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Curves on Endo–Pajitnov Manifolds

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Abstract. Endo–Pajitnov manifolds are generalizations to higher dimensions of the Inoue surfaces S^M. We study the existence of complex submanifolds in Endo–Pajitnov manifolds. We identify a class of these manifolds that do contain compact complex submanifolds and establish an algebraic condition under which an Endo–Pajitnov manifold contains no compact complex curves.

Key words: Inoue surface; Oeljeklaus–Toma manifold; Endo–Pajitnov manifold; foliation

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1 Introduction

Among the surfaces in Kodaira's class VII, the three types of Inoue surfaces [7], play a prominent role. They are compact non-Kähler surfaces with no non-trivial meromorphic functions and without complex curves.

The Inoue surfaces of type S^M are solvmanifolds, quotients of $\mathbb{H}\times\mathbb{C}$, where $\mathbb{H}$ is the Poincaré half-plane, by a group constructed out of a matrix $M \in \mathrm{SL}(3,\mathbb{Z})$ with one real (irrational) eigenvalue $\alpha > 1$ and two complex conjugate ones, β, $\bar{\beta}$. Denoting with (a_i), respectively (b_i), a real eigenvector of α, respectively an eigenvector of β, we can define $g_0(w,z) = (\alpha w, \beta z)$ and $g_i(w,z) = (w + a_i, z + b_i)$, $i = 1,2,3$, and let G_M be the group generated by g_0, g_1, g_2, g_3. Then the Inoue surface S^M is $G_M\backslash\mathbb{H}\times\mathbb{C}$.

In 2005, the surfaces S^M were generalized to higher dimensions by Oeljeklaus and Toma [8]. Each such manifold is covered by $\mathbb{H}^s\times\mathbb{C}^t$ and it is associated to a number field with s real places and t complex ones. The Oeljeklaus–Toma (OT) manifolds are non-Kähler and contain no compact complex curves [11], and no compact complex submanifolds of dimension 2 except Inoue surfaces [10]. Moreover, the OT manifolds which admit locally conformally Kähler metrics do not have non-trivial meromorphic functions and hence they do not admit compact complex submanifolds [9].

In 2019, Endo and Pajitnov [5] proposed another generalization of the Inoue surfaces S^M to higher dimensions, again based on an integer matrix M with special requirements on its eigenvalues, just like the original construction. They proved that these new manifolds are non-Kähler and, if M is diagonalizable, then some of these manifolds are biholomorphic to OT manifolds. Further topological and metric properties of the Endo–Pajitnov manifolds were discussed in [4].

In this note, we study the existence of complex submanifolds in Endo–Pajitnov manifolds. We describe a class of Endo–Pajitnov manifolds which contain complex submanifolds, specifically complex tori (Theorem 3.1). On the other hand, we determine an algebraic condition which prohibits the existence of compact complex curves (Theorem 4.1). We also obtain a result (Proposition 4.5) regarding the existence of surfaces in an Endo–Pajitnov manifold from the class of those without complex curves.

2 Endo–Pajitnov manifolds

In this section, we recall the construction of the Endo–Pajitnov manifolds, as introduced in [5].

Let $n > 1$ and $M \in \mathrm{SL}(2n+1, \mathbb{Z})$ such that the eigenvalues of M are $\alpha, \beta_1, \dots, \beta_k, \overline{\beta}_1, \dots, \overline{\beta}_k$ with $\alpha > 0$, $\alpha \neq 1$ and $\mathrm{Im}(\beta_j) > 0$.

Denote by V the eigenspace corresponding to α and set

$$W(\beta_j) = \{x \in \mathbb{C}^{2n+1} \mid \exists\, N \in \mathbb{N} \text{ such that } (M - \beta_j I)^N x = 0\},$$
$$W = \bigoplus_{j=1}^{k} W(\beta_j), \qquad \overline{W} = \bigoplus_{j=1}^{k} W(\overline{\beta}_j).$$

We then have $\mathbb{C}^{2n+1} = V \oplus W \oplus \overline{W}$. Let $a \in \mathbb{R}^{2n+1}$ be a non-zero eigenvector corresponding to α and fix a basis $\{b_1, \dots, b_n\}$ in W,

$$a = \left(a^1, a^2, \dots, a^{2n+1}\right)^{\mathsf{T}}, \qquad b_i = \left(b_i^1, b_i^2, \dots, b_i^{2n+1}\right)^{\mathsf{T}}, \qquad 1 \leq i \leq n.$$

