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Prolongation of $(8, 15)$-Distribution of Type F_4 by Singular Curves

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Abstract. Cartan gives the model of $(8, 15)$-distribution with the exceptional simple Lie algebra F_4 as its symmetry algebra in his paper (1893), which is published one year before his thesis. In the present paper, we study abnormal extremals (singular curves) of Cartan's model from viewpoints of sub-Riemannian geometry and geometric control theory. Then we construct the prolongation of Cartan's model based on the data related to its singular curves, and obtain the nilpotent graded Lie algebra which is isomorphic to the negative part of the graded Lie algebra F_4.

Key words: exceptional Lie algebra; singular curve; constrained Hamiltonian equation

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1 Introduction

Let M be a manifold of dimension 15 and $D \subset TM$ a distribution, i.e., a vector subbundle of the tangent bundle TM of rank 8. Then D is called an $(8, 15)$-distribution if $\mathcal{D} + [\mathcal{D}, \mathcal{D}] = \mathcal{TM}$ for the sheave $\mathcal{D}$ (resp. $\mathcal{TM}$) of local sections to D (resp. TM). In this paper, we study a special class of $(8, 15)$-distributions related to the simple Lie group F_4.

Distributions are important subjects in manifold theory and global analysis. They are studied also related to the theory of Lie groups, Lie algebras and their representations. Then the theory of prolongations and equivalence problems of distributions are established by many authors (see, for instance, [10, 34, 35, 38]). On symmetries for distributions, there are well-known several powerful and beautiful methods to investigate, based on differential geometry and representation theory; Cartan's prolongation, Tanaka's prolongation, and Kostant's theorem on Bott–Borel–Weil theory and so on [14, 21, 23, 31, 30, 38, 39, 40].

We provide, in this paper, a way of prolongations of $(8, 15)$-distributions of type F_4 via the notion of *abnormal extremals* or *singular curves* and related objects from viewpoints of sub-Riemannian geometry and geometric control theory [3, 32, 33, 34] which recovers several results explicitly. The relations of our constructions with those by the method of representation theory are presented in Remark 4.4 of Section 4 in our paper.

The singular curves or abnormal extremals are extensively used to study distributions by many authors (see, for instance, [4, 11, 16, 17]). In the previous papers (see [24, 25, 27]), we study $(2, 3, 5)$-distributions or Cartan distributions [9, 14] using singular curves. Here a $(2, 3, 5)$-distribution means a distribution D of rank 2 on a 5-dimensional manifold M such that $\mathcal{D}^{(2)} := \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ becomes the sheaf of local sections of a distribution $D^{(2)}$ of rank 3 and that $\mathcal{TM} = \mathcal{D}^{(3)} := \mathcal{D}^{(2)} + \left[\mathcal{D}, \mathcal{D}^{(2)}\right]$. Then we show the prolongation using the cone of singular curves of any $(2,3,5)$-distribution has the nilpotent gradation algebra which is isomorphic to the negative part of the graded simple Lie algebra G_2. Note that the prolongation procedure is a partial case of twistor construction in the general framework of parabolic geometry [6, 12].

In his book [34] on sub-Riemannian geometry, Montgomery gives expositions on $(4,7)$-distributions. In particular, Montgomery classifies $(4,7)$-distributions into elliptic, hyperbolic and parabolic $(4,7)$-distributions and shows the non-existence of non-trivial singular curves for elliptic $(4,7)$-distribution. Moreover, he develops Cartan's approach for $(4,7)$-distributions and studies their symmetry groups. In the previous paper [26], we study hyperbolic $(4,7)$-distributions and their prolongations via the cone of singular curves. Then we observe, contrary to the case of $(2,3,5)$-distributions, the isomorphism classes of the nilpotent graded Lie algebra of prolongations are never unique and then we specifies the class of C_3-$(4,7)$-distributions by the condition that the graded algebra associated to the $(4,7)$-distribution after prolongation is isomorphic to the negative part of the simple Lie algebra C_3.

Cartan, in his paper [14] which is published one year before his thesis [13], gives the model of $(8,15)$-distribution whose infinitesimal symmetry algebra is the simple Lie algebra F_4. The purpose of the present paper is to study Cartan's model of $(8,15)$-distribution from viewpoints of sub-Riemannian geometry and geometric control theory. We construct its prolongation using the data related to abnormal or singular curves, and verify that the prolonged nilpotent graded algebra obtained by our method is isomorphic to the negative part of the simple Lie algebra F_4.

Note that the complex simple Lie algebra F_4 has three real forms; one *compact* type and two *non-compact* types denoted as $F_{4(4)}$ and as $F_{4(-20)}$ (see [14, 13, 15, 18, 29]). In [21], $F_{4(4)}$ (resp. $F_{4(-20)}$) is denoted by F_4I (resp. F_4II), and in [20], by $\widetilde{F}_4$ (resp. F_4'). Cartan's model, which we treat in the present paper, gives the $(8,15)$-distribution corresponding to $F_{4(4)}$, which maybe called the "hyperbolic" F_4-$(8,15)$-distribution. Nurowski [36] has given the explicit models of $(8,15)$-distributions of type F_4 and $(16,24)$-distributions of type E_6. Though we do not touch the details here, it can be observed that the real $(8,15)$-distribution of type $F_{4(-20)}$ in Nurowski's normal form has the canonical definite conformal metric and it has no nontrivial singular curves (cf. Sections 3 and 4 of this paper). Thus Nurowski's $(8,15)$-distribution of type $F_{4(-20)}$ can be called "elliptic" F_4-$(8,15)$-distribution. Refer [36] also for related references and historical remarks. Note also that both $(8,15)$-distributions of type $F_{4(4)}$ and $F_{4(-20)}$ appear, as two cases of real simple Lie algebras, in the classification of certain sub-Riemannian structures in [5, 19].

In Section 2, we recall Cartan's model $\left(\mathbb{K}^{15}, D\right)$ of $(8,15)$-distribution associated to the simple Lie algebra F_4. The basics on sub-Riemannian geometry and geometric control theory which we need in this paper are given in Section 3. We study the singular curves of Cartan's model and show that there exist canonically the conformal metrics on $D \subset T\mathbb{K}^{15}$ and on $D^{\perp} \subset T^*\mathbb{K}^{15}$ in Section 4. In Section 5, we construct null-flag manifold of dimension 9 which prolongs $\left(\mathbb{K}^{15}, D\right)$ to (W, E) so that $\dim(W) = 24$ and E is of rank 4. In Section 6, we show that E has the small growth vector $(4,7,10,13,16,18,20,21,22,23,24)$ and the gradation algebra of E is isomorphic to the negative part of the simple graded Lie algebra with respect to filtration defined by the set of all roots of F_4. In Section 7, we introduce the class of $(8,15)$-distributions of type F_4 regarding the arguments of previous sections and show that also the gradation algebra of the prolongation of any $(8,15)$-distributions of type F_4 is isomorphic to the negative part of the simple Lie algebra F_4 with respect to the filtration defined by the set of all roots of F_4 (Theorem 7.3).

In this paper, all manifolds and maps are supposed to be of class C^{∞} unless otherwise stated.

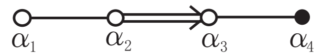
2 Cartan's model of $(8,15)$-distributions of type F_4

We recall Cartan's model of $(8,15)$-distribution [14, 40] which has, as the infinitesimal symmetries, the simple Lie algebra F_4:

The Dynkin diagram of F_4.

As for the exceptional Lie algebra F_4, see, for instance, also [1, 2, 8, 22, 37].

The model of $(8,15)$-distributions found by Cartan is derived from the homogeneous space by the parabolic subgroup of the simple Lie group F_4 which corresponds to $\{\alpha_4\}$ for the simple roots α_1, α_2, α_3, α_4 [14, 40]:



Here we have simply marked the corresponding root in black to the parabolic subgroup, which not meant, say, the Satake diagram. Note that, as the standard way, a cross under the node can be used to indicate a parabolic subgroup as in [6].

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. On $\mathbb{K}^{15}$ with the system of coordinates z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, x_{ij}, $1 \leq i < j \leq 4$, and consider the C^∞ (resp. holomorphic) 1-forms

$$\begin{aligned}
&\omega = \mathrm{d}z - y_1\mathrm{d}x_1 - y_2\mathrm{d}x_2 - y_3\mathrm{d}x_3 - y_4\mathrm{d}x_4,\\
&\omega_{ij} = \mathrm{d}x_{ij} - (x_i\mathrm{d}x_j - x_j\mathrm{d}x_i + y_h\mathrm{d}y_k - y_k\mathrm{d}y_h), \qquad 1 \leq i < j \leq 4,
\end{aligned}$$

where (i,j,h,k) is an even permutation of $(1,2,3,4)$. Let

$$Z,\ X_{12},\ X_{13},\ X_{14},\ X_{23},\ X_{24},\ X_{34},\ X_1,\ X_2,\ X_3,\ X_4,\ Y_1,\ Y_2,\ Y_3,\ Y_4$$

be the dual frame of $T\mathbb{K}^{15}$ to the frame

$$\omega,\ \omega_{12},\ \omega_{13},\ \omega_{14},\ \omega_{23},\ \omega_{24},\ \omega_{34},\ \mathrm{d}x_1,\ \mathrm{d}x_2,\ \mathrm{d}x_3,\ \mathrm{d}x_4,\ \mathrm{d}y_1,\ \mathrm{d}y_2,\ \mathrm{d}y_3,\ \mathrm{d}y_4$$

of $T^*\mathbb{K}^{15}$. Then $D \subset T\mathbb{K}^{15}$ is defined as the distribution generated by X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4. Explicitly the distribution $D \subset T\mathbb{K}^{15}$ has the system of generators

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial x_1} + y_1\frac{\partial}{\partial z} - x_2\frac{\partial}{\partial x_{12}} - x_3\frac{\partial}{\partial x_{13}} - x_4\frac{\partial}{\partial x_{14}},\\
&X_2 = \frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{12}} - x_3\frac{\partial}{\partial x_{23}} - x_4\frac{\partial}{\partial x_{24}},\\
&X_3 = \frac{\partial}{\partial x_3} + y_3\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{13}} + x_2\frac{\partial}{\partial x_{23}} - x_4\frac{\partial}{\partial x_{34}},\\
&X_4 = \frac{\partial}{\partial x_4} + y_4\frac{\partial}{\partial z} + x_1\frac{\partial}{\partial x_{14}} + x_2\frac{\partial}{\partial x_{24}} + x_3\frac{\partial}{\partial x_{34}},\\
&Y_1 = \frac{\partial}{\partial y_1} - y_4\frac{\partial}{\partial x_{23}} + y_3\frac{\partial}{\partial x_{24}} - y_2\frac{\partial}{\partial x_{34}},\\
&Y_2 = \frac{\partial}{\partial y_2} + y_4\frac{\partial}{\partial x_{13}} - y_3\frac{\partial}{\partial x_{14}} + y_1\frac{\partial}{\partial x_{34}},\\
&Y_3 = \frac{\partial}{\partial y_3} - y_4\frac{\partial}{\partial x_{12}} + y_2\frac{\partial}{\partial x_{14}} - y_1\frac{\partial}{\partial x_{24}},\\
&Y_4 = \frac{\partial}{\partial y_4} + y_3\frac{\partial}{\partial x_{12}} - y_2\frac{\partial}{\partial x_{13}} + y_1\frac{\partial}{\partial x_{23}}.
\end{aligned}$$

Moreover, we have that $Z = \frac{\partial}{\partial z}$ and $X_{ij} = \frac{\partial}{\partial x_{ij}}$, $1 \le i < j \le 4$. Then we get the following bracket relations:

$$\begin{aligned}
&[X_1, X_2] = 2X_{12}, \quad [X_1, X_3] = 2X_{13}, \quad [X_1, X_4] = 2X_{14},\\
&[X_2, X_3] = 2X_{23}, \quad [X_2, X_4] = 2X_{24},\\
&[X_3, X_4] = 2X_{34},\\
&[Y_1, Y_2] = 2X_{34}, \quad [Y_1, Y_3] = -2X_{24}, \quad [Y_1, Y_4] = 2X_{23},\\
&[Y_2, Y_3] = 2X_{14}, \quad [Y_2, Y_4] = -2X_{13},\\
&[Y_3, Y_4] = 2X_{12},\\
&[Y_1, X_1] = [Y_2, X_2] = [Y_3, X_3] = [Y_4, X_4] = Z, \quad [Y_i, X_j] = 0, \quad i \ne j.
\end{aligned}$$

