Basis Partitions and Their Signature

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Received July 23, 2025, in final form November 28, 2025; Published online December 11, 2025
https://doi.org/10.3842/SIGMA.2025.104

Abstract. Basis partitions are minimal partitions corresponding to successive rank vectors. We show combinatorially how basis partitions can be generated from primary partitions which are equivalent to the Rogers–Ramanujan partitions. This leads to the definition of a signature of a basis partition that we use to explain certain parity results. We then study a special class of basis partitions which we term as complete. Finally, we discuss basis partitions and minimal basis partitions among partitions with non-repeating odd parts by representing them using 2-modular graphs.

Key words: basis partitions; Rogers–Ramanujan partitions; Durfee squares; sliding operation; signature; partial theta series

2020 Mathematics Subject Classification: 05A17; 05A19; 05A15

Dedicated to Steve Milne on the occasion of his 75th birthday

1 Introduction

For a partition $\pi\colon b_1 \geq b_2 \geq \cdots \geq b_\nu$, its Ferrers graph is an array of left-justified rows with b_i nodes in the i-th row, $i = 1, 2, \dots, \nu$. The conjugate π^* of π is obtained by counting the nodes in the columns of the Ferrers graph of π. Let π^* be $c_1 \geq c_2 \geq \cdots \geq c_{b_1}$.

In every Ferrers graph, there is a largest square of nodes called the *Durfee square* which is the same for the conjugate as well. Given a partition π as above, let it have a $k \times k$ Durfee square. The successive rank vector of π is defined as $\mathbf{r} = (r_1, r_2, \dots, r_k)$, where

$$r_i = b_i - c_i \qquad \text{for } i = 1, 2, \dots, k.$$

Note that $r_1 = b_1 - c_1$ is the *rank* that Dyson [9] used to combinatorially explain two of Ramanujan's congruences. This led to the terminology *successive ranks* for the differences $b_i - c_i$ which are equal to hook differences. Their study has led to the discovery of important partition identities, the results in Andrews et al. [7] being a good example.

Given $\mathbf{r}$, there are, in general, several partitions π which will have $\mathbf{r}$ as the successive rank vector. Among all partitions π having a prescribed successive rank vector, consider the partition for which the sum of the parts $\sigma(\pi) = b_1 + b_2 + \cdots + b_\nu$ (the integer being partitioned) is minimal. Gupta [10] called such minimal partitions as *basis partitions* and proved their existence. He also noticed that if π is the basis partition associated with a rank vector $\mathbf{r} = (r_1, r_2, \dots, r_k)$ and if $n > \sigma(\pi)$, then the number of partitions of n which have the same rank vector $\mathbf{r}$ is equal to the number of partitions of $(n - \sigma(\pi))/2$ into $\leq k$ parts.

We provide here a combinatorial procedure to generate basis partitions from primary partitions which are equivalent to the Rogers–Ramanujan partitions (see Section 3), and use that

This paper is a contribution to the Special Issue on Basic Hypergeometric Series Associated with Root Systems and Applications in honor of Stephen C. Milne's 75th birthday. The full collection is available at https://www.emis.de/journals/SIGMA/Milne.html

for a new study of basis partitions among partitions with non-repeating odd parts (Section 9). Hirschhorn [11] had noticed that the number of basis partitions of an integer n can be expressed as a weighted count over the Rogers–Ramanujan partitions of n, where the weights are powers of 2. He deduced this by suitable interpretation of the generating function of basis partitions obtained by Nolan, Savage, and Wilf [12].

Our approach provides a very natural combinatorial proof of Hirschhorn's weighted identity as well as a refinement of the generating function of basis partitions that leads to a statistic which we call the *signature*. A study of the parity of the signature yields a partial theta series involving the squares (Section 5). This parity result has nice counterparts for partitions with non-repeating odd parts (Section 9).

2 The generating function of basis partitions

The Ferrers graph of any partition π may be split into three parts as follows: (i) the Durfee square $D(\pi)$, (ii) the partition π_r which is the part to the right of the Durfee square, and (iii) the partition π_b which is the portion below the Durfee square. If in a partition there is a column of π_r which is equal in length to a row of π_b, then we may remove this row and column to get a smaller partition which will have the same successive rank vector. Thus, a basis partition, by definition of its minimality, will not have such a redundancy. Thus, we have the following:

Characterization. *π is a basis partition if and only if no column of π_r equals any row of π_b.* Conversely, given a basis partition with rank vector $\mathbf{r}$ of length k, all partitions having the same rank vector can be generated by inserting columns of length $\leq k$ to the right of the Durfee square, and inserting rows below the Durfee square exactly equal in length and multiplicity to the inserted columns. This observation can be used to compute the generating function of basis partitions.

Let $b(n;k)$ denote the number of basis partitions whose rank vector is of length k. Let $p(n;k)$ denote the number of partitions having a $k \times k$ Durfee square. It is well known that

$$\sum_n p(n;k)q^n = \frac{q^{k^2}}{\big\{(1-q)(1-q^2)\cdots(1-q^k)\big\}^2}. \tag{2.1}$$

Also, from the observation in the preceding paragraph, we have

$$\sum_n p(n;k)q^n = \frac{1}{(1-q^2)(1-q^4)\cdots(1-q^{2k})}\sum_n b(n;k)q^n. \tag{2.2}$$

From (2.1) and (2.2), it follows that

$$\sum_n b(n;k)q^n = \frac{q^{k^2}(1+q)(1+q^2)\cdots(1+q^k)}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{2.3}$$

The derivation of (2.3) given above is due to Hirschhorn [11]. In [12], the above characterization of basis partitions is observed, but the derivation of (2.3) is by other means. Andrews [6] has studied basis partitions and a certain polynomial associated with (2.3), but he interpreted (2.3) in terms of overpartitions.

Rogers–Ramanujan partitions $\pi\colon b_1+b_2+\cdots+b_k$ are those whose parts differ by at least 2. That is,

$$b_i - b_{i+1} \geq 2, \qquad \text{for } i = 1, 2, \ldots, k-1.$$

It is well known [5] that the generating function of the Rogers–Ramanujan partitions into exactly k parts is

$$\frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{2.4}$$

The way one realizes this is to note that the minimal partition into k parts differing by ≥ 2 is $1+3+5+\cdots+(2k-1)=k^2$. This accounts for the term q^{k^2} in the numerator in formula (2.4). One can construct all partitions into k parts that differ by ≥ 2 from the minimal partition by inserting columns of length $\leq k$ into the Ferrers graph. This accounts for the factors $(1-q)(1-q^2)\cdots(1-q^k)$ in the denominator in (2.4).

