

Orthogonal Polynomials with Complex Densities and Quantum Minimal Surfaces

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Abstract. We show that the discrete Painlevé-type equations arising from quantum minimal surfaces are equations for recurrence coefficients of orthogonal polynomials for indefinite hermitian products. As a consequence, we obtain an explicit formula for the initial conditions leading to positive solutions.

Key words: orthogonal polynomials; quantum minimal surfaces; random matrices; Painlevé equations

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1 Introduction

A quantum minimal surface [13, 14, 17] in \mathbb{R}^N is described by hermitian operators X_i ($i = 1, \dots, N$) satisfying the double commutator equations

$$\sum_{i=1}^N [X_i, [X_i, X_j]] = 0, \quad j = 1, \dots, N.$$

These equations are a quantization of the equations $\sum_{i=1}^N \{x_i, \{x_i, x_j\}\} = 0$ describing a minimal embedding $x: \Sigma \rightarrow \mathbb{R}^N$ of a Riemann surface Σ with Poisson bracket $\{, \}$ defined by the area form. If $N = 2m$ is even, the double commutator equations are implied by the equations $\sum_{j=1}^m [W_j^\dagger, W_j] = \epsilon 1$, $[W_j, W_k] = 0$ for $W_j = X_{2j-1} + iX_{2j}$, $W_j^\dagger = X_{2j-1} - iX_{2j}$, ($j = 1, \dots, m$). In the classical case these are first order equations implying the second order minimal surface equation. Cornalba and Taylor [8] viewed them as stationary solutions of the membrane matrix model [13]. For $N = 4$, they considered solutions where $W_2 = X_3 + iX_4$ is the square of $W_1 = X_1 + iX_2$, for which an interesting recurrence relation

$$v_n(1 + v_{n+1} + v_{n-1}) = (n+1)\epsilon, \tag{1.1}$$

with a parameter $\epsilon > 0$ arises for the square norms of the matrix elements of W (assumed to be non-zero only on one of the first off-diagonals). In the classical limit, the corresponding minimal surface is a special case of the class of minimal surfaces given by a holomorphic relation $f(x_1 + ix_2, x_3 + ix_4) = 0$ discovered over a century ago [9, 19]. The recurrence relation (1.1) is a type of discrete Painlevé equation: it is called ternary dP_I in [23], where it is shown to be a contiguity relation for the Painlevé differential equation P_V. In [4], apart from discussing several other examples and various aspects of quantum minimal surfaces, such as Hermitian Yang–Mills theory (see also [7, 12] as well as, related in yet other ways, [2, 3]), several properties of the recurrence relations were given, while in [6, 16] the unique positive solution was described.

Another source of recurrence relations such as (1.1) is the theory of random matrices and orthogonal polynomials, where they arise as equations for recurrence coefficients. Indeed, another discrete Painlevé equation, called dP_I , namely $v_n(1 + v_{n+1} + v_n + v_{n-1}) = (n+1)\epsilon$, occurs in hermitian random matrices with quartic potentials and is related to P_{IV} , see the review [18] and references therein. Both dP_I and the ternary dP_I have as a continuum limit the differential Painlevé I equation.

In this paper, we develop a theory of orthogonal polynomials leading to (1.1) and other equations arising in quantum minimal surfaces. The starting point of our research was the observation that in the theory of random *normal* matrices [24] a recurrence relation appears [22] that differs from (1.1) only by a sign change, see also [1] for a recent analysis of the asymptotics of solutions. The orthogonal polynomials are defined by a density on the complex plane. To get the right sign, we consider a variant with complex density which seems to be interesting in its own right. For example, the recurrence relation (1.1) arises if we consider an inner product on polynomials given by an integral over the complex plane

$$(f, g) = \int_{\mathbb{C}} f(z) \overline{g(-z)} e^{-a|z|^2 + i(V(z) + \overline{V(z)})} dx dy, \quad f, g \in \mathbb{C}[z], \quad (1.2)$$

with cubic potential $V(z) = tz^3$. The parameters a, t are real with $a > 0$. The integral is over $z = x + iy \in \mathbb{C}$ and converges absolutely, in contrast to the integrals occurring in normal random matrix models (with t imaginary in this example) [10, 22, 24] which require a regularization, see [5, 11]. The downside is that the integration density is not real making the probabilistic interpretation questionable. Still, this inner product is hermitian $(f, g) = \overline{(g, f)}$ for all $f, g \in \mathbb{C}[z]$, and in particular (f, f) is real for all polynomials f . The inner product is not definite in general, so it is not guaranteed that orthogonal polynomials exist. If they exist (and they do in the case of a cubic potential as follows from [6, 16] as we show in Theorem 3.7), monic orthogonal polynomials are unique and produce positive solutions of (1.1) and thus a quantum minimal surface. Moreover, the theory produces the explicit initial condition that leads to the positive solution of (1.1).

Another hint that minimal surfaces are related to normal random matrices comes from the observation that the eigenvalue distribution of regularized normal random matrices [11, 24] with cubic potential are uniformly distributed on a domain in the complex plane bounded by a hypotrochoid. This is one of the special shapes that when rotated give rise to a minimal surface in Minkowski space [15]. We do not pursue this relation in this paper, but plan to come back to it in the future.

