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Stolz Positive Scalar Curvature Structure Groups, Proper Actions and Equivariant 2-Types

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Abstract. In this note, we study equivariant versions of Stolz' R-groups, the positive scalar curvature structure groups $R_n^{\mathrm{spin}}(X)^G$, for proper actions of discrete groups G. We define the concept of a fundamental groupoid functor for a G-space, encapsulating all the fundamental group information of all the fixed point sets and their relations. We construct classifying spaces for fundamental groupoid functors. As a geometric result, we show that Stolz' equivariant R-group $R_n^{\mathrm{spin}}(X)^G$ depends only on the fundamental groupoid functor of the reference space X. The proof covers at the same time in a concise and clear way the classical non-equivariant case.

Key words: positive scalar curvature; universal space for proper actions; spin bordism; fundamental groupoid

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1 Introduction

In [10], Stolz organized the classification problem of metrics with positive scalar curvature in a long exact sequence

$$\cdots \to R_{n+1}^{\mathrm{spin}}(X) \to \mathrm{Pos}_n^{\mathrm{spin}}(X) \to \Omega_n^{\mathrm{spin}}(X) \to R_n^{\mathrm{spin}}(X) \to \cdots \tag{1.1}$$

of cobordism groups, where X is a CW-complex. This long exact sequence includes the well-known spin cobordism $\Omega_*^{\mathrm{spin}}(X)$; then the structure bordism group of metrics of positive scalar curvature $\mathrm{Pos}_*^{\mathrm{spin}}(X)$, whose cycles $(f\colon M \to X, g)$ are defined like those of $\Omega_*^{\mathrm{spin}}(X)$, but adding the geometric secondary structure given by a positive scalar curvature metric g on the smooth compact manifold M with spin structure; finally, the object of study of this note, $R_*^{\mathrm{spin}}(X)$ which intuitively arises as a sort of mapping cone construction of the forgetful functor

$$\mathrm{Pos}_*^{\mathrm{spin}}(X) \to \Omega_*^{\mathrm{spin}}(X).$$

An explicit calculation of this last group has not yet been achieved in any case of interest. Nonetheless, Stolz has proved that, if $X = B\pi_1(M)$ is the classifying space of the fundamental group of a closed spin manifold M with $\dim(M) \geq 5$, then $R_{\dim M+1}^{\mathrm{spin}}(X)$ acts freely and transitively on the space of concordance classes of positive scalar curvature metrics on M. This action is canonical and hence the space of concordance classes inherits canonically (after the choice of a base point) an abelian group structure and a construction of an explicit model for this is given in [15, Section 4]. This has been used, for instance, in [6, 7, 9, 15] to give a lower bound on the rank of this affine group and of the moduli space of concordance classes of metrics of positive scalar curvature, where moduli space refers to the quotient by the action of the diffeomorphism group of M.

A fundamental step in [9] uses the fact, which can be implicitly deduced from [10], that a 2-connected map between CW-complexes, such as the classifying map $u\colon M \to B\pi_1(M)$, induces an isomorphism between corresponding R_*^{spin} groups for $* \geq 6$.

There is an obvious equivariant reformulation of this story: instead of working with a space X one can work with the a G-cover $\bar{X}$ with its free G-action. One can then go equivalently back and forth between structure on X and G-invariant structure on $\bar{X}$ (pulling back from X to $\bar{X}$ and quotienting by G from $\bar{X}$ to X). This way, for a space X with fundamental group G and universal cover $\tilde{X}$, one can replace equivalently the Stolz sequence (1.1) by its equivariant version

$$\cdots \to R_{n+1}^{\mathrm{spin}}(\tilde{X})^G \to \mathrm{Pos}_n^{\mathrm{spin}}(\tilde{X})^G \to \Omega_n^{\mathrm{spin}}(\tilde{X})^G \to R_n^{\mathrm{spin}}(\tilde{X})^G \to \cdots,$$

where all the cycles are defined as above, by requiring to come with a free and co-compact and (whenever we have metrics) isometric action of G, and all maps are required to be G-equivariant. This equivalent reformulation is important to get information about the groups in the Stolz sequence via higher index theory, which has been successfully implemented in [5] and [16], compare also [14] and the survey [8]. The proofs of lower bounds on the rank of the affine group of concordance classes of positive scalar curvature are based on these techniques.

The aim of this note is two-fold. The first main contribution is an explicit and concise proof of the fact that (non-equivariantly) the group $R_*^{\mathrm{spin}}(X)$ depends for $* \geq 6$ only on the fundamental group information of X. Secondly, we want to analyze the corresponding statement for the case of general proper G-actions. Here, we develop basic tools for this generalization and then study it in the equivariant context of a CW-complex endowed with a proper action of a discrete group. Concretely, we construct an equivariant version of the Stolz exact sequence, compare Proposition 3.3. One contribution is to give a complete proof of exactness, as we are not aware that this is available in the literature. Our main original result is the fact that the equivariant R-groups do only depend on the equivariant 2-type of the space, compare Theorem 4.1. Along the way, we construct an equivariant analog of the space $B\pi_1(X)$, namely a "universal space for a given 2-type", compare Section 5.2.

It should be noted that this relies on surgery constructions requiring enough "room". This is the reason for the dimension restrictions listed above: the manifolds whose positive scalar curvature metrics are controlled have to be spin manifolds of dimension at least 5, and the group $R_*^{\mathrm{spin}}(X)^G$ can be treated efficiently if $* \geq 6$.

2 Proper actions of discrete groups

Let us fix throughout a discrete group G.

Definition 2.1. A G-CW-complex X is a G-space together with a G-invariant filtration

$$\varnothing = X^{(-1)} \subseteq X^{(0)} \subseteq X^{(1)} \subseteq \cdots \subseteq X^{(n)} \subseteq \cdots \subseteq \bigcup_{n\geq 0} X^{(n)} = X$$

such that X carries the colimit topology with respect to this filtration and for each $n \geq 0$ the space $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching equivariant n-dimensional cells, i.e., there exists a G-pushout

$$\begin{tikzcd} \coprod_{i\in I_n} G/H_i \times S^{n-1} \arrow[r] \arrow[d] & X^{(n-1)} \arrow[d] \\ \coprod_{i\in I_n} G/H_i \times D^n \arrow[r] & X^{(n)}. \end{tikzcd}$$

Note that the G-CW-complex X defines its *isotropy family* $\mathcal{I}(X)$ of subgroups of G where H belongs to $\mathcal{I}(X)$ if and only if the fixed point set X^H is non-empty or in other words if and only if X contains a G-cell $G/K \times D^n$ where a conjugate of H is contained in K.

We recall the concept of a family of subgroups which we just have used.

Definition 2.2. Let G be a discrete group. A *family of subgroups* $\mathcal{F}$ is a set of subgroup of G which is closed under conjugation and is closed under passing to smaller subgroups.

Significant examples of such families of subgroups are $\mathcal{FIN}$, the family of all finite subgroups, or $\mathcal{ALL}$, the family of all subgroups.

Let us fix some notation.

Definition 2.3. Let $f\colon X \to Y$ be a continuous G-equivariant map between G-spaces. We will denote by $f^H\colon X^H \to Y^H$ the restriction of f to the H-fixed point sets, with H a subgroup of G.

Definition 2.4. We say that f is cellular if X and Y are G-CW-complexes and, denoting by $X^{(k)}$ the *k-skeleton* of X, one has $f\big(X^{(k)}\big) \subseteq Y^{(k)}$.

The well-known and important cellular approximation theorem extends to the equivariant context (compare [12, Theorem 2.1]).

Theorem 2.5. *Let $f\colon X \to Y$ be a G-map. Then there exists a G-homotopy $h\colon X \times I \to Y$ such that $h_0 := h_{X\times\{0\}} = f$ and $h_1 := h_{X\times\{1\}}$ is cellular.*

We also have an equivariant version of the Whitehead Theorem for G-CW-complexes.

Definition 2.6. Consider a function $\nu\colon \mathcal{ALL} \to \mathbb{N}$. Then we say that f is ν-connected if f^H is $\nu(H)$-connected for all $H \in \mathcal{F}$, namely the induced maps are isomorphisms on the first $\nu(H)-1$ homotopy groups of (all components of) X^H and Y^H and a surjection on the $\nu(H)$-th one. In particular, we say that it is k-connected if ν is constantly equal to k.

Moreover, we say that a relative G-CW-complex (X, A) has dimension less or equal to ν if the cells in $X \setminus A$ are of the form $G/H \times D^k$ with $k \leq \nu(H)$.

Proposition 2.7 (compare [12, Proposition 2.6])**.** *Let $f\colon B \to C$ be a ν-connected map between G-CW-complexes and A another G-CW-complex. Write $[A, B]^G$ for the set of G-homotopy classes of G-maps from A to B. Then*

$$f_*\colon\ [A, B]^G \to [A, C]^G$$

is surjective (or bijective, respectively) if $\dim A \leq \nu$ *(or* $\dim A < \nu$*, respectively).*

Note that the classical Whitehead theorem is a consequence, using $A = C$ and $A = B$ and identity maps.