Moreover, we have

$$[X_i, X_{jk}] = 0, \quad [Y_i, X_{jk}] = 0, \quad [X_i, Z] = 0, \quad [Y_i, Z] = 0 \quad \text{for any } i,\ j,\ k.$$

Remark 2.1. We set, for $1 \le i < j \le 4$, a sub-distribution $D_{ij} = \langle X_i, X_j, Y_h, Y_k, X_{ij} \rangle$ of $D^{(2)}$, where (i, j, h, k) is a permutation of $(1, 2, 3, 4)$. Then we see each D_{ij} is completely integrable and each leaf of the foliation induced by D_{ij} of $\mathbb{K}^{15}$ has a contact structure. Thus we have six contact foliations in $\mathbb{K}^{15}$. For example, for $i = 1$, $j = 2$, then the contact foliation is given by the Pfaff system

$$\begin{aligned}
&\mathrm{d}z - y_1\mathrm{d}x_1 - y_2\mathrm{d}x_2 = 0, \quad \mathrm{d}x_3 = 0, \quad \mathrm{d}x_4 = 0, \quad \mathrm{d}y_1 = 0, \quad \mathrm{d}y_2 = 0,\\
&\mathrm{d}x_{13} + x_3\mathrm{d}x_1 + y_2\mathrm{d}y_4 = 0, \quad \mathrm{d}x_{14} + x_4\mathrm{d}x_1 - y_2\mathrm{d}y_3 = 0, \quad \mathrm{d}x_{23} + x_3\mathrm{d}x_2 + y_1\mathrm{d}y_4 = 0,\\
&\mathrm{d}x_{24} + x_4\mathrm{d}x_2 + y_1\mathrm{d}y_3 = 0, \quad \mathrm{d}x_{34} = 0,
\end{aligned}$$

and with the 1-form

$$\mathrm{d}x_{12} + x_2\mathrm{d}x_1 - x_1\mathrm{d}x_2 + y_4\mathrm{d}y_3 - y_3\mathrm{d}y_4,$$

which gives a contact structure on each leaf of the foliation defined by D_{12}.

3 Abnormal bi-extremals and singular curves of distributions

Here we recall several notions in geometric control theory and sub-Riemannian geometry. For details, consult, for instance, the references [3, 7, 33, 34].

Let M be a real C^∞ manifold, $D \subset TM$ a distribution endowed with a positive definite metric $g\colon D \otimes D \to \mathbb{R}$ on a manifold M, and $\gamma\colon [a, b] \to M$ an absolutely continuous curve satisfying $\dot\gamma(t) \in D_{\gamma(t)}$ for almost all $t \in I$, which is called a *D-integral curve*. Then the arc-length of γ is defined by $L(\gamma) := \int_a^b \sqrt{g(\dot\gamma(t), \dot\gamma(t))}\mathrm{d}t$. A curve γ is called a *D-geodesic* if it minimises the arc-length locally.

Let $\mathrm{rank}(D) = r$ and, just for simplicity, $\xi_1, \dots, \xi_r$ be an orthonormal frame of (D, g) over M. Then we define $F\colon D \cong M \times \mathbb{R}^r \to TM$ by $F(x, u) = \sum_{i=1}^r u_i\xi_i(x)$.

Consider the optimal control problem for the energy function on D defined by

$$e = \frac{1}{2} g\left(\sum_{i=1}^r u_i\xi_i(x),\ \sum_{i=1}^r u_i\xi_i(x)\right) = \frac{1}{2}\sum_{i=1}^r u_i(t)^2.$$

Note that the problem of minimising arc-length and that of minimising energy function are known to be equivalent up to re-parametrisations [3, 34]. Then the Hamiltonian function on $(D \times_M T^*M) \times \mathbb{R}$ is given by

$$H(x, p, u, p^0) = \left\langle p, \sum_{i=1}^r u_i\xi_i(x) \right\rangle + p^0\left(\frac{1}{2}\sum_{i=1}^r u_i^2\right).$$

Here $D\times_M T^*M=\{(x,u),(x',p)\in D\times T^*M \mid x=x'\}\cong T^*M\times\mathbb{R}^r$ and p^0 is an additional parameter.

Regarding the optimal control problem for minimising the energy function of D-integrable curves, we have, by Pontryagin's maximum principle, if γ is a D-geodesic, then, for $\dot\gamma(t)=(x(t),u(t))$, there exists a Lipschitz curve $(x(t),p(t))\in T^*M$ and non-positive constant $p^0\le 0$ such that the following constrained Hamiltonian equation in terms of $H=H\big(x,p,u,p^0\big)$ is satisfied:

$$\dot x_i(t)=\frac{\partial H}{\partial p_i}\big(x(t),p(t),u(t),p^0\big),\qquad 1\le i\le m,$$
$$\dot p_i(t)=-\frac{\partial H}{\partial x_i}\big(x(t),p(t),u(t),p^0\big),\qquad 1\le i\le m,$$

with constraints $\frac{\partial H}{\partial u_j}\big(x(t),p(t),u(t),p^0\big)=0$, $1\le j\le r$, $\big(p(t),p^0\big)\ne 0$.

If $p^0<0$, then the curve $(x(t),p(t))$ (resp. $x(t)$) of a solution of the above constrained Hamiltonian equation is called a *normal bi-extremal* (resp. *normal extremal*) respectively. If $p^0=0$, then bi-extremals and extremals are called *abnormal*. Note that the notion of abnormal (bi-)extremals is independent of the metric g on D and depends only on the distribution D.

The constraint $\frac{\partial H}{\partial u_j}=0$ is equivalent to that $p^0u_j=-\langle p,\xi_j(x)\rangle$. In the normal case, i.e., $p^0<0$, we have $u_j=-\frac{1}{p^0}\langle p,\xi_j(x)\rangle$. Because the Hamiltonian is linear on $\big(p,p^0\big)$, by normalising as $p^0=-1$, we have $H=\frac12\sum_{i=1}^r\langle p,\xi_i(x)\rangle^2$.

For abnormal extremals, the constrained Hamiltonian equation reads as

$$\dot x=\ u_1\xi_1(x)+u_2\xi_2(x)+\cdots+u_r\xi_i(x),$$
$$\dot p=-\left(u_1\frac{\partial H_{\xi_1}}{\partial x}+u_2\frac{\partial H_{\xi_2}}{\partial x}+\cdots+u_r\frac{\partial H_{\xi_r}}{\partial x}\right),$$

with constraints $H_{\xi_1}=0,H_{\xi_2}=0,\dots,H_{\xi_r}=0$ and $p\ne0$, where $H_{\xi_i}(x,p):=\langle p,\xi_i(x)\rangle$.

Given a distribution $D\subset TM$, for any $x\in M$, we define the subbundle $D^\perp\subset T^*M$ by

$$D_x^\perp:=\{\alpha\in T_x^*M\mid\langle\alpha,v\rangle=0,\ \text{for any } v\in D_x\}.$$

Then the above constraints mean that $p(t)\in D^\perp_{x(t)}$.

The notion of abnormal extremals coincides with that of singular curves, i.e., critical points of the end-point mapping [33, 34]. Let $x_0\in M$ and $I=[a,b]$ an interval. Let Ω be the set of Lipschitz continuous curves $\gamma\colon I\to M$ with $\dot\gamma(t)\in D_{\gamma(t)}$ for almost all $t\in I$, which is called a D-integral curve, and $\gamma(a)=x_0$. Then the *endpoint mapping* $\mathrm{End}\colon\Omega\to M$ is defined by $\mathrm{End}(\gamma):=\gamma(b)$. A curve $\gamma\in\Omega$ is called a *D-singular curve* if γ is a critical point of the endpoint mapping, i.e., the differential map $\mathrm{d}_\gamma\mathrm{End}\colon T_\gamma\Omega\to T_{\gamma(b)}M$ is not surjective, for an appropriate manifold structure of Ω (and M).

We introduce the key notion of the present paper.

Definition 3.1. We define the *singular velocity cone* $\mathrm{SVC}(D)\subset TM$ of a distribution $D\subset TM$ by the set of tangent vectors $v\in T_xM$, $x\in M$ such that there exists a D-singular curve $\gamma\colon(\mathbb{R},0)\to(M,x)$ with $\gamma'(0)=v$.

Note that $\mathrm{SVC}(D)$ is a cone field over M, i.e., $\mathrm{SVC}(D)$ is invariant under the fibrewise $\mathbb{R}^\times$-multiplication on TM.

The following lemma is used in the following sections. We have given a proof using coordinates to make sure.

Lemma 3.2 ([3] and [7, Section 4.2])**.** *For a distribution D generated by $\xi_1,\dots,\xi_r$, we have, along abnormal bi-extremals $(x(t),p(t))$ and corresponding $u(t)$, that*

$$\frac{\mathrm{d}}{\mathrm{d}t}H_{\xi_i}(t)=\sum_{j=1}^r u_j(t)H_{[\xi_i,\xi_j]}(t),\qquad 1\le i\le r.$$

Proof. We put $p = \sum_{j=1}^{r} p_j \mathrm{d}x_j$ and $\xi_i = \sum_{k=1}^{r} \xi_{ik} \frac{\partial}{\partial x_k}$. Then $H(x,p,u) = \sum_{1\le i,j\le r} u_i p_j \xi_{ij}(x)$ and $H_{\xi_i} = \sum_{j=1}^{r} p_j \xi_{ij}(x)$. By the Hamiltonian equation, for $1 \le i \le r$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} H_{\xi_i}(t) &= \sum_{j=1}^{r} (p_j' \xi_{ij} + p_j \xi_{ij}') = \sum_{j=1}^{r} \left(p_j' \xi_{ij} + \sum_{\ell=1}^{r} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} x_\ell' \right) \\ &= \sum_{j=1}^{r} \left(-\frac{\partial H}{\partial x_j} \xi_{ij} + \sum_{\ell=1}^{r} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} \frac{\partial H}{\partial p_\ell} \right) = -\sum_{k\ell j} u_k p_\ell \frac{\partial \xi_{k\ell}}{\partial x_j} \xi_{ij} + \sum_{j\ell k} p_j \frac{\partial \xi_{ij}}{\partial x_\ell} u_k \xi_{k\ell} \\ &= -\sum_{k\ell j} u_k p_\ell \frac{\partial \xi_{k\ell}}{\partial x_j} \xi_{ij} + \sum_{\ell jk} p_\ell \frac{\partial \xi_{i\ell}}{\partial x_j} u_k \xi_{kj} = \sum_{k\ell} u_k p_\ell \left(\sum_{j=1}^{r} \left(\xi_{ij} \frac{\partial \xi_{k\ell}}{\partial x_j} - \xi_{kj} \frac{\partial \xi_{i\ell}}{\partial x_j} \right) \right) \\ &= \sum_{k=1}^{r} u_k \langle p, [\xi_i, \xi_k] \rangle = \sum_{j=1}^{r} u_j H_{[\xi_i,\xi_j]}. \end{aligned}$$

■

Remark 3.3. We have defined the notion of abnormal (bi-)extremals and singular curves over the real. In the complex analytic case $\mathbb{K} = \mathbb{C}$, we can (and do) define abnormal (bi-)extremals and singular curves, forgetting about end-point mapping, just by the complex analytic constrained Hamiltonian equation for a complex analytic distribution $D \subset TM$ over a complex analytic manifold M, which is defined similarly as explained in this section.