In Section 2 of this paper, we discuss the case of a cubic potential in details and explain how the relation (1.1) arises from the coefficients of the recurrence relation for orthogonal polynomials. The initial condition is expressed as a modified Bessel function as in [6, 16].

In Section 3, we generalize the construction to the case of monomial potentials, replacing z^3 by z^d for $d \in \mathbb{Z}_{>0}$. The corresponding quantum minimal surfaces are solutions W_1, W_2 such that $W_2 = W_1^{d-1}$. The existence of “integrable” recurrence relations generalizing (1.1) is mentioned in [4] and was written explicitly in [6, 16] for $d = 4$. It turns out that for odd d the same formula (1.2) with 3 replaced by d gives a hermitian inner product and everything generalizes nicely. For even d , a modification is needed, related to the fact that the extension group $\text{Ext}(\mathbb{Z}/d\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is non-trivial for d even. Once this is understood, we get an indefinite hermitian inner product on polynomials of degree $\leq N$ for all N . We conjecture that it is non-degenerate for all N and all values of the parameters. This conjecture implies the existence of orthogonal polynomials and of a positive solution of the recurrence relation.

This solution can be computed explicitly as we show in Section 4. We discuss the initial conditions of the recurrence relation produced by orthogonal polynomials. They are determined by a single function $h(a)$ (the square norm of the polynomial 1) which obeys a linear differential

equation of order $d - 1$, whose solutions are generalized hypergeometric functions. We give explicit formulas for the coefficients and express h in terms of classical functions for small d .

In Section 5, we explain the relation to minimal surfaces, and in Section 6, we discuss some future research directions.

2 Cubic potential

Let $a > 0$, $t \in \mathbb{R}$. Consider the sesquilinear pairing

$$(f, g) = \int_{\mathbb{C}} f(z) \overline{g(-z)} \rho_{a,t}(z) dx dy, \quad \rho_{a,t}(z) = \exp(-a|z|^2 + it(z^3 + \bar{z}^3)), \quad (2.1)$$

on polynomials $f, g \in \mathbb{C}[z]$. The integral is over $z = x + iy \in \mathbb{C}$ and converges absolutely.

Let us assume that (\cdot, \cdot) is non-degenerate. Polynomials P_0, P_1, \dots are called orthogonal for (\cdot, \cdot) if

- (1) $P_n(z) = z^n + \dots$ is monic of degree n ,
- (2) $(P_n, P_m) = 0$ if $n \neq m$.

It is not a priori clear that orthogonal polynomials exist, since the inner product is not definite in general, but if they exist and the inner product is non-degenerate, they are unique. In fact, it is a consequence of our construction, together with existence of the unique positive solution to (1.1) that orthogonal polynomials P_n exist for all n and that the sign of (P_n, P_n) is $(-1)^n$, see Theorem 3.7 below.

If $(P_n)_{n \geq 0}$ is the sequence of orthogonal polynomials, we set $h_n = (-1)^n (P_n, P_n)$. The sign is chosen so that $h_n > 0$ for $t = 0$:

Example 2.1. Let $t = 0$. Then $P_n(z) = z^n$ are orthogonal and

$$h_n = \int_{\mathbb{C}} |z|^{2n} e^{-a|z|^2} dx dy = \frac{\pi n!}{a^{n+1}}.$$

We see that for $t = 0$ the restriction to polynomials of degree $\leq N$ has signature (k, k) if $N = 2k - 1$ is odd and $(k + 1, k)$ if $N = 2k$ is even. This continues to be the case for small t by continuity and also for larger t as long as the inner product stays non-degenerate. We show in the next section that the inner product is non-degenerate for all $a > 0$, $t \in \mathbb{R}$.

2.1 Action of the cyclic group

Let $\zeta = \exp(2\pi i/3)$, T the automorphism of the ring $\mathbb{C}[z]$ such that $z \mapsto \zeta z$. Then $(Tf, Tg) = (f, g)$ for all f, g so that T defines an action of the cyclic group C_3 of order 3 on polynomials. By the uniqueness of orthogonal polynomials, we see that $TP_n = \zeta^n P_n$.

2.2 Recurrence relation for orthogonal polynomials

Integration by parts leads to the identity

$$(\partial_z f + 3itz^2 f, g) = -a(f, zg). \quad (2.2)$$

The following statement shows that we can effectively recursively determine the orthogonal polynomials if we know the h_n .

Proposition 2.2. *Let $L = \partial_z + 3it z^2$. Then $P_0(z) = 1$, $P_1(z) = z$, $P_2(z) = z^2$ and*

$$\begin{aligned} zP_n(z) &= P_{n+1}(z) + \frac{3it}{a} \frac{h_n}{h_{n-2}} P_{n-2}(z), \quad n \geq 2, \\ LP_n(z) &= 3itP_{n+2}(z) + a \frac{h_n}{h_{n-1}} P_{n-1}(z), \quad n \geq 1. \end{aligned}$$

Proof. $1, z, z^2$ are the only monic polynomials of degree ≤ 2 of the correct C_3 -weight.

