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Hilbert Series of S_3-Quasi-Invariant Polynomials in Characteristics 2, 3

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Abstract. We compute the Hilbert series of the space of $n = 3$ variable quasi-invariant polynomials in characteristic 2 and 3, capturing the dimension of the homogeneous components of the space, and explicitly describe the generators in the characteristic 2 case. In doing so we extend the work of the first author in 2023 on quasi-invariant polynomials in characteristic $p > n$ and prove that a sufficient condition found by Ren–Xu in 2020 on when the Hilbert series differs between characteristic 0 and p is also necessary for $n = 3$, $p = 2, 3$. This is the first description of quasi-invariant polynomials in the case, where the space forms a modular representation over the symmetric group, bringing us closer to describing the quasi-invariant polynomials in all characteristics and numbers of variables.

Key words: quasi-invariant polynomials; modular representation theory

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1 Introduction

Let $\Bbbk$ be a field, and consider the action of the symmetric group S_n on the space $\Bbbk[x_1, \dots, x_n]$ of $\Bbbk$-valued polynomials by permuting the variables. A polynomial in $\Bbbk[x_1, \dots, x_n]$ is *symmetric* if it is invariant under this action. Equivalently, since S_n is generated by transpositions, a polynomial K is symmetric if $s_{i_1 i_2} K = K$ or $(1 - s_{i_1 i_2})K = 0$ for all $s_{i_1 i_2} \in S_n$. One may consider generalizations of symmetric polynomials in which this condition is relaxed, so that we only require $(1 - s_{i_1 i_2})K$ be divisible by some large polynomial. This leads to the notion of *quasi-invariant polynomials*.

Definition 1.1. Let $\Bbbk$ be a field. For $m \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{>0}$, a polynomial $K \in \Bbbk[x_1, \dots, x_n]$ is *m-quasi-invariant* if for all $s_{i_1 i_2} \in S_n$ we have that $(x_{i_1} - x_{i_2})^{2m+1}$ divides $(1 - s_{i_1 i_2})K$. We denote the space of m-quasi-invariants by $Q_m(n, \Bbbk)$.

Note that the symmetric polynomials are exactly the polynomials that are m-quasi-invariant for all m. For brevity, we also refer to quasi-invariant polynomials as simply quasi-invariants.

Quasi-invariant polynomials were first introduced by Chalykh and Veselov in 1990 [6] to describe the harmonic, zero eigenvalue eigenfunctions of quantum Calogero–Moser systems. Calogero–Moser systems are a collection of one-dimensional dynamical particle systems that were found to be both solvable [4] and integrable [10]. Due to these properties, they have become extensively studied in mathematical physics, with connections to a number of other fields of mathematics, including representation theory.

Quasi-invariant polynomials were also later found to describe the representation theory of the spherical subalgebra of the rational Cherednik algebra [3]. This subalgebra is Morita equivalent to the entire rational Cherednik algebra [7], so quasi-invariants describe representations of rational Cherednik algebras as well. Such algebras have connections to combinatorics, mathematical physics, algebraic geometry, algebraic topology, and more, leading them to become a central topic in representation theory.

Due to these applications, the quasi-invariant polynomials have been studied extensively in recent years. Of particular interest are properties such as its freeness as a module over the symmetric polynomials and the degrees of its generators. To describe these properties, it is useful to consider the Hilbert series of the quasi-invariants, which encapsulates much of this information.

Definition 1.2. Let $V = \bigoplus_{d=0}^{\infty} V_d$ be a graded vector space. The *Hilbert series* of V is the formal power series

$$\mathcal{H}(V) := \sum_{d=0}^{\infty} \dim(V_d) t^d.$$

In 2003, Felder and Veselov found the Hilbert series of the space of quasi-invariants in characteristic zero [9], proving its freeness in the process. Work on quasi-invariants in characteristic p started in 2020, when Ren and Xu proved a sufficient condition for the Hilbert series of $Q_m(n, \mathbf{F}_p)$ to be different from the Hilbert series of $Q_m(n, \mathbf{Q})$ [11]. They accomplished this by computing non-symmetric polynomial "counterexamples" in characteristic p, where the polynomial has lower degree than any non-symmetric quasi-invariant polynomial in characteristic 0. They also made several conjectures about quasi-invariants in characteristic p, including that the condition they found is also sufficient, the quasi-invariants are free, and that the Hilbert polynomial is palindromic for $p > 2$. In 2023, the first author proved a general form for the Hilbert series of the quasi-invariants for $n = 3$, $p > 3$, proving freeness and the palindromicity of the Hilbert polynomial in the process [13].

We expect the development of the theory of quasi-invariants in characteristic p to be useful in mathematical physics and integrable systems through the theory of q-deformed quasi-invariants. These are certain deformations of quasi-invariants in characteristic zero introduced by Chalykh in 2002 [5] used to describe eigenfunctions of Macdonald difference operators, which are a generalization of elliptic Calogero–Moser systems [12]. We expect the theory of quasi-invariants in characteristic p to be related to the theory of q-deformed quasi-invariants when q is a root of unity, in analog to the classical connection between representations of Lie algebras in characteristic p and quantized enveloping algebras [2]. We note that a few similarities between these two spaces of quasi-invariants have already been found in [13].