4 Conformal metric on Cartan's $(8, 15)$-distribution and singular velocity cone

Let us determine the singular curves of Cartan's model $\left(\mathbb{K}^{15}, D\right)$ explained in Section 2. On the cotangent bundle $T^*\mathbb{K}^{15}$ with base coordinates z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, x_{ij}, $1 \le i < j \le 4$ and fiber coordinates s, p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, r_{ij}, $1 \le i < j \le 4$, we have the Hamiltonian of the distribution $D \subset TX$,

$$H = u_1 H_{X_1} + u_2 H_{X_2} + u_3 H_{X_3} + u_4 H_{X_4} + v_1 H_{Y_1} + v_2 H_{Y_2} + v_3 H_{Y_3} + v_4 H_{Y_4},$$

where

$$\begin{aligned} &H_{X_1} = p_1 + y_1 s - x_2 r_{12} - x_3 r_{13} - x_4 r_{14}, && H_{X_2} = p_2 + y_2 s + x_1 r_{12} - x_3 r_{23} - x_4 r_{24}, \\ &H_{X_3} = p_3 + y_3 s + x_1 r_{13} + x_2 r_{23} - x_4 r_{34}, && H_{X_4} = p_4 + y_4 s + x_1 r_{14} + x_2 r_{24} + x_3 r_{34}, \\ &H_{Y_1} = q_1 - y_4 r_{23} + y_3 r_{24} - y_2 r_{34}, && H_{Y_2} = q_2 + y_4 r_{13} - y_3 r_{14} - y_1 r_{34}, \\ &H_{Y_3} = q_3 - y_4 r_{12} + y_2 r_{14} - y_1 r_{24}, && H_{Y_4} = q_4 + y_3 r_{12} - y_2 r_{13} + y_1 r_{23}. \end{aligned}$$

The constrained Hamiltonian equation is given by

$$\begin{cases} \dot{z} = u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 u_4, \\ \dot{x}_1 = u_1, \quad \dot{x}_2 = u_2, \quad \dot{x}_3 = u_3, \quad \dot{x}_4 = u_4, \quad \dot{y}_1 = v_1, \quad \dot{y}_2 = v_2, \quad \dot{y}_3 = v_3, \quad \dot{y}_4 = v_4, \\ \dot{x}_{12} = -x_2 u_1 + x_1 u_2 - y_4 v_3 + y_3 v_4, \quad \dot{x}_{13} = -x_3 u_1 + x_1 u_3 + y_4 v_2 - y_2 v_4, \\ \dot{x}_{14} = -x_4 u_1 + x_1 u_4 - y_3 v_2 + y_2 v_3, \quad \dot{x}_{23} = -x_3 u_2 + x_2 u_3 - y_4 v_1 + y_1 v_4, \\ \dot{x}_{24} = -x_4 u_2 + x_2 u_4 + y_3 v_1 - y_1 v_3, \quad \dot{x}_{34} = -x_4 u_3 + x_3 u_4 - y_2 v_1 + y_1 v_2, \\ \dot{s} = 0, \ \dots \ \dots \\ \dot{p}_1 = -u_2 r_{12} - u_3 r_{13} - u_4 r_{14}, \quad \dot{p}_2 = u_1 r_{12} - u_3 r_{23} - u_4 r_{24}, \\ \dot{p}_3 = u_1 r_{13} + u_2 r_{23} - u_4 r_{34}, \quad \dot{p}_4 = u_1 r_{14} + u_2 r_{24} + u_3 r_{34}, \\ \dot{q}_1 = -u_1 s - v_2 r_{34} + v_3 r_{24} - v_4 r_{23}, \quad \dot{q}_2 = -u_2 s + v_1 r_{34} - v_3 r_{14} + v_4 r_{13}, \\ \dot{q}_3 = -u_3 s - v_1 r_{24} + v_2 r_{14} - v_4 r_{12}, \quad \dot{q}_2 = -u_4 s + v_1 r_{23} - v_2 r_{13} + v_3 r_{12}, \\ \dot{r}_{12} = 0, \quad \dot{r}_{13} = 0, \quad \dot{r}_{14} = 0, \quad \dot{r}_{23} = 0, \quad \dot{r}_{24} = 0, \quad \dot{r}_{34} = 0, \end{cases} \tag{4.1}$$

with constraints

$$\begin{array}{llll} H_{X_1}=0, & H_{X_2}=0, & H_{X_3}=0, & H_{X_4}=0, \\ H_{Y_1}=0, & H_{Y_2}=0, & H_{Y_3}=0, & H_{Y_4}=0, \end{array}$$

and $s(t)$, $p_1(t)$, $p_2(t)$, $p_3(t)$, $p_4(t)$, $q_1(t)$, $q_2(t)$, $q_3(t)$, $q_4(t)$, $r_{ij}(t)$ are not all zero for any t.

By the constraints, if s, r_{ij} are all zero, then p_i, q_j, $1 \le i, j \le 4$ are also zero. So s, r_{ij}, $1 \le i < j \le 4$ must be not all zero.

Remark 4.1. In Cartan's model, we have that s and r_{ij} are locally constant by the Hamiltonian equation. However, we do not use this property in the following arguments.

For instance, from the constraint $H_{X_1} = 0$, we have, along any solution curve by Lemma 3.2, that

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} H_{X_1} = \sum_{i=1}^{4} u_i H_{[X_1,X_i]} + \sum_{j=1}^{4} v_j H_{[X_1,Y_j]}.$$

Then similarly from the constraint, we have the following equality in a general form:

$$\begin{pmatrix} 0 & H_{[X_1,X_2]} & H_{[X_1,X_3]} & H_{[X_1,X_4]} & H_{[X_1,Y_1]} & H_{[X_1,Y_2]} & H_{[X_1,Y_3]} & H_{[X_1,Y_4]} \\ H_{[X_2,X_1]} & 0 & H_{[X_2,X_3]} & H_{[X_2,X_4]} & H_{[X_2,Y_1]} & H_{[X_2,Y_2]} & H_{[X_2,Y_3]} & H_{[X_2,Y_4]} \\ H_{[X_3,X_1]} & H_{[X_3,X_2]} & 0 & H_{[X_3 X_4]} & H_{[X_3,Y_1]} & H_{[X_3,Y_2]} & H_{[X_3,Y_3]} & H_{[X_3,Y_4]} \\ H_{[X_4,X_1]} & H_{[X_4,X_2]} & H_{[X_4 X_3]} & 0 & H_{[X_4,Y_1]} & H_{[X_4,Y_2]} & H_{[X_4,Y_3]} & H_{[X_4,Y_4]} \\ H_{[Y_1,X_1]} & H_{[Y_1,X_2]} & H_{[Y_1,X_3]} & H_{[Y_1,X_4]} & 0 & H_{[Y_1,Y_2]} & H_{[Y_1,Y_3]} & H_{[Y_1,Y_4]} \\ H_{[Y_2,X_1]} & H_{[Y_2,X_2]} & H_{[Y_2,X_3]} & H_{[Y_2,X_4]} & H_{[Y_2,Y_1]} & 0 & H_{[Y_2,Y_3]} & H_{[Y_2,Y_4]} \\ H_{[Y_3,X_1]} & H_{[Y_3,X_2]} & H_{[Y_3,X_3]} & H_{[Y_3,X_4]} & H_{[Y_3,Y_1]} & H_{[Y_3,Y_2]} & 0 & H_{[Y_3,Y_4]} \\ H_{[Y_4,X_1]} & H_{[Y_4,X_2]} & H_{[Y_4,X_3]} & H_{[Y_4,X_4]} & H_{[Y_4,Y_1]} & H_{[Y_4,Y_2]} & H_{[Y_4,Y_3]} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Explicitly, we have in fact

$$\begin{pmatrix} 0 & 2r_{12} & 2r_{13} & 2r_{14} & -s & 0 & 0 & 0 \\ -2r_{12} & 0 & 2r_{23} & 2r_{24} & 0 & -s & 0 & 0 \\ -2r_{13} & -2r_{23} & 0 & 2r_{34} & 0 & 0 & -s & 0 \\ -2r_{14} & -2r_{24} & -2r_{34} & 0 & 0 & 0 & 0 & -s \\ s & 0 & 0 & 0 & 0 & 2r_{34} & -2r_{24} & 2r_{23} \\ 0 & s & 0 & 0 & -2r_{34} & 0 & 2r_{14} & -2r_{13} \\ 0 & 0 & s & 0 & 2r_{24} & -2r_{14} & 0 & 2r_{12} \\ 0 & 0 & 0 & s & -2r_{23} & 2r_{13} & -2r_{12} & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.2}$$

Equivalently, we have

$$\begin{pmatrix} -v_1 & 2u_2 & 2u_3 & 2u_4 & 0 & 0 & 0 \\ -v_2 & -2u_1 & 0 & 0 & 2u_3 & 2u_4 & 0 \\ -v_3 & 0 & -2u_1 & 0 & -2u_2 & 0 & 2u_4 \\ -v_4 & 0 & 0 & -2u_1 & 0 & -2u_2 & -2u_3 \\ u_1 & 0 & 0 & 0 & 2v_4 & -2v_3 & 2v_2 \\ u_2 & 0 & -2v_4 & 2v_3 & 0 & 0 & -2v_1 \\ u_3 & 2v_4 & 0 & -2v_2 & 0 & 2v_1 & 0 \\ u_4 & -2v_3 & 2v_2 & 0 & -2v_1 & 0 & 0 \end{pmatrix} \begin{pmatrix} s \\ r_{12} \\ r_{13} \\ r_{14} \\ r_{23} \\ r_{24} \\ r_{34} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4.3}$$

Write (4.2) as

$$\begin{pmatrix} A_{11} & -sI \\ sI & A_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

where $u = {}^t(u_1, u_2, u_3, u_4)$, $v = {}^t(v_1, v_2, v_3, v_4)$ and I is the 4×4 unit matrix. We denote by A the skew-symmetric 8×8 matrix $\left(\begin{smallmatrix} A_{11} & -sI \\ sI & A_{22} \end{smallmatrix}\right)$ and by U the 8×7 matrix which appeared in (4.2) and (4.3), respectively.

Then the condition (4.2) is equivalent to that $A_{11}u = sv$, $A_{22}v = -su$. Note that $\det(A_{11}) = \det(A_{22}) = \{4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})\}^2$ and that $A_{11}A_{22} = A_{22}A_{11} = -4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})I$. Then the condition (4.2) implies that

$$\big\{s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})\big\}u = 0, \qquad \big\{s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})\big\}v = 0.$$

Therefore, if $(u, v) \neq (0, 0)$, then we have

$$s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23}) = 0.$$

Suppose $s \neq 0$. Then, since A_{11} is skew-symmetric, we have that ${}^tu \cdot v = \frac{1}{s}\,{}^tu \cdot (A_{11}u) = \frac{1}{s}\,({}^tuA_{11}) \cdot u = \frac{1}{s}\,{}^t({}^tA_{11}u)u = -\frac{1}{s}\,{}^t(A_{11}u)u = -{}^tv \cdot u = -{}^tu \cdot v$. Therefore, we have that

$${}^tu \cdot v = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0.$$

Suppose $s = 0$. Then $A_{11}u = 0$ and $A_{22}v = 0$. Note that $A_{11}A_{22} = A_{22}A_{11} = 0$. Since A_{11} and A_{22} are non-zero and skew-symmetric, we have $\operatorname{rank}(A_{11}) = 2$, $\operatorname{rank}(A_{22}) = 2$, and therefore $\operatorname{Ker}(A_{11}) = \operatorname{Im}(A_{22})$ and $\operatorname{Im}(A_{11}) = \operatorname{Ker}(A_{22})$. Then we have $u = A_{22}\tilde{u}$ and $v = A_{11}\tilde{v}$ for some $\tilde{u}$, $\tilde{v}$, and thus ${}^tu \cdot v = {}^t(A_{22}\tilde{u}) \cdot A_{11}\tilde{v} = {}^t\tilde{u}\,{}^tA_{22}A_{11}\tilde{v} = -{}^t\tilde{u}A_{22}A_{11}\tilde{v} = 0$.