The polynomial zP_n has degree $n+1$ and is therefore a linear combination of P_j with $j \leq n+1$. Similarly, LP_n is a linear combination of P_j with $j \leq n+2$. Thus $(zP_n, P_m) = -a^{-1}(P_n, LP_m)$ vanishes for $m < n-2$ and zP_n is a linear combination of $P_{n+1}, P_n, P_{n-1}, P_{n-2}$. Since z has weight 1, for the C_3 -action $zP_n = a_n P_{n+1} + b_n P_{n-2}$ for some coefficients a_n, b_n . By the condition that P_n is monic, we obtain $a_n = 1$. Similarly, the action of the operator L of weight 2 has the form $LP_n = 3itP_{n+2} + c_n P_{n-1}$. The coefficient b_n is determined by the relation

$$ab_n(P_{n-2}, P_{n-2}) = a(zP_n, P_{n-2}) = -(P_n, LP_{n-2}) = -(P_n, 3itP_n) = 3it(P_n, P_n).$$

For c_n , we obtain

$$c_n(P_{n-1}, P_{n-1}) = (LP_n, P_{n-1}) = -a(P_n, zP_{n-1}) = -a(P_n, P_n). \quad \blacksquare$$

Remark 2.3. Let $V_n = h_{n+1}/h_n$, $n \geq 0$. Then we can write the formulas of Proposition 2.2 as

$$zP_n(z) = P_{n+1}(z) + \frac{3it}{a} V_{n-1} V_{n-2} P_{n-2}(z), \quad LP_n(z) = 3itP_{n+2}(z) + aV_{n-1} P_{n-1}(z).$$

We notice that since $P_n = z^n$ for $n = 0, 1, 2$ these formulas hold for all $n \geq 0$, provided we set $V_n = 0$ for $n < 0$.

2.3 A discrete Painlevé equation

Theorem 2.4. *The ratios $V_n = h_{n+1}/h_n$, $n \geq 0$, obey the recurrence relation*

$$V_n \left(1 + \frac{9t^2}{a^2} (V_{n-1} + V_{n+1}) \right) = \frac{n+1}{a}$$

with initial conditions

$$V_{-1} = 0, \quad V_0 = \frac{\int_{\mathbb{C}} |z|^2 \rho_{a,t}(z) dx dy}{\int_{\mathbb{C}} \rho_{a,t}(z) dx dy}.$$

Proof. Proposition 2.2 gives the matrix elements of multiplication by z and of $L = \partial_z + 3it z^2$ in the basis of orthogonal polynomials. From it, we can deduce the action of ∂_z . The condition that the coefficient of P_{n-1} in the expression for $\partial_z P_n$ must be n gives a condition for the numbers h_n . The calculation goes as follows. By Remark 2.3,

$$\begin{aligned} z^2 P_n &= z \left(P_{n+1} + 3i \frac{t}{a} V_{n-1} V_{n-2} P_{n-2} \right) \\ &= P_{n+2} + 3i \frac{t}{a} V_n V_{n-1} P_{n-1} + 3i \frac{t}{a} V_{n-1} V_{n-2} \left(P_{n-1} + 3i \frac{t}{a} V_{n-3} V_{n-4} P_{n-4} \right) \\ &= P_{n+2} + 3i \frac{t}{a} V_{n-1} (V_n + V_{n-2}) P_{n-1} - (\cdots) P_{n-4}, \end{aligned}$$

(we do not care what the coefficient of P_{n-4} is). Thus

$$\partial_z P_n = LP_n - 3it z^2 P_n$$

$$\begin{aligned}
&= 3itP_{n+2}(z) + aV_{n-1}P_{n-1} - 3it \left(P_{n+2} + 3i\frac{t}{a}V_{n-1}(V_n + V_{n-2})P_{n-1} + (\cdots)P_{n-4} \right) \\
&= \left(aV_{n-1} + 9\frac{t^2}{a}V_{n-1}(V_n + V_{n-2}) \right) P_{n-1} + (\cdots)P_{n-4}.
\end{aligned}$$

Equating the coefficient of P_{n-1} with n yields the recurrence relation. ■

Let $v_n = \frac{9t^2}{a^2}V_n$, $\epsilon = \frac{9t^2}{a^3}$. Then we can write the recurrence relation as $v_n(1 + v_{n+1} + v_{n-1}) = (n+1)\epsilon$. The initial conditions are $v_{-1} = 0$ and a function v_0 of ϵ . This function can be expressed in terms of Bessel functions $v_0(\epsilon) = \epsilon^{\frac{2}{3}}f(\epsilon^{-\frac{1}{3}})$, where

$$f(a) = -h'(a)/h(a), \quad h(a) = \int_{\mathbb{C}} \exp \left(-a|z|^2 + \frac{i}{3}(z^3 + \bar{z}^3) \right) dx dy.$$

Lemma 2.5. *The function $h(a)$ is the solution of the differential equation $h''(a) = ah(a) + a^2h'(a)$ with the boundary condition $\lim_{a \rightarrow \infty} ah(a) = \pi$.*

