\big) = \mathrm{O}(1)$ uniformly for z in the range. Now, let $C' > 0$ be sufficiently

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[9]The proof in [17] was for $\alpha_j = \beta_j = 1/2$ but the same calculations hold in general.

large, but an absolute constant, so that the entry-wise bound from (C.1) holds for $|n^2\xi| \geq C'$. And then for $n^2|z - b_j| \geq 2C'$, there exists n sufficiently large so that $\left|n^2\frac{\varphi(z;b_j)}{4}\right| \geq C'$, and for $C' \leq \left|n^2(z - b_j)\right| \leq C$, we have

$$\left\|\Psi(z)^{\boldsymbol{\sigma}_3}\mathbf{Q}_n(z; b_j)W(z; b_j)^{\boldsymbol{\sigma}_3}\right\| \leq D.$$

The last claim follows from the estimates on Ψ. ∎

Lemma C.5. *Suppose $|z - b_j| < \epsilon$, $z \in \Omega_{j,+} \cup \Omega_{j,-}$. Then as $\left|n^2(b_j - z)\right| \to \infty$*

$$\left\|\mathbf{A}_n(z; b_j)\mathbf{Q}_n(z; b_j)W(z; b_j)^{\boldsymbol{\sigma}_3}\right\| = \mathrm{O}\big(|z - b_j|^{-1/4}\big),$$

$$\left\|W(z; b_j)^{-\boldsymbol{\sigma}_3}\mathbf{Q}_n(z; b_j)^{-1}\Psi(z)^{-\boldsymbol{\sigma}_3}\mathbf{E} - \mathrm{e}^{(\mathrm{sign\,Im}\,z)\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3}\right\| = \mathrm{O}\big(n^{-1}|z - b_j|^{-1/2}\big),$$

$$\left\|\mathbf{E}^{-1}\Psi(z)^{\boldsymbol{\sigma}_3}\mathbf{A}_n(z; b_j)^{-1}\right\| = \mathrm{O}\big(|z - b_j|^{-1/4}\big).$$

Proof. For the first claim, we can write

$$\mathbf{A}_n(z; b_j)\mathbf{Q}_n(z; b_j)W(z; b_j)^{\boldsymbol{\sigma}_3} = \mathbf{G}_n(z)W(z; b_j)^{\boldsymbol{\sigma}_3}\mathrm{e}^{(\mathrm{sign\,Im}\,z)\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3}\mathbf{E}^{-1}\mathbf{E}_{\mathrm{Bessel}}$$
$$\times \left(n^2\frac{\varphi(z; b_j)}{4}; \alpha_j\right)\mathrm{e}^{-(\mathrm{sign\,Im}\,z)\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3}.$$

The claim then follows from the asymptotics of $\mathbf{E}_{\mathrm{Bessel}}$ and the fact that, from Lemma C.3, $\mathbf{G}_n(z)W(z; b_j)^{\boldsymbol{\sigma}_3}$ has quarter-root singularities.

Then we write

$$\mathbf{E}^{-1}\Psi(z)^{\boldsymbol{\sigma}_3}\mathbf{Q}_n(z; b_j)W(z; b_j)^{\boldsymbol{\sigma}_3}\mathrm{e}^{(\mathrm{sign\,Im}\,z)\frac{n}{2}\Delta_j\boldsymbol{\sigma}_3} = \mathbf{I} + \mathrm{O}\big(n^{-1}|z - b_j|^{-1/2}\big),$$

from which the second claim follows. The last claim follows analogously. ∎

C.5 Proof of Proposition A.1

We first consider $y > 0$ and $a_j \leq x \leq b_j$. We write

$$\mathfrak{g}'(z) = \frac{1}{\sqrt{z - a_j}\sqrt{z - b_j}}r(z),$$

where $r(z)$ is analytic in a neighborhood of $[a_j, b_j]$ and is positive on $[a_j, b_j]$. We have

$$\mathrm{Re}\,\mathfrak{g}(z) = -\int_0^y \mathrm{Im}\,\mathfrak{g}'(x + \mathrm{i}y')\,\mathrm{d}y'.$$

Similarly, using M in (3.1),

$$\log|\phi(M(z; a_j, b_j))| = -\int_0^y \mathrm{Im}\,\frac{\mathrm{d}y'}{\sqrt{x + \mathrm{i}y' - b_j}\sqrt{x + \mathrm{i}y' - a_j}}.$$