For any $1 \leq i \leq 2n+1$, we let $u_i = \left(a^i, b_1^i, \dots, b_n^i\right) \in \mathbb{R} \times \mathbb{C}^n \simeq \mathbb{R}^{2n+1}$. Note that $\{u_1, \dots, u_{2n+1}\}$ are linearly independent over $\mathbb{R}$, since $\left\{a, b_1, \dots, b_n, \overline{b}_1, \dots, \overline{b}_n\right\}$ is a basis of $\mathbb{C}^{2n+1}$.

Let now $f_M\colon W \longrightarrow W$ be the restriction of the multiplication by M on W and R the matrix of f_M with respect to the basis $\{b_1, \dots, b_n\}$. Let $\mathbb{H}$ be the Poincaré upper half-plane, and consider the automorphisms $g_0, g_1, \dots, g_{2n+1}\colon \mathbb{H} \times \mathbb{C}^n \longrightarrow \mathbb{H} \times \mathbb{C}^n$,

$$g_0(w, z) = \left(\alpha w, R^{\mathsf{T}} z\right), \qquad g_i(w, z) = (w, z) + u_i, \qquad w \in \mathbb{H}, \quad z \in \mathbb{C}^n, \quad 1 \leq i \leq 2n+1.$$

These automorphisms are well defined because $\alpha > 0$ and the first component of u_i is $a^i \in \mathbb{R}$.

Let G_M be the subgroup of $\mathrm{Aut}(\mathbb{H} \times \mathbb{C}^n)$ generated by $g_0, g_1, \dots, g_{2n+1}$.

Theorem 2.1 ([5])**.** *The action of G_M on $\mathbb{H} \times \mathbb{C}^n$ is free and properly discontinuous. Hence, the quotient $T_M := (\mathbb{H} \times \mathbb{C}^n)/G_M$ is a compact complex manifold of complex dimension $n+1$, with $\pi_1(T_M) \simeq G_M$.*

Definition 2.2. The above quotient $T_M := (\mathbb{H} \times \mathbb{C}^n)/G_M$ is called an *Endo–Pajitnov manifold*.

Remark 2.3. In the same paper, the authors prove that

- If M is diagonalizable, then some T_M are biholomorphic to OT manifolds [5, Proposition 5.3].
- If M is not diagonalizable, then T_M cannot be biholomorphic to any OT manifold [5, Proposition 5.6].

3 A class of Endo–Pajitnov manifolds containing submanifolds

We shall identify a class of Endo–Pajitnov manifolds that admit compact complex submanifolds. The idea is to define a holomorphic submersion from T_M to another complex manifold; the fibers will be the complex submanifolds we look for. More precisely, these submanifolds will be complex tori. The existence of such a structure depends on a suitable choice of the initial matrix M.

Let $n > 1$, $k \geq 1$ and $M \in \mathrm{SL}(2n+1, \mathbb{Z})$ be a matrix which can be written in block form:

$$M = \begin{pmatrix} N & 0 \\ 0 & P \end{pmatrix}$$

where P is a square matrix of dimension $2k$, $N \in \mathrm{SL}(2(n-k)+1,\mathbb{Z})$, such that

$$\mathrm{Spec}(N) = \{\alpha, \beta_1, \dots, \beta_N, \overline{\beta}_1, \dots, \overline{\beta}_N \mid \alpha \in \mathbb{R},\ \alpha > 0,\ \alpha \neq 1,\ \mathrm{Im}(\beta_j) > 0\}$$

and

$$\mathrm{Spec}(P) = \{\beta_{N+1}, \dots, \beta_P, \overline{\beta}_{N+1}, \dots, \overline{\beta}_P \mid \mathrm{Im}(\beta_j) > 0\}.$$

It is clear that M satisfies the conditions required in the construction of Endo–Pajitnov manifolds. In this case, the matrix R (see Section 2) is a block diagonal matrix.

Denote by $W^M(\beta_j)$, $W^N(\beta_j)$, and $W^P(\beta_j)$ the generalised eigenspaces of β_j for M, N, and P respectively. We pick a basis $\{b_1, \dots, b_{2n+1}\}$ in W which comes from bases in each $W^M(\beta_j)$ that in turn come from bases in $W^N(\beta_j)$ and $W^P(\beta_j)$, using the fact that $W^M(\beta_j) = W^N(\beta_j) \oplus W^P(\beta_j)$.