Hirschhorn [11] notes that since

$$\frac{(1+q^j)}{(1-q^j)} = 1+2(q^j+q^{2j}+\cdots+q^{nj}+\cdots),$$

the insertion of the columns of a given length j into the minimal partition increases the size of the gap and the increase is to be counted with weight 2. Thus, he deduces the following.

Theorem 2.1 (Hirschhorn [11])**.** *Let* $\mathbf{R}$ *denote the set of Rogers–Ramanujan partitions. For each partition* $\pi \in \mathbf{R}$*,*

$$\pi:\ b_1+b_2+\cdots+b_k, \qquad b_i - b_{i+1} \geq 2 \quad \text{for } i=1,2,\ldots,k-1, \ \text{and } b_k \geq 1, \tag{2.5}$$

let its weight be $\omega(\pi)=2^t$*, where* t *is the number of strict inequalities in* (2.5)*. Then*

$$b(n,k) = \sum_{\pi\in\mathbf{R},\, \sigma(\pi)=n,\, \nu(\pi)=k} \omega(\pi).$$

Here $\nu(\pi)$ *denotes the number of parts of a partition* π*.*

3 Constructing basis partitions from primary partitions

We define a *primary* partition π to be one for which π_b is empty, that is, there are no nodes below the Durfee square. It is well known [5] that the expression in (2.4) is the generating function for partitions into k parts, each part $\geq k$. This is the same as saying that the expression in (2.4) is the generating function of primary partitions into k parts. The primary partitions are bijectively equivalent to the Rogers–Ramanujan partitions as the following correspondence shows: Given the Ferrers graph of a primary partition π into k parts, consider the partition π' obtained by counting nodes along the hooks of the graph. Then π' is a Rogers–Ramanujan partition, that is, a partition into parts that differ by ≥ 2. This correspondence can be reversed.

The term *primary partitions* is ours [1]. Although it is well known that the expression in (2.4) is the generating function of these partitions, the most common interpretation and use of (2.4) has been its role as the generating function of the Rogers–Ramanujan partitions. We noticed [1, 2], that primary partitions play a crucial role in the theory of weighted partition identities. Indeed, we established a variety of weighted partition identities by performing *sliding operations* (defined below) on primary partitions. This is also what motivated us to coin the term primary partitions.

Note that since a primary partition π has no nodes below the Durfee square, we have trivially that no column of π_r equals a row of π_b. Thus, every primary partition is automatically a basis partition. We will now show how to generate all basis partitions from primary partitions. This will provide a combinatorial proof of Theorem 2.1. Before providing this construction, we reformulate Theorem 2.1 as follows.

Theorem 3.1. *Let* $\mathbf{P}$ *denote the set of primary partitions. For each partition* $\pi \in \mathbf{P}$,

$$\pi: \ b_1 + b_2 + \cdots + b_k, \qquad b_i - b_{i+1} \geq 0 \quad \text{for } i = 1, 2, \ldots, k-1, \ \text{and } b_k \geq k, \tag{3.1}$$

let its weight be $\omega(\pi) = 2^t$, *where* t *is the number of strict inequalities in* (3.1). *Then*

$$b(n,k) = \sum_{\pi \in \mathbf{P},\, \sigma(\pi)=n,\, \nu(\pi)=k} \omega(\pi).$$

Proof. Given a primary partition π, suppose a column of π_r is moved and placed as a row below the Durfee square. We call this a *sliding operation*. On a given primary partition, several sliding operations can be performed. The following are invariants under the sliding operation: (i) $\sigma(\pi)$, the number being partitioned, (ii) $D(\pi)$, the Durfee square of the partition, and (iii) the hook lengths of the partition. This last invariant is important here because the underlying Rogers–Ramanujan partition obtained by counting nodes along hooks remains invariant under sliding operations.

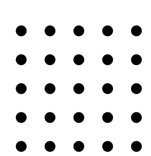
Suppose we consider columns of a certain length ℓ in the portion π_r of the primary partition π. If a column of length ℓ is moved down and placed as a row below the Durfee square, then we will not have a basis partition if there are columns of length ℓ remaining in the portion to the right of the Durfee square, because this violates the characterization of basis partitions (see Section 2). But then, if we move *all* columns of length ℓ and place them as rows below the Durfee square, then the characterization condition is confirmed and we have a basis partition. Thus, under the sliding operation, we have a choice of either not to move any column of length ℓ or move *all* columns of length ℓ to form basis partitions. Thus, we have two choices for every block of columns of a given length in π_r. Finally, we note that under the sliding operation, movement of columns of a certain length is *independent* of the movement of columns of a different length. Thus, if the primary partition π contains t sets of columns of different lengths in π_r, then it spawns 2^t basis partitions (including itself) under the sliding operation. All basis partitions can be obtained from primary partitions this way. The number t of sets of columns of different lengths in π_r corresponds to the number of strict inequalities in (3.1) (and in (2.5)) and so this combinatorial construction provides a proof of Theorem 3.1 (and of Theorem 2.1). ■

4 Constructing a basis partition of a successive rank vector

We will now discuss how to construct a basis partition corresponding to a successive rank vector $\mathbf{r} = (r_1, r_2, \ldots, r_k)$. The construction will be illustrated with the example

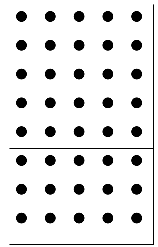
$$\mathbf{r} = (r_1, r_2, r_3, r_4, r_5) = (3, 2, -1, 4, -3).$$

Step 1. First form a $k \times k$ Durfee square.



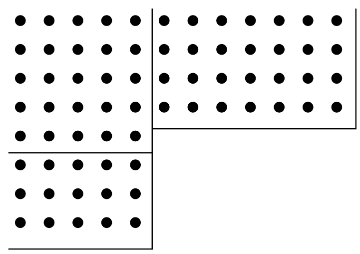
basis partition corresponding to $\mathbf{r} = (0, 0, 0, 0, 0)$

Step 2. If $r_k > 0$, form r_k columns of length k to the right of the Durfee square. If $r_k < 0$, form $|r_k|$ rows of length k below the Durfee square. The resulting partition has all successive ranks equal to r_k. If $r_k = 0$, then move on to r_{k-1}.



$r_5 = -3$, basis partition corresponding to $\mathbf{r} = (-3, -3, -3, -3, -3)$

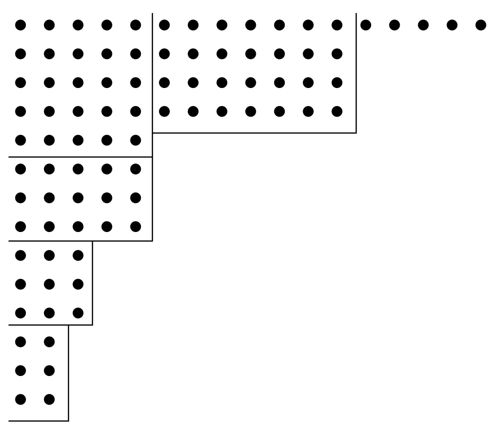
Step 3. Next consider the difference $r_{k-1} - r_k = \delta_{k-1}$. If this difference is > 0, place δ_{k-1} columns of length $k-1$ to the right of the Durfee square. If the difference is < 0, place $|\delta_{k-1}|$ rows of length $k-1$ below the Durfee square. Note that all successive ranks up to the $k-1$-st are equal to r_{k-1}, and the last rank is r_k. If $\delta_{k-1} = 0$, move on to r_{k-2}.