3 The Stolz exact sequence

Definition 3.1. Let X be a G-CW-complex. We then define the following groups:

- $\Omega_n^{\mathrm{spin}}(X)^G$ is the G-equivariant *spin bordism* group: a cycle here is given by a pair (M, f), where M is an n-dimensional spin-manifold with cocompact spin-structure preserving[1] action of G and with a G-equivariant reference map $f\colon M \to X$. Two cycles (M, f) and $\bigl(M', f'\bigr)$ are equivalent if there is a cocompact spin G-bordism W from M to M' and there exists a G-equivariant reference map $F\colon W \to X$ extending f and f'.
- $\mathrm{Pos}_n^{\mathrm{spin}}(X)^G$ consists of cycles (M, f, g), where the pair (M, f) is as before and g is a G-invariant metric with positive scalar curvature on M. Two such cycles (M, f, g) and $\bigl(M', f', g'\bigr)$ are equivalent if there exists a spin bordism (W, F) as before, along with a G-invariant metric g_W on W which is of product type near the boundary which restricts to g on M and to g' on M'.
- $R_n^{\mathrm{spin}}(X)^G$ is the bordism group of spin G-manifolds with boundary (possibly empty) of dimension n, together with a G-invariant Riemannian metric of positive scalar curvature on the boundary. Bordisms are then manifolds with corners. In particular, (M, f, g) and $\bigl(M', f', g'\bigr)$ are equivalent if there exists a G-bordism $(W, F, \bar{g})$, where W is a bordism between M and M' and the resulting bordism $\partial_0 W$ between ∂M and $\partial M'$ carries a G-invariant metric $\bar{g}$ with positive scalar curvature, so that it is a bordism between $\bigl(\partial M, \partial f, g\bigr)$ and $\bigl(\partial M', \partial f', g'\bigr)$ in the sense of $\mathrm{Pos}_{n-1}^{\mathrm{spin}}(X)^G$.

Each of these sets is equipped with an abelian group structure given by disjoint union of manifolds and is covariantly functorial in X as follows: a G-equivariant map of G-CW-Complexes $\varphi\colon X \to Y$ induces a mapping φ_* on these groups by post-composing the reference maps with φ.

Remark 3.2. Note that for each cycle $f\colon M \to X$ in $\Omega_n^{\mathrm{spin}}(X)^G$ the isotropy of M is restricted to belong to the family $\mathcal{I}(X)$ (because the image under the equivariant map f of a point $x \in M$ fixed by a subgroup H of G must also be fixed by H, and is a point in X).

If we want to restrict the isotropy even further, to live in a family $\mathcal{F}$ of subgroups of G, we can replace the space X by $X \times E_{\mathcal{F}}G$ where $E_{\mathcal{F}}G$ is the universal G-CW-complex with isotropy family $\mathcal{F}$. This space is characterized by the property that $E_{\mathcal{F}}G^H$ is empty if $H \notin \mathcal{F}$ and $E_{\mathcal{F}}G^H$ is contractible if $H \in \mathcal{F}$, in particular $\mathcal{I}(E_{\mathcal{F}}G) = \mathcal{F}$. It exists for each family $\mathcal{F}$ and it is unique up to G-equivariant homotopy equivalence. It has the universal property that a G-CW-complex X with isotropy contained in $\mathcal{F}$ has a unique homotopy class of G-maps to $E_{\mathcal{F}}G$. These spaces were introduced and studied in [11, 12].

Proposition 3.3. *The abelian groups defined in Definition 3.1 fit into the following G-equivariant version of the Stolz positive scalar curvature exact sequence:*

$$\cdots \to R_{n+1}^{\mathrm{spin}}(X)^G \to \mathrm{Pos}_n^{\mathrm{spin}}(X)^G \to \Omega_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(X)^G \to \cdots,$$

where the first map sends a manifold to its boundary, the second one is the forgetful map (*i.e., it forgets the metric of positive scalar curvature*) *and the last one considers a closed manifold as a manifold with empty boundary.*

Proof. This is a rather direct consequence of the definitions and well known to the experts, at least non-equivariantly, stated, e.g., in [10, long exact sequence (4.4)] (but without proof). The argument for the G-equivariant case is exactly the same as the classical non-equivariant

[1] That is, with a lift of the action to the Spin principal bundle.

situation. As we are not aware of a treatment available in the journal literature, we follow the referee's suggestion and give a complete account of the argument here.

First, that the composition of two consecutive maps is zero indeed is a direct consequence of the definitions:

- Given $[W,f,g] \in R^{\mathrm{spin}}_{n+1}(X)^G$, its image in $\Omega_n(X)^G$ is represented by $(\partial W, f|_{\partial W})$ which indeed is a boundary, hence represents 0.
- Given $[M,f,g] \in \mathrm{Pos}^{\mathrm{spin}}_n(X)^G$, its image in $R^{\mathrm{spin}}_n(X)^G$ is the manifold M with empty boundary. It is bordant in $R^{\mathrm{spin}}_n(X)^G$ to $\varnothing$ and hence represents 0 via the bordism $M \times [0,1]$, where $(M \times \{1\}, g)$ is a positive scalar curvature bordism of $\varnothing = \partial M$ to $\varnothing = \partial\varnothing$.
- Given $[M,f] \in \Omega^{\mathrm{spin}}_n(X)^G$, its image in $\mathrm{Pos}^{\mathrm{spin}}_{n-1}(X)^G$ is represented by $\partial M = \varnothing$, hence represents 0.

For the opposite inclusions of the kernels in the image, the constructions are almost as straight-forward:

- Assume that $[M,f] \in \Omega^{\mathrm{spin}}_n(X)^G$ is mapped to 0 in $R^{\mathrm{spin}}_n(X)^G$. This means that there is a null-bordism (W,F). In the case at hand, ∂W has two disjoint parts: on the one hand (M,f) and on the other hand $\big(M', f', g'\big)$ which is a bordism from $\partial M = \varnothing$ to $\partial\varnothing = \varnothing$ and which is equipped with a metric g of positive scalar curvature. But this means, by definition, that $[M,f] = \big[M', f'\big] \in \Omega^{\mathrm{spin}}_n(X)^G$ where M' is equipped with a metric g of positive scalar curvature. Hence we have the preimage $\big[M', f', g\big] \in \mathrm{Pos}^{\mathrm{spin}}_n(X)^G$ of the initial equivariant bordism class (M,f).
- Assume that $[M,f,g] \in \mathrm{Pos}^{\mathrm{spin}}_n(X)^G$ is mapped to 0 in $\Omega^{\mathrm{spin}}_n(X)^G$. This means that there is a bordism (W,F) with $\partial(W,F) = (M,f)$. But then $[W,F,g]$ represents a class in $R^{\mathrm{spin}}_{n+1}(X)^G$ which is mapped to $[M,f,g] \in \mathrm{Pos}^{\mathrm{spin}}_n(X)^G$.
- Finally, assume that $[W,f,g] \in R^{\mathrm{spin}}_{n+1}(X)^G$ is mapped to 0 in $\mathrm{Pos}^{\mathrm{spin}}_n(X)^G$. This means that there is a second bordism $\big(W', f', g'\big)$ whose boundary is $(\partial W, f|_{\partial W}, g)$. We now consider the closed G-manifold $Z := W \cup_{\partial W} W'$ with map $F := f \cup_{\partial W} f' \colon Z \to X$ and the bordism $B := Z \times [0,1]$ with map $F \circ \mathrm{pr}_Z \colon B \to X$. The boundary of B consists of three parts: the internal boundary W' between ∂W and $\varnothing = \partial Z$, equipped with the metric g' of positive scalar curvature extending g. The other two boundary parts are W and Z, and by definition $(B, F \circ \mathrm{pr}_Z, g')$ is a bordism in $R^{\mathrm{spin}}_n(X)^G$ between (W,f,g) and $(Z,f,\varnothing)$ and $[Z,f] \in \Omega^{\mathrm{spin}}_n(X)^G$ represents a preimage of $[W,f,g] \in R^{\mathrm{spin}}_n(X)^G$. ∎

Let X be a connected G-CW-complex with $x_0 \in X$ and fundamental group $\pi_1(X,x_0)$. Let

$$\pi\colon \big(\tilde{X}, \tilde{x}_0\big) \to (X, x_0)$$

be the associated universal covering projection. Then the proper G-action on X lifts to a $\widetilde{G}$-action on $\tilde{X}$, where

$$1 \to \pi_1(X,x_0) \to \widetilde{G} \to G \to 1$$

is an extension of discrete groups defined as follows: the elements of $\widetilde{G}$ are pairs $\big(\alpha\colon \tilde{X} \to \tilde{X}, g\big)$ where $g \in G$ and α covers the action map of g on X. We define the multiplication by composition of the maps α and multiplication in G.