We write $r(z) = h(x, y) + \mathrm{i}g(x, y)$, where h and g are real valued. From Taylor's theorem,

$$h(x + \mathrm{i}y) = h(x) + \partial_y h(x, \xi_1)y, \qquad g(x + \mathrm{i}y) = \partial_y g(x, \xi_2)y,$$

for $0 < \xi_1, \xi_2 < y$, or, rather that $|h(x + \mathrm{i}y) - h(x)| \leq C|y|$, $|g(x + \mathrm{i}y)| \leq C|y|$. Then we have that

$$\mathrm{Im}\,\mathfrak{g}'(x + \mathrm{i}y) = h(x + \mathrm{i}y)\,\mathrm{Im}\,\frac{1}{\sqrt{x + \mathrm{i}y - b_j}\sqrt{x + \mathrm{i}y - a_j}}$$
$$+ g(x + \mathrm{i}y)\,\mathrm{Re}\,\frac{1}{\sqrt{x + \mathrm{i}y - b_j}\sqrt{x + \mathrm{i}y - a_j}}.$$

We then claim that

$$\left| \operatorname{Re} \frac{\sqrt{y}}{\sqrt{x+iy-b_j}\sqrt{x+iy-a_j}} \right| \leq C'.$$

Indeed,

$$\frac{\sqrt{y}}{|x+iy-b_j|^{1/2}|x+iy-a_j|^{1/2}} = \frac{1}{\left((x-b_j)^2/y^2+1\right)^{1/4}\left((x-a_j)^2+y^2\right)^{1/4}}$$

$$\leq \frac{1}{\left((x-a_j)^2+y^2\right)^{1/4}}.$$

And we find that

$$\left| \operatorname{Re} \frac{\sqrt{y}}{\sqrt{x+iy-b_j}\sqrt{x+iy-a_j}} \right| \leq \min \left\{ \frac{1}{\left((x-a_j)^2+y^2\right)^{1/4}}, \frac{1}{\left((x-b_j)^2+y^2\right)^{1/4}} \right\}$$

$$\leq \frac{2}{(b_j-a_j)^{1/2}}.$$

This results in, for $c' \geq h(x+iy) \geq c$,

$$c' \operatorname{Im} \frac{1}{\sqrt{x+iy-b_j}\sqrt{x+iy-a_j}} + C''\sqrt{y} \geq \operatorname{Im} \mathfrak{g}'(x+iy)dy$$

$$\geq c \operatorname{Im} \frac{1}{\sqrt{x+iy-b_j}\sqrt{x+iy-a_j}} - C''\sqrt{y},$$

$$c' \log|\phi(M(z;a_j,b_j))| + \frac{2C''}{3}y^{3/2} \geq \int_0^y \operatorname{Im} \mathfrak{g}'(x+iy)\,dy$$

$$\geq c \log|\phi(M(z;a_j,b_j))| - \frac{2C''}{3}y^{3/2}.$$

We then claim for $0 < y < \gamma$ and $a_j \leq x \leq b_j$, that there exists $c'' > 0$ such that

$$I(x,y) = \operatorname{Im} \frac{1}{\sqrt{x+iy-b_j}\sqrt{x+iy-a_j}} \geq c''.$$

Indeed, if this were not the case, then there would exist a convergent sequence $(x_n, y_n)_{n \geq 1}$ such that $I(x_n, y_n) \to 0$. And we must have that $(x_n, y_n) \to (a_j, 0)$ or $(x_n, y_n) \to (b_j, 0)$. But neither can occur as $I(x_n, y_n) \to +\infty$. This implies

$$\log|\phi(M(z;a_j,b_j))| \geq c''y, \qquad -y^{3/2} \geq -\frac{\sqrt{y}}{c''}\log|\phi(M(z;a_j,b_j))|.$$

And therefore

$$\left(c' + \sqrt{y}\frac{2C''}{3c''} \right) \log|\phi(M(z;a_j,b_j))| \geq \int_0^y \operatorname{Im} \mathfrak{g}'(x+iy)dy$$

$$\geq \left(c' - \sqrt{y}\frac{2C''}{3c''} \right) \log|\phi(M(z;a_j,b_j))|,$$

from which the claim follows. Similar considerations work for $\operatorname{Im} y < 0$, establishing the claim for $a_j \leq x \leq b_j$. Continuity of the ratio $\operatorname{Re} \mathfrak{g}(z)/\log|\phi(M(z;a_j,b_j))|$ near a_j, b_j allows the desired bound to hold in neighborhoods of a_j, b_j, completing the proof.