Proposition 4.2. *The singular velocity cone* $\mathrm{SVC}(D)$ *of Cartan's model* D *is given by*

$$\mathrm{SVC}(D) = \left\{ \sum_{i=1}^{4} u_iX_i + \sum_{j=1} v_jY_j \;\middle|\; u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4 = 0 \right\}.$$

Proof. That $\mathrm{SVC}(D)$ is contained in the right hand side is already shown. Let us show the converse inclusion. All columns of the 8×7 matrix U which appeared in (4.3) are null and orthogonal to each other with respect to the metric ${}^tu \cdot v = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$ on $\mathbb{K}^8$. Note that the metric is non-degenerate for $\mathbb{K} = \mathbb{C}$ and is of signature $(4, 4)$ if $\mathbb{K} = \mathbb{R}$. In any case we have that $\operatorname{rank}(U) \leq 4 < 7$, because the subspace generated by all columns of U is a null space in $\mathbb{K}^8$ with respect to the metric ${}^tu \cdot v$. Hence, for any non-zero constant vector (u, v) with ${}^tu \cdot v = 0$, there exists $(s, r_{ij}) \neq 0$ such that (4.3) holds, and therefore that (4.2) holds. Thus we see that, given non-zero (u, v) with ${}^tu \cdot v = 0$, there exist constants s, p_i, $1 \leq i \leq 4$, q_j, $1 \leq j \leq 4$, r_{ij}, $1 \leq i < j \leq 4$, which are not all zero, and functions x_i, $1 \leq i \leq 4$, y_j, $1 \leq j \leq 4$, x_{ij}, $1 \leq i < j \leq 4$ such that the linear ordinary differential equation (4.1) for singular curves is satisfied. Thus we see the required equality. ■

We define a quadratic form Q on $\mathbb{K}^8$ and R on $\mathbb{K}^7$, respectively, by

$$Q(u, v) := u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4, \qquad R(s, r_{ij}) := s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23}).$$

The quadratic form Q induces the bi-linear form

$$\big((u, v), (u', v')\big) = \frac{1}{2}\big(u_1v_1' + v_1u_1' + u_2v_2' + v_2u_2' + u_3v_3' + v_3u_3' + u_4v_4' + v_4u_4'\big)$$

on $\mathbb{K}^8 \times \mathbb{K}^8$. Moreover, the quadratic form R induces the bilinear form

$$\big((s, r_{ij})\big|(s', r_{ij}')\big) = ss' - 2\big(r_{12}r_{34}' + r_{34}r_{12}' - r_{13}r_{24}' + r_{24}r_{13}' + r_{14}r_{23}' + r_{23}r_{14}'\big)$$

on $\mathbb{K}^7 \times \mathbb{K}^7$.

Corollary 4.3. *The distribution $D \subset T\mathbb{K}^{15}$ has the canonical non-degenerate metric $(\,,\,)$ for $\mathbb{K}=\mathbb{C}$ and the canonical conformal $(4,4)$-metric $(\,,\,)$ for $\mathbb{K}=\mathbb{R}$. The distribution $D^\perp \subset T^*\mathbb{K}^{15}$ has the canonical non-degenerate metric $(\,|\,)$ for $\mathbb{K}=\mathbb{C}$ and the canonical conformal $(4,3)$-metric $(\,|\,)$ for $\mathbb{K}=\mathbb{R}$.*

Remark 4.4. Let $G = F_{4(4)}$, $P = P_{\alpha_4}$, the parabolic subgroup of $F_{4(4)}$ corresponding to the root α_4, $X = G/P_{\alpha_4} = \mathbb{O}'P_0^2$, that is the hyperplane section of the split Cayley projective space $\mathbb{O}'P^2$ and $H = \mathrm{Spin}(4,3)$. Then we have the decomposition $TG = T_1 \oplus T_2$ into H-modules, where T_1 (resp. T_2) is regarded as the 8-dimensional spin representation of $\mathrm{Spin}(4,3)$; $T_1 \cong \mathbb{O}'$, (resp. the 7-dimensional vector representation; $T_2 \cong \mathrm{Im}\mathbb{O}'$). Moreover, the closed H-orbit $Y_1 \subset \mathbb{P}(T_1)$ (resp. $Y_2 \subset \mathbb{P}(T_2)$) is a 6-dimensional quadric (resp. is a 5-dimensional quadric) with a conformal structure of type $(3,3)$ (resp. of type $(3,2)$) (see [31, Section 6.3]). See also [31, Section 2] and [6, 28] for general constructions in simple Lie algebras.

Consider the Clifford algebra $\mathrm{Cl}(4,3) \supset T_1$. Let $\mathcal{N}$ be the totality of 3-dimensional null subspaces in T_1. We set $N_s := \{z \in T_2 \mid z(s) = 0\}$ for $s \in T_1$. If $N_s \in \mathcal{N}$, s is called a pure spinor. Denote by $\mathrm{PS}(4,3)$ the set of pure spinors and by $\mathbb{P}(\mathrm{PS}(4,3))$ its projectivisation. Then the correspondence $[s] \in \mathbb{P}(\mathrm{PS}(4,3)) \mapsto N_s \in \mathcal{N}$ turns to be an isomorphism. See, for instance, [20, p. 241 and p. 283].

Now in the left hand side of the equality (4.2) in our argument in this section, the action $\boldsymbol{u} = {}^t(u,v) \mapsto A\boldsymbol{u}$ corresponds to the spinor representation of $T_2 \subset \mathrm{Cl}(4,3)$ to T_1. Moreover, we see that the set D of solutions $\boldsymbol{u}$ to the equation $A\boldsymbol{u} = \mathbf{0}$ is exactly equal to the set $\mathrm{PS}(4,3)$ of pure spinors. Thus we see that $D = T_1$ and that $\mathrm{SVC}(D) \cong \widehat{Y}_1 \cong \mathrm{PS}(4,3)$. Therefore, invariant cone $\widehat{Y}_1$ is constructed from $D = T_1$ algebraically from the viewpoint of representation theory. Further $D^\perp = (TX/T_1)^* = T_2^* (\subset T^*X)$ has the H-invariant $(4,3)$-metric. In this paper, we have characterised these objects known in representation theory by using singular curves from the viewpoint of geometric control theory.

5 Null flags associated to abnormal bi-extremals

We continue to analyse the equation (4.3) appeared in the previous section. Recall the 8×7 matrix U which appeared in (4.3). Write $U = \left(\begin{smallmatrix} U' \\ U'' \end{smallmatrix}\right)$ using 4×7 matrices U', U''. Then we have that

$$
{}^tUU = \begin{pmatrix} {}^tU'' & {}^tU' \end{pmatrix}\begin{pmatrix} U' \\ U'' \end{pmatrix} = \begin{pmatrix}
-2Q & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4Q \\
0 & 0 & 0 & 0 & 0 & -4Q & 0 \\
0 & 0 & 0 & 0 & 4Q & 0 & 0 \\
0 & 0 & 0 & 4Q & 0 & 0 & 0 \\
0 & 0 & -4Q & 0 & 0 & 0 & 0 \\
0 & 4Q & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

where $Q = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$. Note that $\det\left({}^tUU\right) = 2^{13}Q^8$.

If $Q \neq 0$, then $\mathrm{rank}(U) = 7$. If $Q = 0$, then, since ${}^tUU = O$, regarding $U\colon \mathbb{K}^7 \to \mathbb{K}^8$ and ${}^tU\colon \mathbb{K}^8 \to \mathbb{K}^7$, we have that $\mathrm{Im}(U) \subseteq \mathrm{Ker}\left({}^tU\right)$, so that $\mathrm{rank}(U) \leq 8 - \mathrm{rank}(U)$. Thus we have $\mathrm{rank}(U) \leq 4$ again. Moreover, if we set $R = s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})$, then we have that if $(u,v) \neq (0,0)$ and $Q = 0$, then $\mathrm{Ker}(U) \subseteq R^{-1}(0)$. So we have $\mathrm{Ker}(U)$ is a null subspace for the non-degenerate metric R' induced by the quadratic form R and that $\dim \mathrm{Ker}(U) \leq 3$. Thus we have, in fact, $\mathrm{rank}(U) = 4$ and $\dim \mathrm{Ker}(U) = 3$, if $(u,v) \neq (0,0)$ and $Q = 0$. Therefore, we observe that, for any (null) line in $Q^{-1}(0)$, there corresponds a null 3-pace in $R^{-1}(0)$. Conversely, for any null 3-space in $R^{-1}(0)$, there corresponds a null line in $Q^{-1}(0)$. However, for any null line in $R^{-1}(0)$, naturally there corresponds, not a null 3-space, but a null 4-space in $Q^{-1}(0)$ by the equation (4.2), since, on $R^{-1}(0) \setminus \{0\}$, we see $\det(A_{11}) \neq 0$ and the matrix A is of rank 4.

In fact we have

Lemma 5.1. *To any null-flag $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset R^{-1}(0)$ for $R = s^2 - 4(r_{12}r_{34} - r_{13}r_{24} + r_{14}r_{23})$, where $\dim(\Lambda_i) = i$, $i = 1, 2, 3$, there corresponds uniquely, by the equation (4.2), a null-flag $V_1 \subset V_2 \subset V_4 \subset Q^{-1}(0)$ for $Q = u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$, where $\dim(V_k) = k$, $k = 1, 2, 4$.*

Proof. The conformal orthogonal group $\mathrm{CO}(R)$ of the quadratic form R acts transitively on the null Grassmannian $\{(\Lambda_1, \Lambda_2, \Lambda_3)\}$ on the metric space $D_m^{\perp} \cong \mathbb{R}^{4,3}$, $m \in \mathbb{K}^{15}$ defined by R. We take the basis of $D_m^{\perp}$: $\varepsilon_1 = \frac{\partial}{\partial s}$, $\varepsilon_2 = \frac{\partial}{\partial r_{12}}$, $\varepsilon_3 = \frac{\partial}{\partial r_{13}}$, $\varepsilon_4 = \frac{\partial}{\partial r_{14}}$, $\varepsilon_5 = \frac{\partial}{\partial r_{23}}$, $\varepsilon_6 = \frac{\partial}{\partial r_{24}}$, $\varepsilon_7 = \frac{\partial}{\partial r_{34}}$. Then the representation matrix of the $(4, 3)$-metric on R becomes

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then we set the base point $\left(\Lambda_1^0, \Lambda_2^0, \Lambda_3^0\right)$ of the null flag manifold $\mathcal{F}'$, where

$$\Lambda_1^0 = \langle \varepsilon_2 \rangle, \qquad \Lambda_2^0 = \langle \varepsilon_2, \varepsilon_3 \rangle, \qquad \Lambda_3^0 = \langle \varepsilon_2, \varepsilon_3, \varepsilon_4 \rangle,$$

We take the frame

$$\begin{aligned} f_1 &= z_{11}\varepsilon_1 + \varepsilon_2 + z_{13}\varepsilon_3 + z_{14}\varepsilon_4 + z_{15}\varepsilon_5 + z_{16}\varepsilon_6 + z_{17}\varepsilon_7, \\ f_2 &= z_{21}\varepsilon_1 + \varepsilon_3 + z_{24}\varepsilon_4 + z_{25}\varepsilon_5 + z_{26}\varepsilon_6 + z_{27}\varepsilon_7, \\ f_3 &= z_{31}\varepsilon_1 + \varepsilon_4 + z_{35}\varepsilon_5 + z_{36}\varepsilon_6 + z_{37}\varepsilon_7, \end{aligned}$$

associated to a (not necessarily null) flag $(\Lambda_1, \Lambda_2, \Lambda_3)$ with $\Lambda_1 = \langle f_1 \rangle$, $\Lambda_2 = \langle f_1, f_2 \rangle$ and $\Lambda_3 = \langle f_1, f_2, f_3 \rangle$ in a neighbourhood of the base point $\left(\Lambda_1^0, \Lambda_2^0, \Lambda_3^0\right)$.

