Rectangular Recurrence Relations in $\mathfrak{gl}_n$ and $\mathfrak{o}_{2n+1}$ Invariant Integrable Models

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Abstract. A new method is introduced to derive general recurrence relations for off-shell Bethe vectors in quantum integrable models with either type $\mathfrak{gl}_n$ or type $\mathfrak{o}_{2n+1}$ symmetries. These recurrence relations describe how to add a single parameter z to specific subsets of Bethe parameters, expressing the resulting Bethe vector as a linear combination of monodromy matrix entries that act on Bethe vectors which do not depend on z. We refer to these recurrence relations as *rectangular* because the monodromy matrix entries involved are drawn from the upper-right rectangular part of the matrix. This construction is achieved within the framework of the zero mode method.

Key words: Yangians; recurrence relations for Bethe vectors; nested algebraic Bethe ansatz

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1 Introduction

We will consider generic $\mathfrak{g}$-invariant quantum integrable models in the framework of the algebraic Bethe ansatz [4]. In such models the monodromy matrix depends on a spectral parameter and satisfies the RTT relations [33] with a $\mathfrak{g}$-invariant R-matrix. In the Hilbert space of physical states, one can always construct from the local operators of the model a basis of states which are eigenvectors of a set of commuting Hamiltonians. These states form a representation of the finite-dimensional Lie algebra $\mathfrak{g}$. In the algebraic Bethe ansatz, they are constructed from the monodromy matrix entries which depend on spectral parameters satisfying the Bethe equations. These states are called on-shell Bethe vectors. When the spectral parameters are generic (not forced to obey the Bethe equations), the Bethe vectors are called off-shell and their combinatorial properties are defined solely by the RTT relation with a given R-matrix. As a consequence, one can replace the monodromy matrix of a generic model by the fundamental T-operator of the Yangian $Y(\mathfrak{g})$ in its matrix realization [3, 30]. The commutation relation for this T-operator $T(u)$, which depends on a formal spectral parameter u, coincides with the commutation relations of monodromy matrix in a generic $\mathfrak{g}$-invariant integrable model and the generators of the finite dimensional symmetry may be identified with the zero modes of the matrix entries $T_{i,j}(u)$ of the Yangian fundamental T-operator. Below we will explore the Yangian fundamental T-operator $T(u)$ calling it the monodromy matrix of a generic $\mathfrak{g}$-invariant integrable model.

One of the key problems in quantum integrable models is the presentation of the Bethe vector scalar products in a determinant form. For periodic boundary conditions, this problem was investigated and fully solved in [37] for the integrable system associated with the simplest Lie algebra $\mathfrak{gl}_2$, where the structure of the Bethe vectors is quite simple.

In the models associated to higher rank symmetries the structure of the Bethe vectors is rather complicated. In principle, the nested Bethe ansatz ([20, 21] for $\mathfrak{gl}_n$ and [32] for $\mathfrak{o}_n$) ensures that the Bethe vectors can be expressed as a combinatorial expression of the monodromy matrix entries acting on a vacuum vector. However, whether there is a determinant form for scalar products of Bethe vectors in the general case remains an open question and has answers only in the cases $\mathfrak{gl}_3$ [1] and $\mathfrak{gl}_{2|1}$ [10].

It is worth noting that there is another method for studying eigenvectors and correlation functions in integrable systems, which is based on the so-called quantum separation of variables (SoV) method [35, 36]. Recently, for models associated with the Lie algebra $\mathfrak{gl}_n$ was proposed [8] another construction to describe eigenvector using only one creation operator $B(u)$ closely related to SoV. A little later, significant progress has been made in SoV method [28], where it was proposed to describe vectors in terms of the actions of transfer matrices on a certain vector, which is a fairly universal construction. Both groups develop their methods significantly [7, 29, 34] and managed to use this result to describe some overlaps of Bethe vectors. However, the study of correlation functions in this approach is far from being complete and, up to now, applies mostly to $\mathfrak{gl}_n$ models.

On the other hand, if one can find formulas for the action of the monodromy matrix entries on the off-shell Bethe vectors, as well as recurrence relations for them, then one can find expressions for the scalar products of off-shell Bethe vectors, and ultimately obtain recurrence relations for the building blocks of these scalar products. Expressions for the scalar product and the norm of Bethe vectors were obtained in [14, 17] for $\mathfrak{gl}_2$ invariant models and in [31] for $\mathfrak{gl}_3$ invariant models. General expressions for the scalar product, the norm and the recurrence relations were achieved in [11, 12] for models associated to $\mathfrak{gl}(m|n)$. Here by recurrence relations for the off-shell Bethe vectors we mean the possibility to express a Bethe vector with an extended set of Bethe parameters (for example, with an extension by a parameter z) as a linear combination of the action of monodromy matrix entries $T_{i,j}(z)$ with $i < j$ acting on off-shell Bethe vectors which do not depend on z.

The action formulas of the monodromy entries on the off-shell Bethe vectors in $\mathfrak{gl}(m|n)$- and $\mathfrak{o}_{2n+1}$-invariant integrable models were given in [13] and [23], respectively. The existence of recurrence relations is less investigated. Some examples of such relations can be found in the aforementioned papers. One of the goals of this paper is to fill this gap and produce all possible recurrence relations for the Bethe vectors in the models with $\mathfrak{gl}_n$ and $\mathfrak{o}_{2n+1}$ symmetries.

In the present paper, we show that in order to solve this problem, one has to combine a single (simple) action of the monodromy matrix entry with the action of the zero modes of the monodromy matrix, the latter being identified with the simple root generators of the Lie algebra $\mathfrak{g}$. We first perform the calculation of the recurrence relations using this method for the Bethe vectors in $\mathfrak{gl}_n$-invariant models and then extend this approach to the Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant models. The obtained recurrence relations will be then tested in several limiting cases to observe the reductions over rank of $\mathfrak{g}$ and embeddings of $Y(\mathfrak{gl}_\ell) \otimes Y(\mathfrak{gl}_{n-\ell})$ into $Y(\mathfrak{gl}_n)$ and of $Y(\mathfrak{o}_{2\ell+1}) \otimes Y(\mathfrak{gl}_{n-\ell})$ into $Y(\mathfrak{o}_{2n+1})$.

This paper is motivated by two key objectives. First, we would like to generalize the nested Bethe ansatz to the cases when not only extreme nodes of the Dynkin diagram are singularized, but also when the singularized node is inside the Dynkin diagram. In the case of the $\mathfrak{gl}_n$ algebra, some progress in this direction was made in the recent papers [18, 19] in the framework of the trace formula presentation of off-shell Bethe vectors. To the best of our knowledge, for other series, the nested Bethe ansatz method was developed only in the cases when the singularized node is an extreme node of $\mathfrak{gl}_n$-type. In our approach, the recurrence relations for the off-shell Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant integrable models are generalization of the nested Bethe ansatz for arbitrary singularized node in the Dynkin diagram of Lie algebra $\mathfrak{o}_{2n+1}$. The second motivation is to develop methods to investigate the recurrence relations and analytical properties of the highest coefficients in the Bethe vector scalar product in $\mathfrak{g}$-invariant integrable models. An application of our approach to the properties of highest coefficients in $\mathfrak{o}_{2n+1}$-invariant integrable models can be found in the paper [25].

Theorems 3.1 and 4.1 are the main results of the paper. It is surprising that these rectangular relations were not known previously, although some examples of these recurrence relations for the Bethe vectors in $\mathfrak{gl}_n$-invariant integrable models were obtained earlier in [6, 11].

The plan of the paper is as follows. In Section 2, we recall some basic algebraic notions on Yangians, which are at the core of the models under consideration, as well as some definitions on Bethe vectors. Section 3 deals with the rectangular recurrence relations for $\mathfrak{gl}_n$ models. Here we verify also the consistency of the recurrence relations with the embedding $Y(\mathfrak{gl}_\ell)\otimes Y(\mathfrak{gl}_{n-\ell})$ into $Y(\mathfrak{gl}_n)$ and compare our approach with the recent papers [18, 19]. The case of $\mathfrak{o}_{2n+1}$ is studied in Section 4. The main result is contained in Section 4.1, where rectangular recurrence relations for $\mathfrak{o}_{2n+1}$ models are presented. Some particular cases and examples are displayed in Section 4.2. We also show that the recurrence relations are consistent with the embedding of $Y(\mathfrak{gl}_n)$ or $Y(\mathfrak{o}_{2a+1})\otimes Y(\mathfrak{gl}_{n-a})$ in $Y(\mathfrak{o}_{2n+1})$: the Bethe vectors exhibit a nice factorisation property over these subalgebras (see Sections 4.3 and 4.4). We conclude in Section 5, and two appendices are devoted to the technical proofs.

2 Preliminaries

2.1 Algebraic context

Let $\mathfrak{g}$ be either the classical Lie algebra $\mathfrak{gl}_n$ or the orthogonal algebra $\mathfrak{o}_{2n+1}$, where $n=2,3,\dots$. We will use the set of positive integers $I_{\mathfrak{gl}_n}=\{1,\dots,n\}$ to index elements of the operators in $\operatorname{End}(\mathbb{C}^n)$ and the set of integers $I_{\mathfrak{o}_{2n+1}}=\{-n,-n+1,\dots,-1,0,1,2,\dots,n\}$ to index elements of the operators in $\operatorname{End}\big(\mathbb{C}^{2n+1}\big)$. We will use the notation $I_{\mathfrak{g}}$ to describe the two sets of indices simultaneously. Let $N=n$ and $N=2n+1$ for the algebras $\mathfrak{gl}_n$ and $\mathfrak{o}_{2n+1}$ respectively.

***RTT* presentation of the Yangians $\boldsymbol{Y(\mathfrak{gl}_n)}$ and $\boldsymbol{Y(\mathfrak{o}_{2n+1})}$.** Let $R_{\mathfrak{g}}(u,v)$ be the $\mathfrak{g}$-invariant R-matrix [38, 39]

$$R_{\mathfrak{g}}(u,v)=\mathbf{I}\otimes\mathbf{I}+\frac{c\mathbf{P}}{u-v}-\frac{c\mathbf{Q}}{u-v+c\kappa_n}, \tag{2.1}$$

where $\mathbf{I}=\sum_{i\in I_{\mathfrak{g}}}\mathsf{e}_{i,i}$ is the identity operator acting in the space $\mathbb{C}^N$ and $\mathsf{e}_{i,j}\in\operatorname{End}\big(\mathbb{C}^N\big)$ are $N\times N$ matrices with the only nonzero entry equal to 1 at the intersection of the i-th row and j-th column. The operators $\mathbf{P}$ and $\mathbf{Q}$ act in $\mathbb{C}^N\otimes\mathbb{C}^N$. They read

$$\mathbf{P}=\sum_{i,j\in I_{\mathfrak{g}}}\mathsf{e}_{i,j}\otimes\mathsf{e}_{j,i},\qquad \mathbf{Q}=\sum_{i,j\in I_{\mathfrak{g}}}\mathsf{e}_{-i,-j}\otimes\mathsf{e}_{i,j},$$

and[1]

$$\kappa_n=\begin{cases}\infty & \text{for } \mathfrak{g}=\mathfrak{gl}_n,\\ n-1/2 & \text{for } \mathfrak{g}=\mathfrak{o}_{2n+1}.\end{cases}$$

In the $\mathfrak{o}_{2n+1}$-case, for any matrix $M\in\operatorname{End}\big(\mathbb{C}^{2n+1}\big)$, we denote by M^{t} the transposition with respect to the secondary diagonal $\big(M^{\mathrm{t}}\big)_{i,j}=M_{-j,-i}$. In particular, $\mathbf{Q}=\mathbf{P}^{\mathrm{t}_1}=\mathbf{P}^{\mathrm{t}_2}$.

[1]The value of $\kappa_n=\infty$ for the algebra $\mathfrak{gl}_n$ simply means that in this case the $\mathfrak{g}$-invariant R-matrix does not contain the term proportional to operator $\mathbf{Q}$ and so coincides with the Yang R-matrix [38]. The parameter $\kappa_n=n-1/2$ is relevant only for the algebra $\mathfrak{g}=\mathfrak{o}_{2n+1}$.

The generic $\mathfrak{g}$-invariant integrable model is described by the monodromy matrix $T(u)$ which depends on the spectral parameter u and satisfies the commutation relation

$$R_{\mathfrak{g}}(u,v)\,(T(u)\otimes\mathbf{I})\,(\mathbf{I}\otimes T(v)) = (\mathbf{I}\otimes T(v))\,(T(u)\otimes\mathbf{I})\,R_{\mathfrak{g}}(u,v), \tag{2.2}$$

with the $\mathfrak{g}$-invariant R-matrix (2.1).

Equation (2.2) yields the commutation relations of the monodromy matrix entries

$$[T_{i,j}(u),T_{k,l}(v)] = \frac{c}{u-v}(T_{k,j}(v)T_{i,l}(u) - T_{k,j}(u)T_{i,l}(v)) \\ + \frac{c}{u-v+c\kappa_n}\sum_{p=-n}^{n}(\delta_{k,-i}T_{p,j}(u)T_{-p,l}(v) - \delta_{l,-j}T_{k,-p}(v)T_{i,p}(u)). \tag{2.3}$$

The parameter c in (2.1) and (2.3) is a Yangian deformation parameter. In the limit $c\to 0$, the Yangian goes to the Borel subalgebra of the corresponding loop algebra. When non-zero, this parameter can be always changed to the value $c=1$ by rescaling formal spectral parameters u and v but for our convenience we prefer to keep it arbitrary. Note that the second line in (2.3) occurs only in the case of $\mathfrak{o}_{2n+1}$ models, in accordance with the footnote 1. We will assume the following dependence of the monodromy matrix on the spectral parameter

$$T_{i,j}(u) = \chi_i\delta_{ij} + \sum_{m\ge 0} T_{i,j}[m](u/c)^{-m-1}, \tag{2.4}$$

where the parameters χ_i, $i\in I_{\mathfrak{g}}$, are twisting parameters. For $\mathfrak{g}=\mathfrak{gl}_N$, these parameters are all independent, while for $\mathfrak{g}=\mathfrak{o}_{2n+1}$ they satisfy the relation $\chi_i\chi_{-i}=1$ and $\chi_0=1$. For both algebras, the possibility to start the series expansion (2.4) with $\chi_i\delta_{ij}$ instead of δ_{ij} corresponds to the usual twist $T(z)\to DT(z)$ where D is the diagonal matrix with $D_{ii}=\chi_i$. These twisting maps are isomorphisms between two RTT-algebras, since the R-matrix commutes with $D\otimes D$ (with the restriction $D^{\mathrm{t}}=D$ in the $\mathfrak{o}_{2n+1}$ case).

Central element for $\mathfrak{g}=\mathfrak{o}_{2n+1}$. Remark that for $\mathfrak{g}=\mathfrak{o}_{2n+1}$, the pole at $u=v-c\kappa_n$ in the R-matrix (2.1) and in the commutation relation (2.2) implies that the monodromy matrix should satisfy following relations [15] $T(z)^{\mathrm{t}}\cdot T(z+c\kappa_n) = T(z+c\kappa_n)\cdot T(z)^{\mathrm{t}} = \mathcal{C}(z) = \mathbf{I}$, where $\mathcal{C}(z)$ is a central operator in the algebra defined by the relations (2.2). Let $\pi(z)$ be a formal series $\pi(z) = 1+\sum_{m\ge 0}\pi_m(c/z)^{-m-1}$ which solves the equation $\mathcal{C}(z)=\pi(z)\pi(z+c\kappa_n)$. This equation can be solved inductively expressing the coefficient of the formal series $\pi(z)$ through the coefficients of the central element $\mathcal{C}(z)$. Then one can rescale the monodromy matrix $T(z)\to T'(z)=\pi(z)^{-1}T(z)$ in such a way that the rescaled monodromy matrix satisfies the equation

$$T'(z)^{\mathrm{t}}\cdot T'(z+c\kappa_n) = T'(z+c\kappa_n)\cdot T'(z)^{\mathrm{t}} = \mathbf{I}. \tag{2.5}$$

Zero modes. The zero modes $\mathsf{T}_{i,j} := T_{i,j}[0]$ (see (2.4)) will play an important role in our approach. Considering expansion of $T_{i,j}(u)$ and the rational functions in (2.3) as series with respect to $1/u$, the coefficient of u^{-1} in (2.3) yields

$$[\mathsf{T}_{i,j},T_{k,l}(v)] = \chi_i\delta_{i,l}\mathsf{T}_{k,j}(v) - \chi_j\delta_{k,j}T_{i,l}(v) \tag{2.6}$$

for $\mathfrak{g}=\mathfrak{gl}_n$ and

$$[\mathsf{T}_{i,j},T_{k,l}(v)] = \chi_i(\delta_{i,l}T_{k,j}(v) - \delta_{l,-j}T_{k,-i}(v)) - \chi_j(\delta_{k,j}T_{i,l}(v) - \delta_{k,-i}T_{-j,l}(v)) \tag{2.7}$$

for $\mathfrak{g}=\mathfrak{o}_{2n+1}$.

More generally, in each integrable models, the monodromy matrix zero mode operators may be always defined as the operators built from the monodromy matrix entries which satisfy the commutation relations of the finite-dimensional algebra $\mathfrak{g}$.

Embeddings. We describe different embeddings in the Yangians $Y(\mathfrak{gl}_n)$ and $Y(\mathfrak{o}_{2n+1})$ that will be reflected in a factorisation (also called splitting) property of the Bethe vectors, see Sections 3.3 and 4.4 below.

Embedding $Y(\mathfrak{gl}_a)\otimes Y(\mathfrak{gl}_{n-a})\hookrightarrow Y(\mathfrak{gl}_n)$. From the commutation relations (2.3), it is clear that in $Y(\mathfrak{gl}_n)$ the elements $T_{i,j}(z)$, $1\le i,j\le a$ generate a Yangian subalgebra $Y(\mathfrak{gl}_a)$ in $Y(\mathfrak{gl}_n)$, while the elements $T_{a+i,a+j}(z)$, $1\le i,j\le n-a$ generate the Yangian subalgebra $Y(\mathfrak{gl}_{n-a})$. However, these two subalgebras do not commute. To get an embedding of $Y(\mathfrak{gl}_a)\otimes Y(\mathfrak{gl}_{n-a})$ in $Y(\mathfrak{gl}_n)$, one needs to consider quantum minors, see relation (1.84), in [30, Corollaries 1.7.2 and 1.11.4].

Embeddings of $Y(\mathfrak{gl}_n)\hookrightarrow Y(\mathfrak{o}_{2n+1})$. There are several ways to embed $Y(\mathfrak{gl}_n)$ in $Y(\mathfrak{o}_{2n+1})$. We describe here two of them that have some relevance for the study of Bethe vectors. Let us denote the $Y(\mathfrak{gl}_n)$ monodromy matrix embedded in $Y(\mathfrak{o}_{2n+1})$ as $\mathbb{T}(z)$.

- From the commutation relations (2.3), it is clear that in $Y(\mathfrak{o}_{2n+1})$ the elements

$$\mathbb{T}_{i,j}(z):=T_{i,j}(z),\qquad 1\le i,j\le n \tag{2.8}$$

  generate the Yangian $Y(\mathfrak{gl}_n)$, since they obey the commutation relations (2.3) without the second line (all the terms which include the Kronecker symbol δ vanish).