In this paper, we consider the cases $n = 3$, $p = 2, 3$. These cases differ from the $p > 3$ case studied in [13] since in $p = 2, 3$ the representations of S_3 are *modular*, i.e., are not completely reducible. Despite these limitations, we describe the Hilbert series explicitly for all m, proving the following.

Theorem 1.3. *Let $\Bbbk$ be either $\mathbf{F}_2$ or $\mathbf{F}_3$. Then the Hilbert series for $Q_m(3, \Bbbk)$ is given by*

$$\mathcal{H}(Q_m(3, \Bbbk)) = \frac{1 + 2t^d + 2t^{6m+3-d} + t^{6m+3}}{(1-t)(1-t^2)(1-t^3)},$$

where $d = 3m + 1$ if there is no Ren–Xu counterexample and d is the degree of the minimal degree Ren–Xu counterexample otherwise. In particular, the conditions found in [11] for the Hilbert series of $Q_m(3, \Bbbk)$ to be different from the Hilbert series of $Q_m(3, \mathbf{Q})$ are necessary.

Note that this result also implies freeness and the palindromicity of the Hilbert polynomial.

In the case $p = 2$, we also define m-quasi-invariants in the case where m is a half-integer and prove an analogous statement to Theorem 1.3 in this case. Using quasi-invariants at half-integers, we also compute the generators of $Q_m(3, \mathbf{F}_2)$ as an $\mathbf{F}_2[x_1, x_2, x_3]^{S_3}$-module explicitly.

In Section 2, we state some of the basic facts about quasi-invariant polynomials and introduce modular representations of S_3. In Section 3, we compute the generators of $Q_m(3, \mathbf{F}_2)$, proving Theorem 1.3 for $p = 2$ in the process. In Section 4, we begin discussing $p = 3$, and show that some properties of quasi-invariants in 3 variables from [13] carry over to the $p = 3$ case after converting from the standard representation to the sign $-$ triv representation. In Section 5, we show that minimal degree Ren–Xu counterexamples are the lowest degree non-symmetric generators for $Q_m(3, \mathbf{F}_3)$ and show that there is one other higher degree generator belonging to the sign $-$ triv representation. Finally, in Section 6, we consider all other indecomposable representations of S_3 in $Q_m(3, \mathbf{F}_3)$, finishing the proof of Theorem 1.3 for $p = 3$.

2 Preliminaries

We start with some useful properties of the quasi-invariants.

Proposition 2.1 ([8]). *Let $\Bbbk$ be a field.*

1. $\Bbbk[x_1, x_2, x_3]^{S_3} \subset Q_m(3, \Bbbk)$, $Q_0(3, \Bbbk) = \Bbbk[x_1, x_2, x_3]$, *and* $Q_m(3, \Bbbk) \supset Q_{m'}(3, \Bbbk)$, *where* $m' > m$.
2. $Q_m(3, \Bbbk)$ *is a ring.*
3. $Q_m(3, \Bbbk)$ *is a finitely generated* $\Bbbk[x_1, x_2, x_3]^{S_3}$*-module.*

Note that [8] proves Proposition 2.1 in the case, where $\Bbbk = \mathbb{C}$. However, the proofs for the first two assertions work over any field, and the last assertion follows from the Hilbert basis theorem. In view of the structure of $Q_m(3, \Bbbk)$ as a module over the symmetric polynomials, given some $K \in Q_m(3, \Bbbk)$, we will frequently refer to quasi-invariant polynomials that can be obtained via scalar multiplication of Q by a symmetric polynomial. To distinguish these polynomials from the ordinary $\Bbbk$-multiples of Q, we will refer to them as *symmetric polynomial multiples* of Q.

We consider $Q_m(3, \mathbf{F}_2)$ and $Q_m(3, \mathbf{F}_3)$ as representations of S_3, where S_3 permutes the variables x_1, x_2, x_3. Since $Q_m(3, \mathbf{F}_2)$ and $Q_m(3, \mathbf{F}_3)$ are vector spaces over $\mathbf{F}_2$ and $\mathbf{F}_3$ respectively and the characteristics 2 and 3 divide $|S_3|$, $Q_m(3, \mathbf{F}_2)$ and $Q_m(3, \mathbf{F}_3)$ are modular representations of S_3.

Proposition 2.2. $Q_m(3, \mathbf{F}_2)$ *and* $Q_m(3, \mathbf{F}_3)$ *are modular representations of* S_3*.*

First, we consider characteristic 2.

2.1 Preliminary definitions for $p = 2$

We describe the indecomposable and irreducible representations of S_3 for $p = 2$.