$r_4 - r_5 = 7 = \delta_4$, basis partition corresponding to $\mathbf{r} = (4, 4, 4, 4, -3)$

Step 4 (*general*). Consider the difference $r_i - r_{i+1} = \delta_i$. If this difference is > 0, place δ_i columns of length i to the right of the Durfee square. If the difference is < 0, place $|\delta_i|$ rows of length i below the Durfee square. If $\delta_i = 0$, proceed to r_{i-1}. Note that all successive ranks up to the i-th are equal to r_i, but from then on the successive ranks are $r_{i+1}, \ldots, r_k$.

Step 5. Complete the construction by considering the final case $i = 1$ in Step 4. We end up with a basis partition with successive rank vector $(r_1, r_2, \ldots, r_k)$ as desired.



$\delta_1 = 5$, $\delta_2 = -3$, $\delta_3 = -3$, $\delta_4 = 7$, $\delta_5 = -3$, basis partition corresponding to $\mathbf{r} = (3, -2, 1, 4, -3)$

5 The signature of basis partitions and a theta series

The combinatorial proof given in Section 3, and the construction of the basis partition corresponding to successive rank vector given in Section 4, lead to the definition of the *signature* $\Sigma(\pi)$ of a basis partition to be the number of differences $r_i - r_{i+1} = \delta_i$ that are negative. Note that

$$\Sigma(\pi) = \nu_d(\pi_b),$$

where $\nu_d(\pi_b)$ is the number of different parts of π_b. The combinatorial constructions of Sections 3 and 4 enable us to keep track of this signature and obtain a refined generating function of basis partitions. More precisely, if $b(n, k, s)$ denotes the number of basis partitions of n with a $k \times k$ Durfee square and with signature s, then for a fixed k, we get the following refinement of (2.3):

$$\sum_{n,s} b(n,k,s) q^n z^s = \frac{q^{k^2}(1+zq)(1+zq^2)\cdots(1+zq^k)}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{5.1}$$

Observe that $z = -1$ makes the right hand side of (5.1) collapse to q^{k^2}. Thus, we get the partial theta series identity

$$\sum_{n,k,s} b(n,k,s) q^n (-1)^s = \sum_{k=0}^{\infty} q^{k^2}.$$

This has the following partition interpretation.

Theorem 5.1. *For an integer n, consider the difference between the number of basis partitions of n with even signature and the number of basis partitions with odd signature. This difference is 1 if n is a perfect square, and 0 otherwise.*

Actually, the discussion of the signature leads to a refinement of Theorems 2.1 and 3.1, where the weight 2^t is replaced by $(1+z)^t$ so that Theorems 2.1 and 3.1 correspond to the case $z = 1$ and Theorem 5.1 follows as the special case $z = -1$. We state these refinements as Theorem 5.2 (i) and (ii) below.

Theorem 5.2.

(*i*) *Let* $\mathbf{R}$ *denote the set of Rogers–Ramanujan partitions. For each partition* $\pi \in \mathbf{R}$,

$$\pi\colon\ b_1 + b_2 + \cdots + b_k, \qquad b_i - b_{i+1} \geq 2 \quad \text{for } i = 1, 2, \ldots, k-1, \ \text{and } b_k \geq 1, \tag{5.2}$$

let its weight be $\omega_z(\pi) = (1+z)^t$, *where* t *is the number of strict inequalities in* (5.2). *Then for each* n *and* k *we have*

$$\sum_{s} b(n,k,s) z^s = \sum_{\pi \in \mathbf{R},\, \sigma(\pi)=n,\, \nu(\pi)=k} \omega_z(\pi).$$

(*ii*) *Let* $\mathbf{P}$ *denote the set of primary partitions. For each partition* $\pi \in \mathbf{P}$,

$$\pi\colon b_1 + b_2 + \cdots + b_k, \qquad b_i - b_{i+1} \geq 0 \quad \text{for } i = 1, 2, \ldots, k-1, \ \text{and } b_k \geq k, \tag{5.3}$$

let its weight be $\omega_z(\pi) = (1+z)^t$, *where* t *is the number of strict inequalities in* (5.3). *Then for each* n *and* k *we have*

$$\sum_{s} b(n,k,s) z^s = \sum_{\pi \in \mathbf{P},\, \sigma(\pi)=n,\, \nu(\pi)=k} \omega_z(\pi).$$

6 A Franklin type involution for Theorem 5.1

For Euler's celebrated pentagonal number theorem, Fabian Franklin provided an elegant combinatorial proof (see [5]). Franklin's proof utilized an involution on Ferrers graphs of partitions of an integer into distinct parts by considering the *base* and *slope* on these graphs. We will now produce an involution on the Ferrers graphs of basis partitions, in which the base and slope are replaced by *bottom block* and *right block*.

Given a Ferrers graph of a basis partition, we define a *block* to be the full collection of columns of a given length to the right of the Durfee square, or the full collection of rows of a given length below the Durfee square. Thus, the part to the right of the Durfee square is a collection of blocks, as is the portion below the Durfee square. The number of blocks below the Durfee square is equal to the number of different parts below the Durfee square, and therefore equal to the signature of the basis partition.

Next we define the length of a block to be its column length if it is to the right of the Durfee square, and its row length if it is below the Durfee square. Since a basis partition π is characterized by the property that no column of π_r equals a row of π_b, this is equivalent to saying that the blocks in the graph of a basis partition all have different lengths.

Finally, define the *right block* to be the block on the extreme right in π_r, and the *bottom block* to be the block at the very bottom of π_b. Note that the lengths of the right block and the bottom block (if they exist) are different. This leads us to the following definitions.

Definition B. Given the Ferrers graph of a basis partition, we call it a Type B partition, if the bottom block exists, and if its length is less than the length of the right block (if the right block exists). If the right block does not exist, but the bottom block does, it is still a Type B partition.

Definition R. Given the Ferrers graph of a basis partition, we call it a Type R partition, if the right block exists, and if its length is less than the length of the bottom block (if the bottom block exists). If the bottom block does not exist, but the right block does, it is still a Type R partition.

Thus, every Ferrers graph of a basis partition, unless it is just the Durfee square, is either a Type B partition or a Type R partition, but not both.