- Another embedding can be done, considering the elements $T_{i,j}(z)$, $-n\le i,j\le -1$ and defining

$$\hat{\mathbb{T}}_{i,j}(z):=T_{i-n-1,j-n-1}(z),\qquad 1\le i,j\le n. \tag{2.9}$$

  It is easy to see that $\hat{\mathbb{T}}_{i,j}(z)$ also obey the commutation relations (2.3) without the second line. The twisting parameters for the monodromy $\hat{\mathbb{T}}(z)$ in this case are $(\chi_{n+1-i})^{-1}$.

Embedding $Y(\mathfrak{o}_{2a+1})\otimes Y(\mathfrak{gl}_{n-a})\hookrightarrow Y(\mathfrak{o}_{2n+1})$. This type of embedding is more intricate. It uses the concept of quasi-determinants [5] $\widehat{T}_{i,j}(z)$ of the monodromy matrix, and we refer to, e.g., [30] for more details about quasi-determinants in the Yangian case. Indeed, one can show that $\widehat{T}_{i,j}(z)$, $1\le i,j\le a$ generate a Yangian subalgebra $Y(\mathfrak{o}_{2a+1})$, while $T_{k+a,l+a}(z)$, $1\le k,l\le n-a$ generate a Yangian subalgebra $Y(\mathfrak{gl}_{n-a})$. Moreover, we have $\big[\widehat{T}_{i,j}(z),T_{k+a,l+a}(z)\big]=0$ for $1\le i,j\le a$ and $1\le k,l\le n-a$. A detailed presentation of this approach is beyond the scope of the present article, we refer to [15, Theorem 3.7 and Corollary 3.10] for a detailed presentation of the construction. It is remarkable that this type of embedding is reflected in a simple way on the structure of Bethe vectors, see Section 4.4 below.

2.2 Bethe vectors

In the framework of algebraic Bethe ansatz, the states in the Hilbert space of the physical model are defined by the vectors $|0\rangle$ and $\langle 0|$ such that

$$T_{i,j}(z)|0\rangle=0,\quad i>j,\qquad T_{i,i}(z)|0\rangle=\lambda_i(z)|0\rangle,\quad i\in I_{\mathfrak{g}} \tag{2.10}$$

and

$$\langle 0|T_{i,j}(z)=0,\quad i<j,\qquad \langle 0|T_{i,i}(z)=\lambda_i(z)\langle 0|,\quad i\in I_{\mathfrak{g}}. \tag{2.11}$$

In (2.10), the monodromy matrix elements are acting to the right, while in (2.11) they are acting to the left. If they exist, such vectors are called *vacuum vectors*.

The functions $\lambda_i(z)$ are characterizing the physical model under consideration. Since we are considering generic model, we will consider these functions as free functional parameters. For $\mathfrak{gl}_n$-invariant integrable models, the functions $\lambda_i(z)$ are all independent, while for $\mathfrak{o}_{2n+1}$-invariant models and due to (2.5) they satisfy the relations [22]

$$\lambda_{-j}(z) = \frac{1}{\lambda_j(z_j)} \prod_{s=j+1}^{n} \frac{\lambda_s(z_{s-1})}{\lambda_s(z_s)}, \qquad j = 0, 1, \dots, n, \tag{2.12}$$

where we introduced the *shifted spectral parameter*

$$z_s = z - c\left(s - \frac{1}{2}\right), \qquad s = 0, 1, \dots, n. \tag{2.13}$$

Due to our choice (2.4) of dependence of monodromy matrix entries on the formal spectral parameter, the free functional parameters $\lambda_i(z)$ are formal series

$$\lambda_i(z) = \chi_i + \sum_{\ell \ge 0} \lambda_i[\ell](z/c)^{-\ell-1}$$

with respect to the formal parameter z.

The Bethe vectors in the integrable model depends on a collection of sets

$$\bar{t} = \begin{cases} \{\bar{t}^1, \dots, \bar{t}^{n-1}\} & \text{for } \mathfrak{g} = \mathfrak{gl}_n, \\ \{\bar{t}^0, \bar{t}^1, \dots, \bar{t}^{n-1}\} & \text{for } \mathfrak{g} = \mathfrak{o}_{2n+1}. \end{cases}$$

Here, the set $\bar{t}^s = \{t^s_1, \dots, t^s_{r_s}\}$ denotes a collection of r_s Bethe parameters. The non-negative integer r_s is the cardinality $|\bar{t}^s|$ of the set $\bar{t}^s$. The superscripts on the (sets of) Bethe parameters denotes their color. The colors are in correspondence with the simple roots of the algebra $\mathfrak{g}$. This can be formalized through the operators h_i defined by

$$\chi_i \mathsf{h}_i = \mathsf{T}_{i,i} - \lambda_i[0] \tag{2.14}$$

such that the vacuum vectors have zero eigenvalue $\mathsf{h}_i|0\rangle = 0$, $\langle 0|\mathsf{h}_i = 0$. According to (2.6) and (2.7), the monodromy matrix entries $T_{k,l}(z)$ are eigenvector for the adjoint action of the operators h_i $[\mathsf{h}_i, T_{k,l}(z)] = (\delta_{i,l} - \delta_{i,k})T_{k,l}(z)$ for $\mathfrak{gl}_N$-invariant monodromies and

$$[\mathsf{h}_i, T_{k,l}(z)] = (\delta_{i,l} - \delta_{i,k} + \delta_{i,-k} - \delta_{i,-l})T_{k,l}(z)$$

for $\mathfrak{o}_{2n+1}$-invariant monodromies. Using the operators h_i (2.14), one also defines the operators

$$\mathsf{t}_s = \sum_{i=s+1}^{n} \mathsf{h}_i, \tag{2.15}$$

where $s = 1, \dots, n-1$ for $\mathfrak{gl}_N$-invariant monodromies and $s = 0, 1, \dots, n-1$ for $\mathfrak{o}_{2n+1}$-invariant monodromies.

The Bethe vectors themselves are certain polynomials of the non-commutative monodromy entries $T_{i,j}(u)$ for $i < j$ depending on various Bethe parameters acting on the right vacuum vector $|0\rangle$

$$\mathbb{B}(\bar{t}) = \mathcal{P}\big(T_{i<j}(\bar{t})\big)|0\rangle = \mathcal{B}(\bar{t})|0\rangle, \tag{2.16}$$

where the polynomial $\mathcal{B}(\bar{t}) = \mathcal{P}\big(T_{i<j}(\bar{t})\big)$ is called a *pre-Bethe vector*.

Analogously, left or dual Bethe vectors are polynomials of monodromy entries $T_{i,j}(u)$ for $i > j$ acting to the left vacuum vector $|0\rangle$

$$\mathbb{C}(\bar{t}) = \langle 0|\mathcal{P}'\big(T_{i>j}(\bar{t})\big) = \langle 0|\mathcal{C}(\bar{t}). \tag{2.17}$$

The polynomials $\mathcal{P}$ and $\mathcal{P}'$ are related by the transposition antihomomorphism (see Remark 2.1). The ordering of the non-commutative entries $T_{i,j}(t^s_a)$ in the polynomials $\mathcal{P}$ and $\mathcal{P}'$ and the structure of these polynomials can be fixed in the framework of the nested Bethe ansatz [20, 21] or by the method of projections [9, 16]. When the Bethe parameters are generic, we call such Bethe vectors (2.16) and (2.17) *off-shell Bethe vectors*.

The off-shell Bethe vectors $\mathbb{B}(\bar{t})$ and $\mathbb{C}(\bar{t})$ are eigenvectors of the operators $\mathtt{t}_s$ (2.15)

$$\mathtt{t}_s \cdot \mathbb{B}(\bar{t}) = r_s\mathbb{B}(\bar{t}), \qquad \mathbb{C}(\bar{t}) \cdot \mathtt{t}_s = r_s\mathbb{C}(\bar{t}), \qquad i = 1, \dots, n,$$

with $s = 1, \dots, n-1$ for $\mathfrak{gl}_n$ and $s = 0, 1, \dots, n-1$ for $\mathfrak{o}_{2n+1}$. This property can be proved using the recurrence relations for the Bethe vectors and the action of the monodromy matrix entries $T_{i,i}(z)$ on Bethe vectors (see proof of Proposition 3.1 in the case of $\mathfrak{o}_{2n+1}$ in [25]).

Remark 2.1. In what follows, we will consider only the Bethe vectors $\mathbb{B}(\bar{t})$. All the relations for the dual Bethe vectors $\mathbb{C}(\bar{t})$ can be obtained from the corresponding relations for $\mathbb{B}(\bar{t})$ using the transposition antihomomorphism. When we consider different embeddings, we will use the notation $\mathbb{B}_{\mathfrak{gl}}(\bar{t})$ and $\mathbb{B}_{\mathfrak{o}}(\bar{t})$ to distinguish the off-shell Bethe vectors in the models with different symmetries. But most often we will the use notation $\mathbb{B}(\bar{t})$ since it will be clear from the context what type of Bethe vector we are exploring.

Remark 2.2. The commutation relations (2.3) between the monodromy matrix entries do not depend on the parameters χ_i explicitly, nor does the definition of the vacuum state (2.16). Since the Bethe vectors are polynomials in the entries of the monodromy matrix, they do not depend explicitly on the parameters χ_i either. It is only when using the expansion (2.4) that the dependence in the χ_i parameters becomes explicit, as for instance in the relations (2.6) or (2.7) which involve the zero mode action. Therefore, any relation involving only the Bethe vectors, the entries $T_{ij}(z)$ and/or the eigenvalues $\lambda_i(z)$ should not depend on the χ's. In other words, if the χ_i parameters appear explicitly in such relation, each coefficient of the parameters χ_i should be set to zero independently. We will use this property in Appendix B.

On-shell Bethe vectors. Due to the *RTT* commutation relations (2.2), the trace of the monodromy matrix (the transfer matrix)

$$\mathcal{T}(z) = \sum_{i\in I_{\mathfrak{g}}} T_{i,i}(z) \tag{2.18}$$

commutes $\mathcal{T}(z)\mathcal{T}(z') = \mathcal{T}(z')\mathcal{T}(z)$ for two different formal spectral parameters z and z'. Upon expansion in z, the transfer matrix (2.18) generates a family of commuting operators. The Bethe vectors become eigenvectors of the transfer matrix (also known as *on-shell Bethe vectors*) if the Bethe parameters satisfy so called Bethe equations (see below (2.21)). To describe the Bethe equations in the $\mathfrak{g}$-invariant integrable models, we introduce the rational functions

$$f(u,v) = 1 + g(u,v) = h(u,v)g(u,v) = \frac{u-v+c}{u-v}, \qquad \mathfrak{f}(u,v) = \frac{u-v+c/2}{u-v}.$$

We also define the functions

$$\alpha_s(z) = \frac{\lambda_s(z)}{\lambda_{s+1}(z)}, \qquad s = 0, \dots, n-1.$$

We will use the following convention for the products of scalar functions depending on sets of parameters, for example,

$$\lambda_s(\bar t^s)=\prod_{t^s_a\in\bar t^s}\lambda_s(t^s_a),\qquad f\big(\bar t^s,\bar t^{s'}\big)=\prod_{t^s_a\in\bar t^s}\prod_{t^{s'}_b\in\bar t^{s'}}f\big(t^s_a,t^{s'}_b\big),\qquad\text{etc.}\tag{2.19}$$

with $f\big(\varnothing,\bar t\big)=f\big(\bar t,\varnothing\big)=1$.

The recurrence relations for the off-shell Bethe vectors will be written as sums over partitions of the sets of Bethe parameters. A *partition* $\big\{\bar t^s_{\rm I},\bar t^s_{\rm II},\bar t^s_{\rm III}\big\}\vdash\bar t^s$ corresponds to a decomposition into (possibly empty) disjoint subsets $\bar t^s_{\rm I}$, $\bar t^s_{\rm II}$, $\bar t^s_{\rm III}$ such that $\bar t^s=\bar t^s_{\rm I}\cup\bar t^s_{\rm II}\cup\bar t^s_{\rm III}$ and $\bar t^s_{\rm I}\cap\bar t^s_{\rm II}=\bar t^s_{\rm I}\cap\bar t^s_{\rm III}=\bar t^s_{\rm II}\cap\bar t^s_{\rm III}=\varnothing$. The cardinalities of the subsets satisfy the equality $\big|\bar t^s\big|=\big|\bar t^s_{\rm I}\big|+\big|\bar t^s_{\rm II}\big|+\big|\bar t^s_{\rm III}\big|$, where some of the cardinalities $\big|\bar t^s_{\rm I}\big|$, $\big|\bar t^s_{\rm II}\big|$, $\big|\bar t^s_{\rm III}\big|$ can be zero. The partition $\big\{\bar t^s_{\rm I},\bar t^s_{\rm II}\big\}\vdash\bar t^s$ is defined analogously.

The on-shell Bethe vectors are eigenvectors of the transfer matrix

$$\mathcal{T}(z)\cdot\mathbb{B}\big(\bar t\big)=\tau\big(z;\bar t\big)\mathbb{B}\big(\bar t\big)\tag{2.20}$$

if each set $\bar t^s$ of Bethe parameters satisfy *the Bethe equations* [21, 32]

$$\alpha_s\big(\bar t^s_{\rm I}\big)=\frac{\lambda_s\big(\bar t^s_{\rm I}\big)}{\lambda_{s+1}\big(\bar t^s_{\rm I}\big)}=\frac{f_s\big(\bar t^s_{\rm I},\bar t^s_{\rm II}\big)}{f_s\big(\bar t^s_{\rm II},\bar t^s_{\rm I}\big)}\frac{f\big(\bar t^{s+1},\bar t^s_{\rm I}\big)}{f\big(\bar t^s_{\rm I},\bar t^{s-1}\big)}\qquad\text{with}\ \begin{cases}s=1,\dots,n-1 & \text{for }\mathfrak{gl}_n,\\ s=0,1,\dots,n-1 & \text{for }\mathfrak{o}_{2n+1},\end{cases}\tag{2.21}$$

for any disjoint partition $\big\{\bar t^s_{\rm I},\bar t^s_{\rm II}\big\}\vdash\bar t^s$ and with the conditions $\bar t^0=\bar t^n=\varnothing$ for $\mathfrak{gl}_n$ and $\bar t^{-1}=\bar t^n=\varnothing$ for $\mathfrak{o}_{2n+1}$. In (2.21), we have used the convention (2.19) for products of functions and the functions $f_s(u,v)$ are defined as

$$f_s(u,v)=\begin{cases}\mathfrak{f}(u,v), & s=0,\\ f(u,v), & s=1,\dots,n-1.\end{cases}$$

For the $\mathfrak{gl}_n$-invariant models, the eigenvalue $\tau\big(z;\bar t\big)$ in (2.20) is equal to [21]

$$\tau\big(z;\bar t\big)=\sum_{s=1}^{n}\lambda_s(z)f\big(\bar t^s,z\big)f\big(z,\bar t^{s-1}\big).$$

In the case of $\mathfrak{o}_{2n+1}$-invariant models, this eigenvalue is [23, 32]

$$\begin{aligned}\tau\big(z;\bar t\big)={}&\lambda_0(z)f\big(\bar t^0,z_0\big)f\big(z,\bar t^0\big)+\sum_{s=1}^{n}\big(\lambda_s(z)f\big(\bar t^s,z\big)f\big(z,\bar t^{s-1}\big)\\&+\lambda_{-s}(z)f\big(\bar t^{s-1},z_{s-1}\big)f\big(z_s,\bar t^s\big)\big),\end{aligned}$$

where we have used the notations (2.13) and $\lambda_{-s}(z)$ satisfy the relations (2.12).

3 Bethe vectors in $\mathfrak{gl}_n$-invariant models

Denote by $\big\{\bar t^s\big\}_i^j:=\big\{\bar t^i,\bar t^{i+1},\dots,\bar t^j\big\}$ a partial collection of sets $\bar t^s$ of Bethe parameters for $i\le s\le j$. For example, $\bar t=\big\{\bar t^s\big\}_1^{n-1}$ for $\mathfrak{gl}_n$ Bethe vectors. We will always assume that the partial collection of sets $\big\{\bar t^s\big\}_i^j$ is empty if $j<i$.

For $1\le\ell<n-1$ and $1<k\le n$, we introduce the functions

$$\psi_\ell\big(z;\bar t\big)=g\big(z,\bar t^{\ell-1}\big)h\big(\bar t^\ell,z\big),\qquad\phi_k\big(z;\bar t\big)=h\big(z,\bar t^{k-1}\big)g\big(\bar t^k,z\big).$$

Remark that $\psi_1\big(z;\bar t\big)=h\big(\bar t^1,z\big)$ and $\phi_n\big(z;\bar t\big)=h\big(z,\bar t^{n-1}\big)$ because $g\big(z,\bar t^0\big)=g\big(\bar t^n,z\big)=1$ since $\bar t^0=\bar t^n=\varnothing$.

3.1 Action formula for $T_{1,n}(z)$ and $\mathsf{T}_{\ell+1,\ell}$

Denote by $\bar u^s = \{t^s_1,\dots,t^s_{r_s},z\}$ an extended set of the Bethe parameters of the color s. For shortness, we will write $\bar u^s = \{\bar t^s,z\}$. The off-shell Bethe vectors $\mathbb{B}(\bar t)$ in $\mathfrak{gl}_n$-invariant models are normalized in such a way that the action of the monodromy entry $T_{1,n}(z)$ onto off-shell Bethe vector $\mathbb{B}(\bar t)$ has the simple form[2]

$$T_{1,n}(z)\cdot\mathbb{B}(\bar t) = \mu^n_1(z;\bar t)\mathbb{B}(\bar u), \tag{3.1}$$

where $\bar u = \{\bar u^1,\bar u^2,\dots,\bar u^{n-1}\}$. The normalization factor $\mu^k_\ell(z;\bar t)$ is defined as follows for any $1\le \ell<k\le n$

$$\mu^k_\ell(z;\bar t) = \lambda_k(z)\psi_\ell(z;\bar t)\phi_k(z;\bar t) = \lambda_k(z)g(z,\bar t^{\ell-1})h(\bar t^\ell,z)h(z,\bar t^{k-1})g(\bar t^k,z). \tag{3.2}$$

We denote by the symbol $\mathscr{Z}^k_\ell$ the operation of adding a parameter z to the sets $\bar t^\ell,\dots,\bar t^{k-1}$ of Bethe parameters in the off-shell Bethe vectors $\mathbb{B}(\bar t)$

$$\mathscr{Z}^k_\ell\cdot\mathbb{B}(\bar t) = \mathbb{B}\big(\{\bar t^s\}_1^{\ell-1},\{\bar t^s,z\}_\ell^{k-1},\{\bar t^s\}_k^{n-1}\big). \tag{3.3}$$

For example, using this notation the action (3.1) can be written as follows:

$$\mathscr{Z}^n_1\cdot\mathbb{B}(\bar t) = \frac{1}{\mu^n_1(z;\bar t)}T_{1,n}(z)\cdot\mathbb{B}(\bar t). \tag{3.4}$$

We will use relation (3.1) in the form (3.4) as a base relation for the inductive proof of Theorem 3.1.