Proof. We differentiate under the integral and integrate by parts

$$\begin{aligned}
h''(a) &= \int_{\mathbb{C}} z^2 \bar{z}^2 e^{-a|z|^2 + \frac{i}{3}(z^3 + \bar{z}^3)} dx dy = - \int_{\mathbb{C}} e^{-az\bar{z}} \partial_z \partial_{\bar{z}} e^{\frac{i}{3}(z^3 + \bar{z}^3)} dx dy \\
&= - \int_{\mathbb{C}} \partial_z \partial_{\bar{z}} e^{-az\bar{z}} e^{\frac{i}{3}(z^3 + \bar{z}^3)} dx dy = - \int_{\mathbb{C}} (-a + a^2 z \bar{z}) e^{-az\bar{z}} e^{\frac{i}{3}(z^3 + \bar{z}^3)} dx dy \\
&= ah(a) + a^2 h'(a).
\end{aligned}$$

To compute the behavior at infinity, we rescale z by $a^{-\frac{1}{2}}$

$$h(a) = a^{-1} \int_{\mathbb{C}} \exp \left(-|z|^2 + \frac{i}{3}a^{-\frac{3}{2}}(z^3 + \bar{z}^3) \right) dx dy.$$

Thus

$$\lim_{a \rightarrow \infty} ah(a) = \int_{\mathbb{C}} e^{-|z|^2} dx dy = \pi. \quad \blacksquare$$

The change of variables $h(a) = e^{a^3/6} \sqrt{ay} (a^3/6)$ leads to the modified Bessel differential equation

$$x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} - \left(x^2 + \frac{1}{36} \right) y = 0.$$

The solutions vanishing at infinity are proportional to the modified Bessel function of the second kind $K_{1/6}(x)$ behaving asymptotically as $\sqrt{\frac{\pi}{2x}} e^{-x} (1 + O(1/x))$ for $x \rightarrow \infty$. We conclude that

$$h(a) = e^{a^3/6} \sqrt{\frac{\pi a}{3}} K_{1/6}(a^3/6). \quad (2.3)$$

3 Orthogonal polynomials with complex density

We wish to generalize the above construction to the case where the exponent 3 is replaced by an arbitrary natural number d . This is straightforward if d is odd, but for even d a more general construction is needed. Let $d \in \mathbb{Z}_{\geq 0}$ and $\mu_d = \{\zeta \in \mathbb{C} \mid \zeta^d = 1\}$ be the group of d -th roots of unity acting on \mathbb{C} by multiplication. Suppose $\rho: \mathbb{C} \rightarrow \mathbb{C}$ is a measurable complex valued function such that

- (I) $\int_{\mathbb{C}} |z|^n |\rho(z)| dx dy < \infty$ for all $n \in \mathbb{Z}_{\geq 0}$,
 (II) $\rho(\eta z) = \overline{\rho(z)}$ for all $\eta \in \mathbb{C}$ such that $\eta^d = -1$.

Note that (II) implies that ρ is invariant under μ_d , $\rho(\zeta z) = \rho(z)$ for all $\zeta \in \mu_d$. Also any two solutions of $\eta^d = -1$ differ by multiplication by an element of μ_d . So one can replace (II) by

- (II') ρ is μ_d -invariant and $\rho(\eta z) = \overline{\rho(z)}$ for some η such that $\eta^d = -1$.

Property (I) implies that the integrals defining the moments $\int_{\mathbb{C}} z^n \bar{z}^m \rho(z) dx dy$ are absolutely convergent. The group μ_d acts on polynomials via $T_{\zeta} f(z) = f(\zeta z)$, $\zeta \in \mu_d$. The ring of polynomials $\mathbb{C}[z]$ decomposes into a direct sum $\mathbb{C}[z] = \bigoplus_{\ell \in \mathbb{Z}/d\mathbb{Z}} \mathbb{C}[z]_{\ell}$ of weight spaces

$$\mathbb{C}[z]_{\ell} = \{f \mid T_{\zeta} f = \zeta^{\ell} f \text{ for all } \zeta \in \mu_d\}.$$

To a function ρ as above, we associate a hermitian μ_d -invariant inner product. The construction depends on the choice of a right inverse χ of the natural projection $\mathbb{Z}/2d\mathbb{Z} \rightarrow \mathbb{Z}/d\mathbb{Z}$. For definiteness, we take $\chi(\ell \bmod d) = \ell \bmod 2d$ for $\ell = 0, \dots, d-1$.

Proposition 3.1. *Let ρ be a function obeying (I) and (II) for $d \in \mathbb{Z}_{\geq 1}$ and $\eta \in \mathbb{C}$ such that $\eta^d = -1$. Let $\sigma : \mathbb{C}[z] \rightarrow \mathbb{C}[z]$ the linear map such that $\sigma f(z) = (-\eta)^{-\chi(\ell)} f(\eta z)$ if $f \in \mathbb{C}[z]_{\ell}$. Then the inner product*

$$(f, g) = \int_{\mathbb{C}} f(z) \overline{\sigma g(z)} \rho(z) dx dy$$

is independent of the choice of η and hermitian. In particular, $(f, f) \in \mathbb{R}$ for all $f \in \mathbb{C}[z]$. Moreover, $(T_{\zeta} f, T_{\zeta} g) = (f, g)$ for $\zeta \in \mu_d$ and any polynomials f, g .