D   Hyperelliptic Riemann surfaces

Classic references for what follows are [5, 9, 16]. Much of what follows here was closely adapted from that in [18]. Using the intervals $[a_j, b_j]$, $1 \leq j \leq g+1$, we define a Riemann surface by the zero locus of

$$w^2 - \prod_{j=1}^{g+1} (z - a_j)(z - b_j) =: w^2 - P_{2g+2}(z)$$

in $\mathbb{C}^2$. Then define the analytic function

$$R: \hat{\mathbb{C}} \to \mathbb{C}, \qquad R(z)^2 = P_{2g+2}(z), \qquad R(z)z^{-g-1} \to 1, \qquad \text{as} \quad z \to \infty,$$

where

$$\hat{\mathbb{C}} = \mathbb{C} \setminus \bigcup_{j=1}^{g+1} [a_j, b_j].$$

A Riemann surface Γ is built by joining two copies of $\hat{\mathbb{C}}$; see Figure 6. The Riemann surface has a canonical set of cycles, a homology basis. For a cartoon of these $\mathfrak{a}$-cycles and $\mathfrak{b}$-cycles see Figure 6. We have a natural projection operation $\pi: \Gamma \to \mathbb{C}$ defined by $\pi((z, w)) = z$ and its right-inverses $\pi_j^{-1}(z) = (z, (-1)^{j+1} R(z))$, $j = 1, 2$.
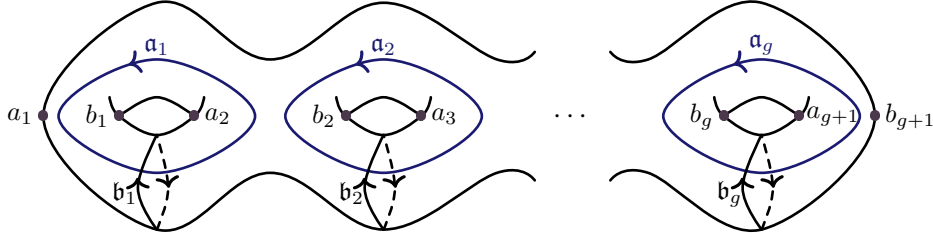
Figure 6. An illustration of the Riemann surface Γ.

It is well-known that (see [9], for example) a basis for holomorphic differentials on Γ is given by

$$\mathrm{d}\nu_j = \frac{z^{j-1}}{R(z)} \, \mathrm{d}z, \qquad j = 1, 2, \ldots, g+1.$$

Then define the $g \times g$ matrix of periods $\mathbf{A} = (A_{ij})$ by $A_{ij} = \oint_{\mathfrak{a}_i} \mathrm{d}\nu_j$. Note that if $\mathbf{c} = \begin{bmatrix} c_1 & c_2 & \cdots & c_g \end{bmatrix}^\mathsf{T} = \mathbf{A}^{-1}\mathbf{e}_j$ for the standard basis vector $\mathbf{e}_j$, then

$$\oint_{\mathfrak{a}_i} \sum_{k=1}^{g} c_k \, \mathrm{d}\nu_k = \sum_{k=1}^{g} c_k A_{ik} = \mathbf{e}_i^\mathsf{T} \mathbf{A} \mathbf{c} = \mathbf{e}_i^\mathsf{T} \mathbf{e}_j = \delta_{ij}.$$

A basis of normalized differentials is then given by

$$\begin{bmatrix} \mathrm{d}\omega_1 \\ \mathrm{d}\omega_2 \\ \vdots \\ \mathrm{d}\omega_g \end{bmatrix} = 2\pi\mathrm{i}\mathbf{A}^{-1} \begin{bmatrix} \mathrm{d}\nu_1 \\ \mathrm{d}\nu_2 \\ \vdots \\ \mathrm{d}\nu_g \end{bmatrix}.$$

These satisfy $\oint_{\mathfrak{a}_i} \mathrm{d}\omega_j = 2\pi\mathrm{i}\delta_{ij}$. The fact that the matrix $\mathbf{A}$ is invertible follows from abstract theory [9].