To find the rectangular recurrence relations for the off-shell Bethe vectors $\mathbb{B}(\bar t)$, we will use the commutation relations (2.6) in the particular case

$$[\mathsf{T}_{\ell+1,\ell},T_{i,j}(z)] = \chi_{\ell+1}\delta_{\ell+1,j}T_{i,\ell}(z) - \chi_\ell\delta_{\ell,i}T_{\ell+1,j}(z) \tag{3.5}$$

and the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ onto off-shell Bethe vectors. These actions as well as the action (3.1) can be calculated in the framework of the projection method. For more details, one can look at the [23, Section 4.1], where the action of the zero modes on the off-shell Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant integrable model was calculated. The calculation of the zero modes action for the $\mathfrak{gl}_n$-invariant integrable model is similar, starting from the results stated in [9]. We will write the action of the zero modes in the form

$$\begin{aligned}\mathsf{T}_{\ell+1,\ell}\cdot\mathbb{B}(\bar t) = \sum_{\rm part}&\big(\chi_{\ell+1}\alpha_\ell(\bar t^\ell_{\rm I})\Omega^L(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1})\\ &-\chi_\ell\Omega^R(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1})\big)\mathbb{B}\big(\{\bar t^s\}_1^{\ell-1},\bar t^\ell_{\rm II},\{\bar t^s\}_{\ell+1}^{n-1}\big),\end{aligned} \tag{3.6}$$

where the sum in (3.6) goes over partitions $\{\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}\}\vdash\bar t^\ell$ such that $|\bar t^\ell_{\rm I}|=1$. The functions Ω^L and Ω^R in (3.6) are defined as follows:

$$\begin{aligned}\Omega^L(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1}) &= \gamma(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I})\frac{h(\bar t^\ell_{\rm I},\bar t^{\ell-1})}{g(\bar t^{\ell+1},\bar t^\ell_{\rm I})},\\ \Omega^R(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1}) &= \gamma(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II})\frac{h(\bar t^{\ell+1},\bar t^\ell_{\rm I})}{g(\bar t^\ell_{\rm I},\bar t^{\ell-1})},\end{aligned} \tag{3.7}$$

where the function $\gamma(u,v)$ is

$$\gamma(u,v) = \frac{f(u,v)}{h(u,v)h(v,u)} = \frac{g(u,v)}{h(v,u)}.$$

[2]Note that normalization of the off-shell Bethe vectors used in (3.1) differs from the normalization used in the paper [13], but it is more suitable for the propose of this paper.

3.2 Rectangular recurrence relations for $\mathfrak{gl}_n$

For $m \in \mathbb{Z}$, we define the step function $\Theta(m)$

$$\Theta(m) = \begin{cases} 1, & m \geq 0, \\ 0, & m < 0. \end{cases}$$

Theorem 3.1 below will be proved in Appendix A by induction starting from (3.4). Theorem 3.1 allows to express off-shell Bethe vectors through the action of the monodromy entries on Bethe vectors with a smaller number of Bethe parameters. This result is new although some examples of this type of recurrence relations were know earlier (see Corollary 3.2).

Theorem 3.1. *For any pair of positive integers $1 \leq \ell < k \leq n$, the off-shell Bethe vector $\mathcal{Z}_\ell^k \cdot \mathbb{B}(\bar{t})$ satisfies the rectangular recurrence relation*

$$\begin{aligned} \mathcal{Z}_\ell^k \cdot \mathbb{B}(\bar{t}) &= \mathbb{B}\big(\{\bar{t}^s\}_1^{\ell-1}, \{\bar{u}^s\}_\ell^{k-1}, \{\bar{t}^s\}_k^{n-1}\big) \\ &= \frac{1}{\mu_\ell^k(z;\bar{t})} \sum_{i=1}^{\ell} \sum_{j=k}^{n} \sum_{\text{part}} \Xi_{i,j}^{\ell,k}(z;\bar{t}_{\scriptscriptstyle \rm I},\bar{t}_{\scriptscriptstyle \rm II},\bar{t}_{\scriptscriptstyle \rm III}) T_{i,j}(z) \cdot \mathbb{B}(\bar{t}_{\scriptscriptstyle \rm II}), \end{aligned} \tag{3.8}$$

where the functions $\Xi_{i,j}^{\ell,k}(z;\bar{t}_{\scriptscriptstyle \rm I},\bar{t}_{\scriptscriptstyle \rm II},\bar{t}_{\scriptscriptstyle \rm III})$

$$\begin{aligned} \Xi_{i,j}^{\ell,k}(z;\bar{t}_{\scriptscriptstyle \rm I},\bar{t}_{\scriptscriptstyle \rm II},\bar{t}_{\scriptscriptstyle \rm III}) &= g\big(z,\bar{t}_{\scriptscriptstyle \rm I}^{\ell-1}\big) g\big(\bar{t}_{\scriptscriptstyle \rm III}^{k},z\big) \\ &\quad\times \prod_{s=i}^{\ell-1} \Omega^R\big(\bar{t}_{\scriptscriptstyle \rm I}^s,\bar{t}_{\scriptscriptstyle \rm II}^s|\bar{t}_{\scriptscriptstyle \rm II}^{s-1},\bar{t}_{\scriptscriptstyle \rm II}^{s+1}\big) \prod_{s=k}^{j-1} \alpha_s\big(\bar{t}_{\scriptscriptstyle \rm III}^s\big) \Omega^L\big(\bar{t}_{\scriptscriptstyle \rm II}^s,\bar{t}_{\scriptscriptstyle \rm III}^s|\bar{t}_{\scriptscriptstyle \rm II}^{s-1},\bar{t}_{\scriptscriptstyle \rm II}^{s+1}\big) \end{aligned} \tag{3.9}$$

depend on the partitions and the sum in (3.8) goes over partitions $\{\bar{t}_{\scriptscriptstyle \rm I}^s,\bar{t}_{\scriptscriptstyle \rm II}^s,\bar{t}_{\scriptscriptstyle \rm III}^s\} \vdash \bar{t}^s$ with cardinalities

$$\big|\bar{t}_{\scriptscriptstyle \rm I}^s\big| = \begin{cases} \Theta(s-i), & s < \ell, \\ 0, & s \geq \ell, \end{cases} \qquad \big|\bar{t}_{\scriptscriptstyle \rm III}^s\big| = \begin{cases} 0, & s \leq k-1, \\ \Theta(j-s-1), & s > k-1. \end{cases} \tag{3.10}$$

The sets $\bar{t}^\ell, \dots, \bar{t}^{k-1}$ are not partitioned and in (3.8) $\bar{t}_{\scriptscriptstyle \rm II}^\ell = \bar{t}^\ell$ and $\bar{t}_{\scriptscriptstyle \rm II}^{k-1} = \bar{t}^{k-1}$.

If the set $\bar{t}^s = \varnothing$ and according to (3.10) one may have $\big|\bar{t}_{\scriptscriptstyle \rm I}^s\big| = 1$ for some i or $\big|\bar{t}_{\scriptscriptstyle \rm III}^s\big| = 1$ for some j, then the terms in the right-hand side of (3.8) for such values of the indices i and j have to be discarded.

The cardinalities of the subsets $\bar{t}_{\scriptscriptstyle \rm I}^s$ and $\bar{t}_{\scriptscriptstyle \rm III}^s$ given by (3.10) for $s = 1, \dots, n-1$ can be visualized in the following table:

s	1	$\cdots$	$i-1$	i	$\cdots$	$\ell-1$	ℓ	$\cdots$	$k-1$	k	$\cdots$	$j-1$	j	$\cdots$	$n-1$
$\big|\bar{t}_{\scriptscriptstyle \rm I}^s\big|$	0	$\cdots$	0	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0
$\big|\bar{t}_{\scriptscriptstyle \rm III}^s\big|$	0	$\cdots$	0	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	1	0	$\cdots$	0

Remark that since $\ell < k$, the cardinalities in (3.10) show that for any color s, either $\bar{t}_{\scriptscriptstyle \rm I}^s$ or $\bar{t}_{\scriptscriptstyle \rm III}^s$ is empty. In other words, we always have partitions into a maximum of two subsets, $\{\bar{t}_{\scriptscriptstyle \rm I}^s,\bar{t}_{\scriptscriptstyle \rm II}^s\} \vdash \bar{t}^s$ or $\{\bar{t}_{\scriptscriptstyle \rm II}^s,\bar{t}_{\scriptscriptstyle \rm III}^s\} \vdash \bar{t}^s$ with cardinalities $\big|\bar{t}_{\scriptscriptstyle \rm I}^s\big| \leq 1$ and $\big|\bar{t}_{\scriptscriptstyle \rm III}^s\big| \leq 1$.

We call the recurrence relation (3.8) a *rectangular recurrence relation* because it contains the monodromy matrix entries $T_{i,j}(z)$ with $1 \leq i \leq l$ and $k \leq j \leq n$. This defines a rectangular submatrix in monodromy matrix with vertices $T_{1,n}(z)$, $T_{l,n}(z)$, $T_{l,k}$ and $T_{1,k}(z)$. It lies above the diagonal of the monodromy matrix because $l < k$.

Corollary 3.2. *Considering the particular case $k=\ell+1$ in the recurrence relation (3.8), we get*

$$\mathbb{B}\big(\{\bar t^s\}_1^{\ell-1},\{\bar t^\ell,z\},\{\bar t^s\}_{\ell+1}^{n-1}\big)$$
$$=\sum_{i=1}^{\ell}\sum_{j=\ell+1}^{n}\sum_{\rm part}\frac{T_{i,j}(z)\cdot\mathbb{B}\big(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^{\ell-1},\bar t^\ell,\{\bar t^s_{\rm II}\}_{\ell+1}^{j-1},\{\bar t^s\}_j^{n-1}\big)}{\lambda_{\ell+1}(z)g\big(z,\bar t^{\ell-1}_{\rm II}\big)h\big(z,\bar t^\ell\big)h\big(\bar t^\ell,z\big)g\big(\bar t^{\ell+1}_{\rm II},z\big)}$$
$$\times\prod_{s=i}^{\ell-1}\gamma\big(\bar t^s_{\rm I},\bar t^s_{\rm II}\big)\frac{h\big(\bar t^{s+1}_{\rm II},\bar t^s_{\rm I}\big)}{g\big(\bar t^s_{\rm I},\bar t^{s-1}_{\rm II}\big)}\prod_{s=\ell+1}^{j-1}\alpha_s\big(\bar t^s_{\rm III}\big)\gamma\big(\bar t^s_{\rm II},\bar t^s_{\rm III}\big)\frac{h\big(\bar t^s_{\rm III},\bar t^{s-1}_{\rm II}\big)}{g\big(\bar t^{s+1}_{\rm II},\bar t^s_{\rm III}\big)}.$$

Here the sum over partition is the same as in Theorem 3.1.

Such type of recurrence relations for arbitrary ℓ were written for the first time in [24] for the case of integrable models associated to $U_q(\mathfrak{gl}_n)$. In the case of $\mathfrak{gl}_n$-invariant integrable models, these recurrence relations for the cases $\ell=1$, $k=2$ and $\ell=n-1$, $k=n$ were presented in [11]. Analogous recurrence relations for the cases $\ell=2$, $k=3$ and $\ell=n-2$, $k=n-1$ can be found in [6]. Examples of this recurrence relation can also be found in [2] for the case of $\mathfrak{gl}_3$.

3.3 Embedding $Y(\mathfrak{gl}_a)\otimes Y(\mathfrak{gl}_{n-a})\hookrightarrow Y(\mathfrak{gl}_n)$

Suppose the Bethe vector does not involve any spectral parameter of some given color $1\le a\le n-1$. In this case, the reduced recurrence relations will imply the following proposition.

Proposition 3.3. *The off-shell Bethe vectors $\mathbb{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1},\varnothing,\{\bar t^s\}_{a+1}^{n-1}\big)$ has a representation as the product of $\mathfrak{gl}_a$ pre-Bethe vectors $\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1}\big)\in Y(\mathfrak{gl}_a)$ and $\mathfrak{gl}_{n-a}$ pre-Bethe vectors $\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_{a+1}^{n-1}\big)\in Y(\mathfrak{gl}_{n-a})$ acting on the vacuum vector $|0\rangle$*

$$\mathbb{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1},\varnothing,\{\bar t^s\}_{a+1}^{n-1}\big)=\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1}\big)\cdot\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_{a+1}^{n-1}\big)|0\rangle$$
$$=\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_{a+1}^{n-1}\big)\cdot\mathcal{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1}\big)|0\rangle. \qquad (3.11)$$

Proof. We will prove only the first equality in (3.11). The second equality can be proved analogously. Let us consider the recurrence relations (3.8) for two different ranges of the indices ℓ and k: $1\le\ell<k\le a$ and $a+1\le\ell<k\le n$. These ranges for k and ℓ ensure that the set $\bar t^a$ remains empty. Since $\mathbb{B}_{\mathfrak{gl}}(\varnothing)=|0\rangle$ (i.e., $\mathcal{B}_{\mathfrak{gl}}(\varnothing)=1$), the recurrence relations will prove the statement of (3.11) by induction on the cardinalities.

- $1\le\ell<k\le a$. The cardinality of the subset $\bar t^a_{\rm III}$ given by (3.10) is equal to $\Theta(j-a-1)$. Since this subset is empty, this restricts the summation over j to $k\le j\le a$. This means that the recurrence relation for $\mathcal{Z}^k_\ell\cdot\mathbb{B}_{\mathfrak{gl}}(\bar t)|_{\bar t^a=\varnothing}$ for $1\le\ell<k\le a$ implies only the monodromy entries $T_{i,j}(z)$ for $1\le i\le\ell$ and $k\le j\le a$. The recurrence relation $\mathcal{Z}^a_1\cdot\mathbb{B}_{\mathfrak{gl}}(\bar t)|_{\bar t^a=\varnothing}$ will have only one term (there is no summation over partitions) corresponding to the action of monodromy entry $T_{1,a}(z)$ on the off-shell Bethe vector. Then the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ for $\ell=1,\dots,a-1$ onto Bethe vector (3.11)

$$\mathsf{T}_{\ell+1,\ell}\cdot\mathbb{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{a-1},\varnothing,\{\bar t^s\}_{a+1}^{n-1}\big)$$
$$=\sum_{\rm part}\big(\chi_{\ell+1}\alpha_\ell\big(\bar t^\ell_{\rm I}\big)\Omega^L\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1}\big)$$
$$-\chi_\ell\Omega^R\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1}\big)\big)\mathbb{B}_{\mathfrak{gl}}\big(\{\bar t^s\}_1^{\ell-1},\bar t^\ell_{\rm II},\{\bar t^s\}_{\ell+1}^{a-1},\varnothing,\{\bar t^s\}_{a+1}^{n-1}\big)$$

does not affect the sets of Bethe parameters $\{\bar t^s\}_{a+1}^{n-1}$ and so coincides with the action of the zero mode operators onto the Bethe vectors in $\mathfrak{gl}_a$-invariant model. This shows that the recurrence relations for $\mathcal{Z}^k_\ell\cdot\mathbb{B}_{\mathfrak{gl}}(\bar t)|_{\bar t^a=\varnothing}$ for $1\le\ell<k\le a$ coincides with the $\mathfrak{gl}_a$ recurrence relation for the vector $\mathbb{B}_{\mathfrak{gl}}\big(\{\bar t\}_1^{a-1}\big)$.

- $a+1 \le \ell < k \le n$. In this case, the cardinality of the empty subset $\bar t^a_{\rm I} = \varnothing$, given by the equalities (3.10), yields the restriction for the index $i \ge a+1$. The summation over i and j in the right-hand side of the recurrence relation for $\mathcal{Z}^k_\ell \cdot \mathbb{B}_{\mathfrak{gl}}(\bar t)|_{\bar t^a=\varnothing}$ will be restricted to the range $a+1 \le i \le \ell < k \le j \le n$. The Bethe vector $\mathcal{Z}^n_{a+1} \cdot \mathbb{B}_{\mathfrak{gl}}(\bar t)$ will be proportional to the action of the monodromy matrix entry $T_{a+1,n}(z) \cdot \mathbb{B}_{\mathfrak{gl}}(\bar t)$ (there is no sum over partitions in this action) and the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ for $\ell = a+1, \dots, n-1$ onto the Bethe vector (3.11) coincides with $\mathfrak{gl}_{n-a}$-type actions (3.6). The recurrence relation (3.8) in this case will not affect the sets of Bethe parameters $\{\bar t^s\}_1^{a-1}$ and will coincide with the $\mathfrak{gl}_{n-a}$-type recurrence relation for the Bethe vector $\mathcal{Z}^k_\ell \cdot \mathbb{B}_{\mathfrak{gl}}(\{\bar t^s\}_{a+1}^{n-1})$, see (3.8). ■

Remark 3.4. Proposition 3.3 is a direct consequence of the projection method, when the off-shell Bethe vectors are expressed in terms of the Cartan–Weyl generators of the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_n)$ [16] or Yangian double $\mathcal{D}Y(\mathfrak{gl}_n)$ [9].

Splitting property of the Bethe vectors. Let us define the operation $\mathcal{Z}^{a+1}_a(t^a_1)$ as the operation which adds the parameter t^a_1 of the color a to the set $\bar t = \{\{\bar t^s\}_1^{a-1}, \varnothing, \{\bar t^s\}_{a+1}^{n-1}\}$ of the Bethe vector $\mathbb{B}_{\mathfrak{gl}}(\{\bar t^s\}_1^{a-1}, \varnothing, \{\bar t^s\}_{a+1}^{n-1})$ according to definition (3.3). A direct consequence of Proposition 3.3 and the rectangular recurrence relations (3.8) is the presentation of the Bethe vector $\mathbb{B}_{\mathfrak{gl}}(\{\bar t^s\}_1^{a-1}, t^a_1, \{\bar t^s\}_{a+1}^{n-1})$ in the form

$$
\begin{aligned}
\mathbb{B}_{\mathfrak{gl}}(\{\bar t^s\}_1^{a-1}, \{t^a_1\}, \{\bar t^s\}_{a+1}^{n-1}) &= \mathcal{Z}^{a+1}_a(t^a_1)\big(\mathcal{B}_{\mathfrak{gl}}(\{\bar t^s\}_1^{a-1}) \cdot \mathcal{B}_{\mathfrak{gl}}(\{\bar t^s\}_{a+1}^{n-1})\big)|0\rangle \\
&= \frac{1}{\mu^{a+1}_a(t^a_1; \bar t)} \sum_{i=1}^{a} \sum_{j=a+1}^{n} \sum_{\rm part} \Xi^{\ell,k}_{i,j}(t^a_1; \bar t_{\rm I}, \bar t_{\rm II}, \bar t_{\rm III}) T_{i,j}(t^a_1) \\
&\quad \times \big(\mathcal{B}_{\mathfrak{gl}}(\{\bar t^s_{\rm II}\}_1^{a-1}) \cdot \mathcal{B}_{\mathfrak{gl}}(\{\bar t^s_{\rm II}\}_{a+1}^{n-1})\big)|0\rangle. \qquad (3.12)
\end{aligned}
$$

Iterating the relation (3.12) one gets the following presentation for the off-shell Bethe vector in the generic $\mathfrak{gl}_n$-invariant integrable model:

$$
\mathbb{B}_{\mathfrak{gl}}(\bar t) = \mathcal{Z}^{a+1}_a(t^a_1) \cdot \mathcal{Z}^{a+1}_a(t^a_2) \cdots \mathcal{Z}^{a+1}_a(t^a_{r_a})\big(\mathcal{B}_{\mathfrak{gl}}(\{\bar t^s\}_1^{a-1}) \cdot \mathcal{B}_{\mathfrak{gl}}(\{\bar t^s\}_{a+1}^{n-1})\big)|0\rangle. \qquad (3.13)
$$

The relation (3.13) was called in the papers [18, 19] the *splitting property* of the off-shell Bethe vectors, see, e.g., [19, Proposition 3.1]. One can also compare the relation (3.11) with the [19, equation (3.9)]. These formulas were proved in these papers by a different approach using the evaluation homomorphism for the Yangian $Y(\mathfrak{gl}_n)$ and the trace formula for the off-shell Bethe vectors.