Then the condition that $(\Lambda_1, \Lambda_2, \Lambda_3)$ is a null flag is equivalent to that

$$\begin{aligned} (f_1|f_1) &= z_{11} - 4z_{17} + 4z_{13}z_{16} - 4z_{14}z_{15} = 0, \\ (f_1|f_2) &= z_{11}z_{21} - 2z_{27} + 2z_{13}z_{26} - 2z_{14}z_{25} - 2z_{15}z_{24} + 2z_{16} = 0, \\ (f_1|f_3) &= z_{11}z_{31} - 2z_{37} + 2z_{13}z_{36} - 2z_{14}z_{35} - 2z_{15} = 0, \\ (f_2|f_2) &= z_{21}^2 + 4z_{26} - 4z_{24}z_{25} = 0, \\ (f_2|f_3) &= z_{21}z_{31} + 2z_{36} - 2z_{24}z_{35} - 2z_{25} = 0, \\ (f_3|f_3) &= z_{31}^2 - 4z_{35} = 0. \end{aligned}$$

Thus the null flag manifold $\mathcal{F}'$ has a system of local coordinates $(z_{11}, z_{13}, z_{14}, z_{15}, z_{16}, z_{21}, z_{24}, z_{25}, z_{31})$ and $\dim \mathcal{F}' = 9$. For $\Lambda_1 = \langle f_1 \rangle$, the equation (4.2) is equivalent to that

$$\begin{aligned} &2u_2 + 2z_{13}u_3 + 2z_{14}u_4 - z_{11}v_1 = 0, && -2u_1 + 2z_{15}u_3 + 2z_{16}u_4 - z_{11}v_2 = 0, \\ &-2z_{13}u_1 - 2z_{15}u_2 + 2z_{17}u_4 - z_{11}v_3 = 0, && -2z_{14}u_1 - 2z_{16}u_2 - 2z_{17}u_3 - z_{11}v_4 = 0, \\ &z_{11}u_1 + 2z_{17}v_2 - 2z_{16}v_3 + 2z_{15}v_4 = 0, && z_{11}u_2 - 2z_{17}v_1 + 2z_{14}v_3 - 2z_{13}v_4 = 0, \\ &z_{11}u_3 + 2z_{16}v_1 - 2z_{14}v_2 + 2v_4 = 0, && z_{11}u_4 - 2z_{15}v_1 + 2z_{13}v_2 - 2v_3 = 0, \end{aligned}$$

and, in fact, to that

$$u_1 = z_{15}u_3 + z_{16}u_4 - \frac{1}{2}z_{11}v_2, \qquad u_2 = -z_{13}u_3 - z_{14}u_4 + \frac{1}{2}z_{11}v_1,$$
$$v_3 = \frac{1}{2}z_{11}u_4 - z_{15}v_2 + z_{13}v_2, \qquad v_4 = -\frac{1}{2}z_{11}u_3 - z_{16}v_1 + z_{14}v_2.$$

Thus we see that the solutions form a null 4-space V_4 in $D_m \cong \mathbb{K}^8$

$$V_4 = \left\langle \begin{matrix} {}^t\big(z_{15}, -z_{13}, 1, 0, 0, 0, 0, -\tfrac{1}{2}z_{11}\big), & {}^t\big(z_{16}, -z_{14}, 0, 1, 0, 0, \tfrac{1}{2}z_{11}, 0\big) \\ {}^t\big(0, \tfrac{1}{2}z_{11}, 0, 0, 1, 0, -z_{15}, -z_{16}\big), & {}^t\big(-\tfrac{1}{2}z_{11}, 0, 0, 0, 0, 1, z_{13}, z_{14}\big) \end{matrix} \right\rangle$$

via the frame $(X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4)$. For $\Lambda_2 = \langle f_1, f_2\rangle$, we get the equation (4.2) for f_1 as above and, in addition, the equation (4.2) applied to f_2,

$$\begin{aligned} &2u_3 + 2z_{24}u_4 - z_{21}v_1 = 0, && -2z_{25}u_3 + 2z_{26}u_4 - z_{21}v_2 = 0, \\ &-2u_1 - 2z_{25}u_2 + 2z_{27}u_4 - z_{21}v_3 = 0, && -2z_{24}u_1 - 2z_{26}u_2 - 2z_{27}u_3 - z_{21}v_4 = 0, \\ &z_{21}u_1 + 2z_{27}v_2 - 2z_{26}v_3 + 2z_{25}v_4 = 0, && z_{21}u_2 - 2z_{27}v_1 + 2z_{24}v_3 - 2v_4 = 0, \\ &z_{21}u_3 + 2z_{26}v_1 - 2z_{24}v_2 = 0, && z_{21}u_4 - 2z_{25}v_1 + 2v_2 = 0. \end{aligned}$$

Then, by the two systems of linear equations for f_1 and f_2, we have

$$\begin{aligned} u_1 &= \left(-z_{15}z_{24} + z_{16} + \frac{1}{4}z_{11}z_{21}\right)u_4 + \left(\frac{1}{2}z_{15}z_{21} - \frac{1}{2}z_{11}z_{25}\right)v_1 \\ u_2 &= (z_{13}z_{24} - z_{14})u_4 + \left(-\frac{1}{2}z_{13}z_{21} + \frac{1}{2}z_{11}\right)v_1, \\ u_3 &= -z_{24}u_4 + \frac{1}{2}z_{21}v_1, \\ v_2 &= -\frac{1}{2}z_{21}u_4 + \frac{1}{2}z_{21}v_1 \\ v_3 &= \left(\frac{1}{2}z_{11} - \frac{1}{2}z_{13}z_{21}\right)u_4 + (-z_{15} + z_{13}z_{25})v_1, \\ v_4 &= \left(\frac{1}{2}z_{11}z_{24} - \frac{1}{2}z_{14}z_{21}\right)u_4 + \left(-\frac{1}{4}z_{11}z_{21} - z_{16} + z_{14}z_{25}\right)v_1. \end{aligned}$$

Thus we see Λ_2 corresponds to the null 2-plane V_2 in $D_m \cong \mathbb{K}^8$, by (4.2), generated by two vectors

$$\begin{gathered} {}^t\big(-z_{15}z_{24} + z_{16} + \tfrac{1}{4}z_{11}z_{21}, z_{13}z_{24} - z_{14}, -z_{24}, 1, 0, -\tfrac{1}{2}z_{21}, \tfrac{1}{2}z_{11} - \tfrac{1}{2}z_{13}z_{21}, \tfrac{1}{2}z_{11}z_{24} - \tfrac{1}{2}z_{14}z_{21}\big), \\ {}^t\big(\tfrac{1}{2}z_{15}z_{21} - \tfrac{1}{2}z_{11}z_{25}, -\tfrac{1}{2}z_{13}z_{21} + \tfrac{1}{2}z_{11}, \tfrac{1}{2}z_{21}, 0, 1, z_{25}, -z_{15} + z_{13}z_{25}, -\tfrac{1}{4}z_{11}z_{21} - z_{16} + z_{14}z_{25}\big). \end{gathered}$$

For $\Lambda_3 = \langle f_1, f_2, f_3\rangle$, we obtain the additional condition (4.2) applied to f_3, which is given by

$$\begin{aligned} &2u_4 - z_{31}v_1 = 0, && 2z_{35}u_3 + 2z_{36}u_4 - z_{31}v_2 = 0, \\ &-2z_{35}u_2 + 2z_{37}u_4 - z_{31}v_3 = 0, && -2u_1 - 2z_{36}u_2 - 2z_{37}u_3 - z_{31}v_4 = 0, \\ &z_{31}u_1 + 2z_{37}v_2 - 2z_{36}v_3 + 2z_{35}v_4 = 0, && z_{31}u_2 - 2z_{37}v_1 + 2v_3 = 0, \\ &z_{31}u_3 + 2z_{36}v_1 - 2v_2 = 0, && z_{31}u_4 - 2z_{35}v_1 = 0. \end{aligned}$$

Then, from the conditions (4.2) for f_1, f_2 and f_3, we have

$$u_1 = \left(-\frac{1}{2}z_{11}z_{25} + \frac{1}{2}z_{16}z_{31} + \frac{1}{8}z_{11}z_{21}z_{31} - \frac{1}{2}z_{15}z_{24}z_{31} + \frac{1}{2}z_{15}z_{21}\right)v_1,$$

$$u_2 = \left(\frac{1}{2}z_{11} - \frac{1}{2}z_{13}z_{21} - \frac{1}{2}z_{14}z_{31} + \frac{1}{2}z_{13}z_{24}z_{31}\right)v_1,$$
$$u_3 = \left(\frac{1}{2}z_{21} - \frac{1}{2}z_{24}z_{31}\right)v_1,$$
$$u_4 = \frac{1}{2}z_{31}v_1,$$
$$v_2 = \left(z_{25} - \frac{1}{4}z_{21}z_{31}\right)v_1,$$
$$v_3 = \left(-z_{15} + \frac{1}{4}z_{11}z_{31} + z_{13}z_{25} - \frac{1}{4}z_{13}z_{21}z_{31}\right)v_1,$$
$$v_4 = \left(-z_{16} - \frac{1}{4}z_{11}z_{21} + z_{14}z_{25} + \frac{1}{4}z_{11}z_{24}z_{31} - \frac{1}{4}z_{14}z_{21}z_{31}\right)v_1.$$

Therefore, if we set, by taking $v_1 = 1$,

$$\eta_1 = \begin{pmatrix} -\frac{1}{2}z_{11}z_{25} + \frac{1}{2}z_{16}z_{31} + \frac{1}{8}z_{11}z_{21}z_{31} - \frac{1}{2}z_{15}z_{24}z_{31} + \frac{1}{2}z_{15}z_{21} \\ \frac{1}{2}z_{11} - \frac{1}{2}z_{13}z_{21} - \frac{1}{2}z_{14}z_{31} + \frac{1}{2}z_{13}z_{24}z_{31} \\ \frac{1}{2}z_{21} - \frac{1}{2}z_{24}z_{31} \\ \frac{1}{2}z_{31} \\ 1 \\ z_{25} - \frac{1}{4}z_{21}z_{31} \\ -z_{15} + \frac{1}{4}z_{11}z_{31} + z_{13}z_{25} - \frac{1}{4}z_{13}z_{21}z_{31} \\ -z_{16} - \frac{1}{4}z_{11}z_{21} + z_{14}z_{25} + \frac{1}{4}z_{11}z_{24}z_{31} - \frac{1}{4}z_{14}z_{21}z_{31} \end{pmatrix},$$

then we see that Λ_3 corresponds to the null line V_1 generated by η_1. Moreover, we set η_2 by ${}^t\big(z_{16} + \frac{1}{4}z_{11}z_{21} - z_{15}z_{24}, -z_{14} + z_{13}z_{24}, -z_{24}, 1, 0, -\frac{1}{2}z_{21}, \frac{1}{2}z_{11} - \frac{1}{2}z_{13}z_{21}, \frac{1}{2}z_{11}z_{24} - \frac{1}{2}z_{14}z_{21}\big)$, η_3 by ${}^t(z_{15}, -z_{13}, 1, 0, 0, 0, 0, -\frac{1}{2}z_{11})$, and η_4 by ${}^t\big(-\frac{1}{2}z_{11}, 0, 0, 0, 0, 1, z_{13}, z_{14}\big)$. Then we have that $(\eta_1, \eta_2, \eta_3, \eta_4)$ is a frame of V_4 satisfying $V_1 = \langle\eta_1\rangle \subset V_2 = \langle\eta_1, \eta_2\rangle \subset V_4 = \langle\eta_1, \eta_2, \eta_3, \eta_4\rangle$. ■

Remark 5.1. The total null flag bundle $\widetilde{\mathcal{F}}$ constructed from D which consists of all null flags $V_1 \subset V_2 \subset V_4 \subset D_m \cong \mathbb{R}^{4,4}$, $m \in \mathbb{K}^{15}$ is of dimension $15 + 11 = 26$. The total null flag bundle $\widetilde{\mathcal{F}}'$ constructed from $D^\perp$ which consists of $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset D_m^\perp \cong \mathbb{R}^{4,3}$, $m \in \mathbb{K}^{15}$, is of dimension $15 + 9 = 24$. Then we have obtained, as above, the embedding $\widetilde{\mathcal{F}}' \to \widetilde{\mathcal{F}}$ of codimension 2.

6 Prolongation of Cartan's model

The theory of prolongations and equivalence problems of distributions are established by many authors (see, for instance, [10, 34, 35, 38]). Here we provide, related to the notion of singular curves of distributions, a way of prolongations from viewpoints of sub-Riemannian geometry and geometric control theory.

We set, as the prolonged space, $W = \widetilde{\mathcal{F}}' \cong \mathbb{K}^{15} \times \mathcal{F}'$ in $\widetilde{\mathcal{F}} \cong \mathbb{K}^{15} \times \mathcal{F}$ by the null flag manifold $\mathcal{F}'$. Note that $\dim(\mathcal{F}') = 9$ and that $\dim(W) = 24$: W has a local coordinate system

$$(z, x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, x_{12}, x_{13}, x_{14}, x_{23}, x_{24}, x_{34}; z_{11}, z_{13}, z_{14}, z_{15}, z_{16}, z_{21}, z_{24}, z_{25}, z_{31}).$$

We are going to define and study the canonical distribution E on $W = \widetilde{\mathcal{F}}'$.