Define

$$\mathbf{u}(z) = \left(\int_{a_1}^{z} \mathrm{d}\omega_j \right)_{j=1}^{g}, \qquad z \notin \mathbb{R},$$

where, for the sake of concreteness, the path of integration is taken to be a straight line connecting a_1 to z. We can treat $\mathbf{u}$ as both a function $\mathbf{u}(z)$ on $\hat{\mathbb{C}}$ and as a function $\mathbf{u}(P)$ on Γ. Then $\mathbf{u}(P)$ is a single-valued analytic function on the Riemann surface Γ provided Γ is cut along the cycles $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_g, \mathfrak{b}_1, \ldots, \mathfrak{b}_g\}$, making it simply connected. Another important fact is for $z \in \hat{\mathbb{C}}$, $\mathbf{u}\big(\pi_1^{-1}(z)\big) = -\mathbf{u}\big(\pi_2^{-1}(z)\big)$.

The associated Riemann matrix of $\mathfrak{b}$ periods is given by

$$\boldsymbol{\tau} = (\tau_{ij}) = \left(\int_{\mathfrak{b}_j} \mathrm{d}\omega_i \right)_{1 \le i, j \le g}.$$

It follows, see [9], that $\boldsymbol{\tau}$ is symmetric, real and negative definite. The vector $\mathbf{k}$ of Riemann constants is defined componentwise via

$$k_j = \frac{2\pi\mathrm{i} + \tau_{jj}}{2} - \frac{1}{2\pi\mathrm{i}} \sum_{\ell \ne j} \oint_{\mathfrak{a}_\ell} u_j \, \mathrm{d}\omega_\ell, \qquad j = 1, 2, \ldots, g.$$

The associated theta function is defined by

$$\theta(\mathbf{z}; \boldsymbol{\tau}) = \sum_{m \in \mathbb{Z}^g} \exp\left(\frac{1}{2}(m, \boldsymbol{\tau}m) + (m, \mathbf{z}) \right), \qquad \mathbf{z} \in \mathbb{C}^g,$$

where $(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\mathsf{T}}\mathbf{b}$. We have

$$\theta(\mathbf{z} + 2\pi\mathrm{i}\mathbf{e}_j; \boldsymbol{\tau}) = \theta(\mathbf{z}; \boldsymbol{\tau}), \qquad \theta(\mathbf{z} + \boldsymbol{\tau}\mathbf{e}_j; \boldsymbol{\tau}) = \exp\left(-\frac{1}{2}\tau_{jj} - z_j \right) \theta(\mathbf{z}; \boldsymbol{\tau}).$$

A divisor $D = \sum_j n_j P_j$ is a formal sum of points $\{P_j\}$ on the Riemann surface Γ. The Abel map of a divisor is defined via $\mathcal{A}(D) = \sum_j n_j \mathbf{u}(P_j)$.

We construct an important function that will have piecewise constant jump conditions. Set

$$\boldsymbol{\Theta}(z; \mathbf{d}; \mathbf{v}) = \boldsymbol{\Theta}(z) := \left[\frac{\theta\left(\mathbf{u}(z) + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau} \right)}{\theta\left(\mathbf{u}(z) - \mathbf{d}; \boldsymbol{\tau} \right)} \quad \frac{\theta\left(-\mathbf{u}(z) + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau} \right)}{\theta\left(-\mathbf{u}(z) - \mathbf{d}; \boldsymbol{\tau} \right)} \right], \qquad z \notin \mathbb{R}. \qquad \text{(D.1)}$$

The first component function is nothing more than $\frac{\theta(\mathbf{u}(P) + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau})}{\theta(\mathbf{u}(P) - \mathbf{d}; \boldsymbol{\tau})}$ restricted to the first sheet. The same is true for the second component function on the second sheet. The vector $\mathbf{v}$ is a free parameter. Then consider

$$\mathbf{u}^+(z) + \mathbf{u}^-(z) = \left(2 \sum_{k=1}^{j-1} \int_{b_k}^{a_{k+1}} \mathrm{d}\omega_\ell \right)_{\ell=1}^{g} = \left(\sum_{k=1}^{j-1} \oint_{\mathfrak{a}_k} \mathrm{d}\omega_\ell \right)_{\ell=1}^{g} = 2\pi\mathrm{i}\mathsf{N}, \qquad z \in [a_j, b_j],$$

for a vector N of zeros and ones. Then we compute

$$\mathbf{u}^+(z) - \mathbf{u}^-(z) = \left(2 \sum_{k=1}^{j} \int_{a_k}^{b_k} \mathrm{d}\omega_\ell \right)_{\ell=1}^{g} = \left(\oint_{\mathfrak{b}_j} \mathrm{d}\omega_\ell \right)_{\ell=1}^{g} = \boldsymbol{\tau}\mathbf{e}_j, \qquad z \in [b_j, a_{j+1}].$$