Note that, by covering theory, because $\tilde{X}$ has trivial fundamental group, indeed for each $\tilde{x}_1 \in \tilde{X}$ with $\pi(\tilde{x}_1) = g \cdot \pi(\tilde{x}_0)$ there is a unique lift of the action map sending $\tilde{x}_0$ to $\tilde{x}_1$.

The projection map $\widetilde{G} \to G$ sends (α, g) to g. By the above consideration, this map is surjective and its kernel consists of the deck transformations which, by covering theory, are identified with $\pi_1(X,x_0)$.

Remark 3.4. The Stolz exact sequence of Proposition 3.3 can always be reduced to a sequence where the space X is connected. This is done in two steps:

(1) For every component X_c of X consider the induced G-invariant subspace $G \cdot X_c$. We then have a disjoint union decomposition into G-invariant subsets $X = \bigsqcup G \cdot X_c$ with X_c connected (choosing one representative in each G-orbit of components). Clearly, every group in the Stolz exact sequence and the whole sequence then split canonically as a direct product, with one factor for each $G \cdot X_c$.

(2) For a G-space $Y = G \cdot X_c$ with X_c connected, consider the subgroup $G_0 \subset G$ of all elements which map X_c to itself. Then X_c is a G_0-space and Y is obtained by "induction": $Y = G \times_{G_0} X_c$. Whenever we have a G-map $f\colon M \to Y$ which could be part of a cycle or a bordism for the groups in the Stolz exact sequence, then we obtain $M_0 := f^{-1}(X_c)$ a union of components of M on which G_0 acts (restricting the action of G on M). Then $f|_{M_0}\colon M_0 \to X_c$ is a G_0-equivariant map and we obtain M and f by induction: $M = G \times_{G_0} M_0$ and $f = \mathrm{id}_G \times_{G_0} f|_{M_0}$. It follows that induction gives an isomorphism (already on the level of cycles and relations)

$$\mathrm{ind}_{G_0}^G\colon\ R_*^{\mathrm{spin}}(X_c)^{G_0} \to R_*^{\mathrm{spin}}(Y)^G$$

and the same for the other groups in the sequence of Proposition 3.3 and the maps between them.

Remark 3.5. The isotropy family $\mathcal{I}(\tilde{X})$ of the action of $\widetilde{G}$ on $\tilde{X}$ is precisely the inverse image of $\mathcal{I}(X)$ under the projection $\widetilde{G} \to G$.

This again follows from covering theory: if we fix $x \in X$ with lift $\tilde{x} \in \tilde{X}$ and a subgroup H of G fixing x we have a canonical split $H \to \widetilde{G}$ sending $h \in H$ to the unique pair $\big(\alpha\colon \tilde{X} \to \tilde{X}, h\big)$ such that $\alpha(\tilde{x}) = \tilde{x}$.

Then we have the following easy identification.

Proposition 3.6. *The G-equivariant Stolz exact sequence associated to X is isomorphic to the $\widetilde{G}$-equivariant Stolz exact sequence associated to $\tilde{X}$.*

Proof. The action of $\pi_1(X) \subset \widetilde{G}$ on $\tilde{X}$ is free. Consequently, for every $\widetilde{G}$-map $f\colon \bar{M} \to \tilde{X}$ the action of $\pi_1(X)$ on $\bar{M}$ is free. We can therefore quotient out this action and obtain a cycle $M := \bar{M}/\pi_1(X) \to X$ with residual action of $G := \widetilde{G}/\pi_1(X)$. Vice versa, given a G-map $f\colon M \to X$ we can pull back $\tilde{X} \to X$ along f and obtain a $\pi_1(X)$-covering $\tilde{M} \to M$ with map $\tilde{f}\colon \tilde{M} \to \tilde{X}$. Because f is a G-map, it is straightforward to pull back also the action of $\widetilde{G}$ to an action on $\tilde{M}$ which covers the action of G on M and such that $\tilde{f}$ is a $\widetilde{G}$-map (indeed, the $\widetilde{G}$-action on $\tilde{M}$ is defined mapping a point $(m, \tilde{x}) \in \tilde{M} \subset M \times \tilde{X}$ by $(\alpha, g) \in \widetilde{G}$ to $(gm, \alpha(\tilde{x}))$).

The same construction gives a bijection between G-invariant metrics on M and $\widetilde{G}$-invariant metrics on $\tilde{M}$, and also works for bordisms. These constructions are clearly inverse to each other and preserve all additional structure and define the required bijections. ■

Remark 3.7. For the paper at hand, the transition to the universal covering as in Proposition 3.6 is not really relevant. However, in other situations this point of view is really fruitful. To our knowledge, essentially all information about non-triviality of the groups $R_n^{\mathrm{spin}}(X)$ and $\mathrm{Pos}_n^{\mathrm{spin}}(X)$ use higher index theory of the Dirac operator. A particularly transparent way to do this is to use the equivariant Dirac operator on the universal covering. This essentially means that one applies the isomorphism between the groups "downstairs" for X and the $\pi_1(X)$-equivariant groups "upstairs" for the universal covering $\widetilde{X}$, as we have already mentioned in the introduction.

It is pretty obvious that the same observation does hold for non-free actions: even if we want to understand symmetric metrics of positive scalar curvature on a compact manifold M with smooth action of a finite group Γ, it will be very useful to pass to the covering space which also sees the symmetries induced by the non-trivial fundamental group. This strategy has already been implemented in [14, Section 5] and [15]. The first paper constructs the map from the equivariant Stolz sequence to analysis (in form of the Higson–Roe exact sequence for the K-theory of the Roe C^*-algebra) in the sense of [5]. The second paper then uses this to explicitly distinguish many bordism classes of equivariant positive scalar curvature metrics.

3.1 Refinements beyond bordism of positive scalar curvature metrics

The Stolz positive scalar curvature exact sequence of Proposition 3.3 gives important information about the existence and classification of metrics of positive scalar curvature. However, by the very definition this information is about *bordism* classes and we are of course also interested in a fixed given manifold M.

Non-equivariantly, the situation here is very satisfactory: for a given connected closed spin manifold M with $\dim(M) \geq 5$, if we use $X = B\pi_1 M$ then M itself admits a metric with positive scalar curvature if (and only if) the image of $[u\colon M \to B\pi_1(M)]$ in $R_n^{\mathrm{spin}}(B\pi_1(M))$ vanishes. Moreover, in this case $R_{n+1}^{\mathrm{spin}}(B\pi_1(M))$ acts freely and transitively on the concordance classes of metrics of positive scalar curvature on M.

Of course, it would be very desirable to extend such results to the equivariant case. Unfortunately, the proof in the non-equivariant case uses as an important tool "handle cancellation" for Morse functions. It is known that equivariantly such cancellation is not even true. Therefore, a general treatment of the equivariant case seems not in reach at the moment. Under very special conditions on the action, positive existence results can be obtained. The strongest results in this direction we are aware of are obtained in [2]. There, also the difficulties are discussed when one attempts to obtain more general results. In the paper at hand, we focus on the bordism context and do not attempt to contribute to obstruction and classification results for G-invariant metrics of positive scalar curvature on a fixed manifold M.

4 Invariance of R-groups under 2-equivalence

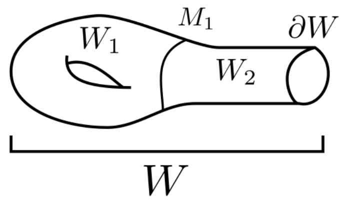
Theorem 4.1. *Let $f\colon X \to Y$ be a continuous, 2-connected G-map between G-CW-complexes (in particular, it induces a bijection between components of fixed point sets and an isomorphism of the fundamental groups of all components of fixed point sets of X and the corresponding ones of Y). Then for $* \geq 6$ the functorially induced map $f_*\colon R_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(Y)^G$ is an isomorphism.*

Proof. *Surjectivity.* We start by showing the surjectivity of the map $f_*\colon R_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(Y)^G$. Let us consider a class $[W, \varphi\colon W \to Y, g] \in R_n^{\mathrm{spin}}(Y)^G$. We want to find a *bordant* cycle whose reference map factors through f.

Consider W as a bordism between its boundary ∂W and the empty set and choose a G-invariant Morse function $\alpha\colon W \to \mathbb{R}$ on it with critical points rearranged as described in [4, Theorem 4.8], namely for any critical points p_i and p_j such that $f(p_i) < f(p_j)$, we have that $\mathrm{Ind}_f(p_i) < \mathrm{Ind}_f(p_j)$, where $\mathrm{Ind}_f(p)$ denotes the Morse index at p of the function f. Notice that we are going to use the enhanced version of this result to the equivariant setting, see for instance [3] or [13].