Example 3.5. Let us consider an example of the recurrence relation (3.12) in the case $n = 4$ and $a = 2$. This relation describes the off-shell Bethe vector $\mathbb{B}_{\mathfrak{gl}}(\bar t^1, t^2, \bar t^3)$ in the form of the actions of the monodromy entries $T_{i,j}(t^2)$ with $i = 1, 2$ and $j = 3, 4$ to the Bethe vector $\mathbb{B}_{\mathfrak{gl}}(\bar t^1_{\rm II}, \varnothing, \bar t^3_{\rm II})$. Let us stress that t^2 denotes a single Bethe parameter, not a set. Using definitions (3.2) and (3.9), we can write the Bethe vector $\mathbb{B}_{\mathfrak{gl}}(\bar t^1, t^2, \bar t^3)$ as follows:

$$
\mathbb{B}_{\mathfrak{gl}}(\bar t^1, t^2, \bar t^3) = \sum_{i=1}^{2} \sum_{j=3}^{4} \sum_{\rm part} \frac{\alpha_3(\bar t^3_{\rm III})}{\lambda_3(t^2)} \frac{g(\bar t^1_{\rm I}, \bar t^1_{\rm II})}{g(t^2, \bar t^1_{\rm II}) h(\bar t^1_{\rm II}, \bar t^1_{\rm I})} \frac{g(\bar t^3_{\rm II}, \bar t^3_{\rm III})}{g(\bar t^3_{\rm II}, t^2) h(\bar t^3_{\rm III}, \bar t^3_{\rm II})} T_{i,j}(t^2) \mathbb{B}_{\mathfrak{gl}}(\bar t^1_{\rm II}, \varnothing, \bar t^3_{\rm II}),
$$

where sum goes over partitions $\{\bar t^1_{\rm I}, \bar t^1_{\rm II}\} \vdash \bar t^1$ with cardinality $|\bar t^1_{\rm I}| = \delta_{i,1}$ and $\{\bar t^3_{\rm II}, \bar t^3_{\rm III}\} \vdash \bar t^3$ with cardinality $|\bar t^3_{\rm III}| = \delta_{j,4}$. In the case when the sets $\bar t^1$ and $\bar t^3$ both have cardinality 1, the Bethe vector $\mathbb{B}_{\mathfrak{gl}}(t^1, t^2, t^3)$ can be obtained from this recurrence relation and is equal to

$$
\begin{aligned}
\mathbb{B}_{\mathfrak{gl}}(t^1, t^2, t^3) &= \frac{1}{\lambda_2(t^1)\lambda_3(t^2)\lambda_4(t^3) g(t^2, t^1) g(t^3, t^2)} \\
&\quad \times \big(T_{2,3}(t^2) T_{1,2}(t^1) T_{3,4}(t^3) + g(t^2, t^1) T_{1,3}(t^2) T_{3,4}(t^3) T_{2,2}(t^1) + g(t^3, t^2) \\
&\quad \times T_{2,4}(t^2) T_{1,2}(t^1) T_{3,3}(t^3) + g(t^2, t^1) g(t^3, t^2) T_{1,4}(t^2) T_{3,2}(t^3) T_{1,1}(t^1)\big)|0\rangle.
\end{aligned}
$$

Taking into account (2.3), this Bethe vector coincides with the example given at the end of Section 3 in [18] up to normalization and the terms which are annihilated on the vacuum vector $|0\rangle$.

4 Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant models

The off-shell Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant models depend on the sets of Bethe parameters $\bar t = \big(\bar t^0, \bar t^1, \dots, \bar t^{n-1}\big)$ with cardinalities $\big|\bar t^s\big| = r_s$ for $s = 0, 1, \dots, n-1$.

We normalize the $\mathfrak{o}_{2n+1}$-invariant off-shell Bethe vectors in such a way that the action of the monodromy matrix element $T_{-n,n}(z)$ onto $\mathbb{B}\big(\bar t\big)$ is given by the equality

$$T_{-n,n}(z)\cdot\mathbb{B}\big(\bar t\big) = \mu^n_{-n}\big(z;\bar t\big)\mathbb{B}(\bar w), \tag{4.1}$$

where $\bar w = \big(\bar w^0, \bar w^1, \dots, \bar w^{n-1}\big)$ is a collection of extended sets of Bethe parameters such that

$$\bar w^s = \big\{\bar t^s, z, z_s\big\}, \qquad z_s = z - c(s - 1/2) \tag{4.2}$$

and the normalization factor $\mu^n_{-n}\big(z;\bar t\big)$ is given by the expression

$$\mu^n_{-n}\big(z;\bar t\big) = -\kappa_n\lambda_n(z)\frac{g\big(z_1,\bar t^0\big)}{h\big(z,\bar t^0\big)}\frac{h\big(z,\bar t^{n-1}\big)}{g\big(z_n,\bar t^{n-1}\big)}. \tag{4.3}$$

Besides the sets $\bar w^s$ of cardinalities r_s+2 which are defined in (4.2), we will also use the sets $\bar u^s$ and $\bar v^s$ of cardinalities r_s+1 given by $\bar u^s = \big\{\bar t^s, z\big\}$, $\bar v^s = \big\{\bar t^s, z_s\big\}$, $\bar w^s = \big\{\bar u^s, z_s\big\} = \big\{\bar v^s, z\big\}$. For $-n \le \ell < k \le n$, let $\mathscr{Z}^k_\ell$ be an operator which extend the sets of Bethe parameters depending on the values of ℓ and k according to the rules

$$\mathscr{Z}^k_\ell\cdot\mathbb{B}\big(\bar t\big) = \begin{cases} \mathbb{B}\big(\{\bar t^s\}_0^{\ell-1}, \{\bar u^s\}_\ell^{k-1}, \{\bar t^s\}_k^{n-1}\big), & 0 \le \ell < k \le n,\\ \mathbb{B}\big(\{\bar t^s\}_0^{-k-1}, \{\bar v^s\}_{-k}^{-\ell-1}, \{\bar t^s\}_{-\ell}^{n-1}\big), & 0 \le -k < -\ell \le n,\\ \mathbb{B}\big(\{\bar w^s\}_0^{-\ell-1}, \{\bar u^s\}_{-\ell}^{k-1}, \{\bar t^s\}_k^{n-1}\big), & 0 \le -\ell \le k \le n,\\ \mathbb{B}\big(\{\bar w^s\}_0^{k-1}, \{\bar v^s\}_k^{-\ell-1}, \{\bar t^s\}_{-\ell}^{n-1}\big), & 0 \le k \le -\ell \le n. \end{cases}$$

The action (4.1) of the monodromy matrix entry $T_{-n,n}(z)$ can be presented as the action of the operator $\mathscr{Z}^n_{-n}$ onto off-shell Bethe vector

$$\mathscr{Z}^n_{-n}\cdot\mathbb{B}\big(\bar t\big) = \frac{1}{\mu^n_{-n}\big(z;\bar t\big)}\, T_{-n,n}(z)\cdot\mathbb{B}\big(\bar t\big). \tag{4.4}$$

The commutation relations (2.7) between the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ and the monodromy matrix elements take the form for $0 \le \ell \le n-1$

$$[\mathsf{T}_{\ell+1,\ell}, T_{i,j}(z)] = \chi_{\ell+1}(\delta_{\ell,j-1} - \delta_{\ell,-j})T_{i,j-1}(z) - \chi_\ell(\delta_{\ell,i} - \delta_{\ell,-i-1})T_{i+1,j}(z). \tag{4.5}$$

The action of the monodromy entry $T_{-n,n}(z)$ (4.1) as well as the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ onto off-shell Bethe vectors can be calculated in the framework of the projection method (see details in [23, Section 4.1]). The action of the zero mode operators is

$$\begin{aligned}\mathsf{T}_{\ell+1,\ell}\cdot\mathbb{B}\big(\bar t\big) = \sum_{\text{part}}\big(&\chi_{\ell+1}\alpha_\ell\big(\bar t^\ell_{\rm I}\big)\Omega^L_\ell\big(\bar t^\ell_{\rm II}, \bar t^\ell_{\rm I}|\bar t^{\ell-1}, \bar t^{\ell+1}\big)\\ &- \chi_\ell\Omega^R_\ell\big(\bar t^\ell_{\rm I}, \bar t^\ell_{\rm II}|\bar t^{\ell-1}, \bar t^{\ell+1}\big)\big)\mathbb{B}\big(\{\bar t^s\}_1^{\ell-1}, \bar t^\ell_{\rm II}, \{\bar t^s\}_{\ell+1}^{n-1}\big),\end{aligned} \tag{4.6}$$

where the sum is over partitions with cardinalities $|\bar t^\ell_{\rm I}| = 1$. In (4.6), the rational functions Ω^L_ℓ and Ω^R_ℓ have expressions similar to (3.7)

$$\Omega^L_\ell\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1}\big)=\gamma_\ell\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}\big)\frac{h\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}\big)}{g\big(\bar t^{\ell+1},\bar t^\ell_{\rm I}\big)},$$
$$\Omega^R_\ell\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1}\big)=\gamma_\ell\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}\big)\frac{h\big(\bar t^{\ell+1},\bar t^\ell_{\rm I}\big)}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}\big)}, \tag{4.7}$$

where

$$\gamma_\ell(u,v)=\begin{cases}\mathfrak{f}(u,v)=\dfrac{u-v+c/2}{u-v}, & \ell=0,\\ \gamma(u,v)=\dfrac{g(u,v)}{h(v,u)}=\dfrac{c^2}{(u-v)(v-u+c)}, & \ell=1,\dots,n-1.\end{cases}$$

4.1 Rectangular recurrence relations for $\mathfrak{o}_{2n+1}$

The main result of the paper is formulated in Theorem 4.1 below. The proof of this theorem is similar to the proof of Theorem 3.1 given in Appendix A. In Appendix B, we will sketch its proof.

For the formulation of the statement of the theorem, we need following notations:

- the sign factor σ_m

$$\sigma_m=2\Theta(m-1)-1=\begin{cases}1, & m>0,\\ -1, & m\le 0,\end{cases}$$

  which satisfy the property $\sigma_{m+1}=-\sigma_{-m}$,

- the functions $\psi_\ell\big(z;\bar t\big)$, $\phi_k\big(z;\bar t\big)$, $\mu^k_\ell\big(z;\bar t\big)$

$$\psi_\ell\big(z;\bar t\big)=\begin{cases}g\big(z,\bar t^{\ell-1}\big)\ h\big(\bar t^\ell,z\big), & 0<\ell<n,\\ g\big(z_0,\bar t^0\big), & \ell=0,\\ \dfrac{g\big(\bar t^{-\ell},z_{-\ell}\big)}{g\big(z_{-\ell},\bar t^{-\ell-1}\big)}, & -n\le\ell<0,\end{cases} \tag{4.8}$$

$$\phi_k\big(z;\bar t\big)=\begin{cases}h\big(z,\bar t^{k-1}\big)\ g\big(\bar t^k,z\big), & 0<k\le n,\\ g\big(z,\bar t^0\big), & k=0,\\ \dfrac{g\big(z_{-k-1},\bar t^{-k-1}\big)}{g\big(\bar t^{-k},z_{-k-1}\big)}, & -n<k<0,\end{cases} \tag{4.9}$$

$$\mu^k_\ell\big(z;\bar t\big)=\sigma_{-\ell-k}(\kappa_k)^{\delta_{k,-\ell}}\lambda_k(z)\psi_\ell\big(z;\bar t\big)\phi_k\big(z;\bar t\big)\left(\frac{g\big(z_1,\bar t^0\big)}{h\big(z,\bar t^0\big)}\right)^{\delta_{\ell<0,k>0}},$$

  where $\kappa_k=k-1/2$ and $\delta_{\rm condition}$ is equal to 1 if "condition" is satisfied and to 0 otherwise.

- the partitions

$$\big|\bar t^s_{\rm I}\big|=\begin{cases}\Theta(\ell)\big(\Theta(s-i)+\Theta(-i-s-1)\big), & s<|\ell|,\\ \Theta(-i-s-1), & s\ge|\ell|,\end{cases} \tag{4.10}$$

$$\big|\bar t^s_{\rm III}\big|=\begin{cases}\Theta(-k)(\Theta(j+s)+\Theta(j-s-1)), & s<|k|,\\ \Theta(j-s-1), & s\ge|k|,\end{cases} \tag{4.11}$$

$$\big|\bar t^s_{\rm II}\big|=|\bar t^s|-\big|\bar t^s_{\rm I}\big|-\big|\bar t^s_{\rm III}\big|, \tag{4.12}$$

- the functions $\Gamma^L_{a,b}(\bar t_{\rm II},\bar t_{\rm I})$ and $\Gamma^R_{a,b}(\bar t_{\rm I},\bar t_{\rm II})$ for $a,b=0,1,\dots,n$ *which depend on the partitions*

$$\Gamma^L_{a,b}(\bar t_{\rm II},\bar t_{\rm I})=\prod_{s=a}^{b-1}\alpha_s(\bar t^s_{\rm I})\Omega^L_s(\bar t^s_{\rm II},\bar t^s_{\rm I}|\bar t^{s-1}_{\rm II},\bar t^{s+1}_{\rm II}),$$
$$\Gamma^R_{a,b}(\bar t_{\rm I},\bar t_{\rm II})=\prod_{s=a}^{b-1}\Omega^R_s(\bar t^s_{\rm I},\bar t^s_{\rm II}|\bar t^{s-1}_{\rm II},\bar t^{s+1}_{\rm II}),$$

and the functions $\Xi^{\ell,k}_{i,j}(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III})$

$$\begin{aligned}\Xi^{\ell,k}_{i,j}(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III})&=\psi_\ell(z,\bar t_{\rm I})\phi_k(z,\bar t_{\rm III})\Gamma^R_{0,n}(\bar t_{\rm I},\bar t_{\rm II,III})\Gamma^L_{0,n}(\bar t_{\rm II},\bar t_{\rm III})\\&=\psi_\ell(z,\bar t_{\rm I})\phi_k(z,\bar t_{\rm III})\Gamma^R_{0,n}(\bar t_{\rm I},\bar t_{\rm II})\Gamma^L_{0,n}(\bar t_{\rm I,II},\bar t_{\rm III}),\end{aligned}\tag{4.13}$$

where the equality between the first and second lines in (4.13) directly follows from the definitions (4.7), with the notation $\bar t_{\rm I,II}=\{\bar t_{\rm I},\bar t_{\rm II}\}$, and $\bar t_{\rm II,III}=\{\bar t_{\rm II},\bar t_{\rm III}\}$. The right hand side of (4.13) depends on i and j through the cardinalities of the subsets $\bar t_{\rm I}$, $\bar t_{\rm II}$, $\bar t_{\rm III}$, see equations (4.10) and (4.11). Note that $\Gamma^L_{a,b}(\bar t_{\rm II},\bar t_{\rm I})=\Gamma^R_{a,b}(\bar t_{\rm I},\bar t_{\rm II})=1$ for $a\ge b$.

Theorem 4.1. *For any ℓ, k such that $-n\le\ell<k\le n$, the off-shell Bethe vector $\mathcal{Z}^k_\ell\cdot\mathbb{B}(\bar t)$ satisfies the rectangular recurrence relation*

$$\mathcal{Z}^k_\ell\cdot\mathbb{B}(\bar t)=\frac{1}{\mu^k_\ell(z;\bar t)}\sum_{i=-n}^{\ell}\sum_{j=k}^{n}\sum_{\rm part}\Xi^{\ell,k}_{i,j}(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III})(\sigma_{-i})^{\delta_{\ell\ge0}}(\sigma_j)^{\delta_{k\le0}}T_{i,j}(z)\cdot\mathbb{B}(\bar t_{\rm II}),\tag{4.14}$$

where the sum is over partitions $\{\bar t^s_{\rm I},\bar t^s_{\rm II},\bar t^s_{\rm III}\}\vdash\bar t^s$ with cardinalities depending on the indices $-n\le i\le\ell<k\le j\le n$ and given by the equalities (4.10), (4.11) and (4.12).

If the cardinality of the set $\bar t^s$ is small and according to (4.10) and (4.11), one may have $|\bar t^s_{\rm I}|+|\bar t^s_{\rm III}|>|\bar t^s|$ for some i and j. Then the terms in the right-hand side of (4.14) for such values of the indices i and j have to be discarded.

Let us stress that, in opposition to the $\mathfrak{gl}_n$ case, for $\mathfrak{o}_{2n+1}$-invariant models, depending on the color s, the subsets $\bar t^s_{\rm I}$ and $\bar t^s_{\rm III}$ satisfy $|\bar t^s_{\rm I}|+|\bar t^s_{\rm III}|\le 2$. Theorem 4.1 can be proved using an inductive approach similar to the one used to prove Theorem 3.1 in Appendix A. Sketch of the proof is given in Appendix B. This result is new, except for two partial cases presented in [23] when $\ell=n-1$.

4.2 Special cases of recurrence relations

We provide subcases of the recurrence relations (4.14) that are relevant for the study of integrable models.

Elementary recurrence relations. For explicit calculations, the most interesting (and simplest) cases of the recurrence relations (4.14) are the $2n$ cases when $k=\ell+1$ and $\ell=-n,\dots,n-1$. These $2n$ cases are gathered in four classes of expressions, depending on the value of the index ℓ.