Proposition 2.3 ([1]). *There are* 3 *irreducible or indecomposable representations of* S_3 *in characteristic* 2*:*

1. triv *is the irreducible representation of* S_3 *that is acted on trivially by* S_3*.*
2. std *is the* 2*-dimensional irreducible representation of* S_3 *obtained by reducing the standard representation in characteristic* 0 *mod* 2*.*
3. triv $-$ triv *is the* 2*-dimensional indecomposable representation that contains a copy of* triv *as a subrepresentation such that the quotient of* triv $-$ triv *by this subrepresentation is* triv*.*

Example 2.4. The polynomial $E_{\text{triv}-\text{triv}} := x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \in \mathbf{F}_2[x_1, x_2, x_3]$ generates a copy of triv $-$ triv. To see this, note that for any i_1, i_2, we have

$$(1 - s_{i_1i_2})E_{\text{triv}-\text{triv}} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2 \in \mathbf{F}_2[x_1, x_2, x_3]^{S_3}.$$

Since the transpositions generate S_3, $E_{\mathrm{triv-triv}}$ generates a two-dimensional representation that contains triv as a subrepresentation. Moreover, since $E_{\mathrm{triv-triv}}$ is not symmetric, this representation is not triv $\oplus$ triv, so it must be triv $-$ triv.

We then study the behaviors of each indecomposable representation in the quasi-invariants. We define $Q_m(3, \mathbf{F}_2)_{\mathrm{triv}}$ and $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ to be the direct sum of all copies of triv and std respectively in the quasi-invariants. We also define $Q_m(3, \mathbf{F}_2)_{\mathrm{triv-triv}}$ to be the direct sum of all copies of triv and triv $-$ triv.

Remark 2.5. We cannot define $Q_m(3, \mathbf{F}_2)_{\mathrm{triv-triv}}$ to exclude copies of triv since we can add elements of $Q_m(3, \mathbf{F}_2)_{\mathrm{triv}}$ to copies of triv $-$ triv and still obtain a copy of triv $-$ triv. For example, $F := E_{\mathrm{triv-triv}} + x_1^3 + x_2^3 + x_3^3$ still satisfies $(1 - s_{i_1i_2})F = (1 - s_{i_1i_2})E_{\mathrm{triv-triv}}$ for all i_1, i_2, so it generates a copy of triv $-$ triv by the same argument as Example 2.4.

Proposition 2.6 ([13])**.** *As an $\mathbf{F}_2[x_1, x_2, x_3]^{S_3}$-module, $Q_m(3, \mathbf{F}_2)_{\mathrm{triv}}$ is freely generated by* 1*.*

Note that by the classification of indecomposables in Proposition 2.3, every extension of std and every extension of a module by std splits. Thus $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ is a direct summand of $Q_m(3, \mathbf{F}_2)$ (whose complement is $Q_m(3, \mathbf{F}_2)_{\mathrm{triv-triv}}$), and we mainly consider $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$. $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ is generated as a $\mathbf{F}_2[x_1, x_2, x_3]^{S_3}$-module by homogeneous copies of std, so following [13], we consider *generating representations* of $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ as homogeneous copies of std in a generators and relations presentation of $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ with a minimal generator set.

2.1.1 Quasi-invariants at half-integers

Note that if $\Bbbk$ is a field with $\operatorname{char}\Bbbk \neq 2$ and $m \in \mathbb{Z}_{\geq 0}$, then for any $K \in \Bbbk[x_1, \ldots, x_n]$, $(x_{i_1} - x_{i_2})^{2m} | (1 - s_{i_1i_2})K$ implies $(x_{i_1} - x_{i_2})^{2m+1} | (1 - s_{i_1i_2})K$ since $(1 - s_{i_1i_2})K$ is $s_{i_1i_2}$-antiinvariant, hence the exponent $2m + 1$ in the definition of quasi-invariant polynomials. But this does not hold in characteristic 2, since there is no concept of antiinvariants. Indeed, one can check that for $K = x_1^2 + x_2^2$, we have $(x_{i_1} - x_{i_2})^2 | (1 - s_{i_1i_2})K$ for all i_1, i_2, but $(x_{i_1} - x_{i_2})^3 \nmid |(1 - s_{i_1i_2})K$ if $i_1 = 1, 2$, $i_2 \neq 1, 2$.

We encapsulate this data by extending the definition of quasi-invariants to half-integers when $p = 2$. For example, $K = x_1^2 + x_2^2$ is $\frac{1}{2}$-quasi-invariant, and this is in fact the minimal degree non-symmetric $\frac{1}{2}$-quasi-invariant polynomial. Proposition 2.1 still holds when m, m' are half-integers, and the definitions of $Q_m(3, \mathbf{F}_2)_{\mathrm{triv}}$, $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ also naturally extend to half-integer m. So from now on, whenever we refer to quasi-invariants in characteristic 2 we let m be a half-integer.

2.2 Preliminary definitions for $p = 3$

Next, we define the indecomposable and irreducible representations of S_3.