Involution. Now we define a mapping as follows. If we have a Type B partition, we move the bottom block, convert its rows into columns, and place it on the extreme right thereby making it the (new) right block. Thus, we create a Type R partition. Conversely, if we have a Type R partition, we move the right block, convert its columns into rows, and place it at the very bottom of the graph to create a Type B partition. Notice that under this mapping, the *parity* of the number of parts below the Durfee square, namely the *parity of the signature*, changes. The only graph on which this mapping does not apply, is the graph consisting of just the Durfee square and so this provides a combinatorial proof of Theorem 5.1.

Conjugation as an involution. It is worth noting that if π is a basis partition, then its (Ferrers) conjugate $\pi^*$is also a basis partition. This is because basis partitions π are characterized by the property that no row of π_b equals a column of π_r, and this property is preserved under conjugation. Also, this characterization implies that π and π^* are different basis partitions unless the graph of π is simply a Durfee square. Thus, by matching every basis basis partition π with its conjugate π^*, we see the number $b(n)$ of basis partitions of n is even unless n is a perfect square. This parity property of $b(n)$ follows from Theorem 5.1 but conjugation as an involution does not prove Theorem 5.1.

7 Basis partitions with a prescribed signature

In this section, we study the generating function of basis partitions with a prescribed signature, and discuss how they are connected to the Rogers–Ramanujan partitions. We will use the standard notation

$$(a;q)_n = (a)_n = \prod_{j=0}^{n-1} \left(1 - aq^j\right),$$

for any complex number a and a non-negative integer n, and when $|q| < 1$

$$(a)_\infty = \lim_{n\to\infty}(a)_n = \prod_{j=0}^{\infty}\left(1 - aq^j\right).$$

The construction in Section 3 showed that we get basis partitions from primary partitions (equivalent to Rogers–Ramanujan partitions) by sliding blocks from the right of the Durfee square and placing them below the Durfee square after converting the columns of these blocks into rows. If there are t blocks in the portion π_r, and j of these have to be moved, then the number of choices is $\binom{t}{j}$. Thus, from this observation, and by collecting the coefficient of z^j in Theorem 5.2, we get the following refinement of Hirschhorn's theorem.

Theorem 7.1. *Let* $\mathbf{R}$ *denote the set of Rogers–Ramanujan partitions. Suppose a partition* $\pi \in \mathbf{R}$,

$$\pi:\ b_1 + b_2 + \cdots + b_k, \qquad b_i - b_{i+1} \geq 2 \quad \text{for } i = 1, 2, \dots, k-1, \ \text{and } b_k \geq 1,$$

has t *strict inequalities in* (5.2). *For a given* j, *define the weight*

$$\omega_j(\pi) = \binom{t}{j}.$$

Then for each n *and* k, *we have*

$$b(n,k,j) = \sum_{\pi\in\mathbf{R},\, \sigma(\pi)=n,\, \nu(\pi)=k} \omega_j(\pi).$$

Theorem 7.1 is what we get combinatorially by looking at the sliding of blocks. But we get a *different* connection between basis partitions of a prescribed signature and Rogers–Ramanujan partitions by evaluating the coefficient of z^j in (5.1).

Denote by $B(n;j)$ the number of basis partitions of n with signature j (note the difference with $b(n;k)$). Thus

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(-zq)_k}{(q)_k} = \sum_j z^j \sum_n B(n;j) q^n. \tag{7.1}$$

But then

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{q^{k^2}(-zq)_k}{(q)_k} &= \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q)_k} \sum_{j=0}^{k} z^j q^{j(j+1)/2} \begin{bmatrix} k \\ j \end{bmatrix} \\ &= \sum_{j=0}^{\infty} \frac{z^j q^{(j(j+1)/2}}{(q)_j} \sum_{i=0}^{\infty} \frac{q^{(i+j)^2}}{(q)_i} = \sum_{j=0}^{\infty} \frac{z^j q^{(3j^2+j)/2}}{(q)_j} \sum_{i=0}^{\infty} \frac{q^{i^2+2ij}}{(q)_i}.\end{aligned} \tag{7.2}$$

By comparing coefficients of z^j in (7.1) and (7.2), we get the generating function of basis partitions with a prescribed signature as

$$\sum_{n=0}^{\infty} B(n;j) q^n = \frac{q^{(3j^2+j)/2}}{(q)_j} \sum_{i=0}^{\infty} \frac{q^{i^2+2ij}}{(q)_i}.$$

We will now interpret this combinatorially.

The term $q^{(3j^2+j)/2}/(q)_j$ is the generating function of partitions into j parts with difference ≥ 3 between the parts and smallest part ≥ 2. The term $q^{i^2+2ij}/(q)_i$ is the generating function for partitions into i parts with difference ≥ 2 between the parts (namely, the Rogers–Ramanujan partitions), but with the condition that the smallest part is $\geq 2j+1$. Thus, we have the following result.

Theorem 7.2. *The number of basis partitions $B(n;j)$ of an integer n with signature j is equal to the number of vector partitions (π_3, π_2) of n, where π_3 is a partition into j parts differing by ≥ 3 and least part ≥ 2, and π_2 is a partition into parts differing by ≥ 2 and least part $\geq 2j+1$.*

It is possible to convert these vector partitions into ordinary partitions, but then we need to count the resulting Rogers–Ramanujan partitions with weights. This is complicated and so we do not pursue it here.

8 Complete basis partitions

One of the fundamental properties of partitions into distinct parts is that if they are represented as Ferrers graphs, then the conjugate partitions have the property that every integer from 1 up to largest part occurs as a part. Motivated by this characterization of partitions into distinct parts, we now define a *complete basis partition* π to be a basis partition with the property that if $|D(\pi)| = k$, then every integer j from 1 to $k-1$ occurs either as a row length below $D(\pi)$ or as a column length to the right of $D(\pi)$, but not both because π is a basis partition ($D(\pi)$ is Durfee square of π). The reason for requiring only row (column) lengths up to $k-1$ will become clear below. From the construction of basis partitions in Section 4, it follows that the an integer j between 1 and $|D(\pi)|$ is missed as either a row length below $D(\pi)$ or a column length to the right of $D(\pi)$ if and only if in the successive rank vector $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, we have $r_j = r_{j+1}$, where we formally set $r_{k+1} = 0$. Thus, we have the following characterization of complete basis partitions.

Characterization. A basis partition with a $k \times k$ Durfee square is complete if and only if its successive rank vector $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ has the property that

$$r_j \neq r_{j+1} \qquad \text{for } j = 1, 2, \ldots, k-1.$$

We note that a row (column) of length k is missing precisely when $r_k = 0$.