- When $0<\ell<n$, the recurrence relation reads

$$\begin{aligned}\mathcal{Z}^{\ell+1}_\ell\cdot\mathbb{B}(\bar t)&=\mathbb{B}(\bar t^0,\dots,\bar t^{\ell-1},\{\bar t^\ell,z\},\bar t^{\ell+1},\dots,\bar t^{n-1})\\&=\sum_{i=-n}^{\ell}\sum_{j=\ell+1}^{n}\sum_{\rm part}\frac{g(z,\bar t^{\ell-1}_{\rm I})h(\bar t^\ell_{\rm I},z)g(\bar t^{\ell+1}_{\rm III},z)}{\lambda_{\ell+1}(z)g(z,\bar t^{\ell-1})h(\bar t^\ell,z)h(z,\bar t^\ell)g(\bar t^{\ell+1},z)}\\&\quad\times\Gamma^R_{[i],\ell_i}(\bar t_{\rm I},\bar t_{\rm II})\Gamma^L_{\ell+1,j}(\bar t_{\rm I,II},\bar t_{\rm III})\sigma_{i+1}T_{i,j}(z)\cdot\mathbb{B}(\bar t_{\rm II}),\end{aligned}$$

where $\ell_i = \max(\ell, |i|)$, $[i] = \frac{1}{2}(i + |i|)$, and the cardinalities of the subsets

$$|\bar{t}^s_{\rm I}| = \begin{cases} \Theta(s-i) + \Theta(-i-s-1), & s < \ell, \\ \Theta(-i-s-1), & s \ge \ell, \end{cases} \qquad |\bar{t}^s_{\rm III}| = \begin{cases} 0, & s < \ell+1, \\ \Theta(j-s-1), & s \ge \ell+1. \end{cases}$$

Remark that for $\ell = n-1$ this recurrence relation reduces to

$$\mathbb{B}\big(\{\bar{t}^s\}_0^{n-2}, \{\bar{t}^{n-1}, z\}\big) = \frac{1}{\lambda_n(z) h\big(z, \bar{t}^{n-1}\big)} \sum_{i=-n}^{n-1} \sum_{\rm part} \frac{\sigma_{i+1} T_{i,n}(z) \cdot \mathbb{B}\big(\bar{t}_{\rm II}\big)}{g\big(z, \bar{t}^{n-2}_{\rm II}\big) h\big(\bar{t}^{n-1}_{\rm II}, z\big)} \times \prod_{s=0}^{n-1} \gamma_s\big(\bar{t}^s_{\rm I}, \bar{t}^s_{\rm II}\big) \prod_{s=1}^{n-1} \frac{h\big(\bar{t}^s_{\rm II}, \bar{t}^{s-1}_{\rm I}\big)}{g\big(\bar{t}^s_{\rm I}, \bar{t}^{s-1}_{\rm II}\big)},$$

which coincides with the recurrence relation (3.16) from the paper [23].

- The cases of $-n \le \ell < -1$ corresponds to so called *shifted* recurrence relation for the off-shell Bethe vectors. To describe these cases, we set $\ell = -l - 1$ with $0 < l < n$. They correspond to an extension of the set of Bethe parameters $\bar{t}^l$ by the shifted parameter $z_l = z - c(l - 1/2)$

$$\begin{aligned} \mathcal{Z}^{-l}_{-l-1} \cdot \mathbb{B}\big(\bar{t}\big) &= \mathbb{B}\big(\bar{t}^0, \dots, \bar{t}^{l-1}, \{\bar{t}^l, z_l\}, \bar{t}^{l+1}, \dots, \bar{t}^{n-1}\big) \\ &= \sum_{i=-n}^{-l-1} \sum_{j=-l}^{n} \sum_{\rm part} \frac{(-1)^{\delta_{j\ge l+1}} g\big(\bar{t}^{l+1}_{\rm I}, z_{l+1}\big) h\big(z_l, \bar{t}^l_{\rm III}\big) g\big(z_{l-1}, \bar{t}^{l-1}_{\rm III}\big)}{\lambda_{-l}(z) g\big(\bar{t}^{l+1}, z_{l+1}\big) h\big(\bar{t}^l, z_l\big) h\big(z_l, \bar{t}^l\big) g\big(z_{l-1}, \bar{t}^{l-1}\big)} \\ &\quad \times \Gamma^R_{l+1,-i}\big(\bar{t}_{\rm I}, \bar{t}_{\rm II,III}\big) \Gamma^L_{[-j],l_j}\big(\bar{t}_{\rm II}, \bar{t}_{\rm III}\big) \sigma_j T_{i,j}(z) \cdot \mathbb{B}\big(\bar{t}_{\rm II}\big), \end{aligned} \tag{4.15}$$

where $l_j = \max(l, |j|)$, $[j] = \frac{1}{2}(j + |j|)$ as above, and the cardinalities of the subsets in the sum over partitions are

$$\begin{aligned} |\bar{t}^s_{\rm I}| &= \begin{cases} 0, & s < l+1, \\ \Theta(-i-s-1), & s \ge l+1, \end{cases} \\ |\bar{t}^s_{\rm III}| &= \begin{cases} \Theta(s+j) + \Theta(j-s-1), & s < l, \\ \Theta(j-s-1), & s \ge l. \end{cases} \end{aligned} \tag{4.16}$$

The sign factor $(-1)^{\delta_{j\ge l+1}}$ in the second line of (4.15) is due to the identity $g\big(\bar{t}^l_{\rm III}, z_{l-1}\big)^{-1} = -h\big(z_l, \bar{t}^l_{\rm III}\big)$, valid when the cardinality of the subset $\bar{t}^l_{\rm III}$ is equal to one. According to (4.16), this happens when $j \ge l+1$.

For $l = n-1$, the recurrence relation (4.15) can be rewritten in the form

$$\begin{aligned} &\mathbb{B}\big(\{\bar{t}^s\}_0^{n-2}, \{\bar{t}^{n-1}, z_{n-1}\}\big) \\ &\quad = \frac{1}{\lambda_{-n+1}(z) h\big(\bar{t}^{n-1}, z_{n-1}\big)} \sum_{j=-n+1}^{n} \sum_{\rm part} \frac{(-1)^{\delta_{j,n}}}{h\big(z_{n-1}, \bar{t}^{n-1}_{\rm II}\big)} \frac{\sigma_j T_{-n,j}(z) \cdot \mathbb{B}\big(\bar{t}_{\rm II}\big)}{g\big(z_{n-2}, \bar{t}^{n-2}_{\rm II}\big)} \\ &\quad\quad \times \prod_{s=0}^{n-1} \alpha_s\big(\bar{t}^s_{\rm III}\big) \gamma_s\big(\bar{t}^s_{\rm II}, \bar{t}^s_{\rm III}\big) \prod_{s=1}^{n-1} \frac{h\big(\bar{t}^s_{\rm III}, \bar{t}^{s-1}_{\rm II}\big)}{g\big(\bar{t}^s_{\rm II}, \bar{t}^{s-1}_{\rm III}\big)} \end{aligned}$$

and coincides exactly with the recurrence relation (3.19) from the paper [23]. The sign factor $(-1)^{\delta_{j\ge l+1}}$ turns in this case to the sign factor $(-1)^{\delta_{j,n}}$.

- When $\ell=0$, the recurrence relation takes the form

$$\begin{aligned}\mathcal{Z}^1_0\cdot\mathbb{B}(\bar t) &= \mathbb{B}\big(\{\bar t^0,z\},\bar t^1,\dots,\bar t^{n-1}\big)\\ &= \sum_{i=-n}^{0}\sum_{j=1}^{n}\sum_{\text{part}}\frac{g\big(z_0,\bar t^0_{\rm I}\big)g\big(\bar t^1_{\rm III},z\big)\sigma_{i+1}T_{i,j}(z)\cdot\mathbb{B}\big(\bar t_{\rm II}\big)}{\lambda_1(z)g\big(z_0,\bar t^0\big)h\big(z,\bar t^0\big)g\big(\bar t^1,z\big)}\\ &\quad\times\prod_{s=0}^{-i-1}\gamma_s\big(\bar t^s_{\rm I},\bar t^s_{\rm II}\big)\frac{h\big(\bar t^{s+1}_{\rm II},\bar t^s_{\rm I}\big)}{g\big(\bar t^s_{\rm I},\bar t^{s-1}_{\rm II}\big)}\prod_{s=1}^{j-1}\alpha_s\big(\bar t^s_{\rm III}\big)\gamma_s\big(\bar t^s_{\rm I,II},\bar t^s_{\rm III}\big)\frac{h\big(\bar t^s_{\rm III},\bar t^{s-1}_{\rm I,II}\big)}{g\big(\bar t^{s+1}_{\rm I,II},\bar t^s_{\rm III}\big)},\end{aligned}\tag{4.17}$$

  where cardinalities of the subsets in the sum over partitions are

$$\big|\bar t^s_{\rm I}\big| = \Theta(-i-s-1)\quad\forall s,\qquad \big|\bar t^s_{\rm III}\big| = \begin{cases}0, & s=0,\\ \Theta(j-s-1), & s\ge 1.\end{cases}$$

- When $\ell=-1$, one gets the recurrence relation

$$\begin{aligned}\mathcal{Z}^0_{-1}\cdot\mathbb{B}(\bar t) &= \mathbb{B}\big(\{\bar t^0,z_0\},\bar t^1,\dots,\bar t^{n-1}\big)\\ &= \sum_{i=-n}^{-1}\sum_{j=0}^{n}\sum_{\text{part}}\frac{g\big(\bar t^1_{\rm I},z_1\big)g\big(z,\bar t^0_{\rm III}\big)\sigma_jT_{i,j}(z)\cdot\mathbb{B}\big(\bar t_{\rm II}\big)}{\lambda_0(z)g\big(\bar t^1,z_1\big)g\big(z_1,\bar t^0\big)^{-1}g\big(z,\bar t^0\big)}\\ &\quad\times\prod_{s=1}^{-i-1}\gamma_s\big(\bar t^s_{\rm I},\bar t^s_{\rm II,III}\big)\frac{h\big(\bar t^{s+1}_{\rm II,III},\bar t^s_{\rm I}\big)}{g\big(\bar t^s_{\rm I},\bar t^{s-1}_{\rm II,III}\big)}\prod_{s=0}^{j-1}\alpha_s\big(\bar t^s_{\rm III}\big)\gamma_s\big(\bar t^s_{\rm II},\bar t^s_{\rm III}\big)\frac{h\big(\bar t^s_{\rm III},\bar t^{s-1}_{\rm II}\big)}{g\big(\bar t^{s+1}_{\rm II},\bar t^s_{\rm III}\big)},\end{aligned}\tag{4.18}$$

  where cardinalities of the subsets in the sum over partitions are

$$\big|\bar t^s_{\rm I}\big| = \begin{cases}0, & s=0,\\ \Theta(-i-s-1), & s\ge 1,\end{cases}\qquad \big|\bar t^s_{\rm III}\big| = \Theta(j-s-1)\quad\forall s\ge 1.$$

Case of $\mathfrak{o}_3$-invariant models. The above recurrence relations were calculated for the algebras $\mathfrak{o}_{2n+1}$ with $n>1$. However, one can verify that they are also valid for the orthogonal algebra $\mathfrak{o}_3$ when $n=1$. This case was investigated in details in the paper [26]. In this case, there are only three cases of rectangular recurrence relations for the Bethe vectors: $\mathcal{Z}^1_{-1}\cdot\mathbb{B}\big(\bar t^0\big)$, $\mathcal{Z}^1_0\cdot\mathbb{B}\big(\bar t^0\big)$ and $\mathcal{Z}^0_{-1}\cdot\mathbb{B}\big(\bar t^0\big)$, that we detail below.

For the Bethe vector $\mathcal{Z}^1_{-1}\cdot\mathbb{B}\big(\bar t^0\big)$, there is no summation over partitions and according to (4.1) and (4.3) it is equal to

$$\mathcal{Z}^1_{-1}\cdot\mathbb{B}\big(\bar t^0\big) = \mathbb{B}\big(\{\bar t^0,z,z_0\}\big) = -\frac{2}{\lambda_1(z)}T_{-1,1}(z)\cdot\mathbb{B}\big(\bar t^0\big).\tag{4.19}$$

For the Bethe vector $\mathcal{Z}^1_0\cdot\mathbb{B}\big(\bar t^0\big)$, the recurrence relation (4.17) becomes

$$\begin{aligned}\mathcal{Z}^1_0\cdot\mathbb{B}\big(\bar t^0\big) &= \mathbb{B}\big(\{\bar t^0,z\}\big)\\ &= \frac{1}{\lambda_1(z)\mathfrak{f}\big(z_0,\bar t^0\big)}\left(T_{0,1}(z)\cdot\mathbb{B}\big(\bar t^0\big)-\sum_{\text{part}}g\big(z_0,\bar t^0_{\rm I}\big)\mathfrak{f}\big(\bar t^0_{\rm I},\bar t^0_{\rm II}\big)T_{-1,1}(z)\cdot\mathbb{B}\big(\bar t^0_{\rm II}\big)\right),\end{aligned}\tag{4.20}$$

while for the Bethe vector $\mathcal{Z}^0_{-1}\cdot\mathbb{B}\big(\bar t^0\big)$ the recurrence relation (4.18) becomes

$$\begin{aligned}\mathcal{Z}^0_{-1}\cdot\mathbb{B}\big(\bar t^0\big) &= \mathbb{B}\big(\{\bar t^0,z_0\}\big) = \frac{1}{\lambda_0(z)\mathfrak{f}\big(\bar t^0,z\big)}\\ &\quad\times\left(-T_{-1,0}(z)\cdot\mathbb{B}\big(\bar t^0\big)+\sum_{\text{part}}g\big(z,\bar t^0_{\rm III}\big)\alpha_0\big(\bar t^0_{\rm III}\big)\mathfrak{f}\big(\bar t^0_{\rm II},\bar t^0_{\rm III}\big)T_{-1,1}(z)\cdot\mathbb{B}\big(\bar t^0_{\rm II}\big)\right).\end{aligned}\tag{4.21}$$

In (4.20) and (4.21), we have $\big|\bar t^0_{\rm I}\big| = \big|\bar t^0_{\rm III}\big| = 1$. Formulas (4.19), (4.20) and (4.21) coincide with [26, formulas (4.15), (4,19) and (4.20)]. This proves that Theorem 4.1 is also valid for $\mathfrak{o}_3$-invariant integrable models.

4.3 Reduction to $\mathfrak{gl}_n$

Still in the framework of the $\mathfrak{o}_{2n+1}$-type monodromy matrix, we consider the particular case of Bethe vectors with $\bar t^0 = \varnothing$. In that case, since the color 0 is absent, one should recover the construction done for the $\mathfrak{gl}_n$-type monodromy matrix. In this subsection, we prove it by showing that the recurrence relations (4.14), coming from the $\mathfrak{o}_{2n+1}$ set-up, provide the recurrence relations (3.8), obtained in the $\mathfrak{gl}_n$ case. Obviously, to stay within the $\mathfrak{gl}_n$ part, we need to keep $\bar t^0 = \varnothing$. Thus, we only consider the recurrence relations (4.14) with either $-n \le \ell < k < 0$ or $0 < \ell < k \le n$.

The case $0 < \ell < k \le n$ and $\bar t^0 = \varnothing$. According to the first line in (4.10), it signifies that $\left|\bar t^0_{\rm I}\right| = 0 = \Theta(-i) + \Theta(-i-1)$ which is possible only for $i > 0$. Since for these values of the index i the step function $\Theta(-i-s-1)$ vanishes for any s, the cardinalities of the partitions (4.10) and (4.11) become the cardinalities (3.10). Moreover, for $i > 0$, the sign $\delta_{-\ell-k}$ compensate the sign factor $\sigma_{-i} = -1$ and $[i] = i$ and $\ell_i = \ell$, so that the function $\Xi^{\ell,k}_{i,j}\big(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III}\big)$ in (4.13) becomes the function (3.9). The recurrence relation (4.14) takes the form

$$\begin{aligned}
&\mathbb{B}\big(\varnothing,\big\{\bar t^s\big\}_1^{\ell-1},\big\{\bar t^s,z\big\}_\ell^{k-1},\big\{\bar t^s\big\}_k^{n-1}\big)\\
&\quad=\sum_{i=1}^{\ell}\sum_{j=k}^{n}\sum_{\rm part}\frac{\mathbb{T}_{i,j}(z)\cdot\mathbb{B}\big(\varnothing,\big\{\bar t^s\big\}_1^{i-1},\big\{\bar t^s_{\rm II}\big\}_i^{\ell-1},\big\{\bar t^s\big\}_\ell^{k-1},\big\{\bar t^s_{\rm II}\big\}_k^{j-1},\big\{\bar t^s\big\}_j^{n-1}\big)}{\lambda_k(z)g\big(z,\bar t^{\ell-1}_{\rm II}\big)h\big(\bar t^\ell,z\big)h\big(z,\bar t^{k-1}\big)g\big(\bar t^k_{\rm II},z\big)}\\
&\qquad\times\prod_{s=i}^{\ell-1}\Omega^R\big(\bar t^s_{\rm I},\bar t^s_{\rm II}|\bar t^{s-1}_{\rm II},\bar t^{s+1}_{\rm II}\big)\prod_{s=k}^{j-1}\alpha_s\big(\bar t^s_{\rm I}\big)\Omega^L\big(\bar t^s_{\rm II},\bar t^s_{\rm I}|\bar t^{s-1}_{\rm II},\bar t^{s+1}_{\rm II}\big),
\end{aligned}\tag{4.22}$$

where the cardinalities of the subsets in the sum over partitions are given by (3.10) and $\mathbb{T}_{i,j}(z) := T_{i,j}(z)$ for $1 \le i,j \le n$ as it was described for the first type of embedding of $Y(\mathfrak{gl}_n)$ in $Y(\mathfrak{o}_{2n+1})$, see (2.8). So we recover the $\mathfrak{gl}_n$-type recurrence relations (3.8).

The case $-n \le \ell < k < 0$ and $\bar t^0 = \varnothing$. This case is more subtle because it is related to the off-shell Bethe vector $\hat{\mathbb{B}}(\bar t)$, connected to $\mathbb{B}(\bar t)$ in the sense of the paper [27]. If the off-shell Bethe vector $\mathbb{B}(\bar t)$ is built from the entries $\mathbb{T}_{i,j}(z) = T_{i,j}(z)$ for $1 \le i < j \le n$ of $\mathfrak{gl}_n$-type according to the embedding (2.8), the off-shell Bethe vector $\hat{\mathbb{B}}(\bar t)$ is built from the $\mathfrak{gl}_n$-type matrix entries $\hat{\mathbb{T}}_{i,j}(z)$ described by the embedding (2.9).

Let us consider the recurrence relations (4.14) at $-n \le \ell < k < 0$ and empty set $\bar t^0 = \varnothing$ in more details. If $\bar t^0 = \varnothing$, then according to the first line in (4.11) $\left|\bar t^0_{\rm III}\right| = 0 = \Theta(j) + \Theta(j-1)$ which is possible only for strictly negative values of the index j. So the recurrence relation (4.14) becomes

$$\begin{aligned}
&\mathcal{Z}^k_\ell\cdot\mathbb{B}(\bar t)\big|_{\bar t^0=\varnothing} = \mathbb{B}\big(\varnothing,\big\{\bar t^s\big\}_1^{-k-1},\big\{\bar t^s,z_s\big\}_{-k}^{-\ell-1},\big\{\bar t^s\big\}_{-\ell}^{n-1}\big)\\
&\qquad\qquad= -\mu^k_\ell\big(z;\bar t\big)^{-1}\sum_{i=-n}^{\ell}\sum_{j=k}^{-1}\sum_{\rm part}\Xi^{\ell,k}_{i,j}\big(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III}\big)T_{i,j}(z)\cdot\mathbb{B}\big(\varnothing,\big\{\bar t^s_{\rm II}\big\}_1^{n-1}\big),
\end{aligned}\tag{4.23}$$

where $\mu^k_\ell\big(z;\bar t\big) = \lambda_k(z)\psi_\ell\big(z;\bar t\big)\phi_k\big(z;\bar t\big)$ and

$$\begin{aligned}
&\Xi^{\ell,k}_{i,j}\big(z;\bar t_{\rm I},\bar t_{\rm II},\bar t_{\rm III}\big) = \psi_\ell\big(z;\bar t_{\rm I}\big)\ \phi_k\big(z;\bar t_{\rm III}\big)\\
&\qquad\times\prod_{s=-\ell}^{-i-1}\Omega^R\big(\bar t^s_{\rm I},\bar t^s_{\rm II}|\bar t^{s-1}_{\rm II},\bar t^{s+1}_{\rm II}\big)\prod_{s=-j}^{-k-1}\alpha_s\big(\bar t^s_{\rm III}\big)\Omega^L\big(\bar t_{\rm II},\bar t_{\rm III}|\bar t^{s-1}_{\rm III},\bar t^{s+1}_{\rm II}\big)
\end{aligned}\tag{4.24}$$

since $\bar t^s_{\rm III} = \varnothing$ for $-\ell \le s \le -i-1$. In (4.24), we dropped the index s of the functions Ω^L_s and Ω^R_s because for positive s these functions does not depend on s and coincide with the functions (3.7).