Proposition 2.7 ([1])**.** *There are* 6 *indecomposable or irreducible representations in S_3 in characteristic* 3*:*

1. triv *is the irreducible representation of S_3 that is acted on trivially by S_3.*
2. sign *is the irreducible representation of S_3 that is acted on by negation by the transpositions.*
3. sign $-$ triv *is the indecomposable representation that contains a copy of* triv *as a subrepresentation, such that the quotient of* sign $-$ triv *by this subrepresentation is* sign*.*
4. triv $-$ sign *is the indecomposable representation that contains a copy of* sign *as a subrepresentation, such that the quotient of* triv $-$ sign *by this subrepresentation is* triv*.*
5. triv $-$ sign $-$ triv *is the indecomposable representation that contains a copy of* sign $-$ triv *as a subrepresentation, such that the quotient of* triv $-$ sign $-$ triv *by this subrepresentation is* triv*.*

6. sign − triv − sign *is the indecomposable representation that contains a copy of* triv − sign *as a subrepresentation, such that the quotient of* sign − triv − sign *by this subrepresentation is* sign.

Provided are some examples of copies of these indecomposable representations:

Example 2.8. The space $W \subset \mathbf{F}_3[x_1, x_2, x_3]$ spanned by $x_1 + x_2 + x_3$ and $x_1 - x_2$ over $\mathbf{F}_3$ is copy of sign − triv. Indeed, the space $T \subset W$ spanned by $x_1+x_2+x_3$ is a copy of triv. One can check $x_1 - x_2 \in W/T$ is acted by negation by all transpositions in S_3 and W/T is 1-dimensional so W/T is a copy of sign. Finally, it is easy to show that there are no copies of triv or sign in W other than T. Since V has a unique irreducible subrepresentation, it is indecomposable, and we conclude that it is a copy of sign − triv.

Example 2.9. The space $V \subset \mathbf{F}_3[x_1, x_2, x_3]$ consisting of homogeneous linear polynomials is a copy of triv − sign − triv. Indeed, $W \subset V$ from Example 2.8 is a copy of sign − triv. Then V/W is one-dimensional, and one can check that it is a copy of triv. Finally, it is easy to show that there are no copies of triv or sign in V other than T, so V has a unique irreducible subrepresentation, it is indecomposable, and we conclude that it is a copy of triv − sign − triv.

Example 2.10. Similarly, one may check that the space U spanned by

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3), \qquad -x_1^2x_2 - x_1^2x_3 + x_1x_2^2 + x_1x_3^2$$

over $\mathbf{F}_3$ is a copy of triv − sign and that the space spanned by

$$(x_1 - x_2)(x_1 - x_3)(x_2 - x_3), \qquad -x_1^2x_2 - x_1^2x_3 + x_1x_2^2 + x_1x_3^2, \qquad (x_1 - x_2)x_1x_2$$

is a copy of sign − triv − sign.

Similarly to the $p = 2$ case, we define $Q_m(3, \mathbf{F}_3)_{\text{sign}}$ and $Q_m(3, \mathbf{F}_3)_{\text{triv}}$ to be the direct sum of all copies of sign and triv in $Q_m(3, \mathbf{F}_3)$, respectively.

Proposition 2.11 ([13])**.** *As* $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}$*-modules,*

1. $Q_m(3, \mathbf{F}_3)_{\text{triv}}$ *is freely generated by* 1.
2. $Q_m(3, \mathbf{F}_3)_{\text{sign}}$ *is freely generated by* $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1}$.

Next we define $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ as the direct sum of all copies of sign, triv, and sign − triv. For this paper we consider generators of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ to be homogeneous polynomials other than 1 and $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1}$ such that they are in the (-1)-eigenspace of s_{12} and are in a generators and relations presentation of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ as an $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}$-module with the least number of generators. Moreover, if K is a generator of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ then it necessarily generates a copy of sign − triv since we assumed K neither generates triv nor sign.

Remark 2.12. Similar to in the $p = 2$ case, we cannot define $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ to exclude copies of sign since we can add elements of $Q_m(3, \mathbf{F}_3)_{\text{sign}}$ to copies of sign − triv and still obtain a copy of sign − triv. For example, the spaces spanned by

$$\left(x_1^6 - x_2^6\right)(x_1 + x_2 + x_3)^3, \qquad \left(x_1^6 + x_2^6 + x_3^6\right)(x_1 + x_2 + x_3)^3$$

and

$$\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^3 + \left(x_1^6 - x_2^6\right)(x_1 + x_2 + x_3)^3, \qquad \left(x_1^6 + x_2^6 + x_3^6\right)(x_1 + x_2 + x_3)^3$$

generate two copies of sign − triv in $Q_1(3, \mathbf{F}_3)$, and their sum contains

$$\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^3 \in Q_1(3, \mathbf{F}_3)_{\text{sign}}.$$

Remark 2.13. One could define subspaces of $Q_m(3, \mathbf{F}_3)$ for triv − sign − triv, sign − triv − sign, triv − sign similar to $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, however this is not particularly helpful, as unlike for $p = 2$, we cannot decompose $Q_m(3, \mathbf{F}_3)$ into a direct sum of subspaces of this form. The space $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ is still relevant, as it is the critical piece to understanding quasi-invariants in characteristic 3, as we see in Sections 4 and 5.