Then there exists a suitable $t \in \mathbb{R}$ such that the subset $W_1 := \alpha^{-1}([0,t]) \subset W$ consists only of G-handles of dimensions 0, 1 and 2. We immediately obtain a decomposition of W as $W_1 \cup W_2$ such that W_1 is a bordism from the empty set to $M_1 := \alpha^{-1}(t)$ and W_2 a bordism from M_1 to ∂W.

Of course, W_2 has only critical points p_i with $\mathrm{Ind}_\alpha(p_i) \geq 3$. Consider now the function $-\alpha$: this is a Morse function on W_2 seen as a bordism from ∂W to M_1 with same critical points p_i but with indices now given by $\mathrm{Ind}_{-\alpha}(p_i) = \dim(W) - \mathrm{Ind}_\alpha(p_i)$. These critical points p_i are then associated to G-equivariant $(\mathrm{Ind}_{-\alpha}(p_i) - 1)$-surgeries, hence of codimension $\mathrm{Index}_\alpha(p_i) + 1$ which is ≥ 3.



This allows us to apply the Gromov–Lawson theorem in its G-equivariant version as it is proved in [2, Theorem 2]. This implies that we can extend the metric with positive scalar curvature g on ∂W to a G-invariant metric with positive scalar curvature $\bar{g}$ on W_2. Let us denote by g_1 its restriction to M_1. Observe that the triad $(W_1, \varphi_{|W_1}, g_1)$ defines a class in $R_n^{\mathrm{spin}}(Y)^G$ and the manifold $W \times [0,1]$ provides a bordism between $(W_1, \varphi_{|W_1}, g_1)$ and (W, φ, g).

Consider now the natural G-equivariant inclusion of the 2-skeleton given by $i\colon Y^{(2)} \hookrightarrow Y$. Here we have the following facts:

- Since the manifold W_1 is obtained from the empty set by attaching G-handles of dimension 0, 1 and 2, it is homotopy equivalent to a 2-dimensional G-CW-complex. It follows from Theorem 2.5 that the map $\varphi_1 := \varphi_{|W_1}$ factors through i up to homotopy.
- Since f is 2-connected, up to G-homotopy we can assume that its restriction to the 2-skeleton $f^{(2)}\colon\ X^{(2)} \to Y^{(2)} \to Y$ has a right inverse, i.e., there exists a G-equivariant map $h\colon Y^{(2)} \to X^{(2)}$ such that $f^{(2)} \circ h$ is homotopic to i. To see this, observe that the existence of such a map h is guaranteed, up to G-homotopy, by Proposition 2.7. In fact, since $f^{(2)}$ is 2-connected and $Y^{(2)}$ has dimension ≤ 2, it suffices to apply Proposition 2.7 with $A = Y^{(2)}$, $B = X^{(2)}$, $C = Y$ and $f = f^{(2)}$ to the map $i \in [A, C]^G$. The surjectivity of f_* then implies the existence of h.

Thus, we obtain the following commutative diagram of G-equivariant maps

$$\begin{tikzcd}
W_1 \arrow[r, "\varphi_1"] \arrow[d, "\varphi_1"'] & Y \\
Y^{(2)} \arrow[ur, "i"] \arrow[d, "h"'] & \\
X^{(2)} \arrow[uur, "f^{(2)}"'] \arrow[r, "j"'] & X \arrow[uu, "f"']
\end{tikzcd}$$

and, if we set $\psi := j \circ h \circ \varphi_1 \colon W_1 \to X$, we obtain by construction that

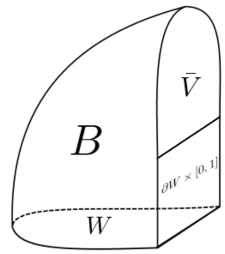
$$f_*[W_1, \psi\colon W_1 \to X, g_1] = [W, \varphi\colon W \to Y, g] \in R_n^{\mathrm{spin}}(Y)^G,$$

which proves that f_* is surjective.

Injectivity. In order to prove the injectivity of $f_*\colon R_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(Y)^G$, let us consider a class $[W, \varphi\colon W \to X, g] \in R_n^{\mathrm{spin}}(X)^G$ such that its image $f_*[W, \varphi\colon W \to X, g]$ is equal to the trivial element in $R_n^{\mathrm{spin}}(Y)^G$. This means that there exists

- an $(n+1)$-dimensional G-manifold with corners B, whose codimension 1 faces are W itself together with a bordism V from ∂W to the empty set, which intersect in the only codimension 2 face $\partial W = W \cap V$;

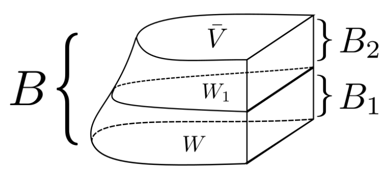
- a G-invariant metric g_V on V of positive scalar curvature of product type near the boundary which restricts to the metric g on ∂W;
- a G-equivariant map $\Psi\colon B \to Y$ which restricts to $f \circ \varphi$ on W.



Consider now a G-invariant collar neighborhood of ∂W inside V such that the boundary of B is made of three faces of codimension 1: W on the bottom, $\partial W \times [0,1]$ vertically and $\bar{V} = V \setminus \partial W \times [0,1)$ on the top.

We want to split the bordism B, as we did in the proof of surjectivity, into the composition of two bordisms, first from W to a manifold with boundary W_1 and then from W_1 to $\bar{V}$, such that the first one involves only handle attachments of dimension less or equal than 2 and the second one only of dimension greater or equal than 3. Since the vertical boundary face $\partial W \times [0,1]$ is a cylinder, B can be obtained from W by attaching all the handles to the interior of W, away from $\partial W \times [0,1]$. Hence we can find a Morse function on B which has all critical points there.

Thus, we can decompose B as desired: B_1 from W to W_1 involving only 0, 1, 2 handle attachments and B_2 from W_1 to $\bar{V}$. We can assume that these two bordisms have vertical boundaries faces equal to $\partial W \times [0,1/2]$ and $\partial W \times [1/2,1]$, respectively, and therefore that W_1 has boundary equal to ∂W.



By construction, the bordism B_2 is the trace of surgeries of codimension ≥ 3. Therefore, we can apply the equivariant version [2, Theorem 2] of the Gromov–Lawson theorem to extend the metric $g_{\bar{V}}$ to a G-invariant metric of positive scalar curvature g_2 on B_2. Let us denote by g_1 the G -invariant metric of positive scalar curvature obtained by restricting g_2 to W_1.

The last fact to prove is that $\Psi_{|B_1}\colon B_1 \to Y$ factors through $f\colon X \to Y$. Indeed, B_1 is obtained form W by attaching, up to homotopy, cells of dimension up to 2. Define a map $h\colon B_1 \times \{0\} \cup W \times [0,1] \to Y$ as the restriction of $\Psi|_{B_1} \circ \mathrm{pr}$ to this subspace of $B_1 \times [0,1]$, where $\mathrm{pr}\colon B_1 \times [0,1] \to B_1$ is the obvious projection. Note that by assumption the restriction of this map to $W \times \{1\}$ equals $f \circ \varphi$. These are exactly the conditions of the relative precursor [12, Proposition 2.5] of Proposition 2.7 which now implies in particular (because f is 2-connected) that there exists an extension $K\colon B_1 \to X$ of φ.

Now observe that (B_1, Φ, g_1) is a bordism between (W, φ, g) and the trivial cycle $(W_1, K|_{W_1}, g_1|_{\partial W})$ in $R_n^{\mathrm{spin}}(X)^G$ (trivial because the metric extends to the metric g_1 of positive scalar curvature on all of W_1), the injectivity of f_* is proved. ■

5 Universal spaces

In algebraic topology, a convenient way to deal with fundamental group information is by using the *classifying space* $B\Gamma$ of a (discrete) group Γ. Let us paraphrase its relevant property:

$B\Gamma$ has the following *universal property*: for every connected CW-complex X with base-point x_0 the fundamental group functor gives a bijection

$$[(X, x_0), (B\Gamma, y_0)] \to \mathrm{Hom}(\pi_1(X, x_0), \Gamma)$$

between the pointed homotopy classes of maps from X to $B\Gamma$ and the group homomorphisms from $\pi_1(X)$ and Γ.

It is known that $B\Gamma$ is characterized by the fact that it is connected, has fundamental group Γ, whereas all higher homotopy groups are trivial.

Our goal now is to achieve the corresponding result for a G-CW-complex X. We observe right away that we have now much richer fundamental group information: not only X, but also each fixed point set X^H for a subgroup H of G (or rather each component of X^H) has a fundamental group and the action of $g \in G$ as well as fixed point set inclusions induce maps between these fundamental groups.