We will now compute the generating function of complete basis partitions by connecting them with partitions into distinct parts. For this, we consider partitions into distinct parts which are primary, that is in their Ferrers graph representation, there is nothing below the Durfee square. So let π be a primary partition into distinct parts with $|D(\pi)| = k$. This Ferrers graph has two properties:

(i) To the right of the Durfee square there is a right-angled isosceles triangle of nodes with the two equal sides of the triangle being of length $\geq k-1$. This guarantees that there are at least $\big(3k^2 - k\big)/2$ nodes in the Ferrers graph.

(ii) Every integer between 1 and $k-1$ occurs as a column length to the right of $D(\pi)$.

In [1], we discussed the construction of all partitions into distinct parts from primary partitions into distinct parts by means of the sliding operation. In doing so we noted in [1] that at most one column of a given length could be slid down because the resulting partition has to have distinct parts. Here we shall focus of generating all complete basis partitions from primary partitions. What this means is that once we slide a column of a certain length j to the right of the Durfee square, we have to slide all columns of length j. Thus, for a column of length j on the right, there are precisely two choices – to slide the entire set of such columns or not to slide at all. Since these choices are independent, and every integer j from 1 to $k-1$ occurs as a column length, each primary partition into distinct parts generates at least 2^{k-1} complete basis partitions. If the graph of π has a column of length k, this will provide two more independent choices. Thus, if $b_c(n)$ denotes the number of complete basis partitions of n, its generating function is

$$\sum_{n=0}^{\infty} b_c(n)q^n = 1 + \sum_{k=1}^{\infty} \frac{2^{k-1}q^{(3k^2-k)/2}(1+q^k)}{(1-q^k)}. \tag{8.1}$$

The sliding operation that yielded the complete basis partitions from primary basis partitions shows that we could count complete basis partitions by keeping track of their signature. More precisely, if $b_c(n;j)$ denotes the number of complete basis partitions with signature j, then

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty} b_c(n;j)z^j q^n = 1 + \sum_{k=1}^{\infty} \frac{(1+z)^{k-1}q^{(3k^2-k)/2}(1+zq^k)}{(1-q^k)}. \tag{8.2}$$

Notice that when $z = -1$, the series in (8.2) reduces to $1 + q$. Thus, we have the following.

Theorem 8.1. *For each integer $n \geq 2$, the number of complete basis partitions of even signature is equal to the number of complete basis partitions of odd signature. In particular, for $n \geq 2$, $b_c(n)$ is always even.*

Theorem 8.1 can also be proved using the Franklin type involution discussed in Section 6.

The hook lengths of primary partitions into distinct parts yield partitions into parts that differ by at least 3, and conversely every partition $\tilde{\pi}$ into parts that differ by at least 3 can be realized in terms of hooks of a primary partition π into distinct parts. If $|D(\pi)| = k$, then $\tilde{\pi}$ has exactly k parts. With this identification, we have the following.

Theorem 8.2. *Let D_3 denote the set of partitions whose parts differ by at least 3. For a partition $\tilde{\pi} \in D_3$, let its weight be $w(\tilde{\pi}) = 2^{\nu(\tilde{\pi})-1}$ if $\tilde{\pi}$ has 1 as a part, and $w(\tilde{\pi}) = 2^{\nu(\tilde{\pi})}$ otherwise, where $\nu(\tilde{\pi})$ is the number of parts of $\tilde{\pi}$. Then*

$$b_c(n) = \sum_{\tilde{\pi}\in D_3, \sigma(\tilde{\pi})=n} w(\tilde{\pi}).$$

Theorem 8.2 is the combinatorial interpretation of (8.1).

Remark. Using (8.1) or Theorem 8.2, one can show that $b_c(n) \equiv 0 \pmod 4$ for $n \geq 5$. More generally, using (8.1) or Theorem 8.2 one can study the values of n for which $b_c(n)$ is a multiple of 2^k.

9 Basis partitions among partitions with non-repeating odd parts

We conclude the paper with a study of basis partitions among the set $P_{\mathrm{o,d}}$ of partitions with non-repeating odd parts. There are similarities with the theory of basis partitions for unrestricted partitions, but there are very interesting differences, and the results quite elegant. That is what motivated this study.

Partitions with non-repeating even parts have been studied extensively owing to the famous Lebesgue identity. Comparatively less is known about partitions with non-repeating odd parts. Our interest in partitions with non-repeating odd parts is due to the fact that if they are represented in terms of 2-modular Ferrers graphs (that is Ferrers graphs in which there is a 2 at every node except possibly at the last node on the right which would be 1 if the part is odd), then the ones can occur only in the corners. Also the conjugate of such a graph will also be a 2-modular graph of a partition with non-repeating odd parts. By exploiting these properties, we established in [3] a Lebesgue type expansion for the two parameter generating function

$$\prod_{m=1}^{\infty} \frac{(1+bzq^{2m-1})}{(1-zq^{2m})}$$

of partitions with non-repeating odd parts and studied connections between this Lebesgue type identity and various fundamental q-hypergeometric identities. More recently in [4], we studied the combinatorial properties of partitions in $P_{\text{o,d}}$ by means of this 2-modular representation. The results in [3] and [4] constitute the first systematic study of partitions in $P_{\text{o,d}}$ by means of their 2-modular graphs. In this section, we will study *minimal basis partitions* and *basis partitions* among the members of $P_{\text{o,d}}$. Results on such partitions in $P_{\text{o,d}}$ were reported in [4], but here we discuss these two types of partitions in $P_{\text{o,d}}$ in greater detail, and also provide a combinatorial method to construct them. Indeed, in [4], reference is made to this paper for details and the combinatorial construction given below.

From now, on by a partition we mean one in $P_{\text{o,d}}$ and by a Ferrers graph, we mean a 2-modular Ferrers graph. We shall from now on refer to the Ferrers graph of unrestricted partitions as ordinary Ferrers graphs.

Given a 2-modular Ferrers graph of a partition $\pi \in P_{\text{o,d}}$, we define the *length* of a row (column) to be the number of nodes in it, and the *size* of a row (column) to be the sum of the nodes in it. Thus, the row 2, 2, 1 will have length 3 and size 5. We do not need this distinction for ordinary Ferrers graphs. Next if $\pi \in P_{\text{o,d}}$ has a $k \times k$ Durfee square $D(\pi)$ in its 2-modular graph, we define its successive rank vector to be $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, where r_j = *size of the* j*-th row* $-$ *size of the* j*-th column*. By a *minimal basis partition*, we mean a partition $\mu \in P_{\text{o,d}}$ for which the sum of its parts $\sigma(\mu)$ is minimal with respect to a given successive rank vector. We will now establish the following statement.