Take any point $w_0 = (m_0, (V_1)_0, (V_2)_0, (V_4)_0)$ of W. Then we define $E_{w_0} \subset T_{w_0}W$ as the set of initial vectors $\big(m'(0), V_1'(0), V_2'(0), V_4'(0)\big)$ of curves $(m(t), V_1(t), V_2(t), V_4(t))\colon (\mathbb{R}, 0) \to W$ in W which satisfy the condition $m'(t) \in V_1(t)$, $\eta_1'(t) \in V_2(t)$, $\eta_2'(t) \in V_4(t)$ for some (so equivalently for any) framing $V_1(t) = \langle\eta_1(t)\rangle$, $V_2(t) = \langle\eta_1(t), \eta_2(t)\rangle$.

Now we calculate the canonical distribution E explicitly. The above condition for $E \subset TW$ reads, at $t=0$, that

$$\begin{aligned} &m'(0) = p\eta_1(m(0)), \qquad \eta_1'(0) = q\eta_1(m(0)) + r\eta_2(m(0)), \\ &f_2'(0) = s\eta_1(m(0)) + u\eta_2(m(0)) + v\eta_3(m(0)) + w\eta_4(m(0)), \end{aligned}$$

for some $p, q, r, s, u, v, w \in \mathbb{R}$.

By the above second condition $\eta_1' = q\,\eta_1 + r\,\eta_2$ at $t=0$, we see $q=0$ and $\frac{1}{2}z_{31}' = r$ at $t=0$. Moreover, after some straightforward calculations, we have

$$\begin{aligned} &z_{11}' - z_{21}z_{13}' - z_{13}z_{21}' - z_{31}z_{14}' + z_{24}z_{31}z_{13}' + z_{13}z_{31}z_{24}' = 0, \\ &z_{21}' - z_{31}z_{24}' = 0, \qquad z_{25}' - \frac{1}{4}z_{31}^2 z_{24}' = 0, \\ &z_{15}' - \frac{1}{4}z_{31}z_{11}' - z_{25}z_{13}' - z_{13}z_{25}' + \frac{1}{4}z_{21}z_{31}z_{13}' + \frac{1}{4}z_{13}z_{31}z_{21}' = 0, \\ &z_{16}' + \frac{1}{4}z_{21}z_{11}' + \frac{1}{4}z_{11}z_{21}' - z_{25}z_{14}' - z_{14}z_{25}' - \frac{1}{4}z_{24}z_{31}z_{11}' - \frac{1}{4}z_{11}z_{31}z_{24}' \\ &\qquad + \frac{1}{4}z_{21}z_{31}z_{14}' + \frac{1}{4}z_{14}z_{31}z_{21}' = 0, \end{aligned}$$

at $t=0$, for the coordinate functions of the curve η_1 on $\mathcal{F}'$. By the above third condition $f_2' = s\,\eta_1 + u\,\eta_2 + v\,\eta_3 + w\,\eta_4$ at $t=0$, we have that $s=u=0$ and that $-z_{24}' = v$, $-\frac{1}{2}z_{21}' = w$ at $t=0$. Moreover, we have that

$$z_{16}' - z_{24}z_{15}' + \frac{1}{4}z_{21}z_{11}' = 0, \qquad z_{14}' - z_{24}z_{13}'0, z_{11}' - z_{21}z_{13}' = 0,$$

at $t=0$. In term of differential 1-forms, the above conditions are reduced to that

$$\begin{aligned} &\mathrm{d}z_{11} - z_{21}\mathrm{d}z_{13} = 0, \qquad \mathrm{d}z_{21} - z_{31}\mathrm{d}z_{24} = 0, \\ &\mathrm{d}z_{14} - z_{24}\mathrm{d}z_{13} = 0, \qquad \mathrm{d}z_{25} - \frac{1}{4}z_{31}^2\mathrm{d}z_{24} = 0, \\ &\mathrm{d}z_{15} - z_{25}\mathrm{d}z_{13} = 0, \qquad \mathrm{d}z_{16} - \left(z_{24}z_{25} - \frac{1}{4}z_{21}^2\right)\mathrm{d}z_{13} = 0, \end{aligned}$$

at $t=0$. To get the frame of E, we set

$$\zeta = A\frac{\partial}{\partial z_{11}} + B\frac{\partial}{\partial z_{13}} + C\frac{\partial}{\partial z_{14}} + D\frac{\partial}{\partial z_{15}} + F\frac{\partial}{\partial z_{16}} + G\frac{\partial}{\partial z_{21}} + H\frac{\partial}{\partial z_{24}} + I\frac{\partial}{\partial z_{25}} + J\frac{\partial}{\partial z_{31}}.$$

The condition that ζ belongs to E is given by

$$\begin{aligned} &A - z_{21}B = 0, \qquad G - z_{31}H = 0, \qquad C - z_{25}B = 0, \\ &I - \frac{1}{4}z_{31}^2 H = 0, \qquad D - z_{25}B = 0, \qquad F - \left(z_{24}z_{25} - \frac{1}{4}z_{21}^2\right)B = 0, \end{aligned}$$

and thus we have, for some $B, H, J \in \mathbb{R}$,

$$\begin{aligned} \zeta = B&\left\{\frac{\partial}{\partial z_{13}} + z_{21}\frac{\partial}{\partial z_{11}} + z_{24}\frac{\partial}{\partial z_{14}} + z_{25}\frac{\partial}{\partial z_{15}} + \left(z_{24}z_{25} - \frac{1}{4}z_{21}^2\right)\frac{\partial}{\partial z_{16}}\right\} \\ &+ H\left(\frac{\partial}{\partial z_{24}} + z_{31}\frac{\partial}{\partial z_{21}} + \frac{1}{4}z_{31}^2\frac{\partial}{\partial z_{25}}\right) + J\frac{\partial}{\partial z_{31}}, \end{aligned}$$

at $t=0$.

Thus, adding the generator which comes from the condition $m'(0) = p\,\eta_1(m(0))$, we have the following lemma.

Lemma 6.1. *We have on the 24-dimensional space $W = \mathbb{R}^{15} \times \mathcal{F}'$ with local coordinates z, x_i, y_j, x_{ij}, $z_{k\ell}$, the prolonged distribution E with the system of generators*

$$\begin{aligned}
\zeta_1 &= \frac{\partial}{\partial z_{13}} + z_{21}\frac{\partial}{\partial z_{11}} + z_{24}\frac{\partial}{\partial z_{14}} + z_{25}\frac{\partial}{\partial z_{15}} + \left(z_{24}z_{25} - \frac{1}{4}z_{21}^2\right)\frac{\partial}{\partial z_{16}},\\
\zeta_2 &= \frac{\partial}{\partial z_{24}} + z_{31}\frac{\partial}{\partial z_{21}} + \frac{1}{4}z_{31}^2\frac{\partial}{\partial z_{25}},\\
\zeta_3 &= \frac{\partial}{\partial z_{31}},\\
\zeta_4 &= \left(-\frac{1}{2}z_{11}z_{25} + \frac{1}{2}z_{15}z_{21} + \frac{1}{2}z_{16}z_{31} + \frac{1}{8}z_{11}z_{21}z_{31} - \frac{1}{2}z_{15}z_{24}z_{31}\right)X_1\\
&\quad + \left(\frac{1}{2}z_{11} - \frac{1}{2}z_{13}z_{21} - \frac{1}{2}z_{14}z_{31} + \frac{1}{2}z_{13}z_{24}z_{31}\right)X_2 + \left(\frac{1}{2}z_{21} - \frac{1}{2}z_{24}z_{31}\right)X_3 + \frac{1}{2}z_{31}X_4\\
&\quad + Y_1 + \left(z_{25} - \frac{1}{4}z_{21}z_{31}\right)Y_2 + \left(-z_{15} + \frac{1}{4}z_{11}z_{31} + z_{13}z_{25} - \frac{1}{4}z_{13}z_{21}z_{31}\right)Y_3\\
&\quad + \left(-z_{16} - \frac{1}{4}z_{11}z_{21} + z_{14}z_{25} + \frac{1}{4}z_{11}z_{24}z_{31} - \frac{1}{4}z_{14}z_{21}z_{31}\right)Y_4.
\end{aligned}$$

Note that the vector field ζ_4 in Lemma 6.1 is induced from η_1 obtained in the previous section. We have chosen the above system of generators regarding the F_4-Dynkin diagram (see Remark 6.3). Now we have the following.

Lemma 6.2. *The growth vector of the distribution E defined in the previous Lemma 6.1 is given by $(4, 7, 10, 13, 16, 18, 20, 21, 22, 23, 24)$ and the following bracket relations for the generators ζ_1, ζ_2, ζ_3, ζ_4 of E given in Lemma 6.1:*

$$\begin{array}{llll}
[\zeta_1,\zeta_2] = \zeta_5, & [\zeta_1,\zeta_3] = 0, & [\zeta_1,\zeta_4] = 0, & [\zeta_2,\zeta_3] = \zeta_6,\\
[\zeta_2,\zeta_4] = 0, & [\zeta_3,\zeta_4] = \zeta_7 \quad \textit{in } E^{(2)}; & & \\
[\zeta_1,\zeta_5] = 0, & [\zeta_1,\zeta_6] = \zeta_8, & [\zeta_1,\zeta_7] = 0, & [\zeta_2,\zeta_5] = 0,\\
[\zeta_2,\zeta_6] = 0, & [\zeta_2,\zeta_7] = \zeta_9, & [\zeta_3,\zeta_5] = -\zeta_8, & [\zeta_3,\zeta_6] = \zeta_{10},\\
[\zeta_3,\zeta_7] = 0, & [\zeta_4,\zeta_5] = 0, & [\zeta_4,\zeta_6] = -\zeta_9, & [\zeta_4,\zeta_7] = 0 \quad \textit{in } E^{(3)};\\
[\zeta_1,\zeta_8] = 0, & [\zeta_1,\zeta_9] = \zeta_{11}, & [\zeta_1,\zeta_{10}] = \zeta_{12}, & [\zeta_2,\zeta_8] = 0,\\
[\zeta_2,\zeta_9] = 0, & [\zeta_2,\zeta_{10}] = 0, & [\zeta_3,\zeta_8] = \zeta_{12}, & [\zeta_3,\zeta_9] = \zeta_{13},\\
[\zeta_3,\zeta_{10}] = 0, & [\zeta_4,\zeta_8] = -\zeta_{11}, & [\zeta_4,\zeta_9] = 0, & [\zeta_4,\zeta_{10}] = -2\zeta_{13} \quad \textit{in } E^{(4)};\\
[\zeta_1,\zeta_{11}] = 0, & [\zeta_1,\zeta_{12}] = 0, & [\zeta_1,\zeta_{13}] = \zeta_{14}, & [\zeta_2,\zeta_{11}] = 0,\\
[\zeta_2,\zeta_{12}] = \zeta_{15}, & [\zeta_2,\zeta_{13}] = 0, & [\zeta_3,\zeta_{11}] = \zeta_{14}, & [\zeta_3,\zeta_{12}] = 0,\\
[\zeta_3,\zeta_{13}] = 0, & [\zeta_4,\zeta_{11}] = 0, & [\zeta_4,\zeta_{12}] = -2\zeta_{14}, & [\zeta_4,\zeta_{13}] = \zeta_{16} \quad \textit{in } E^{(5)};\\
[\zeta_1,\zeta_{14}] = 0, & [\zeta_1,\zeta_{15}] = 0, & [\zeta_1,\zeta_{16}] = 0, & [\zeta_2,\zeta_{14}] = \zeta_{17},\\
[\zeta_2,\zeta_{15}] = 0, & [\zeta_2,\zeta_{16}] = 0, & [\zeta_3,\zeta_{14}] = 0, & [\zeta_3,\zeta_{15}] = 0,\\
[\zeta_3,\zeta_{16}] = 0, & [\zeta_4,\zeta_{14}] = \zeta_{18}, & [\zeta_4,\zeta_{15}] = -2\zeta_{17}, & [\zeta_4,\zeta_{16}] = 0 \quad \textit{in } E^{(6)};\\
[\zeta_1,\zeta_{17}] = 0, & [\zeta_1,\zeta_{18}] = 0, & [\zeta_2,\zeta_{17}] = 0, & [\zeta_2,\zeta_{18}] = \zeta_{19},\\
[\zeta_3,\zeta_{17}] = \zeta_{20}, & [\zeta_3,\zeta_{18}] = 0, & [\zeta_4,\zeta_{17}] = \zeta_{19}, & [\zeta_4,\zeta_{18}] = 0 \quad \textit{in } E^{(7)};\\
[\zeta_1,\zeta_{19}] = 0, & [\zeta_1,\zeta_{20}] = 0, & [\zeta_2,\zeta_{19}] = 0, & [\zeta_2,\zeta_{20}] = 0,\\
[\zeta_3,\zeta_{19}] = \zeta_{21}, & [\zeta_3,\zeta_{20}] = 0, & [\zeta_4,\zeta_{19}] = 0, & [\zeta_4,\zeta_{20}] = \dfrac{1}{2}\zeta_{21} \quad \textit{in } E^{(8)};\\
[\zeta_1,\zeta_{19}] = 0, & [\zeta_1,\zeta_{20}] = 0, & [\zeta_2,\zeta_{19}] = 0, & [\zeta_2,\zeta_{20}] = 0,
\end{array}$$