As a first step, we define the *fundamental groupoid functor* of a G-CW-complex as a natural generalization of the fundamental group of a CW-complex. In order to do that, let us first recall the definition of fundamental groupoid $\Pi_1(X)$ of a topological space X: it is the groupoid whose objects are the points of X; whose arrows, from x to y for instance, are equivalence classes of paths starting at x and ending at y, where the equivalence relation is given by homotopy of paths with fixed starting and ending points; the composition is given by the concatenation of paths. Observe that the set of arrows starting and ending at a same point $x \in X$ is just the fundamental group of X at x.

Definition 5.1. Define for a family of subgroup $\mathcal{F}$ of G the following orbit category $\mathrm{Orb}(G, \mathcal{F})$, whose objects are all subgroups in $\mathcal{F}$ and morphisms from $H \in \mathcal{F}$ to $K \in \mathcal{F}$ are G-maps $G/H \to G/K$ for $H, K \in \mathcal{F}$. Note[2] that any such map is of the form $xH \mapsto xgK$ for a well defined coset $[g] \in G/K$, where we also have that $g^{-1}Hg \subset K$ —precisely the condition for this map to be well defined.

As an abbreviation, define $\mathrm{Orb}(G) := \mathrm{Orb}(G, \mathcal{ALL})$ for the family of all subgroups of G.

Definition 5.2. Let X be a G-CW-complex. The fundamental groupoid functor of X is the contravariant functor

$$\Pi_1(X; G)\colon\ \ \mathrm{Orb}(G, \mathcal{I}(X)) \to \text{groupoids},$$

which associates

- to a group $H \in \mathcal{I}(X)$ (the isotropy family) the fundamental groupoid of X^H restricted to the 0-skeleton of X^H (meaning that we take the full subgroupoid), which we denote by $\Pi_1\big(X^H\big)_{|X^H_{(0)}}$,
- to a morphism from H to K in the category $\mathrm{Orb}(G, \mathcal{I}(X))$ given as G-map $G/H \mapsto G/K; xH \mapsto xgK$ the morphism of groupoids between $\Pi_1(X^K)_{|X^K_{(0)}}$ and $\Pi_1\big(X^H\big)_{|X^H_{(0)}}$ induced by the map $X^K \to X^H$ defined as $x \mapsto gx$.

This extends canonically to a functor

$$\Pi_1(X; G)\colon\ \ \mathrm{Orb}(G) \to \text{groupoids},$$

assigning to $H \notin \mathcal{I}(X)$ the empty groupoid. Note that the empty groupoid has a unique map to any other groupoid, but is not the target of any map from a non-empty groupoid. The former is

[2] To be explicit: as G/H consists of a single G-orbit, the map is determined by the image of the coset H. Say this image is the coset gK. Then by G-equivariance for each $x \in G$ we must have that xH is mapped to xgK. To be well defined, for every $x \in G$ and $h \in H$ we require that $xhgK$ and xgK are in the same K-coset, i.e., $g^{-1}x^{-2}xhg \in K$ for each $h \in H$, which is the condition $g^{-1}Hg \subset K$.

good because it determines the value of $\Pi_1(X;G)$ on morphisms in $\mathrm{Orb}(G)$ with domain H. The latter is no problem because families are closed under conjugation and under taking subgroups and therefore if $K \in \mathcal{I}(X)$ and $H \notin \mathcal{I}(X)$ there never is a morphism from K to H in $\mathcal{ALL}$.[3]

Below, when dealing with different G-spaces with potentially different isotropy, we are using this version of the fundamental groupoid functor.

Remark 5.3. More conceptually, to construct the fundamental groupoid functor, we can observe that we have the canonical morphism $\mathrm{map}^G(G/H, X) \to X^H$ between the mapping space of G-equivariant maps and the fixed point set, sending $f\colon G/H \to X$ to $f(eH)$. The functoriality is then just given by precomposition.

Observe that the construction of the fundamental groupoid functor is itself functorial. This means that if $\varphi\colon Y \to X$ is a G-equivariant cellular map between G-CW-complexes, then there is an induced natural transformation $\varphi_\#\colon \Pi_1(Y;G) \to \Pi_1(X;G)$ whose component at H is the homomorphism of groupoids $\varphi_\#(H)\colon \Pi_1\big(Y^H\big)_{|Y^H_{(0)}} \to \Pi_1\big(X^H\big)_{|X^H_{(0)}}$ induced by $\varphi_{|Y^H}\colon Y^H \to X^H$.

5.1 Fundamental groupoid functor realization

We know that every discrete group is the fundamental group of a 2-dimensional CW-complex. The corresponding result holds for our fundamental groupoid functors:

Proposition 5.4. *Let G be a discrete group and $\Pi\colon \mathrm{Orb}(G) \to$ Groupoids be a functor whose image is given by discrete groupoids. Then there is a G-CW-complex X with $\Pi_1(X;G) \cong \Pi$.*

Proof. The proof is somewhat parallel to the one in the non-equivariant case. One has to be careful to work canonically (without choices) as one has to achieve compatibility between the different fixed point set data.

To simplify, we make use of some well established constructions (classifying space/simplicial set of a small category and its geometric realization), we also make use of one very special case of the co-end construction, called "tensor product of space valued functors over the orbit category" in [1, Section 1].

Since the proof is rather long, we summarize here the steps we are going to follow:

- **Step 1:** we first construct the 0-skeleton $X^{(0)}$ of the candidate G-CW-complex X;
- **Step 2:** we check that, for each subgroup H of G, $\big(X^{(0)}\big)^H$ is in bijection with the units of $\Pi(H)$;
- **Step 3:** we construct the G-CW-complex X;
- **Step 4:** in order to check that, for each subgroup H in G, we have that $\Pi_1\big(X^H\big)_{|(X^{(0)})^H} = \Pi(H)$, we construct an intermediate subspace Z of X which allows to facilitate this computation and we do it;
- **Step 5:** finally, we check that for each morphism α in $\mathrm{Hom}_{\mathrm{Orb}}(H, K)$, the associated map $X^K \to X^H$ induces the groupoid morphism $\Pi(\alpha)$ under the identification of $\Pi_1\big(X^H\big)_{|(X^0)^H}$ with $\Pi(H)$ and the similar one for the subgroup K.

Step 1. The 0-skeleton of any G-CW-complex X with fundamental groupoid functor Π is directly determined by Π itself, more precisely by the units of $\Pi(H)$ for the subgroups H of G. Considering an orbit G/H as a discrete topological space with G-action, the "identity" functor defines a covariant functor $E\colon \mathrm{Orb}(G) \to$ G-TOP sending G/H to the discrete topological G-space G/H. Then we define (and are required to do so) $X^{(0)} := E \otimes_{\mathrm{Orb}} \Pi$. Concretely, by

[3] Recall that a morphism from H to K exists if and only if $\exists g \in G$ with $g^{-1}Hg \subset K$.

definition of $\otimes_{\mathrm{Orb}}$ in [1], this is the (discrete) G-space obtained as quotient space

$$\coprod_{H\in\mathrm{Ob}(\mathrm{Orb}(G))} G/H\times\Pi(H)_0/\!\sim, \tag{5.1}$$

where the equivalence relation is generated by declaring for all morphisms $\alpha\in\mathrm{Orb}(G)$, say $\alpha\colon G/H\to G/K$, that

$$(\alpha(gH),x)\sim(gH,\Pi(\alpha)(x))\qquad\forall x\in\Pi(K)_0.$$

Note that we write $\Pi(K)_0$ for the units in the groupoid $\Pi(K)$. The G-action is induced by the G-action on the orbits G/H, which is well defined because the maps α are G-equivariant.

Step 2. Let us show that this produces the desired 0-skeleton of the space X to construct. For this, we have to compute the fixed point sets say for the subgroup H of G. It is straightforward to see that $gK\in G/K$ is fixed by H if and only if $g^{-1}Hg\subset K$. At the same time, from such an element gK, we then get a well defined G-map $G/H\to G/K$, $uH\mapsto ugK$ which sends the coset $1H\in G/H$ to $gK\in G/K$, where 1 denotes the unit in G. Thus, all the points $(gK,x)\in G/K\times\Pi(K)_0$ in $X^{(0)}$ which are fixed by H are identified with a point in the single copy $\{1H\}\times\Pi(H)_0$ and therefore the H-fixed set of $X^{(0)}$ is a quotient of $\Pi(H)_0$. We are done once we have shown that no further identifications occur. This follows from the functoriality of Π: whenever two H-fixed points are identified with each other, they are also identified with a single point in $\{1H\}\times\Pi(H)_0$.

Step 3. As a next step, we produce spaces with the correct fundamental groupoids for the fixed point sets. For this, we rely on the well established construction of the classifying space (as simplicial set) of a small category: $|\Pi(H)|$ is the geometric realization of a simplicial set associated to the groupoid $\Pi(H)$ with a canonical identification $\Pi_1(|\Pi(H)|)|_{\Pi(H)|^{(0)}}=\Pi(H)$. In particular, for its zero skeleton we have $|\Pi(H)|^{(0)}=\Pi(H)_0$.