The diffeomorphisms $g_0, g_1, \dots, g_{2n+1} \colon \mathbb{H}\times\mathbb{C}^n \longrightarrow \mathbb{H}\times\mathbb{C}^n$ can be written explicitly as follows:

$$g_0(w,(z_1,\dots,z_n)) = (\alpha w, R^{\mathsf{T}} z), \tag{3.1}$$

$$g_i(w,z) = (w,z) + \begin{cases} (a^i, b^i_1, \dots, b^i_{n-k}, 0, \dots, 0), & 1 \le i \le 2(n-k)+1, \\ (0,0,\dots,0,b^i_{n-k+1},\dots,b^i_n), & i > 2(n-k)+1, \end{cases} \qquad w \in \mathbb{H}, z \in \mathbb{C}^n.$$

Since R is a block diagonal matrix and g_i acts independently on each component for any $1 \le i \le 2n+1$, and owing to the special form of the automorphisms (3.1), there exist $\tilde{g}_i \colon \mathbb{H}\times\mathbb{C}^{n-k} \longrightarrow \mathbb{H}\times\mathbb{C}^{n-k}$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathbb{H}\times\mathbb{C}^n & \xrightarrow{\ g_i\ } & \mathbb{H}\times\mathbb{C}^n \\ \downarrow{\scriptstyle \mathrm{pr}_{n-k}} & & \downarrow{\scriptstyle \mathrm{pr}_{n-k}} \\ \mathbb{H}\times\mathbb{C}^{n-k} & \xrightarrow{\ \tilde{g}_i\ } & \mathbb{H}\times\mathbb{C}^{n-k}. \end{array}$$

Let Γ_N the subgroup of $\mathrm{Aut}(\mathbb{H}\times\mathbb{C}^{n-k})$ generated by $\tilde{g}_0, \dots, \tilde{g}_{2(n-k)+1}$. Since the matrix N satisfies the conditions in the construction in Section 2, the action of Γ_N generates an Endo–Pajitnov manifold T_N of dimension $n-k+1$, $T_N = (\mathbb{H}\times\mathbb{C}^{n-k})/\Gamma_N$.

It is known that T_M has a solvmanifold structure, $T_M \simeq G/\Gamma$ [4, Theorem 3.1]. Let us denote by $\mathfrak{g}$ the Lie algebra of G. Our goal is to show that T_M is the total space of a holomorphic fiber bundle with base T_N. To achieve this, we use the fact that both T_M and T_N are solvmanifolds and we analyze the structure of $\mathfrak{g}$. The idea is to write $\mathfrak{g}$ as a semidirect product between an abelian ideal that will correspond to the fiber, and a complementary subalgebra corresponding to the base T_N.

This structure naturally extends to the corresponding Lie group level. We also describe the lattice Γ as a semidirect product between a lattice generating T_N and a lattice consisting only of translations. The compatibility between the semidirect product structures of G and Γ allows us to define a holomorphic submersion π between T_M and T_N, and the fiber will be a complex torus.

Consider the subspace

$$\mathfrak{h} = \left\langle Y_{2n-k+1} + \mathrm{i}Y_{n-k+1}, \dots, Y_{2n} + \mathrm{i}Y_n, \overline{Y_{2n-k+1} + \mathrm{i}Y_{n-k+1}}, \dots, \overline{Y_{2n} + \mathrm{i}Y_n} \right\rangle_{\mathbb{C}}.$$

A direct computation shows that $\mathfrak{h}$ is an abelian ideal of $\mathfrak{g}$. Indeed, from the structure equations of $\mathfrak{g}$, it is sufficient to prove that $[A, Y_{2n-k+j} + \mathrm{i}Y_{n-k+j}] \in \mathfrak{h}$, for any $1 \le j \le k$.