Directly using this relation, it follows that, for $z \in [b_j, a_{j+1}]$,

$$\frac{\theta\left(\pm\mathbf{u}^+(z) + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau} \right)}{\theta\left(\pm\mathbf{u}^+(z) - \mathbf{d}; \boldsymbol{\tau} \right)} = \frac{\theta\left(\pm\mathbf{u}^-(z) \pm \boldsymbol{\tau}\mathbf{e}_j + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau} \right)}{\theta\left(\pm\mathbf{u}^-(z) \pm \boldsymbol{\tau}\mathbf{e}_j - \mathbf{d}; \boldsymbol{\tau} \right)} = \mathrm{e}^{\mp v_j} \frac{\theta\left(\pm\mathbf{u}^-(z) + \mathbf{v} - \mathbf{d}; \boldsymbol{\tau} \right)}{\theta\left(\pm\mathbf{u}^-(z) - \mathbf{d}; \boldsymbol{\tau} \right)}.$$

On the interval $(-\infty, a_1)$, $\mathbf{u}^+(z) = \mathbf{u}^-(z)$. And on (b_{g+1}, ∞),

$$\mathbf{u}^+(z) - \mathbf{u}^-(z) = \left(\oint_{\mathcal{C}} d\omega_j \right)_{j=1}^g ,$$

where $\mathcal{C}$ is a clockwise-oriented simple contour that encircles $[a_1, b_{g+1}]$. Because all the differentials $d\omega_j$ are of the form $P(z)/R(z)$ where P is a degree $g-1$ polynomial and $R(z)/z^{g+1} \to 1$ as $z \to \infty$, it follows that $\oint_{\mathcal{C}} d\omega_j = 0$. Thus, ignoring any poles Θ may have, we find that Θ satisfies the following jump conditions:

$$\Theta^+(z) = \begin{cases} \Theta^-(z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & z \in (a_j, b_j), \\[2ex] \Theta^-(z) \begin{bmatrix} e^{-v_j} & 0 \\ 0 & e^{v_j} \end{bmatrix}, & z \in (b_j, a_{j+1}), \\[2ex] \Theta^-(z), & z \in (-\infty, a_1) \cup (b_{g+1}, \infty). \end{cases}$$

Also, note that since $\mathbf{u}(\infty)$ is finite, Θ is analytic at infinity.

Next, we must understand the poles of Θ. It is known that (see [9], for example) if for $D = P_1 + \cdots + P_g$, $\theta(\mathbf{u}(P) - \mathcal{A}(D) - \mathbf{k})$ is not identically zero,[10] then, counting multiplicities, $\theta(\mathbf{u}(P) - \mathcal{A}(D) - \mathbf{k})$, has g zeros on Σ. These zeros are then given by the points in the divisor D. Consider the function

$$\gamma(z) = \left[\prod_{j=1}^{g+1} \left(\frac{z - b_j}{z - a_j} \right) \right]^{1/4} ,$$

analytic on $\mathbb{C} \setminus \bigcup_j [a_j, b_j]$, with $\gamma(z) \sim 1, z \to \infty$. It can be shown that $\gamma - \gamma^{-1}$ has a single root z_j in (b_j, a_{j+1}) for $j = 1, 2, \ldots, g$, while $\gamma + \gamma^{-1}$ does not vanish on $\mathbb{C} \setminus \bigcup_j [a_j, b_j]$. So, define two divisors

$$D_1 = \sum_{j=1}^g \pi_1^{-1}(z_j), \qquad D_2 = \sum_{j=1}^g \pi_2^{-1}(z_j).$$

From [21] (see also [37, Lemma 11.10]), these divisors are nonspecial so that the θ functions we consider do not vanish identically.

Note that for $\mathbf{d}_1 := \mathcal{A}(D_1) + \mathbf{k}$, the function $z \mapsto \theta(\mathbf{u}(z) - \mathbf{d}_1; \boldsymbol{\tau})$ has zeros at z_j, while the function $z \mapsto \theta(-\mathbf{u}(z) - \mathbf{d}_1; \boldsymbol{\tau})$ is non-vanishing. Similarly, for $\mathbf{d}_2 := \mathcal{A}(D_2) + \mathbf{k}$, the function $z \mapsto \theta(-\mathbf{u}(z) - \mathbf{d}_2; \boldsymbol{\tau})$ has zeros at z_j, while the function $z \mapsto \theta(\mathbf{u}(z) - \mathbf{d}_2; \boldsymbol{\tau})$ is non-vanishing.