Note, as a remark, that we can not glue together these spaces in the same way as we glued together the 0-skeleta in (5.1) to produce the desired G-space as the gluing process could destroy the fundamental groups of the smaller fixed point sets. Instead, we have to carry out a homotopy version of the co-end construction which has to the correct 0-skeleton to obtain the following G-space W:

$$\left(\coprod_{H\in\mathrm{Ob}(\mathrm{Orb}(G))} G/H\times|\Pi(H)|\amalg\coprod_{\substack{\alpha\colon G/H\to G/K\\ \in\mathrm{Hom}_{\mathrm{Orb}(G)}(H,K)}} G/H\times[0,1]\times|\Pi(K)|\right)\Big/\!\sim,$$

where the equivalence relation is now generated by a multiple mapping cylinder construction, gluing the ends of the spaces associated to morphisms appropriately to the spaces associated to the objects. Concretely, this is done as follows: let us use the notation $(gH,t,x)_\alpha$ to denote an element of the summand $G/H\times[0,1]\times|\Pi(K)|$ associated to $\alpha\colon G/H\to G/K\in\mathrm{Hom}_{\mathrm{Orb}(G)}$. Then we declare for all morphisms $\alpha\in\mathrm{Mor}(\mathrm{Orb}(G))$, say $\alpha\colon G/H\to G/K$

$$\begin{aligned}(gH,0,x)_\alpha&\sim(gH,\Pi(\alpha)(x)), &&\forall gH\in G/H,\quad x\in|\Pi(K)|,\\ (gH,1,x)_\alpha&\sim(\alpha(gH),x), &&\forall gH\in G/H,\quad x\in|\Pi(K)|.\end{aligned}\tag{5.2}$$

Note that the G-space W contains the G-subspace W_0 obtained when performing this "homotopy co-end construction" just to the 0-skeleton $|\Pi(\cdot)|^{(0)}=\Pi(\cdot)_0$ of $|\Pi(\cdot)|$, namely

$$W_0:=\left(\coprod_{\substack{H\in\\ \mathrm{Ob}(\mathrm{Orb}(G))}} G/H\times\Pi(H)_0\amalg\coprod_{\substack{\alpha\colon G/H\to G/K\\ \in\mathrm{Hom}_{\mathrm{Orb}(G)}(H,K)}} G/H\times[0,1]\times\Pi(K)_0\right)\Big/\!\sim$$

with the same equivalence relation as in (5.2).

Note that there is the evident projection map $p\colon W_0 \to X^{(0)}$ sending the class of (gH, x) in W to the class of (gH, x) in $X^{(0)}$ and sending the class of $(gH, t, x)_\alpha$ to the class of (gH, x).

Now we define the desired G-CW-complex X by identifying in W all the points in W_0 with their image points under the surjective map p, i.e., as the pushout

$$\begin{tikzcd} W_0 \arrow[r, hook] \arrow[d, "p"] & W \arrow[d] \\ X^{(0)} \arrow[r] & X. \end{tikzcd}$$

Step 4. We next have to analyze X and compute in particular $\Pi_1(X;G)$. In order to do that, we define an intermediate G-CW-complex Z in the following way. Note that the space W contains canonically as subspace $\coprod_{H\in\mathrm{Orb}(G)} G/H \times |\Pi(H)|$. We therefore obtain as a subspace of X the pushout

$$\begin{tikzcd} W_0 \arrow[r, hook] \arrow[d, "p"] & W_0 \cup \left(\coprod_{H\in\mathrm{Orb}(G)} G/H \times |\Pi(H)| \right) \arrow[d] \\ X^{(0)} \arrow[r] & Z, \end{tikzcd}$$

where we glue the union of the $G/H \times |\Pi(H)|$ along their 0-skeleta with identifications. Observe that we obtain the same space by simply attaching the spaces $|\Pi(H)|$ to the already constructed 0-skeleton $X^{(0)}$ as the following pushout:

$$\begin{tikzcd} \coprod_{H\in\mathcal{ALL}} G/H \times |\Pi(H)|^{(0)} \arrow[r, hook] \arrow[d, "p"] & \coprod_{H\in\mathcal{ALL}} G/H \times |\Pi(H)| \arrow[d] \\ X^{(0)} \arrow[r] & Z. \end{tikzcd}$$

In particular, it is clear that the 0-skeleton of X is the 0-skeleton of Z, i.e., precisely $X^{(0)}$.

Observe that all 1-cells of X are already contained in the subspace Z. This space Z is obtained from a disjoint union by identifying along 0-cells. The fundamental groupoids of X and of the subspaces X^H, for H subgroup of G, are then generated by the arrows given by 1-cells, which are all contained in the constituent subspaces of Z. By the van Kampen theorem, the relations in the fundamental groupoid are then generated precisely by the 2-cells. There are two types of 2-cells: first the 2-cells contained in the constituent subspaces of Z, giving rise to the fundamental groupoids $\Pi(H)$ of $|\Pi(H)|$ and second 2-cells $\{gH\}\times[0,1]\times c_\alpha$ for $\alpha\colon G/H \to G/K$ and 1-cells c_α in $|\Pi(K)|$.

Using these considerations, let us now compute for a subgroup H of G the fundamental groupoid $\Pi_1\big(X^H\big)|_{X^H_{(0)}}$ of the H-fixed set. We follow the arguments which lead to the identification of

$$\big(\Pi_1\big(\big(X^{(0)}\big)^H\big)|_{X^{(0)}}\big)_0 = \big(\Pi_1\big(X^H\big)|_{(X^{(0)})^H}\big)_0 = \Pi(H)_0.$$

Indeed, the H-fixed set of Z is given as pushout

$$\begin{tikzcd} (W_0)^H \arrow[r, hook] \arrow[d, "p"] & (W_0)^H \cup \left(\coprod_{\substack{K\in\mathrm{Orb}(G);\\ \{gK | g^{-1}Hg\subset K\}}} gK \times |\Pi(K)| \right) \arrow[d] \\ \big(X^{(0)}\big)^H \arrow[r] & Z^H \end{tikzcd}$$

with one 1-cell for each 1-cell $\{gK\} \times c$ in $\{gK\} \times |\Pi(K)|^H$ for each subgroup K of G and coset $gK \in G/K$ with $g^{-1}Hg \subset K$ and 1-cell c of $|\Pi(K)|$.

Step 5. Now, each such gK gives rise to $\alpha\colon G/H \to G/K;\ uH \mapsto ugK$ as above which finally gives rise to a 2-cell in X^H which identifies the class of $\{gK\} \times c$ and of $\{1H\} \times \alpha^*(c)$ in $\Pi_1(Z^H)$. As before, the other morphisms in $\mathrm{Orb}(G)$ do not add further relations by functoriality of α. Next, the 2-cells in $|\Pi(H)|$ add precisely the required further relations which imply that $\Pi_1(X^H)|_{X^H_{(0)}} = \Pi(H)$. Note that the fact that $\alpha^*\colon \Pi_1(|\Pi(K)|) \to \Pi_1(|\Pi(H)|)$ are groupoid homomorphisms implies that no further relations will be created. We obtain the desired result $\Pi_1(X^H)|_{X^H_{(0)}} = \Pi(H)$ with a canonical identification.

From the construction, we also obtain that the induced map coming from a morphism $\alpha\colon G/H \to G/K$ of the form $uH \mapsto ugK$, given by fixed point inclusion and translation by $g \in G$ produces on the fundamental groupoids of the fixed point sets X^K and X^H exactly the morphism $\Pi(\alpha)$.

To summarize, we have constructed the G-CW-complex X such that $\Pi_1(X;G) = \Pi$ exactly as required. ■

5.2 Construction of a universal space

In the following, given a functor $\Pi\colon \mathrm{Orb}(G) \to \text{groupoid}$ (thought of as an abstract fundamental groupoid functor) we construct a universal space $B\Pi$ such that canonically $\Pi_1(B\Pi;G) = \Pi$, but all possible higher homotopy groups vanish. It has the universal property that there is a bijection between algebraic maps between fundamental groupoid functors and homotopy classes of maps between G-spaces, compare Proposition 5.7.

We start with the G-CW-complex X of Proposition 5.4 such that canonically $\Pi_1(X;G) = \Pi$. Inductively on $k \geq 3$ we construct larger G-CW-complexes X_k by attaching G-equivariant k-cells to X_{k-1} to kill π_{k-1}. We start by setting $X_2 := X$.

We assume as induction hypothesis that for each subgroup H of G for each component of X^H_{k-1} the π_j-th homotopy groups vanishes for $2 \leq j < k-2$. Note that for $k=3$ this is an empty condition and hence the induction start for $k=3$ is trivially satisfied.