Let us prove the case $j = 1$. We have

$$\begin{aligned}[A, Y_{2n-k+1} + \mathrm{i}Y_{n-k+1}] = &- \sum_{i \le n-k} \mathrm{Im}\,\Delta_{i,n-k+1} \cdot Y_i + \sum_{i \le n-k} \mathrm{Re}\,\Delta_{i,n-k+1} \cdot Y_{n+i} \\ &+ \mathrm{i}\Bigg(\sum_{i \le n-k} \mathrm{Re}\,\Delta_{i,n-k+1} \cdot Y_i + \sum_{i \le n-k} \mathrm{Im}\,\Delta_{i,n-k+1} \cdot Y_{n+i} \Bigg),\end{aligned}$$

where $\Delta = \log R^{\mathsf{T}}$ (see [4, Theorem 3.1]).

Since R is a block diagonal matrix, Δ inherits this structure and, thus $\Delta_{i,n-k+1} = 0$, for all $i \leq n-k$. Therefore,

$$[A, Y_{2n-k+1} + \mathrm{i}Y_{n-k+1}] = \Delta_{n-k+1,n-k+1}(Y_{2n-k+1} + \mathrm{i}Y_{n-k+1}) \in \mathfrak{h}.$$

Analogously, we can prove the same thing for any j, hence, $\mathfrak{h}$ is ideal of $\mathfrak{g}$ and from the structure equations it is clear that $\mathfrak{h}$ is abelian.

Consider $\mathfrak{l} := \mathfrak{g}/\mathfrak{h}$. Then $\mathfrak{l}$ is the Lie algebra corresponding to the Endo–Pajitnov manifold T_N. Moreover,

$$\mathfrak{l} = \left\langle A + \mathrm{i}X, Y_{n+1} + \mathrm{i}Y_1, \ldots, Y_{2n} + \mathrm{i}Y_n, \overline{A + \mathrm{i}X}, \ldots, \overline{Y_{2n} + \mathrm{i}Y_n} \right\rangle_{\mathbb{C}}$$

is a subalgebra of $\mathfrak{g}$, and $\mathfrak{l}$ acts on $\mathfrak{h}$ via the adjoint representation, giving $\mathfrak{g} = \mathfrak{l} \rtimes_{\varphi} \mathfrak{h}$.

Let $H := \mathbb{C}^k$ be the simply-connected complex Lie group corresponding to $\mathfrak{h}$ and $L := \mathbb{H} \times \mathbb{C}^{n-k}$ the simply-connected complex Lie group corresponding to $\mathfrak{l}$. As differentiable manifolds, $G = L \times H$. Since H is simply-connected, we can identify $\mathrm{Aut}(\mathfrak{h}) = \mathrm{Aut}(H)$ and by [2, Theorem 3], the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{l} & \xrightarrow{\ \varphi\ } & \mathrm{Der}(\mathfrak{h}) \\ {\scriptstyle \exp}\big\downarrow & & \big\downarrow{\scriptstyle \exp} \\ L & \xrightarrow{\ \tilde{\varphi}\ } & \mathrm{Aut}(H), \end{array}$$

where $\tilde{\varphi}$ is given by conjugation. Thus, L acts on H and by Lie's third theorem we have that $G = L \rtimes_{\tilde{\varphi}} H$ with the composition group law

$$(l_1, h_1) \cdot (l_2, h_2) = (l_1 l_2, \tilde{\varphi}(l_1)(h_2)).$$

Let $\Gamma_P := \langle g_{2(n-k)+2}, \ldots, g_{2n+1} \rangle$. Then Γ_P is a lattice of $\mathbb{C}^k$. We will prove that $\Gamma = \Gamma_N \rtimes \Gamma_P$. We define an action $\rho \colon \Gamma_N \longrightarrow \mathrm{Aut}(\Gamma_P)$ by $\rho(g)(h) = ghg^{-1}$, for any $g \in \Gamma_N$ and $h \in \Gamma_P$. This map is well defined. To verify this, it suffices to show that for every g_i with $0 \leq i \leq 2(n-k)+1$ and every g_j with $2(n-k)+2 \leq j \leq 2n+1$, the conjugate $g_i g_j g_i^{-1} \in \Gamma_P$.

For $i > 0$, it is trivial. For $i = 0$, using [5, Lemma 2.3], we have

$$g_0 g_j g_0^{-1} = g_1^{m_{j,1}} \cdots g_{2n+1}^{m_{j,2n+1}}.$$

Given the block form of M, it follows that $m_{j,l} = 0$, for any $l \leq 2(n-k)+1$, and thus

$$g_0 g_j g_0^{-1} = g_{2(n-k)+2}^{m_{j,2(n-k)+2}} \cdots g_{2n+1} m^{j,2n+1} \qquad \text{for all } 2(n-k)+2 \leq j \leq 2n+1.$$

Hence, we obtain the semidirect product structure, $\Gamma = \Gamma_N \rtimes \Gamma_P$.