Following [14], consider

$$\mathbf{L}(z; \mathbf{v}) = \begin{bmatrix} \left(\frac{\gamma(z) + \gamma(z)^{-1}}{2} \right) \Theta_1(z; \mathbf{d}_2; \mathbf{v}) & \left(\frac{\gamma(z) - \gamma(z)^{-1}}{2i} \right) \Theta_2(z; \mathbf{d}_2; \mathbf{v}) \\[2ex] \left(\frac{\gamma(z)^{-1} - \gamma(z)}{2i} \right) \Theta_1(z; \mathbf{d}_1; \mathbf{v}) & \left(\frac{\gamma(z) + \gamma(z)^{-1}}{2} \right) \Theta_2(z; \mathbf{d}_1; \mathbf{v}) \end{bmatrix}, \tag{D.2}$$

which is analytic in $\mathbb{C} \setminus \bigcup_j [a_j, b_j]$, with a limit as $z \to \infty$ and satisfies the jumps

$$\mathbf{L}^+(z; \mathbf{v}) = \begin{cases} \mathbf{L}^-(z; \mathbf{v}) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & z \in (a_j, b_j), \\[3ex] \mathbf{L}^-(z; \mathbf{v}) \begin{bmatrix} e^{-v_j} & 0 \\ 0 & e^{v_j} \end{bmatrix}, & z \in (b_j, a_{j+1}), \\[3ex] \mathbf{L}^-(z; \mathbf{v}), & z \in (-\infty, a_1) \cup (b_{g+1}, \infty). \end{cases}$$

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[10] This holds if D is nonspecial.

This follows because $\gamma^+(z) = i\gamma^-(z)$ for $z \in (a_j, b_j)$ and therefore

$$\gamma^+(z) + \left(\gamma(z)^{-1}\right)^+ = i\left(\gamma^-(z) - \left(\gamma(z)^{-1}\right)^-\right),$$
$$\gamma^+(z) - \left(\gamma(z)^{-1}\right)^+ = i\left(\gamma^-(z) + \left(\gamma(z)^{-1}\right)^-\right).$$

It is important to note that (D.2) was first used in [14] and subsequently by many, see [11, 13, 16], for example.

E   Estimates of some exponential integrals

In this appendix, we discuss how to estimate integrals of the form

$$\int_{E_\rho(a,b)} |z-a|^{-\gamma}|z-b|^{-\gamma}\left|e^{-m\mathfrak{g}(z)}\right| dA(z),$$
$$E_\rho = \{z \mid |z + \sqrt{z+1}\sqrt{z-1}| < \rho\}, \qquad \rho > 1,$$
$$E_\rho(a,b) = M^{-1}(E_\rho), \qquad M^{-1}(z) = M^{-1}(z; a, b) = \frac{b-a}{2}x + \frac{b+a}{2},$$

as $m \to \infty$. From Proposition A.1, it follows that we can consider, for a new m,

$$I_m := \int_{E_\rho} \left|z^2 - 1\right|^{-\gamma}|\phi(z)|^{-m} dA(z).$$

We then set $z = 1/2\left(w + w^{-1}\right)$ for $1 < |w| < \rho$ and change variables. We have

$$dA(z) = \frac{1}{4}\frac{\left|w^2 - 1\right|^2}{|w|^4} dA(w).$$

To finish the change of variables, we note that

$$\varphi(z) = w, \qquad z^2 - 1 = \left[\frac{1}{2}\left(w - w^{-1}\right)\right]^2 = w^{-2}\left[\frac{1}{2}\left(w^2 - 1\right)\right]^2.$$

Therefore,

$$I_m = \int_{1<|w|<\rho} |w|^{2\gamma-4-m}2^{2\gamma-2}\left|w^2 - 1\right|^{2-2\gamma} dA(w).$$

For $\gamma \leq 1$, we have $I_m = O\left(m^{-1}\right)$, and therefore

$$\int_{E_\rho(a,b)} |z-a|^{-\gamma}|z-b|^{-\gamma}|e^{-m\mathfrak{g}(z)}| dA(z) = O\left(m^{-1}\right). \tag{E.1}$$

Now, suppose $0 \geq 2 - 2\gamma = -\sigma > -1$. We write $w = \rho e^{i\theta}$ and $\theta \in [-\pi/2, \pi/2]$, following [41], write

$$|w - 1| = \sqrt{(\rho-1)^2 + 4\rho\sin^2(\theta/2)} \geq C[(\rho-1) + |\theta|],$$

for some $C > 0$. Consider

$$\int_1^\rho \int_{-\pi/2}^{\pi/2} \rho^{-m}|w-1|^{-\sigma} d\rho d\theta \leq C^{-\sigma}\int_1^\rho \rho^{-m}\left(\int_{-\pi/2}^{\pi/2} \frac{d\theta}{((\rho-1) + |\theta|))^\sigma}\right) d\rho.$$