Theorem 9.1. *Given any vector of integers $\mathbf{r} = (r_1, r_2, \ldots, r_k)$, there exists a unique minimal basis partition μ for which $\mathbf{r}$ is the successive rank vector.*

We will establish Theorem 9.1 by actually constructing μ from a given $\mathbf{r}$. Before proceeding with the construction, we note that the Durfee square $D(\pi)$ of any partition $\pi \in P_{\text{o,d}}$ will have all entries as twos, except possibly at the right hand bottom corner where there could be a 1. If there is a 1 in the Durfee square, then in the graph there will be no nodes to its right in the same row and no nodes below it in the same column.

We will illustrate the construction of μ in a sequence of steps with the successive rank vector $\mu = (1, -4, 1, 2, -5)$, but the construction applies in general as in Section 4. The construction, although similar to that in Section 4 has an important differences involving parity which we will point out below. That is why we give this construction even though there is overlap with Section 4.

Construction of the minimal basis partition. Let the given successive rank vector be $\mathbf{r} = (r_1, r_2, \ldots, r_k)$.

Step 1. Form a $k \times k$ Durfee square with a 1 at the bottom right node and twos everywhere else. This is the minimal basis partition corresponding to the vector

2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**1**

$\mathbf{r} = (0, 0, 0, 0, 0)$

Step 2. If $r_k = 0$, retain the 1 in the Durfee square and move to r_{k-1}. If $r_k > 0$ (resp. < 0), change the 1 to a 2 in the Durfee square, and represent r_k (resp. $|r_k|$) in 2-modular form as a row (resp. column) to the right of (resp. below) the k-th row of the Durfee square. Fill every thing above (resp. to the left of) the newly formed row (resp. column) with twos. The resulting partition will have the k-th successive rank as r_k, and all other successive ranks will be equal to r_k if r_k is even; if r_k is odd, then since the end of the k-th row (resp. column) will have a 1 with twos elsewhere, all other successive ranks will be $r_k + 1$ if $r_k > 0$ or $r_k - 1$ if $r_k < 0$. The important thing is that we now have r_k as the k-th successive rank. We will represent this new vector as $\left(r_k^*, r_k^*, \ldots, r_k^*, r_k\right)$. This is an important difference between the situation here and what we had in Section 4. In our example, we have constructed the minimal basis partition corresponding to

2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**1**

$$\mathbf{r} = (-6, -6, -6, -6, -5)$$

Step 3. Next consider the difference $r_{k-1} - r_k^* = \delta_{k-1}$. If this difference is > 0, write δ_{k-1} in 2-modular form to the right of the $(k-1)$-st row and fill everything above that with twos. If $\delta_{k-1} < 0$, write $|\delta_{k-1}|$ in 2-modular form as a column below the $(k-1)$-st column and fill everything to its left with twos. If $\delta_{k-1} = 0$, move on to r_{k-2}. Thus, we have formed the minimal basis partition who last two successive ranks are r_{k-1} and r_k. The remaining successive ranks will all be equal, their values being r_{k-1} if it is even, or $r_{k-1}+1$ if $r_{k-1} > 0$ and odd, or $r_{k-1} - 1$ if $r_{k-1} < 0$ and odd. We represent these remaining successive ranks by r_{k-1}^*. Thus, we have the minimal basis partition corresponding to $\left(r_{k-1}^*, r_{k-1}^*, \ldots, r_{k-1}^*, r_{k-1}, r_k\right)$. In our example, we have formed the minimal basis partition corresponding to

2	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**	**2**	**2**	**2**	**2**
2	**2**	**2**	**2**	**2**				
2	**2**	**2**	**2**	**2**				
2	**2**	**2**	**2**	**2**				
2	**2**	**2**	**2**	**1**				

$$\mathbf{r} = (2, 2, 2, 2, -5)$$

Step 4 (*general*). Consider the difference $r_j - r_{j+1}^* = \delta_j$. If this difference is > 0, represent δ_j in 2-modular form as a row to the right of the j-th row, and fill everything above with twos. If the difference is < 0, represent $|\delta_j|$ in 2-modular form as a column below the j-th column, and fill everything to its left with twos. If $\delta_j = 0$, proceed to r_{j-1}. Note that all successive ranks from r_j on are $r_j, r_{j+1}, \ldots, r_k$. the remaining $j-1$ successive ranks are all equal and we represent them by r_j^*. So we have formed the minimal basis partition corresponding to $\left(r_j^*, r_j^*, \ldots, r_j^*, r_j, r_{j+1}, \ldots, r_k\right)$.

Step 5. Complete the construction by considering the final case $j = 1$ in Step 4. we then have the minimal basis partition μ corresponding to the given successive rank vector $\mathbf{r}$. In our example, we have the minimal basis partition corresponding to

2	2	2	2	2	2	2	2	2	2	2	1
2	2	2	2	2	2	2	2	2			
2	2	2	2	2	2	2	2	2			
2	2	2	2	2	2	2	2	2			
2	2	2	2	2							
2	2	2	2	2							
2	2	2	2	2							
2	2	2	2	1							
2	2	1									
2	2										
2	2										

$$\mathbf{r} = (1, -4, 1, 2, -5)$$

Our construction has shown that minimal basis partitions are characterized by the following.

Theorem 9.2. *The partition μ is a minimal basis partition if and only if no row below the Durfee square is equal in length to a column to the right of the Durfee square, and the right hand bottom entry in the Durfee square is* 1 *if the last successive rank is* 0*.*

Remark 9.3.

(1) It is very interesting that even though minimal basis partitions were defined in terms of *size*, the characterization is in terms of *length*. In view of this, we now define a *basis partition* to be a partition in $P_{o.d}$ for which no row below the Durfee square is equal *in size* to a column to the right of the Durfee square. For example, in a basis partition we could have 2, 2, 1 as a row and 2, 2, 2 as a column, but this is not permissible in a minimal basis partition. Thus, every minimal basis partition is a basis partition, but the converse is not true.

(2) Berkovich and Garvan [8] have studied successive ranks among members of $P_{\mathrm{o,d}}$, but they defined the rank as the difference between the lengths of the row and the column, not in terms of the differences in sizes as we have considered here. But Berkovich–Garvan did not study basis partitions with their definition.

Construction of basis partitions from minimal basis partitions. The set of all basis partitions can be constructed from the set of all minimal basis partitions as we show now.

Suppose we are given a minimal basis partition μ. If the last successive rank is 0, then in the Durfee square of μ we have a 1. If we replace this 1 by a 2, the last rank would still be 0 and so we would get a basis partition corresponding to the same successive rank vector.