The functions $\psi_\ell\big(z;\bar t\big)$ and $\phi_k\big(z;\bar t\big)$ are

$$\psi_\ell\big(z;\bar t\big) = \frac{g\big(\bar t^{-\ell},z_{-\ell}\big)}{g\big(z_{-\ell},\bar t^{-\ell-1}\big)},\qquad \phi_k\big(z;\bar t\big) = \frac{g\big(z_{-k-1},\bar t^{-k-1}\big)}{g\big(\bar t^{-k},z_{-k-1}\big)}$$

and the cardinalities of the partitions in (4.23) are

$$|\bar t^s_{\rm I}| = \begin{cases} 0, & s < -\ell, \\ \Theta(-i-s-1), & s \geq -\ell, \end{cases} \qquad |\bar t^s_{\rm III}| = \begin{cases} \Theta(j+s), & s < -k, \\ 0, & s \geq -k. \end{cases} \tag{4.25}$$

Note again that there are no summations in the right-hand side of (4.23) in case when $\ell = -n$ and $k = -1$.

Let us perform the following transformation of the recurrence relation (4.23). First we replace the strictly negative indices $-n \leq \ell < k < 0$ by the strictly positive indices $0 < \ell' < k' \leq n$ defined as $\ell' = n + 1 + \ell$, $1 \leq \ell' \leq n-1$, $k' = n+1+k$, $2 \leq k' \leq n$. Analogously, we replace the summation over strictly negative indices $-n \leq i \leq \ell$ and $k \leq j \leq -1$ in (4.23) by the summation over strictly positive indices i' and j' such that $i' = i + n + 1$, $1 \leq i' \leq \ell'$, $j' = j + n + 1$, $k' \leq j' \leq n$. Moreover, instead of the sets of Bethe parameters $\bar t^s$ we consider the shifted sets of Bethe parameters

$$\bar\tau^s = \bar t^{n-s} - cs = \bigl\{t^{n-s}_1 - cs, \dots, t^{n-s}_{r_{n-s}} - cs\bigr\}, \qquad s = 1, \dots, n-1. \tag{4.26}$$

These sets will be partitioned in the sum of the partitions $\{\bar\tau^s_{\rm I}, \bar\tau^s_{\rm II}, \bar\tau^s_{\rm III}\} \vdash \bar\tau^s$ with cardinalities according to (4.25) equal to

$$|\bar\tau^s_{\rm I}| = \begin{cases} \Theta(s-i'), & s < \ell', \\ 0, & s \geq \ell', \end{cases} \qquad |\bar\tau^s_{\rm III}| = \begin{cases} 0, & s < k', \\ \Theta(j'-s-1), & s \geq k'. \end{cases}$$

For $2 \leq k' \leq n$, we introduce the functions $\hat\lambda_{k'}(u)$ and $\hat\alpha_{k'}(u)$ by the equations

$$\hat\lambda_{k'}(u) = \frac{1}{\lambda_{n+1-k'}(u + c(k'-1))} \prod_{s=1}^{k'-1} \frac{\lambda_{n+1-s}(u+cs)}{\lambda_{n+1-s}(u + c(s-1))},$$

$$\hat\alpha_{k'}(u) = \frac{\hat\lambda_{k'}(u)}{\hat\lambda_{k'+1}(u)} = \frac{\lambda_{n-k'}(u+ck')}{\lambda_{n+1-k'}(u+ck')} = \alpha_{n-k'}(u+ck').$$

Then according to the relation (2.12) which connects the values of the functions $\lambda_k(u)$ for negative and positive values of the index k and definition (4.26) one gets

$$\lambda_k(z) = \hat\lambda_{k'}(z_n) \qquad \text{and} \qquad \hat\alpha_s(\bar\tau^s) = \alpha_{n-s}\bigl(\bar t^{n-s}\bigr). \tag{4.27}$$

Using all these definitions, one can calculate

$$\frac{1}{\psi_\ell(z;\bar t_{\rm II})} \prod_{s=-\ell}^{-i-1} \Omega^R\bigl(\bar t^s_{\rm I}, \bar t^s_{\rm II} | \bar t^{s-1}_{\rm II}, \bar t^{s+1}_{\rm II}\bigr) = \frac{(-1)^{|\bar\tau^{i'-1}|-|\bar\tau^{i'}_{\rm II}|}}{g\bigl(z_n, \bar\tau^{\ell'-1}_{\rm II}\bigr) h\bigl(\bar\tau^{\ell'}, z_n\bigr)} \prod_{s=i'}^{\ell'-1} \Omega^R\bigl(\bar\tau^s_{\rm I}, \bar\tau^s_{\rm II} | \bar\tau^{s-1}_{\rm II}, \bar\tau^{s+1}_{\rm II}\bigr),$$

$$\frac{1}{\phi_k(z;\bar t_{\rm II})} \prod_{s=-j}^{-k-1} \alpha_s\bigl(\bar t^s_{\rm III}\bigr) \Omega^L\bigl(\bar t^s_{\rm I}, \bar t^s_{\rm II} | \bar t^{s-1}_{\rm II}, \bar t^{s+1}_{\rm II}\bigr)$$
$$= \frac{(-1)^{|\bar\tau^{j'-1}_{\rm II}|-|\bar\tau^{j'}|}}{h\bigl(z_n, \bar\tau^{k'-1}\bigr) g\bigl(\bar\tau^{k'}_{\rm II}, z_n\bigr)} \prod_{s=k'}^{j'-1} \hat\alpha_s\bigl(\bar\tau^s_{\rm III}\bigr) \Omega^L\bigl(\bar\tau^s_{\rm II}, \bar\tau^s_{\rm III} | \bar\tau^{s-1}_{\rm II}, \bar\tau^{s+1}_{\rm II}\bigr),$$

and the recurrence relations (4.23) for the indices ℓ and k such that $-n \leq \ell < k < 0$ becomes

$$\mathbb{B}'\bigl(\varnothing, \bigl\{\bar t^s\bigr\}_1^{-k-1}, \bigl\{\bar t^s, z_s\bigr\}_{-k}^{-\ell-1}, \bigl\{\bar t^s\bigr\}_{-\ell}^{n-1}\bigr)$$
$$= \sum_{i'=1}^{\ell'} \sum_{j'=k'}^{n} \sum_{\rm part} \frac{T_{i,j}(z) \cdot \mathbb{B}'\bigl(\varnothing, \bigl\{\bar t^s\bigr\}_1^{-j-1}, \bigl\{\bar t^s_{\rm II}\bigr\}_{-j}^{-k-1}, \bigl\{\bar t^s\bigr\}_{-k}^{-\ell-1}, \bigl\{\bar t^s_{\rm II}\bigr\}_{-\ell}^{-i-1}, \bigl\{\bar t^s\bigr\}_{-i}^{n-1}\bigr)}{\hat\lambda_{k'}(z_n) g\bigl(z_n, \bar\tau^{\ell'-1}_{\rm II}\bigr) h\bigl(\bar\tau^{\ell'}, z_n\bigr) h\bigl(z_n, \bar\tau^{k'-1}\bigr) g\bigl(\bar\tau^{k'}_{\rm II}, z_n\bigr)}$$
$$\times \prod_{s=i'}^{\ell'-1} \Omega^R\bigl(\bar\tau^s_{\rm I}, \bar\tau^s_{\rm II} | \bar\tau^{s-1}_{\rm II}, \bar\tau^{s+1}_{\rm II}\bigr) \prod_{s=k'}^{j'-1} \hat\alpha_s\bigl(\bar\tau^s_{\rm III}\bigr)\ \Omega^L\bigl(\bar\tau^s_{\rm II}, \bar\tau^s_{\rm III} | \bar\tau^{s-1}_{\rm II}, \bar\tau^{s+1}_{\rm II}\bigr), \tag{4.28}$$

where $\mathbb{B}'(\bar t) = (-1)^{\sum_{s=1}^{n-2} |\bar t^s||\bar t^{s+1}| + \sum_{s=1}^{n-1} |\bar t^s|} \mathbb{B}(\bar t)$.

Let us define the monodromy matrix entries $\hat{\mathbb{T}}_{i',j'}(z)$ for all $i',j'=1,\dots,n$ defined by embedding (2.9) together with a shift of the spectral parameter as

$$\hat{\mathbb{T}}_{i',j'}(z_n) = T_{i,j}(z) \qquad \text{with} \quad \begin{cases} i = i' - n - 1, \\ j = j' - n - 1. \end{cases} \tag{4.29}$$

The corresponding eigenvalues of $\hat{\mathbb{T}}_{k'k'}(z_n)$ on the vacuum vector $|0\rangle$ coincide with $\hat{\lambda}_{k'}(z_n)$ from (4.27).

Note that the coefficients in the recursion relation (4.28) have exactly the same form as (4.22), if one replaces $z_n \to z$, $\bar{\tau}^s \to \bar{t}^s$, $\hat{\lambda}_{k'}(z_n) \to \lambda_{k'}(z)$, $\hat{\mathbb{T}}_{i',j'}(z_n) \to \mathbb{T}_{i',j'}(z)$.

Thus, the Bethe vector $\mathbb{B}(\bar{t})$ relates to Bethe vectors $\hat{\mathbb{B}}(\bar{t})$ constructed from the monodromy matrix $\hat{\mathbb{T}}(z)$ (4.29) by relation

$$\hat{\mathbb{B}}\big(\varnothing, \big\{\bar{t}^{n-s} - cs - c\kappa_n\big\}_1^{n-1}\big) = (-1)^{\sum_{s=1}^{n-2}|\bar{t}^s||\bar{t}^{s+1}| + \sum_{s=1}^{n-1}|\bar{t}^s|}\mathbb{B}\big(\varnothing, \big\{\bar{t}^s\big\}_1^{n-1}\big).$$

Such relation was studied in the paper [27], where it was shown that the vector Bethe $\hat{\mathbb{B}}(\bar{t})$ is constructed from an inverted and transposed $\mathfrak{gl}_n$-type monodromy matrix, i.e., $\hat{\mathbb{T}}(z_n) = \big(\mathbb{T}(z)^{-1}\big)^{\mathrm{t}}$.

4.4 Embedding $Y(\mathfrak{o}_{2a+1}) \otimes Y(\mathfrak{gl}_{n-a}) \hookrightarrow Y(\mathfrak{o}_{2n+1})$

In this subsection, we study the recurrence relations (4.14) in the case when the set $\bar{t}^a = \varnothing$ for some $a > 0$.

Proposition 4.2. *The off-shell Bethe vectors* $\mathbb{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{a-1}, \varnothing, \big\{\bar{t}^s\big\}_{a+1}^{n-1}\big)$ *factorizes into the product of* $\mathfrak{o}_{2a+1}$*-type pre-Bethe vector* $\mathcal{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{a-1}\big)$ *and* $\mathfrak{gl}_{n-a}$*-type pre-Bethe vectors* $\mathcal{B}_{\mathfrak{gl}}\big(\big\{\bar{t}^s\big\}_{a+1}^{n-1}\big)$

$$\mathbb{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{a-1}, \varnothing, \big\{\bar{t}^s\big\}_{a+1}^{n-1}\big) = \mathcal{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{a-1}\big) \cdot \mathcal{B}_{\mathfrak{gl}}\big(\big\{\bar{t}^s\big\}_{a+1}^{n-1}\big)|0\rangle. \tag{4.30}$$

Proof. The proof of this proposition is analogous to the proof of Proposition 3.3. We consider the recurrence relations (4.14) for three different ranges of the indices ℓ and k: $-a \le \ell < k \le a$, $a+1 \le \ell < k \le n$ and $-n \le \ell < k \le -a-1$. These ranges for k and ℓ ensure that the set $\bar{t}^a$ remains empty. Since $\mathcal{B}_{\mathfrak{o}}(\varnothing) = \mathcal{B}_{\mathfrak{gl}}(\varnothing) = 1$, the recurrence relations will prove the statement of the proposition by induction on the cardinalities.

- $-a \le \ell < k \le a$. According to (4.10), the cardinality of the empty subset $\bar{t}^a_{\text{I}}$ is given by the step function $\Theta(-i-a-1)$, which must vanish. It immediately results that the summation in the recurrence relation (4.14) is restricted to the range $-a \le i \le \ell$. Analogously, the cardinality of the empty subset $\bar{t}^a_{\text{III}}$ given by (4.11) is equal to $\Theta(j-a-1)$ which restricts the range in the summation over j to $k \le j \le a$. This means that the recurrence relation for $\mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar{t})|_{\bar{t}^a=\varnothing}$ implies only the monodromy entries $T_{i,j}(z)$ for $-a \le i \le \ell$ and $k \le j \le a$. The recurrence relation $\mathcal{Z}^a_{-a} \cdot \mathbb{B}(\bar{t})|_{\bar{t}^a=\varnothing}$ will have only one term corresponding to the action of monodromy entry $T_{-a,a}(z)$ on the off-shell Bethe vector.

  Moreover, the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ for $\ell = 0,1,\dots,a-1$ onto the Bethe vector (4.30)

  $$\begin{aligned} &\mathsf{T}_{\ell+1,\ell} \cdot \mathbb{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{a-1}, \varnothing, \big\{\bar{t}^s\big\}_{a+1}^{n-1}\big) \\ &\quad = \sum_{\text{part}} \big(\chi_{\ell+1}\alpha_\ell\big(\bar{t}^\ell_{\text{I}}\big)\Omega^L_\ell\big(\bar{t}^\ell_{\text{II}}, \bar{t}^\ell_{\text{I}}|\bar{t}^{\ell-1}, \bar{t}^{\ell+1}\big) - \chi_\ell \Omega^R_\ell\big(\bar{t}^\ell_{\text{I}}, \bar{t}^\ell_{\text{II}}|\bar{t}^{\ell-1}, \bar{t}^{\ell+1}\big)\big) \\ &\qquad \times \mathbb{B}_{\mathfrak{o}}\big(\big\{\bar{t}^s\big\}_0^{\ell-1}, \bar{t}^\ell_{\text{II}}, \big\{\bar{t}^s\big\}_{\ell+1}^{a-1}, \varnothing, \big\{\bar{t}^s\big\}_{a+1}^{n-1}\big) \end{aligned}$$

  does not affect the sets of Bethe parameters $\big\{\bar{t}^s\big\}_{a+1}^{n-1}$ and so coincides with the action of the zero mode operators onto the Bethe vectors in $\mathfrak{o}_{2a+1}$-invariant model. This shows that the recurrence relations for $\mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar{t})|_{\bar{t}^a=\varnothing}$ for $-a \le \ell < k \le a$ coincides with the $\mathfrak{o}_{2a+1}$ recurrence relation for the vector $\mathbb{B}_{\mathfrak{o}}\big(\big\{\bar{t}\big\}_0^{a-1}\big)$.

- $a+1 \le \ell < k \le n$. In this case, the cardinalities of the empty subsets $\bar{t}^a_{\text{I}} = \bar{t}^a_{\text{III}} = \varnothing$ given by the equalities (4.10) and (4.11) yield a restriction only for the index $i \ge a+1$. The summation over i and j in the right-hand side of the recurrence relation for $\mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar{t})|_{\bar{t}^a=\varnothing}$ in this case will be restricted to the range $a+1 \le i \le \ell < k \le j \le n$. The Bethe vector $\mathcal{Z}^n_{a+1} \cdot \mathbb{B}(\bar{t})$ will be proportional to the action of the monodromy matrix entry $T_{a+1,n}(z)\cdot\mathbb{B}(\bar t)$ and the action of the zero mode operators $\mathsf{T}_{\ell+1,\ell}$ for $\ell=a+1,\dots,n-1$ onto Bethe vectors (4.30) coincide with $\mathfrak{gl}_{n-a}$-type actions (3.6). The recurrence relation (4.14) in this case will not affect the sets of Bethe parameters $\{\bar t^s\}_0^{a-1}$ and will coincides with the $\mathfrak{gl}_{n-a}$-type recurrence relation for the Bethe vector $\mathcal{Z}_\ell^k\cdot\mathbb{B}_{\mathfrak{gl}}(\{\bar t^s\}_{a+1}^{n-1})$, see (4.22).

- $-n\le\ell<k\le-a-1$. Using again (4.10) and (4.11), we can observe that the summation over i and j in the right-hand side of the recurrence relation for the Bethe vector $\mathcal{Z}_\ell^k\cdot\mathbb{B}(\bar t)|_{\bar t^a=\varnothing}$ will be restricted to the range $-n\le i\le\ell<k\le j\le-a-1$. Using arguments of Section 4.3, one can show that the corresponding recurrence relation for $-n\le\ell<k\le-a-1$ will be equivalent to the $\mathfrak{gl}_{n-a}$-type recurrence relation for $\hat{\mathbb{B}}(\bar t)$, where $\hat{\mathbb{B}}(\bar t)$ is the Bethe vector symmetric to $\mathbb{B}(\bar t)$ in the sense of the symmetry introduced in the paper [27]. ■

Remark 4.3. When $a=n-1$, the recurrence relations described in the second and third items are absent and the recurrence relations described in the first item yield the reduction over rank $n\to n-1$ of the recurrence relations for the off-shell Bethe vectors in $\mathfrak{o}_{2n-1}$-invariant integrable models.

Remark 4.4 (splitting property)**.** Using the statement of Proposition 4.2 and the rectangular recurrence relations (4.14) for the off-shell Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant integrable models, one can prove a splitting property for these vectors, similar to (3.13). Again, the splitting property is a direct consequence of the presentation of the Bethe vectors within the projection method [23].

Example 4.5. Proposition 4.2 for the case when $a=0$ allows to get Bethe vectors in $\mathfrak{o}_{2n+1}$-invariant integrable model from the Bethe vectors of the $\mathfrak{gl}_n$-invariant model. Below we illustrate this in the case of $\mathfrak{o}_5$-invariant model. According to the general recurrence relation (4.17) for the Bethe vector $\mathcal{Z}_0^1\cdot\mathbb{B}(\bar t)$ in the case when $n=2$ and $\bar t^0=\varnothing$, we obtain for the Bethe vector $\mathbb{B}_{\mathfrak{o}}(z,\bar t^1)$ the presentation

$$\mathbb{B}_{\mathfrak{o}}(z,\bar t^1)=\frac{1}{\lambda_1(z)g(\bar t^1,z)}\left(T_{0,1}(z)\cdot\mathbb{B}_{\mathfrak{gl}}(\varnothing,\bar t^1)+\sum_{a=1}^{r_1}g(t_a^1,z)\alpha_1(t_a^1)\frac{g(\bar t_a^1,t_a^1)}{h(t_a^1,\bar t_a^1)}T_{0,2}(z)\cdot\mathbb{B}_{\mathfrak{gl}}(\varnothing,\bar t_a^1)\right),$$

where set $\bar t_a^1=\bar t^1\setminus\{t_a^1\}$ and $\mathbb{B}(\varnothing,\bar t^1)$ are Bethe vectors in $\mathfrak{gl}_2$-integrable model. Here we restrict the summation over the index i to the case when cardinality of the set $\bar t^0$ is equal to 0. When cardinality $|\bar t^1|=1$ and $\bar t^1=\{t^1\}$ this formula yields a presentation of the Bethe vector $\mathbb{B}(t^0,t^1)$

$$\mathbb{B}_{\mathfrak{o}}(t^0,t^1)=\frac{1}{\lambda_1(t^0)\lambda_2(t^1)g(t^1,t^0)}\big(T_{0,1}(t^0)T_{1,2}(t^1)+g(t^1,t^0)T_{0,2}(t^0)T_{1,1}(t^1)\big)|0\rangle, \tag{4.31}$$

where we rename $z\to t^0$. This formula has the same form as for the analogous Bethe vector $\mathbb{B}_{\mathfrak{gl}}(t^1,t^2)$.