Then, for each subgroup H of G we first attach cells $H/H \times D^k$ of dimension k to X^H_{k-1} in order to make $\pi_{k-1}(X^H)$ trivial. To actually remain in the world of G-CW-complexes we induce these attaching constructions up and attach $G/H \times D^k$ to $G \cdot X^H_{k-1} \subset X_{k-1}$. Note that these G-k-cells affect also other fixed point sets, but there will be no change of π_j for $j < k-1$, and π_{k-1} can only get smaller, but as we make sure that it is trivial, this property will not be affected. After attaching all these G-cells we therefore get a new G-CW-complex X_k containing X_{k-1} such that $\pi_j(X^H_k, x) = 0$ for all subgroups H of G and for all $2 \leq j \leq k-1$ and for all basepoints $x \in X^H_k$.

Definition 5.5. The union of all X_k (with the colimit topology) is then called $B\Pi$.

It has the following characteristic property.

Lemma 5.6. *The G-space $B\Pi$ just constructed contains X as a subcomplex and has the same 2-skeleton as X, and satisfies that $\pi_j(B\Pi^H, x) = 0$ for all $j \in \mathbb{N}$ and for all subgroups H of G and all $x \in B\Pi^H$.*

This classifying space $B\Pi$ has the following universal property.

Proposition 5.7. *For every G-CW-complex Y and for every natural transformation $\Phi\colon \Pi_1(Y;G) \to \Pi$, there exists, unique up to G-equivariant homotopy, a G-equivariant cellular map $\varphi\colon Y \to B\Pi$ such that $\varphi_\# = \Phi$.*

Proof. We begin by defining the map φ on the G-equivariant 0-skeleton of Y by putting $\varphi_{|Y_{(0)}} := \Phi(\{e\})_{|Y_{(0)}}$. Now, we proceed to define φ on the 1-skeleton of Y. For a G 1-cell c of the form $G/H \times [0,1]$ (with isotropy H) pick the ordinary 1-cell $c_H := H/H \times D^1$ contained in Y^H. Then c_H defines (if we choose an orientation of it) an element $\gamma \in \Pi_1\big(Y^H\big)_{|Y^H_{(0)}}$. Define now $\phi\colon c_H \to \big(B\Pi^H\big)^{(1)}$ such that this map represents the element $\Phi(H)(\gamma)$ in $\Pi_1\big(B\Pi^H\big)|_{(B\Pi^{(0)})^H}$. Since H acts trivially on the target, this extends uniquely to a G-map on the G-cell c with values in the 1-skeleton of $B\Pi$. Of course, we could have picked a different base cell in the G-cell c. But because Φ is a natural transformation, the resulting map is independent of this —up to the choice of the representative in the homotopy class of $\Phi(H)(\gamma)$ which had already to be made anyway. This defines the map ϕ on the 1-skeleton of Y.

To extend ϕ to $Y^{(2)}$, let c be a G-2-cell of the form $G/H \times D^2$. Pick the corresponding ordinary 2-cell $H/H \times D^2$ (with isotropy H) which is contained in Y^H. Its attaching map $\psi\colon S^1 \to Y^H$ is obviously contractible in Y^H and hence trivial (i.e., a unit) in the fundamental groupoid $\Pi_1\big(B\Pi^H\big)|_{(B\Pi^{(0)})^H}$ (we conjugate with a path in Y^H to the 0-skeleton to make ψ represent an element of $\Pi_1\big(Y^H\big)_{B\Pi^H_{(0)}}$, the triviality does not depend on the choice of such a path). Consequently, $\Phi(H)$ being a map of groupoids, also the image under $\Phi(H)$ of this element of the fundamental groupoid is trivial (a unit). By construction of ϕ on the 1-skeleton, this image element is represented by $\phi \circ \psi$ (up to the chosen conjugation with a path to the 0-skeleton), which is hence contractible in $B\Pi^H$. Extend ϕ over $H/H \times D^2$ using this contraction and then extend it G-equivariantly to $c = G/H \times D^2$.

Inductively, we then extend ϕ over the k-skeleta of Y. The extension property now follows because each attaching map has contractible image by the vanishing of all higher homotopy groups of all components of all fixed point sets of $B\Pi$ for the various subgroups of H.

By the very construction of ϕ on the 1-skeleton, we have $\varphi_\# = \Phi$, because the morphism sets of $\Pi(H) = \Pi_1\big(B\Pi^H\big)|_{(B\Pi^{(0)})^H}$ are generated by the 1-cells.

The last step of the proof concerns uniqueness. Choose therefore a G-equivariant map $\varphi'\colon Y \to B\Pi$ such that $\varphi'_\# = \Phi$. We have to show that φ' is G-homotopic to φ.

It is immediate from the definition that if $\varphi'_\# = \varphi_\#$, then their restrictions to $Y_{(0)}$ are equal. The construction of the desired G-homotopy is now done inductively over the skeleta and follows the pattern in the non-equivariant case and for the construction part, making use of the condition that $\phi'_\# = \phi_\#$ for the extension over the 1-skeleton and of the vanishing of higher homotopy groups for the further extensions. ■

Example 5.8. Consider the 2-dimensional torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ with the following CW-complex structure: the first factor is given by two vertices v_1 and v_2; two edges e_1 and e_2 both with extremes v_1 and v_2 (this gives the first factor); two loops l_1 and l_2 attached to v_1 and v_2 respectively (these give two copies of the second factor); finally two 2-cells c_1 and c_2 suitably attached, the first one to e_1, l_1, e_1^{-1}, and l_2, the second one to e_2, l_1, e_2^{-1}, and l_2. Let then $\mathbb{Z}_2$ act on $\mathbb{T}^2$ by flipping the first factor (in particular, swapping e_1 and e_2) and also c_1 and c_2, and then fixing l_1 and l_2.

In Figure 1, we represent the torus as a square where we identify opposite sides. Because the fixed-point sets are the aspherical spaces $\mathbb{S}^1$ for $\mathbb{Z}_2$ and $\mathbb{T}^2$ for $\{e\}$, this is a classifying space for the functor $\Pi_1\big(T^2;\mathbb{Z}_2\big)$, isomorphically given as $\Pi\colon \mathrm{Orb}(\mathbb{Z}_2) \to \mathrm{Groupoids}$ which is defined as follows. We have to specify which groupoid it assigns to each of the two possible $\mathbb{Z}_2$-orbits, namely to the trivial orbit $\mathbb{Z}_2/\mathbb{Z}_2$ and the free orbit $\mathbb{Z}_2/\{0\}$. Moreover, in the category $\mathrm{Orb}(\mathbb{Z}_2)$ there are only two non-identity morphisms, the collapse map $\mathbb{Z}_2/\{0\} \to \mathbb{Z}_2/\mathbb{Z}_2$ and the non-identity bijection $\tau\colon \mathbb{Z}_2/\{0\} \to \mathbb{Z}_2/\{0\}$ and we have to specify the associated morphisms of groupoids.

Concretely, we define Π as follows:

- We assign to $\mathbb{Z}_2$ the groupoid $\mathbb{Z} \times \{v_1, v_2\}$ over $\{v_1, v_2\}$, where $\{S\}$ denotes the trivial groupoid of the set S with only identity morphisms.

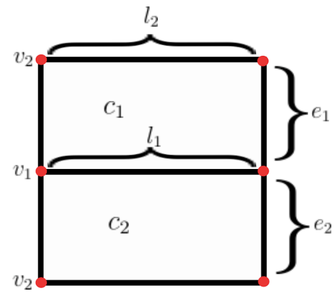


Figure 1. Torus with action of $\mathbb{Z}_2$.

- We assign to $\{0\} \subset \mathbb{Z}_2$ the group $\mathbb{Z}^2$.
- We assign to the collapse map $\mathbb{Z}_2/\{0\} \to \mathbb{Z}_2/\mathbb{Z}_2$ the morphism of groupoids which first projects $\mathbb{Z} \times \{v_1, v_2\}$ to the factor $\mathbb{Z}$ and then injects $\mathbb{Z}$ into $\mathbb{Z}^2$ as the subgroup $\{0\} \times \mathbb{Z}$. Note that, if we identify $\mathbb{Z}^2$ with the fundamental group of the torus T^2 we constructed above, we think of this as the subgroup generated by each of the two loops l_1, or l_2 (the two loops are homotopic).
- We assign to the non-identity map $\tau\colon \mathbb{Z}_2/\{0\} \to \mathbb{Z}_2/\{0\}$ the automorphism of $\mathbb{Z}^2$ which sends the generator of the first factor to its inverse and the second generator to itself.

Note that indeed this is precisely the fundamental groupoid of the $\mathbb{Z}_2$-space T^2 with the chosen CW-structure, and by the contractibility of S^1 and T^2 it is a model for $B\Pi_1(T^2; \mathbb{Z}_2)$.