Let us consider $p \colon G \longrightarrow L$ the projection $p(h,l) = l$. Since $p(\Gamma) = \Gamma_N$, this descends to a well-defined map on the quotients:

$$\pi \colon\ G/\Gamma \simeq T_M \longrightarrow L/\Gamma_N \simeq T_N, \qquad \pi(g\Gamma) = p(g)\Gamma_N.$$

To see that π is well-defined, suppose $g\Gamma = g'\Gamma$, which implies $g^{-1}g' \in \Gamma$. For $g = (l,h)$, $g' = (l', h')$, we have

$$g^{-1}g' = \left(l^{-1}, l^{-1}h^{-1}l\right) \cdot (l', h') = \left(l^{-1}l', l^{-1}h^{-1}l \cdot \tilde{\varphi}\left(l^{-1}\right)(h')\right) \in \Gamma.$$

Hence, $l^{-1}l' \in \Gamma_N$. So $p(g)^{-1}p(g') \in \Gamma_N$, and therefore $p(g)\Gamma_N = p(g')\Gamma_N$.

Since π is induced by the projection p on the first coordinates, it is clearly a holomorphic submersion. Also, it is a proper map. By Ehresmann theorem [6, Corollary 6.2.3], it follows that

$\pi\colon T_M \longrightarrow T_N$ is a locally trivial fibration. Since holomorphic local trivializations exist, π defines a holomorphic fiber bundle.

We will prove that the fibers of π, which are complex submanifolds of T_M, are complex tori, by constructing an explicit isomorphism between each fiber and a complex torus. Fix $l\Gamma_N \in L/\Gamma_N$. Then

$$\pi^{-1}(l\Gamma_N) = \{g\Gamma \in G/\Gamma \mid p(g)\Gamma_N = l\Gamma_N\}.$$

Define a map

$$\psi_l\colon\; H \longrightarrow \pi^{-1}(l\Gamma_N), \psi_l(h) = (l, h)\Gamma.$$

Since Γ_P is generated only by translations, it is a normal subgroup in Γ. We obtain an induced map

$$\bar{\psi}_l\colon\; H/\Gamma_P \simeq \mathbb{T}^k \longrightarrow \pi^{-1}(l\Gamma_N), \qquad \bar{\psi}_l(h\Gamma_P) = (l, h)\Gamma.$$

It is clear that $\bar{\psi}_l$ is a biholomorphism. Thus, the fiber of π is a complex torus.

In conclusion, we can state the following.

Theorem 3.1. *Let X be an Endo–Pajitnov manifold associated to a block diagonal matrix such that one of the blocks produces a (smaller dimensional) Endo–Pajitnov manifold Y. Then X admits the structure of a holomorphic fiber bundle over Y. In particular, X contains complex tori, as complex submanifolds.*

In the following, we provide a numerical example in the lowest possible dimension.

Example 3.2. We give an example[1] of an Endo–Pajitnov manifold that contains complex curves. Let $n = 2$, $k = 1$, and a diagonalizable matrix M

$$M = \begin{pmatrix} N & 0 \\ 0 & P \end{pmatrix}, \qquad \text{where} \qquad N = \begin{pmatrix} 1 & 2 & -1 \\ -1 & 0 & -2 \\ 0 & 1 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $M \in \mathrm{SL}(5, \mathbb{Z})$ and satisfies the special conditions from construction of Endo–Pajitnov manifold. Thus, we obtain T_M, an Endo–Pajitnov manifold of complex dimension 3.

On the other hand, $N \in \mathrm{SL}(3, \mathbb{Z})$ has a single real eigenvalue, α, and two complex conjugate eigenvalues β_1, $\overline{\beta}_1$, and hence it defines an Inoue surface of type S^N, call it $T_N = \mathbb{H} \times \mathbb{C}/G_N$.