$$[\zeta_3,\zeta_{19}]=\zeta_{21},\qquad [\zeta_3,\zeta_{20}]=0,\qquad [\zeta_4,\zeta_{19}]=0,\qquad [\zeta_4,\zeta_{20}]=\frac{1}{2}\zeta_{21}\quad \text{in } E^{(8)};$$
$$[\zeta_1,\zeta_{21}]=0,\qquad [\zeta_2,\zeta_{21}]=0,\qquad [\zeta_3,\zeta_{21}]=\zeta_{22},\qquad [\zeta_4,\zeta_{21}]=0\quad \text{in } E^{(9)};$$
$$[\zeta_1,\zeta_{22}]=0,\qquad [\zeta_2,\zeta_{22}]=\zeta_{23},\qquad [\zeta_3,\zeta_{22}]=0,\qquad [\zeta_4,\zeta_{22}]=0\quad \text{in } E^{(10)};$$
$$[\zeta_1,\zeta_{23}]=\zeta_{24},\qquad [\zeta_2,\zeta_{23}]=0,\qquad [\zeta_3,\zeta_{23}]=0,\qquad [\zeta_4,\zeta_{23}]=0\quad \text{in } E^{(11)}=TW,$$

which are calculated explicitly in the proof. In particular, the distribution E is isomorphic to the $(8,15)$-distribution on the quotient space by the parabolic subgroup associated to the root α_4 of F_4 in the complex case (resp. of $F_{4(4)}$ in the real case).

Remark 6.3. Between the simple roots α_1, α_2, α_3, α_4 of F_4 (see, for instance, [8]) and the generators ζ_1, ζ_2, ζ_3, ζ_4 of E, there exists the correspondence $\zeta_i \longleftrightarrow -\alpha_i$, $i=1,2,3,4$,

$$\zeta_5 \longleftrightarrow -(\alpha_1+\alpha_2),\qquad \zeta_6 \longleftrightarrow -(\alpha_2+\alpha_3),$$
$$\zeta_7 \longleftrightarrow -(\alpha_3+\alpha_4),\qquad \zeta_8 \longleftrightarrow -(\alpha_1+\alpha_2+\alpha_3),$$
$$\zeta_9 \longleftrightarrow -(\alpha_2+\alpha_3+\alpha_4),\qquad \zeta_{10} \longleftrightarrow -(\alpha_2+2\alpha_3),$$
$$\zeta_{11} \longleftrightarrow -(\alpha_1+\alpha_2+\alpha_3+\alpha_4),\qquad \zeta_{12} \longleftrightarrow -(\alpha_1+\alpha_2+2\alpha_3),$$
$$\zeta_{13} \longleftrightarrow -(\alpha_2+2\alpha_3+\alpha_4),\qquad \zeta_{14} \longleftrightarrow -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4),$$
$$\zeta_{15} \longleftrightarrow -(\alpha_1+2\alpha_2+2\alpha_3),\qquad \zeta_{16} \longleftrightarrow -(\alpha_2+2\alpha_3+2\alpha_4),$$
$$\zeta_{17} \longleftrightarrow -(\alpha_1+\alpha_2+2\alpha_3+\alpha_4),\qquad \zeta_{18} \longleftrightarrow -(\alpha_1+\alpha_2+2\alpha_3+2\alpha_4),$$
$$\zeta_{19} \longleftrightarrow -(\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4),\qquad \zeta_{20} \longleftrightarrow -(\alpha_1+2\alpha_2+3\alpha_3+\alpha_4),$$
$$\zeta_{21} \longleftrightarrow -(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4),\qquad \zeta_{22} \longleftrightarrow -(\alpha_1+2\alpha_2+4\alpha_3+2\alpha_4),$$
$$\zeta_{23} \longleftrightarrow -(\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4),\qquad \zeta_{24} \longleftrightarrow -(2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4).$$

Proof of Lemma 6.2. In fact, we have for the vector fields ζ_1, ζ_2, ζ_3, ζ_4 in Lemma 6.1:

$$[\zeta_1,\zeta_2]=-\frac{\partial}{\partial z_{14}}-z_{31}\frac{\partial}{\partial z_{11}}-\frac{1}{4}z_{31}^2\frac{\partial}{\partial z_{15}}+\left(-z_{25}+\frac{1}{2}z_{21}z_{31}-\frac{1}{4}z_{24}z_{31}^2\right)\frac{\partial}{\partial z_{16}}=:\zeta_5,$$
$$[\zeta_1,\zeta_3]=0,\qquad [\zeta_1,\zeta_4]=0,$$
$$[\zeta_2,\zeta_3]=-\frac{\partial}{\partial z_{21}}-\frac{1}{2}z_{31}\frac{\partial}{\partial z_{25}}=:\zeta_6,\qquad [\zeta_2,\zeta_4]=0,$$
$$\begin{aligned}[\zeta_3,\zeta_4]=&\left(\frac{1}{2}z_{16}+\frac{1}{8}z_{11}z_{21}-\frac{1}{2}z_{15}z_{24}\right)X_1+\left(-\frac{1}{2}z_{14}+\frac{1}{2}z_{13}z_{24}\right)X_2-\frac{1}{2}z_{24}X_3+\frac{1}{2}X_4\\&-\frac{1}{4}z_{21}Y_2+\left(\frac{1}{4}z_{11}-\frac{1}{4}z_{13}z_{21}\right)Y_3+\left(\frac{1}{4}z_{11}z_{24}-\frac{1}{4}z_{14}z_{21}\right)Y_4=:\zeta_7.\end{aligned}$$

So far, we have $\operatorname{rank} E^{(2)}=7$.

Moreover, we have

$$[\zeta_1,\zeta_5]=0,\qquad [\zeta_1,\zeta_6]=\frac{\partial}{\partial z_{11}}+\frac{1}{2}z_{31}\frac{\partial}{\partial z_{15}}+\left(-\frac{1}{2}z_{21}+\frac{1}{2}z_{24}z_{31}\right)\frac{\partial}{\partial z_{16}}=:\zeta_8,$$
$$[\zeta_1,\zeta_7]=0,$$
$$[\zeta_2,\zeta_5]=0,\qquad [\zeta_2,\zeta_6]=0,$$
$$\begin{aligned}[\zeta_2,\zeta_7]=&\left(-\frac{1}{2}z_{15}+\frac{1}{8}z_{11}z_{31}\right)X_1+\frac{1}{2}z_{13}X_2-\frac{1}{2}X_3-\frac{1}{4}z_{31}Y_2-\frac{1}{4}z_{13}z_{31}Y_3\\&+\left(\frac{1}{4}z_{11}-\frac{1}{4}z_{14}z_{31}\right)Y_4=:\zeta_9\end{aligned}$$
$$[\zeta_3,\zeta_5]=-\frac{\partial}{\partial z_{11}}-\frac{1}{2}z_{31}\frac{\partial}{\partial z_{15}}+\left(\frac{1}{2}z_{21}-\frac{1}{2}z_{24}z_{31}\right)\frac{\partial}{\partial z_{16}}=-[\zeta_1,\zeta_6]=-\zeta_8,$$

$$[\zeta_3, \zeta_6] = -\frac{1}{2}\frac{\partial}{\partial z_{25}} =: \zeta_{10}, \qquad [\zeta_3, \zeta_7] = 0,$$
$$[\zeta_4, \zeta_5] = 0, \qquad [\zeta_4, \zeta_6] = -[\zeta_2, \zeta_7] = -\zeta_9, \qquad [\zeta_4, \zeta_7] = 0.$$

Then we have $\operatorname{rank} E^{(3)} = 10$.

Further we have

$$[\zeta_1, \zeta_8] = 0,$$
$$[\zeta_1, \zeta_9] = \left(-\frac{1}{2}z_{25} + \frac{1}{8}z_{21}z_{31}\right)X_1 + \frac{1}{2}X_2 - \frac{1}{4}z_{31}Y_3 + \left(\frac{1}{4}z_{21} - \frac{1}{4}z_{24}z_{31}\right)Y_4 =: \zeta_{11},$$
$$[\zeta_1, \zeta_{10}] = \frac{1}{2}\frac{\partial}{\partial z_{15}} + \frac{1}{2}z_{24}\frac{\partial}{\partial z_{16}} =: \zeta_{12},$$
$$[\zeta_2, \zeta_8] = 0, \qquad [\zeta_2, \zeta_9] = 0, \qquad [\zeta_2, \zeta_{10}] = 0,$$
$$[\zeta_3, \zeta_8] = [\zeta_1, \zeta_{10}] = \zeta_{12}, \qquad [\zeta_3, \zeta_9] = \frac{1}{8}z_{11}X_1 - \frac{1}{4}Y_2 - \frac{1}{4}z_{13}Y_3 - \frac{1}{4}z_{14}Y_4 =: \zeta_{13},$$
$$[\zeta_3, \zeta_{10}] = 0,$$
$$[\zeta_4, \zeta_8] = -[\zeta_1, \zeta_9] = -\zeta_{11}, \qquad [\zeta_4, \zeta_9] = 0, \qquad [\zeta_4, \zeta_{10}] = -2[\zeta_3, \zeta_9] = -2\zeta_{13}.$$

We get that $\operatorname{rank} E^{(4)} = 13$.

Further we have

$$[\zeta_1, \zeta_{11}] = 0, \qquad [\zeta_1, \zeta_{12}] = 0, \qquad [\zeta_1, \zeta_{13}] = \frac{1}{8}z_{21}X_1 - \frac{1}{4}Y_3 - \frac{1}{4}z_{24}Y_4 =: \zeta_{14},$$
$$[\zeta_2, \zeta_{11}] = 0, \qquad [\zeta_2, \zeta_{12}] = \frac{1}{2}\frac{\partial}{\partial z_{16}} =: \zeta_{15}, \qquad [\zeta_2, \zeta_{13}] = 0,$$
$$[\zeta_3, \zeta_{11}] = [\zeta_1, \zeta_{13}] = \zeta_{14}, \qquad [\zeta_3, \zeta_{12}] = 0, \qquad [\zeta_3, \zeta_{13}] = 0,$$
$$[\zeta_4, \zeta_{11}] = 0, \qquad [\zeta_4, \zeta_{12}] = -\frac{1}{4}z_{21}X_1 + \frac{1}{2}Y_3 + \frac{1}{2}z_{24}Y_4 = -2[\zeta_1, \zeta_{13}] = -2\zeta_{14},$$
$$[\zeta_4, \zeta_{13}] = \left(-\frac{1}{8}z_{11}^2 - \frac{1}{2}z_{13}z_{16} + \frac{1}{2}z_{14}z_{15}\right)X_{12} + \frac{1}{2}z_{16}X_{13} - \frac{1}{2}z_{15}X_{14} - \frac{1}{2}z_{14}X_{23}$$
$$+ \frac{1}{2}z_{13}X_{24} - \frac{1}{2}X_{34} + \frac{1}{4}z_{11}Z =: \zeta_{16}.$$

Therefore, we have $\operatorname{rank} E^{(5)} = 16$.