Proof. Any two choices of η differ by multiplication by a root of unity $\zeta \in \mu_d$. But σ does not change if we replace η by $\eta\zeta$, so the inner product is unchanged.

Also σ commutes with the action of μ_d , ρ and the measure $dx dy$ are μ_d -invariant. implying that last claim. In particular, $(f, g) = 0$ if f and g have different weight. To check the hermitian property, we can thus assume that f and g have the same weight ℓ . We use (II) and that $\eta^2 \in \mu_d$,

$$\begin{aligned} \overline{(g, f)} &= \int_{\mathbb{C}} \overline{g(z)} (-\eta)^{-\chi(\ell)} f(\eta z) \rho(\eta z) dx dy = \int_{\mathbb{C}} \overline{g(\eta^{-1} z)} (-\eta)^{-\chi(\ell)} f(z) \rho(z) dx dy \\ &= \int_{\mathbb{C}} \overline{g(\eta^{-2} \eta z)} (-\eta)^{-\chi(\ell)} f(z) \rho(z) dx dy = \int_{\mathbb{C}} \overline{\eta^{-2\ell} g(\eta z)} (-\eta)^{-\chi(\ell)} f(z) \rho(z) dx dy \\ &= \int_{\mathbb{C}} \overline{(-1)^{-\chi(\ell)} \eta^{-2\ell + \chi(\ell)} g(\eta z)} f(z) \rho(z) dx dy \\ &= \int_{\mathbb{C}} \overline{(-\eta)^{-\chi(\ell)} g(\eta z)} f(z) \rho(z) dx dy = (f, g). \end{aligned}$$

In the last step, we used the fact that $2\ell \equiv 2\chi(\ell) \bmod 2d$ so that $\eta^{-2\ell} = \eta^{-2\chi(\ell)}$ for $\eta \in \mu_{2d}$. ■

Remark 3.2. To construct the inner product, we chose a right inverse of the natural $2 : 1$ projection $p : \mathbb{Z}/2d\mathbb{Z} \rightarrow \mathbb{Z}/d\mathbb{Z}$. Any of the other 2^d choices would also lead to a hermitian inner product. In fact, for any map $\lambda : \mathbb{Z}/d\mathbb{Z} \rightarrow \{\pm 1\}$, the inner product $(\ , \)_s$ whose restriction to the weight space $\mathbb{C}[z]_{\ell}$ is $(f, g)_{\lambda} = \lambda(\ell)(f, g)$, if $f \in \mathbb{C}[z]_{\ell}$ is also a μ_d invariant hermitian inner product.

Here is the class of examples we have in mind.

Example 3.3. Let $\rho(z) = \exp(-a|z|^2 + i(V(z) + \overline{V(z)}))$, where $V(z) = U(z^d)$ for some odd entire function U . If d is odd, we can take $\eta = -1$ and the inner product is given by the same formula (2.1).

The integration by part formula (2.2) for this class of examples generalizes, but in the case of even d there is a twist by a sign.

Lemma 3.4. Let ρ be as in Example 3.3 with polynomial V and set $L = \partial_z + iV'(z)$. Let $\tau: \mathbb{Z}/d\mathbb{Z} \rightarrow \{\pm 1\}$ be such that

$$\tau(\ell) = \begin{cases} (-1)^{d-1} & \text{if } \ell \equiv -1 \pmod{d}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(Lf, g) = -\tau(\ell)a \cdot (f, zg)$ for any polynomials f, g such that g has weight ℓ ,

Again, orthogonal polynomials are not guaranteed to exist. A sufficient condition for existence of monic orthogonal polynomials P_0, \dots, P_N up to degree N is the non-vanishing of the determinants of the Gram matrices up to degree N . The N -th Gram matrix of the inner product is the matrix G_N with entries (z^n, z^m) , $n, m = 0, \dots, N$. If all Gram matrices G_j for j up to N are invertible, then (\cdot, \cdot) admits orthogonal polynomials $P_j = z^j + \dots$ up to degree N and they are unique.

Example 3.5. Let $\rho(z) = \exp(-a|z|^2)$. Then $1, z, z^2, \dots$ are orthogonal polynomials for any d and

$$(z^n, z^n) = \rho(n) \frac{\pi n!}{a^{n+1}},$$

where $\rho(n) = (-1)^n \prod_{j=0}^{n-1} \tau(j)$. Explicitly, $\rho(n) = (-1)^n$ for $n = 0, \dots, d-1$ and $\rho(n+d) = -\rho(n)$ for general n .

By continuity of the determinant of the Gram matrices, it is still true that orthogonal polynomials P_0, \dots, P_N exist for ρ as in Example 3.3 for any fixed N , provided V is a polynomial with sufficiently small coefficients (depending on N). The sign of (P_n, P_n) is then still $\rho(n)$. It is an interesting question to find criteria for the potential $U(z)$ ensuring the non-degeneracy of the Gram matrices and thus the existence of orthogonal polynomials.