3 Quasi-invariants in characteristic 2

In this section, we write down explicit generators for $Q_m(3, \mathbf{F}_2)$ and prove Theorem 1.3 for $p = 2$. Note that we already know the structure of $Q_m(3, \mathbf{F}_2)_{\text{triv}}$ from Proposition 2.6. We start by extending this to $Q_m(3, \mathbf{F}_2)_{\text{triv}-\text{triv}}$.

Proposition 3.1. *As an $\mathbf{F}_2[x_1, x_2, x_3]^{S_3}$-module, $Q_m(3, \mathbf{F}_2)_{\text{triv}-\text{triv}}$ is freely generated by 1 and $E_{\text{triv}-\text{triv}} \prod (x_{i_1} - x_{i_2})^{2m}$.*

Proof. Let K be a nonsymmetric element of $Q_m(3, \mathbf{F}_2)_{\text{triv}-\text{triv}}$ so that $(x_{i_1} - x_{i_2})^{2m+1}$ divides $(1 + s_{i_1 i_2})K$. Because

$$(1 + s_{12})K = (1 + s_{13})K = (1 + s_{23})K,$$

we have $(1 + s_{i_1 i_2})K = P \prod (x_{i_1} - x_{i_2})^{2m+1}$ for some symmetric polynomial P. Letting $G = E_{\text{triv}-\text{triv}} \prod (x_{i_1} - x_{i_2})^{2m}$ yields $(1 + s_{i_1 i_2})G = \prod (x_{i_1} - x_{i_2})^{2m+1}$. Thus $(1 + s_{i_1 i_2})PG = (1 + s_{i_1 i_2})K$ and $(1 + s_{i_1 i_2})(PG - K) = 0$, so $PG - K$ is symmetric and K is generated by G and 1. Moreover, since G is not symmetric, P and G have no relation implying freeness. ■

We have an explicit description of $Q_m(3, \mathbf{F}_2)_{\text{triv}-\text{triv}}$, so it remains to compute the generators and relations of $Q_m(3, \mathbf{F}_2)_{\text{std}}$. A number of the properties of $Q_m(3, \mathbf{F}_p)$ for $p > 3$ found in [13] are true for $Q_m(3, \mathbf{F}_2)$. We prove these first.

If V is a copy of std, then we denote by $V_{i_1 i_2}$ the 1-eigenspace of $s_{i_1 i_2}$ in V.

Lemma 3.2. *Let V be a copy of* std *in $Q_m(3, \mathbf{F}_2)_{\text{std}}$, and let $K \in V_{i_1 i_2}$. Then we have $K + sK + s^2 K = 0$, where $s = (1\,2\,3) \in S_3$ and $K = (x_{i_1} - x_{i_2})^{2m+1} K'$ for some polynomial K' that is invariant under the action of $s_{i_1 i_2}$. Conversely, let K' be an s_{12}-invariant polynomial such that*

$$(x_1 - x_2)^{2m+1} K' + (x_2 - x_3)^{2m+1} sK' + (x_3 - x_1)^{2m+1} s^2 K' = 0.$$

Then $(x_1 - x_2)^{2m+1} K'$ belongs to the 1-eigenspace of s_{12} in some copy of std *inside $Q_m(3, \mathbf{F}_2)_{\text{std}}$.*

Proof. For the first statement, $K + sK + s^2 K = 0$ holds for any copy of std. For the next, suppose $\{i_1, i_2, i_3\} = \{1, 2, 3\}$ for some integer i_3. Then $(1 - s_{i_1 i_3})K = s_{i_2 i_3} K$, so $(x_{i_1} - x_{i_3})^{2m+1} | s_{i_2 i_3} K$, implying $(x_{i_1} - x_{i_2})^{2m+1} | K$. The second statement follows from the proof in [13]. ■

Corollary 3.3. *Let V be a generating representation of $Q_m(3, \Bbbk)_{\text{std}}$ and let $K \in V_{i_1 i_2}$. Let us write $K = (x_{i_1} - x_{i_2})^{2m+1} K'$. Then K' is not divisible by any nonconstant symmetric polynomial.*

The proof of this statement is identical to the one in [13].

Lemma 3.4. *Let V, W be distinct generating representations of $Q_m(3, \Bbbk)_{\text{std}}$. Let $K \in V_{12}$, $L \in W_{12}$. For $\sigma K \sigma L := (\sigma K)(\sigma L)$, we have that $KL + s_{13} K s_{23} L$ is a nonsymmetric element of $Q_m(3, \Bbbk)_{\text{triv}-\text{triv}}$ and $\deg V + \deg W \geq 6m + 3$.*

Proof. $KL + s_{13}Ks_{23}L$ is an element of $Q_m(3, \mathbf{F}_2)$ since the quasi-invariants form a ring by Proposition 2.1. Using that $s_{12}K = K$ and $s_{12}L = L$, we have that

$$\begin{aligned}
(1 + s_{12})(KL + s_{13}Ks_{23}L) &= s_{23}Ks_{13}L + s_{13}Ks_{23}L,\\
(1 + s_{13})(KL + s_{13}Ks_{23}L) &= KL + s_{13}Ks_{23}L + s_{13}Ks_{13}L + Ks_{23}L = Ks_{13}L + s_{13}KL,\\
(1 + s_{23})(KL + s_{13}Ks_{23}L) &= KL + s_{13}Ks_{23}L + s_{23}Ks_{23}L + s_{13}KL = Ks_{23}L + s_{23}KL.
\end{aligned}$$