As in the general case, we define the projection $\pi\colon T_M \longrightarrow T_N$

$$\pi([w, (z_1, z_2)]) = [[w, z_1]], \qquad w \in \mathbb{H}, \quad z_1, z_2 \in \mathbb{C}.$$

Since π is a holomorphic submersion, Theorem 3.1 assures that T_M projects over an Inoue surface, with complex curves as fibres.

4 Curves on Endo–Pajitnov manifolds

In this section, we derive a necessary condition that the matrix M must satisfy so that the manifold T_M does not contain complex curves. The condition we find is algebraic, expressed in terms of the components of the eigenvector a associated with the real eigenvalue α of the matrix M. The proof is similar to the one in [11], where it was shown that no OT manifold can contain complex curves.

[1] We are much grateful to Alexandru Gica for offering us this example.

Theorem 4.1. *Let T_M be an Endo–Pajitnov manifold. If the components of the eigenvector a associated to the real eigenvalue α of the matrix M are linearly independent over $\mathbb{Z}$, then there are no compact complex curves on T_M.*

Proof. The idea of the proof is the following. We construct an exact, semipositive (1,1)-form ω on T_M whose integral over any compact complex curve of T_M will be necessarily nonnegative and such that any complex curve in T_M should stay in a leaf of the null foliation of ω. We then show that the stated condition implies that these leaves are isomorphic to $\mathbb{C}^n$, which contains no compact curves.

Here are the details.

Step 1. *Construction of the form ω.* We start by constructing a semipositive $(1,1)$-form $\tilde{\omega}$ on the universal cover $\tilde{T}_M := \mathbb{H} \times \mathbb{C}^n$ of T_M, invariant by the action of the deck group G_M.

Let $(w, z_1, \dots, z_n)$ be the complex coordinates on $\tilde{T}_M$ and define $\varphi\colon \tilde{T}_M \longrightarrow \mathbb{R}$, by

$$\varphi(w, z_1, \dots, z_n) = \frac{1}{\operatorname{Im} w}, \qquad w \in \mathbb{H}, \quad z_1, \dots, z_n \in \mathbb{C}.$$

It is clear that $\varphi(w, z_1, \dots, z_n) > 0$ on $\tilde{T}_M$.

Define $\tilde{\omega} := \mathrm{i}\partial\bar{\partial}\log(\varphi)$. In the above coordinates on $\tilde{T_M}$, $\tilde{\omega}$ is expressed as

$$\tilde{\omega} = \mathrm{i}\frac{1}{4(\operatorname{Im} w)^2}\mathrm{d}w \wedge \mathrm{d}\bar{w}.$$

Note that using the d and d^c operators, we can rewrite

$$\tilde{\omega} = \frac{1}{2}\mathrm{d}\mathrm{d}^c \log\varphi,$$

and hence $\tilde{\omega}$ is an exact form on $\tilde{T}_M$.

Let us consider $\omega_{\mathbb{H}}$, the Poincaré metric on $\mathbb{H}$ and $\mathrm{pr}_1 : \mathbb{H} \times \mathbb{C}^n \longrightarrow \mathbb{H}$ the projection onto the first factor. Then, we have

$$\tilde{\omega} = \mathrm{pr}_1^*(\omega_{\mathbb{H}}).$$

Since pr_1 is a holomorphic submersion, it follows that $\tilde{\omega}$ is semipositive definite. Moreover, since $\omega_{\mathbb{H}}$ is invariant under translations and multiplications by real numbers, we obtain that $\tilde{\omega}$ is invariant under the action of G_M. Since $\tilde{\omega}$ is G_M-invariant, it is the pullback of an $(1,1)$-form ω on $T_M := \tilde{T}_M/G_M$. Clearly, ω is an exact, semipositive $(1,1)$-form on T_M.

Step 2. *The action of the deck group on the leaves of the null foliation of $\tilde{\omega}$.* If $V = Z + A$, where $Z \in T\mathbb{H}$, $A \in T\mathbb{C}^n$ and $Z = X + \mathrm{i}Y$, then

$$\tilde{\omega}(V, JV) = \frac{\mathrm{i}}{4(\operatorname{Im} w)^2} \cdot (-2\mathrm{i})\,\mathrm{d}w(Z)\,\mathrm{d}\bar{w}(Z) = \frac{2}{4(\operatorname{Im} w)^2}\big(|X|^2 + |Y|^2\big). \tag{4.1}$$

From (4.1), we obtain that any (maximal) leaf of the zero foliation of $\tilde{\omega}$ on $\tilde{T}_M$ is isomorphic to $\mathbb{C}^n$.