We can apply the recurrence relation (4.17) to the Bethe vector (4.31) once again to obtain the presentation for the Bethe vector $\mathbb{B}_{\mathfrak{o}}(\{t_1^0,t_2^0\},t^1)$ in the form

$$\begin{aligned}\mathbb{B}_{\mathfrak{o}}(\{t_1^0,t_2^0\},t^1)&=\big(\lambda_1(t_1^0)\lambda_1(t_2^0)\lambda_2(t^1)\mathfrak{f}(t_1^0+c/2,t_2^0)g(t^1,t_1^0)g(t^1,t_2^0)\big)^{-1}\Big(T_{0,1}(t_1^0)T_{0,1}(t_2^0)T_{1,2}(t^1)\\&+g(t^1,t_2^0)\Big(T_{0,1}(t_1^0)T_{0,2}(t_2^0)T_{1,1}(t^1)+g(t_2^0,t_1^0+c/2)\Big(T_{-2,1}(t_1^0)T_{1,1}(t_2^0)T_{2,2}(t^1)\\&+h(t^1,t_2^0)T_{-1,1}(t_1^0)T_{1,2}(t^1)T_{1,1}(t_2^0)\Big)+g(t^1,t_1^0)h(t^1,t_2^0)\Big(T_{0,2}(t_1^0)\\&\times T_{0,1}(t_2^0)T_{1,1}(t^1)+g(t_2^0,t_1^0+c/2)T_{-1,2}(t_1^0)T_{1,1}(t_2^0)T_{1,1}(t^1)\Big)\Big)\Big)|0\rangle.\end{aligned}$$

This Bethe vector already differs greatly from the corresponding example for the Bethe vector in the case of $\mathfrak{gl}_3$. Indeed, due to the recurrence relations given in the Corollary 3.2, the latter can be written in the following form

$$\begin{aligned}\mathbb{B}_{\mathfrak{gl}}(\{t_1^1,t_2^1\},t^2)={}&\big(\lambda_2(t_1^1)\lambda_2(t_2^1)\lambda_3(t^2)h(t_1^1,t_2^1)h(t_2^1,t_1^1)g(t^2,t_1^1)g(t^2,t_2^1)\big)^{-1}\\&\times\big(T_{1,2}(t_2^1)T_{1,2}(t_1^1)T_{2,3}(t^2)+\lambda_2(t^2)g(t^2,t_1^1)f(t_1^1,t_2^1)T_{1,3}(t_1^1)T_{1,2}(t_2^1)\\&+\lambda_2(t^2)g(t^2,t_2^1)f(t_2^1,t_1^1)T_{1,3}(t_2^1)T_{1,2}(t_1^1)\big)|0\rangle.\end{aligned}$$

5 Conclusion

Using the zero modes method, we describe in this paper the rectangular recurrence relations for the off-shell Bethe vectors in $\mathfrak{gl}_n$- and $\mathfrak{o}_{2n+1}$-invariant integrable models. These relations are presented as sums of the actions of the monodromy entries $T_{i,j}(z)$ with $i<j$ on the off-shell Bethe vectors $\mathbb{B}(\bar t)$ and sums over partitions of the sets of Bethe parameters. When one of the set $\bar t^s$ becomes empty, the recurrence relations given by Theorem 4.1 describe a factorization of the Bethe vectors and relates to the symmetry of $\mathfrak{gl}_n$-type Bethe vectors found in the paper [27].

Now that the recurrence relations are obtained, one can deduce recurrence relations for scalar products, and for their building block, the so-called highest coefficients. One can also compute the norm of the on-shell Bethe vectors and express it as a determinant. Calculations are under review, and the results will appear soon.

Finally, the outcome obtained in this paper can be generalized for the Bethe vectors in $\mathfrak{sp}_{2n}$- and $\mathfrak{so}_{2n}$-invariant integrable models. The procedure is similar to the one presented here and is currently under investigation. The issue will be published elsewhere.

A Proof of Theorem 3.1

The proof of this theorem is based on the following lemma.

Lemma A.1. *We recall that $\bar u^s=\{\bar t^s,z\}$. The off-shell Bethe vector $\mathbb{B}(\{\bar t^s\}_1^{\ell-1},\{\bar u^s\}_\ell^{n-1})$ for $1\le\ell<n$ can be presented in the form*

$$\mathcal{Z}^n_\ell\cdot\mathbb{B}(\bar t)=\frac{1}{\mu^n_\ell(z;\bar t)}\sum_{i=1}^{\ell}\sum_{\text{part}}g(z,\bar t^{\ell-1}_{\rm I})\prod_{p=i}^{\ell-1}\Omega^R(\bar t^p_{\rm I},\bar t^p_{\rm II}|\bar t^{p-1}_{\rm II},\bar t^{p+1}_{\rm II})$$
$$\times T_{i,n}(z)\cdot\mathbb{B}(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^{\ell-1},\{\bar t^s\}_\ell^{n-1}),\tag{A.1}$$

where the sum goes over partitions $\{\bar t^s_{\rm I},\bar t^s_{\rm II}\}\vdash\bar t^s$ with cardinalities $|\bar t^s_{\rm I}|=1$ for all $s=i,\dots,\ell-1$, $\bar t^\ell$ is not partitioned with $\bar t^\ell_{\rm II}=\bar t^\ell$, $\bar t^{i-1}_{\rm II}=\bar t^{i-1}$, $\bar t^{i-1}_{\rm I}=\varnothing$, and $\bar t^0_{\rm I}=\varnothing$.

Proof. We prove relation (A.1) through an induction over ℓ. Equality (3.4) can be considered as the base of this induction since for $\ell=1$ there are no partitions in the right-hand side of (A.1) and it becomes identical to (3.4).

Assuming that equality (A.1) is valid for some fixed $\ell<n-1$, we multiply it by the normalization factor $\mu^n_\ell(z;\bar t)$ and apply to both sides the zero mode operator $\mathsf{T}_{\ell+1,\ell}$. Using the action of the zero modes (3.6), we get from the left-hand side of (A.1)

$$-\chi_\ell\mu^n_{\ell+1}(z;\bar t)\mathcal{Z}^n_{\ell+1}\cdot\mathbb{B}(\bar t)+\sum_{\text{part}}\big(\chi_{\ell+1}\alpha_\ell(\bar t^\ell_{\rm I})\Omega^L(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1})-\chi_\ell f(\bar t^\ell_{\rm I},z)\Omega^R(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1})\big)$$
$$\times\mu^n_\ell(z;\bar t^{\ell-1},\bar t^\ell_{\rm II},\bar t^{n-1})\mathcal{Z}^n_\ell\cdot\mathbb{B}(\{\bar t^s\}_1^{\ell-1},\bar t^\ell_{\rm II},\{\bar t^s\}_{\ell+1}^{n-1}).\tag{A.2}$$

Here the sum goes over partitions $\{\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}\}\vdash\bar t^\ell$ with cardinality $|\bar t^\ell_{\rm I}|=1$. In the last line of (A.2), we wrote explicitly the arguments of the function $\mu^n_\ell(z;\bar t^{\ell-1},\bar t^\ell_{\rm II},\bar t^{n-1})$ for clarification. Using the induction assumption, relation (A.2) rewrites

$$-\chi_\ell\mu^n_{\ell+1}(z;\bar t)\mathcal{Z}^n_{\ell+1}\cdot\mathbb{B}(\bar t)+\sum_{\text{part}}\big(\chi_{\ell+1}\alpha_\ell(\bar t^\ell_{\rm I})\Omega^L(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1})-\chi_\ell f(\bar t^\ell_{\rm I},z)\Omega^R(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1})\big)$$
$$\times\sum_{i=1}^{\ell}g(z,\bar t^{\ell-1}_{\rm I})\prod_{p=i}^{\ell-1}\Omega^R(\bar t^p_{\rm I},\bar t^p_{\rm II}|\bar t^{p-1}_{\rm II},\bar t^{p+1}_{\rm II})T_{i,n}(z)\cdot\mathbb{B}(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^{\ell},\{\bar t^s\}_{\ell+1}^{n-1}).\tag{A.3}$$

Here the sum goes over partitions $\{\bar t^s_{\rm I},\bar t^s_{\rm II}\}\vdash\bar t^s$ with cardinalities $|\bar t^s_{\rm I}|=1$ for all $s=i,\dots,\ell$, $\bar t^{i-1}_{\rm II}=\bar t^{i-1}$, and $\bar t^{i-1}_{\rm I}=\varnothing$.

To get (A.2) and (A.3), we have used the equalities

$$\Omega^L(\bar t^\ell,z|\bar t^{\ell-1},\bar u^{\ell+1})=0,\qquad \Omega^R(z,\bar t^\ell|\bar t^{\ell-1},\bar u^{\ell+1})=\frac{g(z,\bar t^\ell)}{h(\bar t^\ell,z)}\frac{h(\bar t^{\ell+1},z)}{g(z,\bar t^{\ell-1})},$$

$$\Omega^L\big(\{\bar t^\ell_{\rm II},z\},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar u^{\ell+1}\big)=\frac{1}{h\big(\bar t^\ell_{\rm I},z\big)}\Omega^L\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1},\bar t^{\ell+1}\big),$$
$$\Omega^R\big(\bar t^\ell_{\rm I},\{z,\bar t^\ell_{\rm II}\}|\bar t^{\ell-1},\bar u^{\ell+1}\big)=g\big(\bar t^\ell_{\rm I},z\big)\Omega^R\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1},\bar t^{\ell+1}\big),$$
$$\mu^n_\ell\big(z;\bar t\big)\frac{g\big(z,\bar t^\ell\big)}{h\big(\bar t^\ell,z\big)}\frac{h\big(\bar t^{\ell+1},z\big)}{g\big(z,\bar t^{\ell-1}\big)}=\mu^n_{\ell+1}\big(z;\bar t\big),\qquad \mu^n_\ell\big(z;\bar t\big)=h\big(\bar t^\ell_{\rm I},z\big)\mu^n_\ell\big(z;\bar t^{\ell-1},\bar t^\ell_{\rm II},\bar t^{n-1}\big).$$

Using the commutation relations $[\mathsf{T}_{\ell+1,\ell},T_{i,n}(z)]=-\chi_\ell\ \delta_{\ell,i}\ T_{\ell+1,n}(z)$, the action of the zero mode $\mathsf{T}_{\ell+1,\ell}$ on the right-hand side of the induction assumption (A.1) reads

$$\begin{aligned}&-\chi_\ell T_{\ell+1,n}(z)\cdot\mathbb{B}\big(\bar t\big)+\sum_{i=1}^{\ell}\sum_{\rm part}g\big(z,\bar t^{\ell-1}_{\rm I}\big)\prod_{p=i}^{\ell-2}\Omega^R\big(\bar t^p_{\rm I},\bar t^p_{\rm II}|\bar t^{p-1}_{\rm II},\bar t^{p+1}_{\rm II}\big)\Omega^R\big(\bar t^{\ell-1}_{\rm I},\bar t^{\ell-1}_{\rm II}|\bar t^{\ell-2}_{\rm II},\{\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}\}\big)\\&\quad\times\big(\chi_{\ell+1}\alpha_\ell\big(\bar t^\ell_{\rm I}\big)\Omega^L\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1}_{\rm II},\bar t^{\ell+1}\big)-\chi_\ell\Omega^R\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1}_{\rm II},\bar t^{\ell+1}\big)\big)\\&\quad\times T_{i,n}(z)\cdot\mathbb{B}\big(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^\ell,\{\bar t^s\}_{\ell+1}^{n-1}\big)\big).\end{aligned}\tag{A.4}$$

Here the sum over partitions is the same as in (A.3).

Let us compare the coefficients of the twisting parameters $\chi_{\ell+1}$ and χ_ℓ in (A.3) and (A.4). Due to the relations

$$\Omega^L\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\{\bar t^{\ell-1}_{\rm I},\bar t^{\ell-1}_{\rm II}\},\bar t^{\ell+1}\big)=h\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)\Omega^L\big(\bar t^\ell_{\rm II},\bar t^\ell_{\rm I}|\bar t^{\ell-1}_{\rm II},\bar t^{\ell+1}\big),$$
$$\Omega^R\big(\bar t^{\ell-1}_{\rm I},\bar t^{\ell-1}_{\rm II}|\bar t^{\ell-2}_{\rm II},\{\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}\}\big)=h\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)\Omega^R\big(\bar t^{\ell-1}_{\rm I},\bar t^{\ell-1}_{\rm II}|\bar t^{\ell-2}_{\rm II},\bar t^\ell_{\rm II}\big),$$

the terms proportional to $\chi_{\ell+1}$ in (A.3) and in (A.4) are equal, and cancel each other when we equate (A.3) and (A.4). Then χ_ℓ factorizes globally, and we get a relation with no explicit dependence of the χ_i parameters: this is a manifestation of the principle described in Remark 2.2.

On the other hand, using in (A.3) the equality

$$\Omega^R\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\{\bar t^{\ell-1}_{\rm I},\bar t^{\ell-1}_{\rm II}\},\bar t^{\ell+1}\big)=\frac{1}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)}\Omega^R\big(\bar t^\ell_{\rm I},\bar t^\ell_{\rm II}|\bar t^{\ell-1}_{\rm II},\bar t^{\ell+1}\big)$$

we conclude that the equality between (A.3) and (A.4) is equivalent to the relation (A.1) at $\ell\to\ell+1$ due to the identities

$$\begin{aligned}&g\big(z,\bar t^{\ell-1}_{\rm I}\big)\left(h\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)-\frac{f\big(\bar t^\ell_{\rm I},z\big)}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)}\right)\\&\quad=\frac{g\big(z,\bar t^{\ell-1}_{\rm I}\big)}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)}\big(f\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)-f\big(\bar t^\ell_{\rm I},z\big)\big)\\&\quad=\frac{g\big(z,\bar t^{\ell-1}_{\rm I}\big)}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)}\big(g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)-g\big(\bar t^\ell_{\rm I},z\big)\big)=\frac{g\big(z,\bar t^{\ell-1}_{\rm I}\big)g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)g\big(\bar t^\ell_{\rm I},z\big)}{g\big(\bar t^\ell_{\rm I},\bar t^{\ell-1}_{\rm I}\big)g\big(\bar t^{\ell-1}_{\rm I},z\big)}=g\big(z,\bar t^\ell_{\rm I}\big)\end{aligned}\tag{A.5}$$

for $i<\ell$ and $1-f(\bar t^i_{\rm I},z)=g(z,\bar t^i_{\rm I})$ for $i=\ell$. This ends the inductive proof of relation (A.1). ■

End of theorem's proof. To finish the proof of Theorem 3.1 one has to perform an inductive proof over k for the recurrence relation (3.8). We will consider the induction step $k+1\to k$ taking as induction base the just proved equality (A.1).

Let us assume that equality (3.8) is valid for $\mathcal{Z}^{k+1}_\ell$ for some $k<n$. The induction proof means that this assumption should lead to the equality (3.8) for $\mathcal{Z}^k_\ell$. To perform the induction step, we multiply both sides of (3.8) at $k+1$ by the function $\mu^{k+1}_\ell\big(z;\bar t\big)$ and act on this relation by the zero mode operator $\mathsf{T}_{k+1,k}$. Using the equalities

$$\Omega^L\big(\bar t^k,z|\bar u^{k-1},\bar t^{k+1}\big)=\frac{g\big(\bar t^k,z\big)}{h\big(z,\bar t^k\big)}\frac{h\big(z,\bar t^{k-1}\big)}{g\big(\bar t^{k+1},z\big)},\qquad \Omega^R\big(z,\bar t^k|\bar u^{k-1},\bar t^{k+1}\big)=0,$$
$$\Omega^L\big(\{\bar t^k_{\rm II},z\},\bar t^k_{\rm III}|\bar u^{k-1},\bar t^{k+1}\big)=g\big(z,\bar t^k_{\rm III}\big)\Omega^L\big(\bar t^k_{\rm II},\bar t^k_{\rm III}|\bar t^{k-1},\bar t^{k+1}\big),$$
$$\Omega^R\big(\bar t^k_{\rm III},\{\bar t^k_{\rm II},z\}|\bar u^{k-1},\bar t^{k+1}\big)=\frac{1}{h\big(z,\bar t^k_{\rm III}\big)}\Omega^R\big(\bar t^k_{\rm III},\bar t^k_{\rm II}|\bar t^{k-1},\bar t^{k+1}\big),$$

$$\alpha_k(z)\mu_\ell^{k+1}(z;\bar t)\frac{g(\bar t^k,z)}{h(z,\bar t^k)}\frac{h(z,\bar t^{k-1})}{g(\bar t^{k+1},z)}=\mu_\ell^k(z;\bar t),\qquad \mu_\ell^{k+1}(z;\bar t)=h(z,\bar t^k_{\rm III})\mu_\ell^{k+1}(z;\bar t^{l-1},\bar t^\ell,\bar t^k_{\rm II},\bar t^{k+1}),$$

one can write the left-hand side of the resulting relation as follows:

$$\begin{aligned}&\mu_\ell^{k+1}(z;\bar t)\mathsf{T}_{k+1,k}\cdot\mathbb{B}\big(\{\bar t^s\}_1^{\ell-1},\{\bar u^s\}_\ell^k,\{\bar t^s\}_{k+1}^{n-1}\big)\\ &\quad=\chi_{k+1}\mu_\ell^k(z;\bar t)\mathcal{Z}_\ell^k\cdot\mathbb{B}(\bar t)+\sum_{i=1}^{\ell}\sum_{j=k+1}^{n}\sum_{\rm part}\big(\chi_{k+1}\alpha_k(\bar t^k_{\rm III})\Omega^L(\bar t^k_{\rm II},\bar t^k_{\rm III}|\bar t^{k-1},\bar t^{k+1})f(z,\bar t^k_{\rm III})\\ &\qquad-\chi_k\Omega^R(\bar t^k_{\rm III},\bar t^k_{\rm II}|\bar t^{k-1},\bar t^{k+1})\big)\Xi_{i,j}^{\ell,k+1}\big(z;\{\bar t^s\}_1^{k-1},\bar t^k_{\rm II},\{\bar t^s\}_{k+1}^{n-1}\big)\\ &\qquad\times T_{i,j}(z)\cdot\mathbb{B}\big(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^{\ell-1},\{\bar t^s\}_\ell^{k-1},\{\bar t^s_{\rm II}\}_k^{j-1},\{\bar t^s\}_j^{n-1}\big).\end{aligned}\tag{A.6}$$

Here for further convenience, the partition resulting from the action of the zero mode operator $\mathsf{T}_{k+1,k}$ is noted $\{\bar t^k_{\rm II},\bar t^k_{\rm III}\}\vdash\bar t^k$ with cardinality $|\bar t^k_{\rm III}|=1$.