One can check that each polynomial is a transposition of another and that they are symmetric due to the structure of triv − triv, so they are all the same symmetric polynomial. Thus $KL + s_{13}Ks_{23}L$ lies in a quotient of a copy of triv − triv. Note that by the same argument as in [13], we have $Ks_{23}L + s_{23}KL \neq 0$, so $KL + s_{13}Ks_{23}L$ is nonsymmetric and must generate a copy of triv − triv.

By Proposition 3.1, $KL + s_{13}Ks_{23}L$ has degree at least $6m + 3$, so $\deg V + \deg W \geq 6m + 3$ as desired. ■

Lemma 3.5. *Assume that there exist generating representations V, W of $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ such that $\deg V + \deg W = 6m + 3$. Then $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ is a free module over $\Bbbk[x_1, x_2, x_3]^{S_3}$ generated by V and W.*

Proof. Assume for the sake of contradiction there exists another generator U of $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$. Supposing $\deg W \geq \deg V$, by Lemma 3.4, $\deg U \geq \deg W$. By Lemma 3.4, if $K \in V_{12}$, $L \in W_{12}$, and $T \in U_{12}$ then $KL + s_{13}Ks_{23}L$ and $KT + s_{13}Ks_{23}T$ are both nonsymmetric elements of $Q_m(3, \mathbf{F}_2)_{\mathrm{triv-triv}}$. Moreover, we have

$$(1 + s_{12})(KL + s_{13}Ks_{23}L) = s_{23}Ks_{13}L + s_{13}Ks_{23}L = \prod (x_{i_1} - x_{i_2})^{2m+1},$$

and

$$(1 + s_{12})(KT + s_{13}Ks_{23}T) = s_{23}Ks_{13}T + s_{13}Ks_{23}T = Q \prod (x_{i_1} - x_{i_2})^{2m+1}$$

for some symmetric polynomial Q. From there we may proceed identically to [13]. ■

Now, we are ready to prove Theorem 1.3 for $p = 2$.

Theorem 3.6. *Let a be the largest natural number such that $2^a < 2m + 1$. Then $Q_m(3, \mathbf{F}_2)_{\mathrm{std}}$ is freely generated by $(x_1 - x_2)^{2^{a+1}}$ and $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2m+1-2^a}$.*

Remark 3.7. Note that when m is an integer, the degrees of the generators in this theorem agree with the degrees conjectured in [11]. In particular, when 2^{a+1} is one of $3m + 1$, $3m + 2$, we actually have that the Hilbert series of $Q_m(3, \mathbf{F}_2)$ and $Q_m(3, \mathbf{Q})$ agree, so $(x_1 - x_2)^{2^{a+1}}$, $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2m+1-2^a}$ are the reductions modulo 2 of the generators of $Q_m(3, \mathbf{Q})$, when written as integer polynomials with coprime coefficients.

Proof of Theorem 3.6. We prove this by induction on m.

The generators of $Q_0(3, \mathbf{F}_2)_{\mathrm{std}}$ are $(x_1 - x_2)$ and $(x_1 - x_2)^2$, completing our base case.

Let j be a half-integer, and suppose that $Q_{j-\frac{1}{2}}(3, \mathbf{F}_2)_{\mathrm{std}}$ is freely generated by $(x_1 - x_2)^{2^{a+1}}$ and $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j-2^a}$, where 2^a is the greatest such power of 2 less than $2j$. If $2j \neq 2^{a+1}$, then 2^a is the largest power of 2 less than $2j+1$, so $(x_1 - x_2)^{2^{a+1}}$ and $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j+1-2^a}$ are both in $Q_j(3, \mathbf{F}_2)$. Further, $(x_1 - x_2)^{2^{a+1}}$ must be a generator and if $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j+1-2^a}$ is a not a generator, by Lemma 3.4, $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j+1-2^a}$ is generated by $(x_1 - x_2)^{2^{a+1}}$ which implies a relation between $(x_1 - x_2)^{2^{a+1}}$ and $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j-2^a}$. Because they freely generate $Q_{j-\frac{1}{2}}(3, \mathbf{F}_2)$, this is impossible. Thus $(x_1 - x_2)^{2^{a+1}}$ and $(x_1 - x_2)^{2^a} \prod (x_{i_1} - x_{i_2})^{2j+1-2^a}$ freely generate $Q_j(3, \mathbf{F}_2)_{\mathrm{std}}$ by Lemma 3.5.