It is a nice exercise to carry out the construction of the space X of Proposition 5.4 for this Π. One observes that one attaches quite a few 2-cells and 3-cells and that way produces a space X not homotopy equivalent to $\mathbb{T}^2$. Specifically, the 0-skeleton of X is the 0-skeleton of T^2. To obtain the 2-skeleton, we use the classifying spaces T^2 of $\Pi(\{0\}) = \mathbb{Z}^2$ and $S^1 \times \{v_1, v_2\}$ of $\Pi(\mathbb{Z}_2) = \mathbb{Z} \times \{v_1, v_2\}$.

By construction of X, we have to take the disjoint union of the 2-skeleta of $\mathbb{Z}_2/\{0\} \times T^2$ and of $\mathbb{Z}_2/\mathbb{Z}_2 \times S^1 \times \{v_1, v_2\}$ which is the disjoint union of two copies of T^2 and of S^1. To this, we have to glue 4 cylinders for the 4 morphisms in $\mathrm{Orb}(\mathbb{Z}_2)$

$$\mathbb{Z}_2/\{0\} \times [0,1] \times T^2, \qquad \mathbb{Z}_2/\{0\} \times [0,1] \times T^2, \qquad \mathbb{Z}_2/\{0\} \times [0,1] \times S^1 \times \{v_1, v_2\},$$
$$[0,1] \times S^1 \times \{v_1, v_2\}.$$

Gluing in the first cylinder, corresponding to $\mathrm{id}_{\mathbb{Z}_2/\{0\}}$, produces $\mathbb{Z}_2/\{0\} \times T^3$. The second cylinder produces another copy of T^3, glued with the previous two along embedded copies of T^2. The third cylinder, corresponding to the collapse map, homotopically and $\mathbb{Z}_2$-equivariantly glues the 2 copies of S^1 into this space (without changing the homotopy type). The forth cylinder, corresponding to the identity of $\mathbb{Z}_2/\mathbb{Z}_2$, glues in two more copies of T^2 into the space obtained so far along a homotopically non-trivial circle in each. From the cellular chain complex we can read off that $H^3(X; \mathbb{Z}) \cong \mathbb{Z}^3$, generated by the fundamental classes of the three copies of T^3 the first two cylinders produced. In particular, this space is not homotopy equivalent to T^2.

Remark 5.9. Observe that when $G = \{e\}$ is the trivial group and $\Pi(\{e\})$ is a discrete group Γ (a groupoid with only one object) then $B\Pi$ is a standard CW-complex with a single 0-cell. It follows that the space obtained in this way is an Eilenberg–Mac Lane space $B\Gamma = K(\Gamma, 1)$.

Now we combine the results obtained in this section and Theorem 4.1, obtaining as a corollary the fact that the Stolz G-equivariant R-groups depend only on the equivalence class of the fundamental groupoid functor.

Here, we define a natural equivalence between two groupoid valued functors as follows.

Definition 5.10. Let $\mathcal{C}$ be a small category and $F, G\colon \mathcal{C} \to \mathit{Groupoid}$ be two functors. Let T be a natural transformation between F and G. We say that T is an *equivalence* if and only if for each object c of $\mathcal{C}$, the functor $T(c)\colon F(c) \to G(c)$ between the groupoids $F(c)$ and $G(c)$ is an equivalence of groupoids. Recall that the latter condition means that there are functors $S(c)\colon G(c) \to F(c)$ and the compositions $T(c) \circ S(c)$ and $S(c) \circ T(c)$ admit natural transformations to the identity functor.

Remark 5.11. Let $T\colon \mathcal{G} \to \mathcal{H}$ be a morphism of groupoids. Recall the standard two facts:

(1) If T is an equivalence then T induces a bijection between the sets of orbits and for each unit x of $\mathcal{G}$ an isomorphism of isotropy groups $\mathcal{G}_x^x \to \mathcal{H}_{T(x)}^{T(x)}$.

(2) If $\mathcal{G}$ has a single orbit then for each object x of $\mathcal{G}$ the inclusion $\mathcal{G}_x^x \to \mathcal{G}$ induces an equivalence between the isotropy group of x in $\mathcal{G}$ (considered as a groupoid with a single object) and the full groupoid $\mathcal{G}$.

Lemma 5.12. *Let $T\colon \mathcal{G} \to \mathcal{H}$ be a morphism of groupoids such that it induces a bijection between the sets of orbits and for each object $x \in \mathcal{G}$ an isomorphism of isotropy groups $\mathcal{G}_x^x \to \mathcal{H}_{T(x)}^{T(x)}$. Then T is an equivalence of groupoids.*

Proof. The assertion follows directly from the following commutative diagram:

$$\begin{tikzcd} \coprod_{[x]\in\pi_0(\mathcal{G})} \mathcal{G}_x^x \arrow[r, "\cong"] \arrow[d, "\simeq"] & \coprod_{[x]\in\pi_0(\mathcal{H})} \mathcal{H}_{T(x)}^{T(x)} \arrow[d, "\simeq"] \\ \mathcal{G} \arrow[r, "T"] & \mathcal{H}. \end{tikzcd}$$

Here, we denote $\pi_0(\mathcal{G})$ the set of orbits of $\mathcal{G}$ and we use that T induces a bijection between $\pi_0(\mathcal{G})$ and $\pi_0(\mathcal{H})$ and that a disjoint union of equivalences of groupoids is again an equivalence of groupoids. ■

Lemma 5.13. *Let X and Y be G-CW-complexes and $f\colon X \to Y$ be a cellular G-map. Then the induced transformation $f_\#\colon \Pi_1(X;G) \to \Pi_1(Y;G)$ is an equivalence in the sense of Definition 5.10 if and only if for each subgroup H of G and for each $x_0 \in X^H$ f induces isomorphisms*

$$\pi_j\big(f^H\big)\colon\ \pi_j(X^H, x_0) \to \pi_j\big(Y^H, f(x_0)\big) \qquad \textit{for } j = 0, 1.$$

Proof. First, assume that $f_\#$ is an equivalence. By definition, this means that f^H induces an equivalence of groupoids between $\Pi_1\big(X^H\big)$ and $\Pi_1\big(Y^H\big)$. This, in turn, by Remark 5.11 implies that $\pi_j\big(f^H\big)$ is an isomorphism for $j = 0$ and $j = 1$. The other implication is a special case of Lemma 5.12. ■

We now show that the equivariant R-groups of G-CW-complexes depend only on the equivalence class of the fundamental groupoid functor. More precisely, we have the following result.

Proposition 5.14. *Let $\Pi\colon \operatorname{Orb}(G) \to \mathit{Groupoids}$ be a functor (thought of as an abstract fundamental groupoid functor). Let X be a G-CW complex and $\Phi\colon \Pi_1(X;G) \to \Pi$ be an equivalence as in Definition 5.10. Then for the classifying G-map $\varphi\colon X \to B\Pi$ as in Proposition 5.7 which induces Φ on the level of fundamental groupoid functors, we have that for $n \geq 6$*

$$\varphi_*\colon R_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(B\Pi)^G$$

is an isomorphism.

Proof. By Lemma 5.13, the map $f\colon X \to Y$ induces isomorphisms of π_0 and π_1 for each fixed point set X^H and each basepoint. Moreover, as all higher homotopy groups of fixed point sets of $B\Pi$ are trivial, this map indeed is 2-connected in the sense of Definition 2.6. Therefore, the assertion is exactly the one of Theorem 4.1. ■

Using this proposition, we arrive at a slight strengthening of Theorem 4.1.

Corollary 5.15. *Let $f\colon X \to Y$ be a G-map between G-CW-complexes such that for each subgroup H of G the induced map $f^H\colon X^H \to Y^H$ induces an isomorphism*

$$\pi_j\big(f^H\big)\colon\ \pi_j\big(X^H, x\big) \to \pi_j\big(Y^H, f(x)\big)$$

for all $x \in X^H$ and $j = 0, 1$. Then for $n \geq 6$ the induced map

$$f_*\colon\ R_n^{\mathrm{spin}}(X)^G \to R_n^{\mathrm{spin}}(Y)^G$$

is an isomorphism.

Note that the assumption on f in the corollary is close to the condition to be a 2-connected, but we do not have any condition on π_2.

Proof. This improvement relies on the existence of the universal space $B\Pi$. In fact, we can post-compose with the classifying map $u\colon Y \to B\Pi_1(Y;G)$ of Proposition 5.7. Then u and $u \circ f$ both are automatically 2-connected and hence both induce isomorphisms between the R-groups. Then also the third map f_* is an isomorphism. ■

Remark 5.16. Note that Proposition 5.14 and Corollary 5.15 of course also hold if G is trivial and just state that $R_n^{\mathrm{spin}}(X)$ depends only on the fundamental group of the CW-complex X.

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