For $\sigma < 1$, the θ integral produces a continuous function of ρ. Since it therefore must be bounded, we can conclude the integral is $O\left(m^{-1}\right)$. This argument can then be used for w in the right and left half-planes separately, bounding $|w \pm 1|^{-\sigma} \leq 1$ for $\pm\operatorname{Re} w \geq 0$ to also conclude (E.1) for $\gamma < 3/2$.

Acknowledgments

References

[1] Akhiezer N.I., Orthogonal polynomials on several intervals, *Dokl. Akad. Nauk SSSR* **134** (1960), 9–12.

[2] Akhiezer N.I., Tomčuk Yu.Ya., On the theory of orthogonal polynomials over several intervals, *Soviet Math. Dokl.* **2** (1961), 687–690.

[3] Astala K., Iwaniec T., Martin G., Elliptic partial differential equations and quasiconformal mappings in the plane, *Princeton Math. Ser.*, Vol. 48, Princeton University Press, Princeton, NJ, 2009.

[4] Atkinson K., Han W., Theoretical numerical analysis. A functional analysis framework, 3rd ed., *Texts Appl. Math.*, Vol. 39, Springer, Dordrecht, 2009.

[5] Baik J., Kriecherbauer T., McLaughlin K.T.-R., Miller P.D., Discrete orthogonal polynomials. Asymptotics and applications, *Ann. of Math. Stud.*, Vol. 164, Princeton University Press, Princeton, NJ, 2007.

[6] Ballew C., Trogdon T., A Riemann–Hilbert approach to computing the inverse spectral map for measures supported on disjoint intervals, *Stud. Appl. Math.* **152** (2024), 31–72, arXiv:2302.12930.

[7] Ballew C., Trogdon T., The Akhiezer iteration, in Recent Developments in Orthogonal Polynomials, *Contemp. Math.*, Vol. 822, American Mathematical Society, Providence, RI, 2025, 1–34, arXiv:2312.02384.

[8] Baratchart L., Yattselev M., Convergent interpolation to Cauchy integrals over analytic arcs with Jacobi-type weights, *Int. Math. Res. Not.* **2010** (2010), 4211–4275, arXiv:0911.3850.

[9] Belokolos E.D., Bobenko A.I., Enol'skii V.Z., Its A.R., Matveev V.B., Algebro-geometric approach to nonlinear integrable equations, Springer, Berlin, 1994.

[10] Canuto C., Hussaini M.Y., Quarteroni A., Zang T.A., Spectral methods. Fundamentals in single domains, *Sci. Comput.*, Springer, Berlin, 2006.

[11] Chen Y., Its A.R., A Riemann–Hilbert approach to the Akhiezer polynomials, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **366** (2008), 973–1003, arXiv:2007.2058.

[12] Deift P., Orthogonal polynomials and random matrices: a Riemann–Hilbert approach, *Courant Lect. Notes Math.*, Vol. 3, American Mathematical Society, Providence, RI, 1999.

[13] Deift P., Its A., Kapaev A., Zhou X., On the algebro-geometric integration of the Schlesinger equations, *Comm. Math. Phys.* **203** (1999), 613–633.

[14] Deift P., Its A., Zhou X., A Riemann–Hilbert approach to asymptotic problems arising in the theory of random matrix models, and also in the theory of integrable statistical mechanics, *Ann. of Math.* **146** (1997), 149–235.

[15] Deift P., Kriecherbauer T., McLaughlin K.T.-R., Venakides S., Zhou X., Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* **52** (1999), 1491–1552.

[16] Deift P., Kriecherbauer T., McLaughlin K.T.-R., Venakides S., Zhou X., Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* **52** (1999), 1335–1425.

[17] Ding X., Trogdon T., The conjugate gradient algorithm on a general class of spiked covariance matrices, *Quart. Appl. Math.* **80** (2022), 99–155, arXiv:2106.13902.

[18] Ding X., Trogdon T., A Riemann–Hilbert approach to the perturbation theory for orthogonal polynomials: applications to numerical linear algebra and random matrix theory, *Int. Math. Res. Not.* **2024** (2024), 3975–4061, arXiv:2112.12354.