If $2j = 2^{a+1}$, then both $(x_2-x_3)^{2^{a+1}}$ and $(x_2-x_3)^{2^{a+2}}\prod(x_{i_1}-x_{i_2})^{2j+1-2^a}$ lie in $Q_j(3,\mathbf{F}_2)_{\mathrm{std}}$. The former is a generator by our inductive hypothesis. Since $2^{a+1}+2^{a+2}+3 = 6j+3$, if the latter is not a generator, then by Lemma 3.4, $(x_2-x_3)^{2^{a+2}\prod(x_{i_1}-x_{i_2})^{2j+1-2^a}}$ is generated by $(x_2-x_3)^{2^{a+1}}$, which is false. Thus $(x_2-x_3)^{2^{a+1}}$ and $(x_2-x_3)^{2^{a+2}}\prod(x_{i_1}-x_{i_2})^{2j+1-2^a}$ freely generate $Q_j(3,\mathbf{F}_2)_{\mathrm{std}}$ by Lemma 3.5 as desired. ■

4 Properties of 3 variable quasi-invariants

Similarly to the $p=2$ case, we can adapt many of the properties of $Q_m(3,\mathbf{F}_p)$ for $p>3$ found in [13] to the $p=3$ case. We accomplish this by converting std to sign − triv. For example, in $Q_0(3,\mathbf{F}_p)$ for $p>3$, the space spanned by x_1-x_2, x_1-x_3 is a copy of std. However, in $Q_0(3,\mathbf{F}_3)$, the space spanned by x_1-x_2, x_1-x_3 becomes a copy of sign − triv. Using this, we may show that there are equivalents of Lemmas 3.2–3.5 from [13] in characteristic 3.

We define $V^-_{i_1i_2}$ to be the (-1)-eigenspace of $s_{i_1i_2}$ in V, where V is a copy of std or sign − triv. Note that if $v\in V^-_{i_1i_2}$ we have $v = s_{23}v + s_{13}v$. The following lemma and corollary correspond to Lemma 3.2 and Corollary 3.3 from [13], respectively.

Lemma 4.1. *Let V be a copy of* sign − triv *in $Q_m(3,\mathbf{F}_3)_{\mathrm{sign-triv}}$, and let $K\in V^-_{i_1i_2}$. Then we have $K+sK+s^2K=0$, where $s=(1\,2\,3)\in S_3$ and $K=(x_{i_1}-x_{i_2})^{2m+1}K'$ for some polynomial K' that is invariant under the action of $s_{i_1i_2}$. Conversely, let K' be an s_{12}-invariant polynomial such that*

$$(x_1-x_2)^{2m+1}K' + (x_2-x_3)^{2m+1}sK' + (x_3-x_1)^{2m+1}s^2K' = 0.$$

Then $(x_1-x_2)^{2m+1}K'$ either belongs to $Q_m(3,\mathbf{F}_3)_{\mathrm{sign}}$ or the (-1)-eigenspace of s_{12} in some copy of sign − triv *inside $Q_m(3,\mathbf{F}_3)_{\mathrm{sign-triv}}$.*

Proof. The proof is largely the same as in [13]; the only difference is in the last step. Namely, now we have 2 2-dimensional indecomposable representations sign − triv and triv − sign, but an element in the (-1)-eigenspace of s_{12} in triv − sign must be in a copy of sign. ■

Corollary 4.2. *Let K be a generator of $Q_m(3,\mathbf{F}_3)_{\mathrm{sign-triv}}$ in $V^-_{i_1i_2}$ for some copy V of* sign − triv *and write $K=(x_{i_1}-x_{i_2})^{2m+1}K'$. Then K' is not divisible by any nonconstant symmetric polynomial.*

The proof of this corollary is identical to the proof of [13, Corollary 3.3].

We define generators of $Q_m(3,\mathbf{F}_3)$ to be "distinct" if they are either in different degrees, or if no linear combination of them over $\mathbf{F}_3$ is generated by lower degree generators.

Lemma 4.3. *Let K and L be distinct generators of $Q_m(3,\Bbbk)_{\mathrm{sign-triv}}$, and let V and W be the copies of* sign − triv *generated by K and L respectively such that $K\in V^-_{i_1i_2}$ and $L\in W^-_{i_1i_2}$. Then $Ks_{23}L - Ls_{23}K$ is a nonzero element of $Q_m(3,\mathbf{F}_3)_{\mathrm{sign}}$ and $\deg V + \deg W \geq 6m+3$.*

Noting that $\wedge^2(\mathrm{sign}-\mathrm{triv}) = \mathrm{sign}$, the proof of this lemma is also identical to the proof of [13, Lemma 3.4].

Lemma 3.5 from [13] does not completely hold in characteristic 3. A very similar and useful version does, however, and we have the following.