3.1 Monomial potentials

In the same way as in the case of cubic monomial potentials, we can consider the case of monomials of general degree d . The density of the inner product (2.1) is replaced by

$$\rho_a(z) = \exp\left(-a|z|^2 + \frac{i}{d}(z^d + \bar{z}^d)\right),$$

where d is a positive integer, $a > 0$.¹ Then the inner product is hermitian and we can define orthogonal monic polynomials P_n with square norms $(P_n, P_n) = \rho(n)h_n$.

Conjecture 3.6. The Gram matrices $((z^n, z^m))_{n,m=0}^N$, $N = 0, 1, 2, \dots$, are invertible for all $a \geq 0$.

For small d , the conjecture holds as a consequence of the results of [6, 16].

Theorem 3.7. The conjecture holds for $d = 1, 2, 3$.

¹For simplicity, we set $t = 1/d$. The case with a general t can be recovered by rescaling variables.

Proof. Let us fix the maximal degree N . Then the inner product multiplied by a power of a converges for $a \rightarrow \infty$ to the inner product with density $\exp(-|z|^2)$ for which the z^n are orthogonal with non-vanishing squared norms $\rho(n)\pi n!$. Thus the Gram matrices up to N are non-degenerate for large a , orthogonal polynomials exist, and the determinant of the Gram matrix is (up to sign) the product $\prod_{n=0}^N h_n(a)$. Assume by contradiction that for some $a > 0$ a Gram matrix is degenerate and let a_* be the largest a for which this happens. This means that $h_{n+1}(a_*) = 0$ for some $n \geq 0$, which can be chosen as small as possible so that

$$\begin{aligned} h_j(a) &> 0 && \text{for all } a \geq a_* \text{ and } j \leq n, \\ h_{n+1}(a) &> 0 && \text{for all } a > a_*, \\ h_{n+1}(a_*) &= 0. \end{aligned}$$

This means that there is an $n \geq 1$ and $a_* > 0$ such that $V_n(a)$ vanishes at a_* and is positive for $a > a_*$ while $V_0(a), \dots, V_{n-1}(a)$ are positive for all $a \geq a_*$. For $d = 1, 2$, the V_n can be computed explicitly, see below, and they are positive for all a . For $d = 3$, it is shown in [6, 16] that the initial condition $V_0(a) = -h'(a)/h(a)$ with h given by (2.3) gives rise to a solution for which all $V_n(a) > 0$ for all $a > 0$, so no such a_* can exist. We reproduce here the argument of [16] to prove this in the language of orthogonal polynomials. It relies on a first order differential equation for $V_n(a)$ which can be obtained by observing that the derivative of (P_n, P_n) with respect to a is proportional to (zP_n, zP_n) which can be expressed in terms of (P_{n+1}, P_{n+1}) and (P_{n-2}, P_{n-2}) by using the relation (3.1). The result is

$$\partial_a h_n = -h_{n+1} + \frac{1}{a^2} \frac{h_n^2}{h_{n-d+1}},$$

where we take $d = 3$. This translates to a differential equation for $V_n = h_{n+1}/h_n$

$$\partial_a V_n = -V_n V_{n+1} + V_n \left(V_n + \frac{1}{a^2} (V_n - V_{n-2}) V_{n-1} \right).$$

Now we can eliminate V_{n+1} using the recurrence relation $V_n V_{n+1} = -V_n V_{n-1} - a^2 V_n + (n+1)a$ and obtain a differential equation of the form

$$\partial_a V_n(a) = -(n+1)a + V_n(a)P(V_{n-2}(a), V_{n-1}(a), V_n(a))$$

for a polynomial P . The point is that at $a = a_*$, where V_n vanishes, the derivative $\partial_a V_n(a_*) = -(n+1)a_*$ is negative, in contradiction with $V_n(a)$ being positive for $a > a_*$. ■

In the following, we assume the validity of this conjecture for general d . It implies that orthogonal polynomials exist and that $h_n > 0$ for all n .

The polynomials P_n have weight n under the unitary action of μ_d , meaning that $P_n(\zeta z) = \zeta^n P_n(z)$ for d -th roots of unity ζ . The first d polynomials are $1, \dots, z^{d-1}$. The relations of Proposition 2.2 become

$$zP_n(z) = P_{n+1}(z) + \frac{i}{a} \frac{h_n}{h_{n-d+1}} P_{n-d+1}(z), \quad n \geq d-1, \quad (3.1)$$

$$LP_n(z) = iP_{n+d-1}(z) + a \frac{h_n}{h_{n-1}} P_{n-1}(z), \quad n \geq 1, \quad (3.2)$$

where $L = \partial_z + iz^{d-1}$. By the calculation of Theorem 2.4, the ratios $V_n = h_{n+1}/h_n$ obey the recurrence relations

$$V_n + \frac{1}{a^2} \sum_{j=0}^{d-2} \prod_{k=0}^{d-2} V_{n+j-k} = \frac{n+1}{a}, \quad (3.3)$$

with initial conditions $V_{-1} = 0$ and

$$V_j = \frac{\int_{\mathbb{C}} |z|^{2j+2} \rho_a(z) dx dy}{\int_{\mathbb{C}} |z|^{2j} \rho_a(z) dx dy}, \quad j = 0, \dots, d-3.$$