[19] Ding X., Yang F., Spiked separable covariance matrices and principal components, *Ann. Statist.* **49** (2021), 1113–1138, arXiv:1905.13060.

[20] Ditzian Z., Totik V., Moduli of smoothness, *Springer Ser. Comput. Math.*, Vol. 9, Springer, New York, 1987.

[21] Dubrovin B.A., Theta-functions and nonlinear equations (with an appendix by I.M. Krichever), *Russian Math. Surveys* **36** (1981), 11–80.

[22] Fokas A.S., Its A.R., Kitaev A.V., The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* **147** (1992), 395–430.

[23] Folland G.B., Real analysis. Modern techniques and their applications, 2nd ed., *Pure Appl. Math.*, John Wiley & Sons, New York, 1999.

[24] Geronimo J.S., Van Assche W., Orthogonal polynomials on several intervals via a polynomial mapping, *Trans. Amer. Math. Soc.* **308** (1988), 559–581.

[25] Gragg W.B., Harrod W.J., The numerically stable reconstruction of Jacobi matrices from spectral data, *Numer. Math.* **44** (1984), 317–335.

[26] Kriecherbauer T., McLaughlin K.T.-R., Strong asymptotics of polynomials orthogonal with respect to Freud weights, *Int. Math. Res. Not* **1999** (1999), 299–333.

[27] Kuijlaars A.B.J., Riemann–Hilbert analysis for orthogonal polynomials, in Orthogonal Polynomials and Special Functions, *Lecture Notes in Math.*, Vol. 1817, Springer, Berlin, 2003, 167–210.

[28] Kuijlaars A.B.J., McLaughlin K.T.-R., Van Assche W., Vanlessen M., The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1, 1]$, *Adv. Math.* **188** (2004), 337–398.

[29] Levin E., Lubinsky D., Orthogonal polynomials for exponential weights $x^{2\rho}e^{-2Q(x)}$ on $[0, d)$. II, *J. Approx. Theory* **139** (2006), 107–143.

[30] McLaughlin K.T.-R., Miller P.D., The $\overline{\partial}$ steepest descent method and the asymptotic behavior of polynomials orthogonal on the unit circle with fixed and exponentially varying nonanalytic weights, *Int. Math. Res. Pap.* **2006** (2006), 48673, 77 pages, arXiv:math.CA/0406484.

[31] McLaughlin K.T.-R., Miller P.D., The $\overline{\partial}$ steepest descent method for orthogonal polynomials on the real line with varying weights, *Int. Math. Res. Not.* **2008** (2008), 075, 66 pages, arXiv:0805.1980.

[32] Olver F.W.J., Lozier D.W., Boisvert R.F., Clark C.W., NIST handbook of mathematical functions, Cambridge University Press, Cambridge, 2010.

[33] Saff E.B., Totik V., Logarithmic potentials with external fields, *Grundlehren Math. Wiss.*, Vol. 316, Springer, Berlin, 1997.

[34] Szegőo G., Orthogonal polynomials, *Amer. Math. Soc. Colloq. Publ.*, Vol. 23, American Mathematical Society, New York, 1939.

[35] Townsend A., Trogdon T., Olver S., Fast computation of Gauss quadrature nodes and weights on the whole real line, *IMA J. Numer. Anal.* **36** (2016), 337–358, arXiv:1410.5286.

[36] Trogdon T., Olver S., A Riemann–Hilbert approach to Jacobi operators and Gaussian quadrature, *IMA J. Numer. Anal.* **36** (2016), 174–196, arXiv:1311.5838.

[37] Trogdon T., Olver S., Riemann–Hilbert problems, their numerical solution, and the computation of nonlinear special functions, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2016.

[38] Van Assche W., Asymptotics for orthogonal polynomials, *Lecture Notes in Math.*, Vol. 1265, Springer, Berlin, 1987.

[39] Walsh J.L., Interpolation and approximation by rational functions in the complex domain, 3rd ed., *Amer. Math. Soc. Colloq. Publ.*, American Mathematical Society, Providence, RI, 1960.

[40] Yattselev M.L., Nuttall's theorem with analytic weights on algebraic S-contours, *J. Approx. Theory* **190** (2015), 73–90, arXiv:1406.0832.

[41] Yattselev M.L., On smooth perturbations of Chebyshëv polynomials and $\overline{\partial}$-Riemann–Hilbert method, *Canad. Math. Bull.* **66** (2023), 142–155, arXiv:2202.10374.