Note that in any hook of μ, there can be at most one 1 – either at the northeastern end or at the southwestern end but not both. Also, if there is a 1 in a hook of μ, then the 2 at the other end cannot be a corner, because it is were, then μ would have a row and column of equal length. So if there is a 1 in a hook of μ, we could replace that by a 2, and add a 1 to the other end of the hook *provided* there is a 2 immediately to the left of (or above) the position where are introducing the new 1. We call such a hook as an allowable hook. If a minimal basis partition has α allowable hooks (including the special case where the Durfee square has a 1 in it), then with each such hook we have the choice of making such a change or not. Thus, each minimal basis partition with α allowable hooks, will spawn 2^{α} basis partitions, including μ, all with the same successive rank vector. This procedure will generate all basis partitions from minimal basis partitions.

Constructing basis partitions by the sliding operation. There is a nice way to get all basis partitions from primary partitions in $P_{\mathrm{o,d}}$ using the sliding operation.

In $P_{\mathrm{o,d}}$ we define a primary partition to be one such that in its 2-modular Ferrers graph there are no nodes below the Durfee square. In [4] the construction of all partitions in $P_{\mathrm{o,d}}$ from primary partitions is discussed. This involves the sliding operation along with the creation of secondary partitions generated by primary partitions. For the purpose of constructing basis partitions, only primary partitions and the sliding operation are needed and we discuss this now.

Given a primary partition, we consider a column on length j only of twos to the right of the Durfee square. If there are several such columns of the same length j, we either can slide all the columns or none, thereby giving us two choices for each collection of columns of twos of a given length. If we have a column of length j but whose size is odd, then such a column can also be slid down giving us once again two choices. Through such sliding operations, all basis partitions can be generated. Since the weights now will be powers of 2, there is an analogue to Hirschhorn's Theorem 2.1, but to determine these weights we need to consider separately columns having only twos, and columns of twos ending in a one. But before stating this, we will consider the hook operation on primary partitions.

Given a primary partition in $P_{\mathrm{o,d}}$, consider the partition $\tilde{\pi}$ generated by summing the entries on the hooks of π. This will be a partition whose parts will differ by ≥ 4 with strict inequality if a part is odd. We call such partitions as *special*, and denote the set of all special partitions by Ψ. Conversely given a special partition, we can find a primary partition whose hook lengths will correspond to the given special partition. Thus, the hook operation

$$\pi \mapsto \tilde{\pi} \tag{9.1}$$

is a bijection between primary partitions and special partitions. It is to be noted that the existence of a column of length j to the right of the Durfee square of a primary partition π is equivalent to saying that the difference between the j-th part and the $(j+1)$-st part of $\tilde{\pi}$ is > 4. Here we adopt the convention that if $\tilde{\pi}\colon b_1 + b_2 + \cdots + b_k$, then $b_{k+1} = -2$. The construction of basis partitions from primary partitions in the preceding paragraph together with (9.1) yields the following analogue to Hirschhorn's theorem:

Theorem 9.4. *If $\tilde{\pi}\colon b_1 + b_2 + \cdots + b_k$ is a special partition, then let*

$$b_i - b_{i+1} \geq m(b_{i+1}) := 4 + [b_{i+1}]_2,$$

where $[n]_2 = 0$ if n is even, and $[n]_2 = 1$ if n is odd. Define the weight $w(\tilde{\pi}) = 2^{\ell}$, where

$$\begin{aligned}\ell = \{&\textit{number of gaps } b_i - b_{i+1} > m(b_{i+1})\}\\ &+ \{\textit{number of gaps } b_i - b_{i+1} > m(b_{i+1}) + 2 \textit{ when } b_i \textit{ is odd}\},\end{aligned}$$

with the convention $b_{k+1} = -2$. If $b(n)$ is the number of basis partitions of n in $P_{\mathrm{o,d}}$, then

$$b(n) = \sum_{\tilde{\pi}\in\Psi,\, \sigma(\tilde{\pi})=n} w(\tilde{\pi}).$$

More generally, if $b(n, j)$ is the number of basis partitions in $P_{\mathrm{o,d}}$ with signature j, then

$$\sum_j b(n;j) z^j = \sum_{\tilde{\pi}\in\Psi,\, \sigma(\tilde{\pi})=n} w_z(\tilde{\pi}),$$

where $w_z(\tilde{\pi}) = (1+z)^{\ell}$, if $\tilde{\pi}$ with ℓ as above.

The generating function of basis partitions and minimal basis partitions. We will begin with the generating function of basis partitions which is easier to derive.

To get the generating function of basis partitions, we start with a $k \times k$ Durfee square of all twos which we call Case 2, and a $k \times k$ Durfee square of twos with a 1 at the bottom right hand corner, which we call Case 1. For Case 2, consider a collection of integers all equal to $2j$ represented as a row of twos j in number, where $1 \le j \le k$. We have either a choice of placing all of them to the right of the Durfee square as columns, or all of them as rows below the Durfee square. This gives two choices for each set of such $2j$. Similarly, if we have an integer $2j-1$ occurring singly, we could either place represent it as a column to the right of the Durfee square or as a row below the Durfee square. This yields the generating function

$$q^{2k^2}\frac{(-q^2;q^2)_k}{(q^2;q^2)_k}(-2q;q^2)_k. \tag{9.2}$$

For Case 1 with a $k \times k$ Durfee square, the argument is the same as above except that we can only have $1 \le j \le k-1$, since $j = k$ is not allowed. This yields the generating function

$$q^{2k^2-1}\frac{(-q^2;q^2)_{k-1}}{(q^2;q^2)_{k-1}}(-2q;q^2)_{k-1}. \tag{9.3}$$

To get the generating function of $b(n)$, the number of basis partitions of n, we need to sum the expressions in (9.2) and (9.3) over all k to get

$$\sum_{n=0}^{\infty} b(n)q^n = \sum_{k=0}^{\infty}\frac{q^{2k^2}(-q^2;q^2)_k(-2q;q^2)_k}{(q^2;q^2)_k} + \sum_{k=1}^{\infty}\frac{q^{2k^2-1}(-q^2;q^2)_{k-1}(-2q;q^2)_{k-1}}{(q^2;q^2)_{k-1}}. \tag{9.4}$$

The above combinatorial derivation of the generating function actually allows us to refine (9.4) by introducing parameters z and b whose powers keep track of the signature and the number of odd parts of the partition, respectively. That is if $b(n;j,i)$ denotes the number of basis partitions of n with signature j and the number of odd parts equal to i, then (9.4) refines to

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{q^{2k^2}(-zq^2;q^2)_k(-b(1+z)q;q^2)_k}{(q^2;q^2)_k} &+ \sum_{k=1}^{\infty}\frac{bq^{2k^2-1}(-zq^2;q^2)_{k-1}(-b(1+z)q;q^2)_{k-1}}{(q^2;q^2)_{k-1}} \\ &= \sum_{n,j,i} b(n;j,i)b^i z^j q^n.\end{aligned} \tag{9.5}$$