On the other hand, the right-hand side of the same equality takes the form

$$\begin{aligned}&\chi_{k+1}\sum_{i=1}^{\ell}\sum_{\rm part}\Xi_{i,k+1}^{\ell,k+1}(z,\bar t)T_{i,k}(z)\cdot\mathbb{B}(\bar t_{\rm II})+\sum_{i=1}^{\ell}\sum_{j=k+1}^{n}\sum_{\rm part}\Xi_{i,j}^{\ell,k+1}(z;\bar t)\big|_{\{\bar t^k_{\rm II},\bar t^k_{\rm III}\}\vdash\bar t^k}\\ &\times\big(\chi_{k+1}\alpha_k(\bar t^k_{\rm III})\Omega^L(\bar t^k_{\rm II},\bar t^k_{\rm III}|\bar t^{k-1},\bar t^{k+1}_{\rm II})-\chi_k\Omega^R(\bar t^k_{\rm III},\bar t^k_{\rm II}|\bar t^{k-1},\bar t^{k+1}_{\rm II})\big)\\ &\times T_{i,j}(z)\cdot\mathbb{B}\big(\{\bar t^s\}_1^{i-1},\{\bar t^s_{\rm II}\}_i^{\ell-1},\{\bar t^s\}_\ell^{k-1},\{\bar t^s_{\rm II}\}_k^{j-1},\{\bar t^s\}_j^{n-1}\big),\end{aligned}\tag{A.7}$$

where we used the commutation relation (3.5) and took into account that the summation index i in (A.7) satisfies the inequalities $1\le i\le\ell<k$. Due to the formulas

$$\begin{aligned}&\Xi_{i,k+1}^{\ell,k+1}(z;\bar t)=\Xi_{i,k}^{\ell,k}(z;\bar t)=g(z,\bar t^{\ell-1}_{\rm I})\prod_{p=i}^{\ell-1}\Omega^R(\bar t^p_{\rm I},\bar t^p_{\rm II}|\bar t^{p-1}_{\rm II},\bar t^{p+1}_{\rm II}),\\ &\Xi_{i,j}^{\ell,k+1}(z;\bar t)\big|_{\{\bar t^k_{\rm II},\bar t^k_{\rm III}\}\vdash\bar t^k}=h(\bar t^{k+1}_{\rm III},\bar t^k_{\rm III})\Xi_{i,j}^{\ell,k+1}\big(z;\{\bar t^s\}_1^{k-1},\bar t^k_{\rm II},\{\bar t^s\}_{k+1}^{n-1}\big),\\ &\Omega^R\big(\bar t^k_{\rm III},\bar t^k_{\rm II}|\bar t^{k-1},\{\bar t^{k+1}_{\rm II},\bar t^{k+1}_{\rm III}\}\big)=h(\bar t^{k+1}_{\rm III},\bar t^k_{\rm III})\Omega^R(\bar t^k_{\rm III},\bar t^k_{\rm II}|\bar t^{k-1},\bar t^{k+1}_{\rm II}),\\ &\Omega^L\big(\bar t^k_{\rm II},\bar t^k_{\rm III}|\bar t^{k-1},\{\bar t^{k+1}_{\rm II},\bar t^{k+1}_{\rm III}\}\big)=\frac{1}{g(\bar t^{k+1}_{\rm III},\bar t^k_{\rm III})}\Omega^L(\bar t^k_{\rm II},\bar t^k_{\rm III}|\bar t^{k-1},\bar t^{k+1}_{\rm II})\end{aligned}$$

and the identity for the rational functions

$$g(\bar t^{k+1}_{\rm III},z)\left(h(\bar t^{k+1}_{\rm III},\bar t^k_{\rm III})-\frac{f(z,\bar t^k_{\rm III})}{g(\bar t^{k+1}_{\rm III},\bar t^k_{\rm III})}\right)=g(\bar t^k_{\rm III},z),$$

in the equality between (A.6) and (A.7) coefficients at χ_k cancel each other and coefficients at χ_{k+1} yield (3.8). This finishes the inductive proof of Theorem 3.1. ■

Alternative proof. Theorem 3.1 can also be proven using an induction $\ell\to\ell+1$ and taking as induction base the relation (A.8), coming from following alternative lemma.

Lemma A.2. *The off-shell Bethe vector* $\mathcal{Z}_1^k\cdot\mathbb{B}(\bar t)=\mathbb{B}\big(\{\bar u^s\}_1^{k-1},\{\bar t^s\}_k^{n-1}\big)$ *for* $1<k\le n$ *can be presented in the form*

$$\begin{aligned}\mathcal{Z}_1^k\cdot\mathbb{B}(\bar t)&=\frac{1}{\mu_1^k(z;\bar t)}\sum_{j=k}^{n}\sum_{\rm part}g(\bar t^k_{\rm III},z)\prod_{p=k}^{j-1}\alpha_p(\bar t^p_{\rm III})\Omega^L(\bar t^p_{\rm II},\bar t^p_{\rm III}|\bar t^{p-1}_{\rm II},\bar t^{p+1}_{\rm II})\\ &\quad\times T_{1,j}(z)\cdot\mathbb{B}\big(\{\bar t^s\}_1^{k-1},\{\bar t^s_{\rm II}\}_k^{j-1},\{\bar t^s\}_j^{n-1}\big),\end{aligned}\tag{A.8}$$

where the sum goes over partitions $\{\bar t^s_{\rm II},\bar t^s_{\rm III}\}\vdash\bar t^s$ *with cardinalities* $|\bar t^s_{\rm III}|=1$ *for all* $s=k,\dots,j-1$, *the set* $\bar t^{k-1}$ *is not partitioned,* $\bar t^{k-1}_{\rm II}=\bar t^{k-1}$, *and* $\bar t^n_{\rm III}=\varnothing$.

The proofs of Lemma A.2 and the end of recursion for Theorem 3.1 are similar as above, and we do not reproduce them in the present paper.

B Sketch of the proof of Theorem 4.1

The proof of this theorem follows the method described in Appendix A. The starting point is the simple recurrence relation (4.4) for the Bethe vector $\mathbb{B}\big(\{\bar{w}^s\}_0^{n-1}\big)$.

The recurrence relations (4.14) become the relations (4.19), (4.20), and (4.21) for $n=1$. They were proved in [26]. To prove Theorem 4.1, it is sufficient to consider the cases when $n>1$.

Applying the zero mode operator $\mathsf{T}_{n,n-1}$ to (4.4) and using the commutation relations between the zero modes and the monodromy entries (4.5) as well as the action of the zero mode operators on the off-shell Bethe vectors (4.6), we get a relation which involves terms proportional to χ_n and terms proportional to χ_{n-1}. Since the relation involves only Bethe vectors, monodromy matrix entries $T_{ij}(z)$ and eigenvalues $\lambda_i(z)$, due to Remark 2.2, the coefficients of these two independent twisting parameters yield two recurrence relations, one for the Bethe vectors $\mathbb{B}\big(\{\bar{w}^s\}_0^{n-2},\bar{u}^{n-1}\big)=\mathcal{Z}^n_{-n+1}\cdot\mathbb{B}(\bar{t})$ and one for the Bethe vector $\mathbb{B}\big(\{\bar{w}^s\}_0^{n-2},\bar{v}^{n-1}\big)=\mathcal{Z}^{n-1}_{-n}\cdot\mathbb{B}(\bar{t})$. These relations are

$$\mathcal{Z}^n_{-n+1}\cdot\mathbb{B}(\bar{t})=\frac{1}{\mu^n_{-n+1}(z;\bar{t})}\sum_{i=-n}^{-n+1}\sum_{\text{part}}g\big(\bar{t}^{n-1}_{\mathrm{I}},z_{n-1}\big)\Omega^R\big(\bar{t}^{n-1}_{\mathrm{I}},\bar{t}^{n-1}_{\mathrm{II}}|\bar{t}^{n-2},\varnothing\big)$$
$$\times T_{i,n}(z)\cdot\mathbb{B}\big(\{\bar{t}^s\}_0^{n-2},\bar{t}^{n-1}_{\mathrm{II}}\big), \tag{B.1}$$

$$\mathcal{Z}^{n-1}_{-n}\cdot\mathbb{B}(\bar{t})=\frac{1}{\mu^{n-1}_{-n}(z;\bar{t})}\sum_{j=n-1}^{n}\sum_{\text{part}}g\big(\bar{t}^{n-1}_{\mathrm{III}},z\big)\alpha_{n-1}\big(\bar{t}^{n-1}_{\mathrm{III}}\big)\Omega^L\big(\bar{t}^{n-1}_{\mathrm{II}},\bar{t}^{n-1}_{\mathrm{III}}|\bar{t}^{n-2},\varnothing\big)$$
$$\times T_{-n,j}(z)\cdot\mathbb{B}\big(\{\bar{t}^s\}_0^{n-2},\bar{t}^{n-1}_{\mathrm{II}}\big), \tag{B.2}$$

where $\big|\bar{t}^{n-1}_{\mathrm{I}}\big|=\Theta(-i-n)$ and $\big|\bar{t}^{n-1}_{\mathrm{III}}\big|=\Theta(j-n)$.

These recurrence relations allows to get an inductive proof of the formula (4.14) similarly to the one presented in Appendix A. The only differences for the different steps of this inductive proof are in the different ranges for the indices ℓ and k that one has to consider, and for which different identities are needed.

Some of these identities will be different from those used in the proof of the recurrence relations for $\mathfrak{gl}_n$ Bethe vectors because of the following mechanism. To consider the inductive step $k+1\to k$ or $\ell\to\ell+1$ in the recurrence relation (4.14) we will apply the zero mode operator $\mathsf{T}_{k+1,k}$ or $\mathsf{T}_{\ell+1,\ell}$ to the inductive assumption recurrence relations which correspond to the indices $k+1$ and ℓ respectively. Since the action of the zero mode operator $\mathsf{T}_{k+1,k}$ also parts the set of Bethe parameters $\big\{\bar{t}^k_{\mathrm{i}},\bar{t}^k_{\mathrm{ii}}\big\}\vdash\bar{t}^k$, the partition of $\bar{t}^k$ first by the induction assumption and then by the zero mode action, or vice-versa, may lead to different splittings in the left and right hand sides of the resulting recurrence relation. In the left-hand side the set $\bar{t}^k$ first parts into subsets $\big\{\bar{t}^k_{\mathrm{i}},\bar{t}^k_{\mathrm{ii}}\big\}$ through the action of the zero mode, and then the subset $\bar{t}^k_{\mathrm{ii}}$ is partitioned into subsets $\bar{t}^k_{\mathrm{ii}}\vdash\big\{\bar{t}^k_{\mathrm{I}},\bar{t}^k_{\mathrm{II}}\big\}$ according to the induction assumption. On the other hand, in the right-hand side the set $\bar{t}^k$ first parts into subsets $\big\{\bar{t}^k_{\mathrm{I}},\bar{t}^k_{\mathrm{II}}\big\}$ through the induction assumption, and then the subset $\bar{t}^k_{\mathrm{II}}$ is partitioned into subsets $\big\{\bar{t}^k_{\mathrm{i}},\bar{t}^k_{\mathrm{ii}}\big\}\vdash\bar{t}^k_{\mathrm{II}}$ by the action of the zero mode. If the value of the index j is such that according to (4.11) the subset $\bar{t}^k_{\mathrm{I}}$ is not empty, then the resulting equality after action of $\mathsf{T}_{k+1,k}$ should be symmetrized over the subsets $\bar{t}^k_{\mathrm{i}}$ and $\bar{t}^k_{\mathrm{I}}$ both having the cardinality 1. It will make appear the subset $\bar{t}^k_{\mathrm{i}}\cup\bar{t}^k_{\mathrm{I}}$, which may have cardinality 2, hence the cardinality 2 subset $\bar{t}^k_{\mathrm{I}}$ in the sum over partitions in the recurrence relation (4.14). This phenomena does not happen for $\mathfrak{gl}_n$ Bethe vectors but is present for the inductive proof of the recurrence relations for the Bethe vectors of other algebra series.

Referring to the calculations presented in Appendix A for more details, we describe below the different ranges of the indices ℓ, k and the corresponding identities which should be used at each of the inductive step. Let us divide the whole domain $-n\le\ell<k\le n$ of the values of the indices ℓ and k into three subdomains: $1\le-\ell,k\le n$, $0\le\ell<k\le n$, and $-n\le\ell<k\le 0$.

For the first subdomain $1\le-\ell,k\le n$, the calculations can be performed as a sequence of the following steps

- $\mathcal{Z}^n_{-n}\cdot\mathbb{B}(\bar{t})\to\mathcal{Z}^n_{-n+1}\cdot\mathbb{B}(\bar{t})$ and $\mathcal{Z}^n_{-n}\cdot\mathbb{B}(\bar{t})\to\mathcal{Z}^{n-1}_{-n}\cdot\mathbb{B}(\bar{t})$. The recurrence relations for the Bethe vectors $\mathbb{B}\big(\{\bar{w}^s\}_0^{n-2},\bar{u}^{n-1}\big)=\mathcal{Z}^n_{-n+1}\cdot\mathbb{B}(\bar{t})$ and $\mathbb{B}\big(\{\bar{w}^s\}_0^{n-2},\bar{v}^{n-1}\big)=\mathcal{Z}^{n-1}_{-n}\cdot\mathbb{B}(\bar{t})$ are given by the formulas (B.1) and (B.2) above.
- $\mathcal{Z}^n_{-n+1}\cdot\mathbb{B}(\bar{t})\to\mathcal{Z}^n_{\ell}\cdot\mathbb{B}(\bar{t})$, $1\le-\ell\le n-1$. The recurrence relations (4.14) is proved in these cases by induction over ℓ starting from the recurrence relation (B.1). In this case, we will not need to perform the symmetrization described above and the only identity which will be necessary to consider these cases is the simple identity (A.5).

- $\mathcal{Z}^n_\ell \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar t)$, $-\ell \le k \le n-1$. These cases are proved by induction over k starting from the recurrence relation for the Bethe vector $\mathcal{Z}^n_\ell \cdot \mathbb{B}(\bar t)$ which is already proved at the previous step. This proof will require to use the simple identity (A.5) and a more complicated identity

$$\operatorname*{Sym}_{y_1,y_2}\left(\frac{h(q,y_1)}{g(y_1,x)}\frac{g(y_1,y_2)}{h(y_2,y_1)} - \frac{h(y_1,x)}{g(q,y_1)}\frac{g(y_2,y_1)}{h(y_1,y_2)}\right) = 0, \tag{B.3}$$

where for any expression $E(y_1,y_2)$ we define

$$\operatorname*{Sym}_{y_1,y_2} E(y_1,y_2) = E(y_1,y_2) + E(y_2,y_1).$$

- $\mathcal{Z}^{n-1}_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_{-n} \cdot \mathbb{B}(\bar t)$, $1 \le k \le n-1$. The induction proof of the recurrence relations will require only the simple identity (A.5).
- $\mathcal{Z}^k_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar t)$, $k \le -\ell \le n-1$. For these cases, the starting point will be the recurrence relation for the Bethe vector $\mathcal{Z}^k_{-n} \cdot \mathbb{B}(\bar t)$ proved at the previous step. We will need for these cases the identities (A.5) and (B.3).

For the second domain $0 \le l < k \le n$, the most simple way to prove the recurrence relations (4.14) can be depicted as the sequence of the following steps.

- The first step of the induction $\mathcal{Z}^n_{-1} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^n_0 \cdot \mathbb{B}(\bar t)$ is simple. It will not require any complicated rational functions identities, but fixes the function $\psi_0(z;\bar t)$ in (4.8).
- The next step $\mathcal{Z}^n_0 \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^n_1 \cdot \mathbb{B}(\bar t)$ is also particular. It is the first step for which partitions with $|\bar t^0_{\rm I}| = 2$ appear. To get the recurrence relation, we use the identities

$$f(t,z) - \frac{f(z,t)}{f(z_1,t)} = -g(z_0,t) \tag{B.4}$$

and

$$\operatorname*{Sym}_{y_1,y_2}\left(\frac{g(y_2,z_0)}{g(x,y_1)}\left(\mathfrak{f}(y_1,y_2)f(x,y_1)\frac{f(z,y_1)}{f(z_1,y_1)} - \mathfrak{f}(y_2,y_1)\right)\right) = g(z,\bar y)h(x,z), \tag{B.5}$$

where $\bar y = \{y_1,y_2\}$.
- Then, to prove $\mathcal{Z}^n_\ell \cdot \mathbb{B}(\bar t)$, $\ell > 1$, starting from $\mathcal{Z}^n_1 \cdot \mathbb{B}(\bar t)$, we use the identities

$$g(z,\bar x)\left(h(t,\bar x) - \frac{f(t,z)}{g(t,\bar x)} - h(t,z)\right) = g(z,t) \tag{B.6}$$

and

$$\operatorname*{Sym}_{y_1,y_2}\left(h(y_2,z)\left(\frac{h(y_1,\bar x)}{g(q,y_1)}\frac{g(y_2,y_1)}{h(y_1,y_2)} - f(y_1,z)\frac{h(q,y_1)}{g(y_1,\bar x)}\frac{g(y_1,y_2)}{h(y_2,y_1)}\right)\right) = \frac{g(z,\bar y)h(q,z)}{g(z,\bar x)}, \tag{B.7}$$

where $\bar x = \{x_1,x_2\}$ and $\bar y = \{y_1,y_2\}$.
- For the steps $\mathcal{Z}^n_\ell \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar t)$, $0 \le \ell < k \le n$, besides the simple identity (A.5), the identity (B.3) should be used.

Finally, for the third domain $-n \le l < k \le 0$, the recurrence relations (4.14) is proved in several steps, involving the two particular cases corresponding to $k=0$ and $k=-1$. Among the recurrence relations corresponding to this domain there are the so called shifted recurrence relations, when the sets of Bethe parameters $\bar t^s$ are extended by the shifted parameter $z_s = z - c(s-1/2)$.

- $\mathcal{Z}^1_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^0_{-n} \cdot \mathbb{B}(\bar t)$. This step is simple. No complicated identities should be used, but it fixes the function $\phi_0(z;\bar t)$ in (4.9).
- $\mathcal{Z}^0_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^{-1}_{-n} \cdot \mathbb{B}(\bar t)$. Here the proof relies on the identities (B.4) and an identity equivalent to (B.5).
- For $\mathcal{Z}^{-1}_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_{-n} \cdot \mathbb{B}(\bar t)$, $-n < k < -1$, we need identities which appear to be equivalent to (B.6) and (B.7).
- $\mathcal{Z}^k_{-n} \cdot \mathbb{B}(\bar t) \to \mathcal{Z}^k_\ell \cdot \mathbb{B}(\bar t)$, $-n \le \ell < k \le 0$. This final step will require identities (B.3).

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