Again the dependence of the initial conditions $V_j = V_j(a)$, $j \leq d-3$, on the parameter a is controlled by a differential equation for $h = (1, 1)$. The initial conditions are

$$V_0(a) = -\frac{h'(a)}{h(a)}, \quad V_1(a) = -\frac{h''(a)}{h'(a)}, \quad \dots, \quad V_{d-3}(a) = -\frac{h^{(d-2)}(a)}{h^{(d-3)}(a)},$$

with

$$h(a) = \int_{\mathbb{C}} \exp \left(-a|z|^2 + \frac{i}{d}(z^d + \bar{z}^d) \right) dx dy. \quad (3.4)$$

As in Lemma 2.5, we see that this function is a solution of the linear differential equation of order $d-1$,

$$(-1)^{d-1} h^{(d-1)}(a) = ah(a) + a^2 h'(a), \quad (3.5)$$

such that $\lim_{a \rightarrow \infty} ah(a) = \pi$.

Finally, the rescaled versions (for $d \neq 2$) $v_n = a^{-\frac{2}{d-2}} V_n$ obey the recurrence relations in standard form [4, 8, 16]

$$v_n + \sum_{j=0}^{d-2} \prod_{k=0}^{d-2} v_{n+j-k} = (n+1)\epsilon, \quad \epsilon = a^{-\frac{d}{d-2}}.$$

The initial conditions are $v_{-1} = 0$ and

$$v_j = -a^{-\frac{2}{d-2}} \frac{h^{(j+1)}(a)}{h^{(j)}(a)}, \quad j = 0, \dots, d-3.$$

4 Explicit construction of the orthogonal polynomials

The relation (3.1) or the relation (3.2) can be used to determine the orthogonal polynomials recursively from $P_j(z) = z^j$ for $j = 0, \dots, d-1$, once we know the V_n . For example, we can write (3.1) as

$$P_{n+1}(z) = zP_n(z) - \frac{i}{a} \left(\prod_{j=1}^{d-1} V_{n-j} \right) P_{n-d+1}(z). \quad (4.1)$$

The coefficients are obtained from the recurrence relation (3.3). There remains to compute the initial conditions V_0, \dots, V_{d-3} . As we have seen, they are determined by the function $h(a)$ given by the integral (3.4), solution to the differential equation

$$(-1)^{d-1} h^{(d-1)}(a) = ah(a) + a^2 h'(a)$$

decaying at infinity. If $d > 1$, $a = 0$ is a regular point of the differential equation, so the power series method applies. As a result, a basis of local solutions is formed by the generalized hypergeometric functions

$$\phi_m(a) = a^m \cdot {}_1F_{d-3} \left(\frac{m+1}{d}; \frac{m+2}{d}, \dots, \frac{d-1}{d}, \frac{d+1}{d}, \dots, \frac{m+d}{d}; -(-d)^{-d+2} a^d \right). \quad (4.2)$$

Here m runs over $0, \dots, d-2$. For $d \geq 3$, these series have an infinite radius of convergence.

To compute the coefficients of h as a linear combination of basis elements, we rewrite the integral as a Laplace transform of the Bessel function J_0

$$h(a) = \pi \int_0^\infty e^{-at} J_0\left(\frac{2t^{\frac{d}{2}}}{d}\right) dt.$$

This formula can be obtained by expanding the exponential of $i(z^d + \bar{z}^d)/d$ and performing the angular integration. Now we can expand the exponential series e^{-at} and use the identity (see [20, formula (10.22.43)])

$$\int_0^\infty t^\mu J_0(t) dt = 2^\mu \frac{\Gamma(\frac{1}{2}(1+\mu))}{\Gamma(\frac{1}{2}(1-\mu))}.$$

We obtain the following statement.

Proposition 4.1. *Let $d \geq 2$. The function h is given by the linear combination*

$$h(a) = \pi d^{\frac{2}{d}-1} \sum_{m=0}^{d-2} \frac{(-d^{\frac{2}{d}})^m}{m!} \frac{\Gamma(\frac{m+1}{d})}{\Gamma(1 - \frac{m+1}{d})} \phi_m(a)$$

of the solutions $\phi_0, \dots, \phi_{d-2}$, see (4.2).

For small d , h can be expressed in terms of classical functions.

- For $d = 1$, $a = 0$ is not a regular point, not even a regular singular point, of the differential equation (3.5), but we can compute the integral explicitly

$$h(a) = \frac{\pi}{a} e^{-\frac{1}{a}}$$

In this case, $V_n = (n+1)a$ and the orthogonal polynomials are $P_n(z) = (z - \frac{i}{a})^n$,

- For $d = 2$, the solution is

$$h(a) = \frac{\pi}{\sqrt{a^2 + 1}}.$$

The recurrence relation (3.3) reduces to $(a^2 + 1)V_n = (n+1)a$ and the recurrence relation (4.1) for the orthogonal polynomials reduces up to a linear change of variables to the three-term relations for Hermite polynomials $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$. Thus

$$P_n(z) = \left(\frac{\zeta}{\sqrt{2(a^2 + 1)}} \right)^n H_n \left(\zeta^{-1} \sqrt{\frac{a^2 + 1}{2}} z \right), \quad \zeta^2 = i.$$

Notice that orthogonal polynomials arising in normal matrix models with real Gaussian density, see [21], are also related to Hermite polynomials. The new feature here is the rotation of the argument by an eighth root of unity.