Furthermore,

$$[\zeta_1, \zeta_{14}] = 0, \qquad [\zeta_1, \zeta_{15}] = 0, \qquad [\zeta_1, \zeta_{16}] = 0,$$
$$[\zeta_2, \zeta_{14}] = \frac{1}{8}z_{31}X_1 - \frac{1}{4}Y_4 =: \zeta_{17}, \qquad [\zeta_2, \zeta_{15}] = 0, \qquad [\zeta_2, \zeta_{16}] = 0,$$
$$[\zeta_3, \zeta_{14}] = 0, \qquad [\zeta_3, \zeta_{15}] = 0, \qquad [\zeta_3, \zeta_{16}] = 0,$$
$$[\zeta_4, \zeta_{14}] = \left(-\frac{1}{2}z_{16} - \frac{1}{4}z_{11}z_{21} + \frac{1}{2}z_{14}z_{25} + \frac{1}{2}z_{15}z_{24} + \frac{1}{8}z_{13}z_{21}^2 - \frac{1}{2}z_{13}z_{24}z_{25}\right)X_2$$
$$+ \left(-\frac{1}{8}z_{21}^2 + \frac{1}{2}z_{24}z_{25}\right)X_{13} - \frac{1}{2}z_{25}X_{14} - \frac{1}{2}z_{24}X_{23} + \frac{1}{2}X_{24} + \frac{1}{4}z_{12}Z =: \zeta_{18},$$
$$[\zeta_4, \zeta_{15}] = -2[\zeta_2, \zeta_{14}] = -2\zeta_{17}, \qquad [\zeta_4, \zeta_{16}] = 0.$$

Thus we see $\operatorname{rank} E^{(6)} = 18$.

Furthermore, we have

$$[\zeta_1, \zeta_{17}] = 0, \qquad [\zeta_1, \zeta_{18}] = 0, \qquad [\zeta_2, \zeta_{17}] = 0,$$

$$[\zeta_2, \zeta_{18}] = \left(\frac{1}{2}z_{15} - \frac{1}{4}z_{11}z_{31} - \frac{1}{2}z_{13}z_{25} + \frac{1}{8}z_{14}z_{31}^2 + \frac{1}{4}z_{13}z_{21}z_{31} - \frac{1}{8}z_{13}z_{24}z_{31}^2\right)X_{12}$$
$$+ \left(\frac{1}{2}z_{25} - \frac{1}{4}z_{21}z_{31} + \frac{1}{8}z_{24}z_{31}^2\right)X_{13} - \frac{1}{8}z_{31}^2 X_{14} - \frac{1}{2}X_{23} + \frac{1}{4}z_{31}Z =: \zeta_{19},$$
$$[\zeta_3, \zeta_{17}] = \frac{1}{8}X_1 =: \zeta_{20}, \qquad [\zeta_3, \zeta_{18}] = 0,$$
$$[\zeta_4, \zeta_{17}] = [\zeta_2, \zeta_{18}] = \zeta_{19}, \qquad [\zeta_4, \zeta_{18}] = 0.$$

Thus we have $\operatorname{rank} E^{(7)} = 20$.

Moreover,

$$[\zeta_1, \zeta_{19}] = 0, \qquad [\zeta_1, \zeta_{20}] = 0,$$
$$[\zeta_2, \zeta_{19}] = 0, \qquad [\zeta_2, \zeta_{20}] = 0,$$
$$[\zeta_3, \zeta_{19}] = \left(-\frac{1}{4}z_{11} + \frac{1}{4}z_{14}z_{31} + \frac{1}{4}z_{13}z_{21} - \frac{1}{4}z_{13}z_{24}z_{31}\right)X_{12} + \left(-\frac{1}{4}z_{21} + \frac{1}{4}z_{24}z_{31}\right)X_{13}$$
$$- \frac{1}{4}z_{31}X_{14} + \frac{1}{4}Z =: \zeta_{21},$$
$$[\zeta_3, \zeta_{20}] = 0,$$
$$[\zeta_4, \zeta_{19}] = 0, \qquad [\zeta_4, \zeta_{20}] = \frac{1}{2}[\zeta_3, \zeta_{19}] = \frac{1}{2}\zeta_{21}.$$

We obtain that $\operatorname{rank} E^{(8)} = 21$.

We have

$$[\zeta_1, \zeta_{21}] = 0, \qquad [\zeta_2, \zeta_{21}] = 0,$$
$$[\zeta_3, \zeta_{21}] = \left(\frac{1}{4}z_{14} - \frac{1}{4}z_{13}z_{24}\right)X_{12} + \frac{1}{4}z_{24}X_{13} - \frac{1}{4}X_{14} =: \zeta_{22}$$

and we have $[\zeta_4, \zeta_{21}] = 0$. So we get that $\operatorname{rank} E^{(9)} = 22$.

Also we have

$$[\zeta_1, \zeta_{22}] = 0, \qquad [\zeta_2, \zeta_{22}] = -\frac{1}{4}z_{13}X_{12} + \frac{1}{4}X_{13} =: \zeta_{23},$$
$$[\zeta_3, \zeta_{22}] = 0, \qquad [\zeta_4, \zeta_{22}] = 0.$$

Then we have $\operatorname{rank} E^{(10)} = 23$.

Lastly, we have

$$[\zeta_1, \zeta_{23}] = -\frac{1}{4}X_{12} =: \zeta_{24}, \qquad [\zeta_2, \zeta_{23}] = 0,$$
$$[\zeta_3, \zeta_{23}] = 0, \qquad [\zeta_4, \zeta_{23}] = 0.$$

We have that $\operatorname{rank} E^{(11)} = 24$. This shows the claim. ■

Remark 6.4. By the calculations in the proof of Lemma 6.2, we observe that $\pi_*^{-1}(D) \subset E^{(7)}$ for the projection $\pi\colon W \to M$, $\pi(m, (V_1, V_2, V_4)) = m$.

7 (8, 15)-distributions of type F_4

Inspired by our study on singular curves for Cartan model performed in the previous sections, it would be natural to introduce the class of (8, 15)-distributions of type F_4 including Cartan's model.

Definition 7.1. Let $D \subset TM$ be a complex (resp. a real) $(8,15)$-distribution. Then we call D *of type* F_4 (resp. *of type* $F_{4(4)}$) if, for each point $x_0 \in M$, there exists a local frame $\{X_1, X_2, X_3, X_4, Y_1, Y_2, Y_3, Y_4\}$ of D over an open neighbourhood of x_0 such that, modulo $\mathcal{D}$,

$$\begin{aligned} &[X_1, X_2] \equiv [Y_3, Y_4], \qquad [X_1, X_3] \equiv -[Y_2, Y_4], \qquad [X_1, X_4] \equiv [Y_2, Y_3], \\ &[X_2, X_3] \equiv [Y_1, Y_4], \qquad [X_2, X_4] \equiv -[Y_1, Y_3], \qquad [X_3, X_4] \equiv [Y_1, Y_2], \\ &[X_1, Y_1] \equiv [X_2, Y_2] \equiv [X_3, Y_3] \equiv [X_4, Y_4], \qquad \text{and} \qquad [X_i, Y_j] \equiv 0 \quad i \neq j,\ 1 \leq i, j \leq 4, \end{aligned}$$

and, if we set

$$\begin{aligned} X_{12} &= \frac{1}{2}[X_1, X_2], \qquad X_{13} = \frac{1}{2}[X_1, X_3], \qquad X_{14} = \frac{1}{2}[X_1, X_4], \\ X_{23} &= \frac{1}{2}[X_2, X_3], \qquad X_{24} = \frac{1}{2}[X_2, X_4], \qquad X_{34} = \frac{1}{2}[X_3, X_4], \end{aligned}$$

and $Z = [Y_1, X_1]$, then the vector fields

$$X_1,\ X_2,\ X_3,\ X_4,\ Y_1,\ Y_2,\ Y_3,\ Y_4\, X_{12},\ X_{13},\ X_{14},\ X_{23},\ X_{24},\ X_{34},\ Z,$$

form a local frame of TM.

Remark 7.2. Comparing with the relations on generators of Cartan's model in Section 2, the relations in Definition 7.1 are given modulo $\mathcal{D}$. The class of $(8,15)$-distributions of type F_4 in Definition 7.1 coincides with the class of regular differential system of type $\mathfrak{m}_F$ in the sense of Tanaka [38, 39, 40].

Then we have the following theorem.

Theorem 7.3. *Let (M, D) be a complex (resp. real) $(8,15)$-distribution of type F_4 (resp. $F_{4(4)}$). Then there exist uniquely the conformal non-degenerate bilinear form (resp. $(4,4)$-metric) on D and the conformal non-degenerate bilinear form (resp. $(4,3)$-metric) on $D^\perp$ obtained from the abnormal bi-extremals of D such that the null-cone $C \subset D$ coincides with the singular velocity cone* SVC(D). *Moreover, the flag manifold of null-subspaces $\{\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset D^\perp \subset T^*M\}$ corresponds to a subclass of flags by null-subspaces $\{V_1 \subset V_2 \subset V_4 \subset C \subset D \subset TM\}$ in D. The prolongation (W, E) of (M, D) by the above null-flags of D turns out to be a $(4,7,10,13,16,18,20,21,22,23,24)$-distribution such that its symbol algebra is isomorphic to the negative part of the nilpotent algebra for the gradation by the full set $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of simple roots of simple Lie algebra F_4 (resp. $F_{4(4)}$).*

Proof of Theorem 7.3. We re-examine the arguments on Cartan's model of $(8,15)$-distribution defined in Section 2 and performed in Sections 4–6 for general $(8,15)$-distributions of type F_4.

Let $D \subset TM$ be an $(8,15)$-distribution of type F_4. Reversing the correspondence in Section 2, we take the local frame

$$\beta_1,\ \beta_2,\ \beta_3,\ \beta_4,\ \gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \omega_{12},\ \omega_{13},\ \omega_{14},\ \omega_{23},\ \omega_{24},\ \omega_{34},\ \sigma$$

of T^*M which is dual to the local frame

$$X_1,\ X_2,\ X_3,\ X_4,\ Y_1,\ Y_2,\ Y_3,\ Y_4,\ X_{12},\ X_{13},\ X_{14},\ X_{23},\ X_{24},\ X_{34},\ Z$$

of TM in Definition 7.1. Then $D^\perp$ is generated by $\omega_{12}, \omega_{13}, \omega_{14}, \omega_{23}, \omega_{24}, \omega_{34}$ and σ. Any $\alpha \in D^\perp$ is expressed uniquely as $\alpha = \sum_{1 \leq i < j \leq 4} r_{ij}\omega_{ij} + s\sigma$. Then we have $\langle \alpha, X_{ij} \rangle = r_{ij}$ and $\langle \alpha, \sigma \rangle = s$. The functions r_{ij} and s with local coordinates of the base manifold M form a system of local coordinates of the submanifold $D^{\perp} \subset T^*M$. Then the equations (4.2) and (4.3) are obtained other (linear) algebraic arguments in Section 4 work as well also for general $(8,15)$-distributions of type F_4. Thus we have the same conclusion of Corollary 4.3 and moreover our discussions on the correspondence of null-flags in D and $D^{\perp}$ performed in Section 5 and the same proofs of the results such as Lemma 6.2 which concern on the prolongations of D in Section 6 works well also for any $(8,15)$-distribution of type F_4. This shows Theorem 7.3. ∎

Remark 7.4. The above statement on $(8,15)$-distribution of type F_4 (resp. $F_{4(4)}$) means that the gradation sheaf, i.e., the sheaf of nilpotent graded Lie algebras $\mathfrak{m} := \bigoplus_{i=1}^{11} \big(\mathcal{D}^{(i)}/\mathcal{D}^{(i-1)}\big)$ is isomorphic to that for the model derived from the simple Lie algebra F_4, which is described in Section 2. It is stated in [40] (see Proposition 5.5 and the arguments in pp. 482–483) that any $(8,15)$-distribution of type F_4 (resp. $F_{4(4)}$) is isomorphic to Cartan's model over $\mathbb{C}$ (resp. $\mathbb{R}$) in fact by Tanaka theory on simple graded Lie algebras. Note that we have proved our Theorem 7.3 without using this fact.

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