Let $L = \{w\} \times \mathbb{C}^n$, for some fixed w, be such a leaf. We look at the image of the action of G_M on L and we determine its intersection with L. By the description of L, for any $\sigma \in G_M$ such that $L \cap \sigma(L) \neq \varnothing$, the first coordinate of the points in L coincide with the first coordinate of the points in $\sigma(L)$. In general, σ contains all generators of G_M. Its most general form is $\sigma = g_0^{s_0} \circ g_1^{s_1} \circ \cdots \circ g_{2n+1}^{s_{2n+1}}$, where $s_i \in \mathbb{Z}$.

We show that g_0 cannot appear. Indeed, the above mentioned coincidence of the first coordinates translates in the following equation:

$$\alpha^{s_0} w + \sum_{i=1}^{2n+1} s_i a^i = w,$$

which is equivalent to

$$(\alpha^{s_0} - 1)w = -\sum_{i=1}^{2n+1} s_i a^i.$$

Taking imaginary parts in the equation and using the fact that $w \in \mathbb{H}$, we necessarily obtain that $s_0 = 0$. We conclude that σ cannot contain the generator g_0. It is then obtained only from translations, $\sigma = g_1^{s_1} \circ \cdots \circ g_{2n+1}^{s_{2n+1}}$, and we have

$$s_1 a^1 + \cdots + s_{2n+1} a^{2n+1} = 0,$$

a linear dependence relation over $\mathbb{Z}$ which contradicts the hypothesis. Thus, we showed that $L \cap \sigma(L) = \varnothing$, for all $\sigma \in G_M$.

Step 3. *The zero foliation of ω on T_M.* Recall that ω is semipositive on T_M (Step 1) and hence its integral on any compact complex curve $\gamma \subset T_M$ is nonnegative. By Stokes theorem, since ω is exact, this integral vanishes. Thus, ω vanishes on all closed complex curves in T_M. Equivalently, any compact complex curve in T_M stays in a leaf of the zero foliation of ω.

On the other hand, since $\tilde{\omega}$ is G_M-invariant, each leaf of the zero foliation of ω on T_M is isomorphic to a component of the leaf of the zero foliation of $\tilde{\omega}$ on $\tilde{T}_M$. Therefore, it is isomorphic with $\mathbb{C}^n$, which does not contain any compact complex submanifold. ■

Remark 4.2. Clearly, Example 3.2 does not satisfy the condition in Theorem 4.1. At the moment, we cannot prove that the condition is also sufficient. However, we dare to propose the following:

Conjecture 4.3. *Let T_M be an Endo–Pajitnov manifold, with real eigenvalue α. Then T_M admits complex curves if and only if the components of the eigenvector a are not linearly independent over $\mathbb{Z}$.*

Remark 4.4. In [4, Example 5.8], we constructed an example of a 4-dimensional Endo–Pajitnov manifold admitting both pluriclosed and astheno-Kähler metrics. Moreover, the condition in Theorem 4.1 is satisfied, so the manifold contains no compact complex curves.

In a manner similar to [10], we obtain a result concerning the existence of complex surfaces in Endo–Pajitnov manifolds.

Proposition 4.5. *Let T_M be an Endo–Pajitnov manifold without compact complex curves. Then T_M does not contain any closed complex surfaces except Inoue surfaces.*

Proof. The idea of the proof is based on a result by Brunella about classification of surfaces of Kähler rank one.

It was shown in [5] that Endo–Pajitnov manifolds are non-Kähler. In the previous proof, we constructed a non-trivial closed semipositive $(1,1)$-form ω on T_M. The restriction of ω to any complex surface in T_M yields a non-trivial closed semipositive $(1,1)$-form on that surface. Therefore, any compact complex surface in T_M must have Kähler rank one.

Compact surfaces of Kähler rank 1 have been classified in [3] and [1]. They can be

1. Non-Kählerian elliptic fibrations;
2. certain Hopf surfaces, and their blow-ups;
3. Inoue surfaces, and their blow-ups.

By definition, elliptic fibrations contain curves, as do Hopf surfaces and their blow-ups. Since T_M does not contain any compact complex curves by hypothesis, it cannot contain any surface from the above classification except for Inoue surfaces. ■

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