- For $d = 3$, as we have seen,

$$h(a) = \sqrt{\frac{\pi a}{3}} e^{a^3/6} K_{\frac{1}{6}}(a^3/6),$$

and the initial condition leading to the positive solution of (3.3) is $V_{-1} = 0$, $V_0 = -h'(a)/h(a)$.

- For $d = 4$, the third order equation $h'''(a) + ah(a) + a^2h'(a) = 0$ reduces with the change of variables $h(a) = ay(a^2/4)$ to the case $\alpha = 1/4$ of the differential equation

$$x^2 \frac{d^3 y}{dx^3} + 3x \frac{d^2 y}{dx^2} + (1 + 4(x^2 - \alpha^2)) \frac{dy}{dx} + 4xy = 0$$

for products of Bessel functions of order α . In fact,

$$h(a) = \frac{\pi^2 a}{4} \left(J_{-\frac{1}{4}} \left(\frac{a^2}{4} \right)^2 - \sqrt{2} J_{-\frac{1}{4}} \left(\frac{a^2}{4} \right) J_{\frac{1}{4}} \left(\frac{a^2}{4} \right) + J_{\frac{1}{4}} \left(\frac{a^2}{4} \right)^2 \right).$$

The initial conditions are $V_{-1} = 0$, $V_0 = -h'(a)/h(a)$, $V_1 = -h''(a)/h'(a)$.

5 Relation to quantum minimal surfaces

A positive solution of the recurrence relation (3.3) gives rise to a quantum minimal surface given by operators W_1, W_2 acting on the space of polynomials. Let $|n\rangle = \frac{P_n}{\sqrt{h_n}}$ be the normalized orthogonal polynomials and let $w_n = \sqrt{v_n} = a^{-\frac{1}{d-2}} \sqrt{V_n}$. Then the operator L and the operator M of multiplication by z and L can be expressed in terms of the operators W, W^\dagger such that

$$W|n\rangle = w_n|n+1\rangle, \quad W^\dagger|n+1\rangle = w_n|n\rangle \quad \text{for all } n \geq 0,$$

and $W^\dagger|0\rangle = 0$. The relations (3.1), (3.2) can namely be written as

$$M = a^{\frac{1}{d-2}} (W + iW^{\dagger d-1}), \quad L = a^{\frac{d-1}{d-2}} (iW^{d-1} + W^\dagger)$$

The relation $[L, M] = 1$ between the differential operators $\partial_z + iz^{d-1}$ and z translates to

$$[W^\dagger, W] + [W^{\dagger d-1}, W^{d-1}] = \epsilon 1, \quad \epsilon = a^{-\frac{d}{d-2}}.$$

6 Conclusion and outlook

We have introduced a class of indefinite hermitian inner product on polynomials for which orthogonal polynomials lead to solutions of discrete Painlevé-type equations occurring in quantum minimal surfaces. These surfaces are quantizations of the algebraic curves $w_2 = w_1^{d-1}$ in $\mathbb{C}^2 = \mathbb{R}^4$. More generally, one expects a similar story to hold for the algebraic curves $w_1^p = w_2^q$ for positive integers p, q . One may expect that the corresponding orthogonal polynomials are defined by a density of the form $\rho = |z|^b \exp(-a|z|^{2p} + it(z^{p+q} + \bar{z}^{p+q}))$.

Our results motivate considering the corresponding normal matrix model given by the complex measure

$$d\mu_N(M) = \exp(-a \operatorname{tr}(M^\dagger M) + i \operatorname{tr}(V(M) + V(M^\dagger))) dM dM^\dagger$$

on normal $N \times N$ matrices for monomial (or more generally entire) potentials V . Here M^\dagger denotes the matrix adjoint (conjugate transposed) to M and $dM dM^\dagger$ is the natural $U(N)$ -invariant measure on complex normal matrices, see [11, 24]. As in the random normal matrix model, the induced complex measure on eigenvalues is

$$d\mu_N(z) = \exp \left(-a \sum_{j=1}^N |z_j|^2 + 2i \sum_{j=1}^N \operatorname{Re} V(z_j) \right) \prod_{j < k} |z_j - z_k|^2 \prod_{j=1}^N d \operatorname{Re} z_j d \operatorname{Im} z_j.$$

For monomial potential V , the partition function $Z_N = \int d\mu(M)$ is then up to sign the product of the h_n up to N and is the determinant of the Gram matrix, so Conjecture 3.6 is equivalent to the non-vanishing of Z_N for all N . For general polynomial potentials, the complex measure $d\mu_N$ is absolutely integrable, as opposed to the real measure of the normal matrix model of [24], which requires regularization [5, 11] to normalize it to a probability measure. One should expect that, in spite of the fact that the probabilistic interpretation is not possible, the results of [24], such as the relation to Laplacian growth models and the description in terms of a real form of the 2D Toda integrable system, hold for the complex measure $d\mu_N$.

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