To determine the generating function of minimal basis partitions, we need to observe that in Case 1, when the Durfee square has dimension $k \times k$, if a collection of parts all equal to $2j$ are represented as a set of columns to the right of the Durfee square or as a set of rows below the Durfee square, but not both, then the integer $2j-1$ if it is to be included in the graph, *has to be placed alongside* the collection of $2j$. There is no choice as to where to place the $2j-1$ if the $2j$ occurs. But then, if $2j$ *does not occur*, we could have $2j-1$ represented either as a row below the Durfee square or as a column to the right of the square, thereby giving two choices. With regard to Case 2, the above observations in this paragraph all hold. In addition, we need to note that the last entry in the successive rank vector has to be non-zero, thereby forcing the graph to have either a row of *length* k below the Durfee square or a column of *length* k to the right of the square, but not both. This row or column of length k could represent either $2k-1$ or $2k$. With these observations, we can suitably modify (9.4) and write down the generating function of $b_m(n)$, the number of minimal basis partitions of n, namely,

$$\sum_{n=0}^{\infty} b_m(n)q^n = 1 + \sum_{k=1}^{\infty} q^{2k^2-1}\prod_{j=1}^{k-1}\left\{1 + 2q^{2j-1} + \frac{2q^{2j}(1+q^{2j-1})}{(1-q^{2j})}\right\}$$

$$+\sum_{k=1}^{\infty} q^{2k^2} \prod_{j=1}^{k-1}\left\{1+2q^{2j-1}+\frac{2q^{2j}\left(1+q^{2j-1}\right)}{\left(1-q^{2j}\right)}\right\} \times\left(2q^{2k-1}+\frac{2q^{2k}\left(1+q^{2k-1}\right)}{\left(1-q^{2k}\right)}\right). \quad (9.6)$$

As in the case of the generating function of basis partitions, we can refine (9.6), but here it is best to keep track of the number of different lengths below the Durfee square, which we call as *ℓ-signature*, which, for the sake of clarity, we will keep track by a parameter ζ. Thus, we have the following refinement of (9.6):

$$\begin{aligned}\sum_{n,j} b_m(n;j)\zeta^j q^n = 1 &+ \sum_{k=1}^{\infty} q^{2k^2-1} \prod_{j=1}^{k-1}\left\{1+(1+\zeta)q^{2j-1}+\frac{(1+\zeta)q^{2j}\left(1+q^{2j-1}\right)}{\left(1-q^{2j}\right)}\right\}\\ &+\sum_{k=1}^{\infty} q^{2k^2} \prod_{j=1}^{k-1}\left\{1+(1+\zeta)q^{2j-1}+\frac{(1+\zeta)q^{2j}\left(1+q^{2j-1}\right)}{\left(1-q^{2j}\right)}\right\}\\ &\times\left[(1+\zeta)q^{2k-1}+\frac{(1+\zeta)q^{2k}\left(1+q^{2k-1}\right)}{\left(1-q^{2k}\right)}\right], \end{aligned} \quad (9.7)$$

where $b_m(n;j)$, the number of minimal basis partitions of n with ℓ-signature equal to j.

Parity results. The series in (9.5) collapses to

$$\sum_{k=0}^{\infty} q^{2k^2} + b\sum_{k=1}^{\infty} q^{2k^2-1} \quad (9.8)$$

when $z=-1$ and so it yields a nice parity result with a parameter b. To state this, let us denote by $\nu_0(\beta)$ and $\psi(\beta)$ the number of odd parts of β and the signature of β, respectively. Here β is a basis partition in $P_{\mathrm{o,d}}$. Also, let

$$B_{\mathrm{e}}(n,b)=\sum_{\beta\in P_{\mathrm{o,d}},\,\sigma(\beta)=n,\,\psi(\beta)=\mathrm{even}} b^{\nu_{\mathrm{o}}(\beta)}, \qquad B_{\mathrm{o}}(n,b)=\sum_{\beta\in P_{\mathrm{o,d}},\,\sigma(\beta)=n,\,\psi(\beta)=\mathrm{odd}} b^{\nu_{\mathrm{o}}(\beta)},$$

where the sums are over basis partitions β. Then the collapse of (9.4) to (9.8) has the following partition interpretation.

Theorem 9.5. *We have $B_{\mathrm{e}}(n,b)-B_{\mathrm{o}}(n,b)=1$ if $n=2k^2$, b if $n=2k^2-1$, and 0 otherwise.*

Finally, we note that (9.7) collapses to

$$1+\sum_{k=1}^{\infty} q^{2k^2-1}, \quad (9.9)$$

when $\zeta=-1$. To interpret this collapse, we denote by $\lambda(\mu)$ the ℓ-signature of a minimal basis partition μ, and by

$$M_{\mathrm{e}}(n)=\sum_{\mu\in P_{\mathrm{o,d}},\,\sigma(\mu)=n,\,\lambda(\mu)=\mathrm{even}} 1 \qquad \text{and} \qquad M_{\mathrm{o}}(n)=\sum_{\mu\in P_{\mathrm{o,d}},\,\sigma(\mu)=n,\,\lambda(\mu)=\mathrm{odd}} 1,$$

where the sum is over minimal basis partitions μ. Then the collapse of (9.7) to (9.9) has the following interpretation.

Theorem 9.6. *We have $M_{\mathrm{e}}(n)-M_{\mathrm{o}}(n)=1$ if $n=2k^2-1$, and 0 otherwise.*

Concluding remark. We would like to point out that Andrews [6], motivated by the generating function (2.3) of basis partitions, has investigated the more general series

$$G(a, z; q) := \sum_{k=0}^{\infty} \frac{a^k q^{k^2} (-zq)_k}{(q)_k},$$

which gives the generating function of Rogers–Ramanujan partitions in the special case $z = 0$. Using $G(a, z; q)$, he studied certain polynomials which he called *basis partition polynomials*, and employed these to obtain a more rapidly convergent generating function for basis partitions as well as a new proof of the Rogers–Ramanujan identities. Note that we have combinatorially interpreted $G(1, z; q)$ in Section 7 as the generating function of basis partitions, where the power of z keeps track of the signature. It would be worthwhile to study connections between the combinatorics of basis partitions established in this paper and the results obtained by Andrews.

Acknowledgements

This work which was begun in 2007 was completed only in Fall 2013 when I visited The Pennsylvania State University, at which time the results of Section 9 were established. I would like to thank George Andrews for arranging that visit and for urging me to write this manuscript which was long overdue. But this paper remained unpublished since 2013. I thank Gaurav Bhatnagar, Frank Garvan, Christian Krattenthaler, and Michael Schlosser, for including this paper in the special volume of SIGMA in honor of Steve Milne's 75th birthday. Finally, I thank the referees for their careful reading of the manuscript and for their helpful suggestions. Research supported in part by NSA Grants MSPF-06G-150 and MSPF-08G-154.

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