Lemma 4.4. *Assume that there exists generators K and L of $Q_m(3,\mathbf{F}_3)_{\mathrm{sign-triv}}$ such that $\deg K + \deg L = 6m+3$. Then $Q_m(3,\mathbf{F}_3)_{\mathrm{sign-triv}}$ is freely generated by K, L, and 1 over $\mathbf{F}_3[x_1,x_2,x_3]^{S_3}$.*

Proof. We note that $(L+s_{23}L)K-(K+s_{23}K)L = c\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m+1}$ for some $c \neq 0$ by Lemma 4.3. Moreover, $L+s_{23}L$ and $K+s_{23}K$ are symmetric because K and L are both acted on by negation by s_{12}, so elements in $Q_m(3,\mathbf{F}_3)_{\text{sign}}$ are generated by K and L. From there, the fact that $Q_m(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ is generated by K, L, and 1 over $\mathbf{F}_3[x_1,x_2,x_3]^{S_3}$ follows from the first part of the proof from [13].

To prove freeness, assume for the sake of contradiction that there was a relation $PK+QL+S=0$ for symmetric polynomials P, Q, and S. PK and QL are both in the (-1)-eigenspace of s_{12} while S is not, so $S=0$. Thus we have $PK=-QL$ and from there we can proceed the same as [13].

■

5 Ren–Xu counterexamples

We aim to explicitly describe the Hilbert series of $Q_m(3,\mathbf{F}_3)$. To do so we wish to identify the generators of $Q_m(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$.

In [11], Ren and Xu found polynomials of the form $P_k^{3^a}\prod(x_{i_1}-x_{i_2})^{2b}$ in $Q_m(3,\mathbf{F}_3)$ with degree strictly less than $3m+1$, where P_k is the map of the $3k+1$ degree generator of $Q_k(3,\mathbf{Q})$ into characteristic 3 and where a, k, and b are natural numbers. We refer to these polynomials as Ren–Xu counterexamples as they demonstrate the Hilbert series of $Q_m(3,\mathbf{F}_3)$ differs from that of $Q_m(3,\mathbf{Q})$ for certain m.

Definition 5.1. Let $\overline{P_k}$ be the generator of $Q_k(3,\mathbf{Q})$ of degree $3k+1$ in the (-1)-eigenspace of s_{12}, expressed as an integer polynomial with coprime coefficients. Let P_k be the image of $\overline{P_k}$ under the quotient map $\mathbb{Z}[x_1,x_2,x_3]\to\mathbf{F}_3[x_1,x_2,x_3]$. Define the set X as the set of all natural numbers m such that $Q_m(3,\mathbf{F}_3)$ has a Ren–Xu counterexample. Let R_m be a lowest degree Ren–Xu counterexample in $Q_m(3,\mathbf{F}_3)$ for all $m\in X$.

A key step in describing the Hilbert series of $Q_m(3,\mathbf{F}_3)$ is proving Ren–Xu's conjecture [11] for $n=3$ and $p=3$.

Conjecture 5.2 ([11])**.** *If the Hilbert series of $Q_m(n,\mathbf{F}_p)$ differs from that of $Q_m(n,\mathbf{Q})$, then there exists integers $a\geq 0$ and $k\geq 0$ such that*

$$\frac{mn(n-2)+\binom{n}{2}}{n(n-2)k+\binom{n}{2}-1}\leq p^a\leq\frac{mn}{nk+1}.$$

The main step for proving the conjecture for $n=3$, $p=3$ is the following theorem.

Theorem 5.3. *$Q_m(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ is either freely generated by a generator of degree $3m+1$, $3m+2$, and the polynomial 1, or it is freely generated by R_m, another generator in degree $6m+3-\deg R_m$, and the polynomial 1.*

To prove this theorem, we first describe the Ren–Xu counterexamples.

Lemma 5.4. *If $m\in X$, we must have $R_m=P_k^{3^a}\prod(x_{i_1}-x_{i_2})^{2b}$, where a, b, k are natural numbers and $k\notin X$.*

Proof. Assume for contradiction that there exists a nonnegative integer $m\in X$ such that $R_m=P_k^{3^a}\prod(x_{i_1}-x_{i_2})^{2b}$, where a, b, k are natural numbers and $k\in X$. Then if $R_k=P_l^{3^c}\prod(x_{i_1}-x_{i_2})^{2d}$, the polynomial

$${R_k}^{3^a}\prod(x_{i_1}-x_{i_2})^{2b}=P_l^{3^{a+c}}\prod(x_{i_1}-x_{i_2})^{2d\cdot 3^a+2b}$$

has a strictly smaller degree than R_m since $\deg R_k<3k+1=\deg P_k$. Moreover, it is at least m-quasi-invariant, so it is a Ren–Xu counterexample for $Q_m(3,\mathbf{F}_3)$. Yet R_m is a minimal counterexample, giving a contradiction. ■

This lemma allows us to consider only counterexamples $P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2b}$ such that $Q_k(3, \mathbf{F}_3)$ does not contain a Ren–Xu counterexample.

From [11], the Hilbert series for $Q_m(3, \mathbf{F}_3)$ differs from characteristic 0 when there exists $a \in \mathbf{N}_0$ such that

$$\frac{1}{3} \leq \left\{ \frac{m}{3^a} \right\} \leq \frac{2}{3} - \frac{1}{3^a}.$$

Notice this is equivalent to $m \pmod{3^a}$ being in $\{3^{a-1}, 3^{a-1}+1, \ldots, 2 \cdot 3^{a-1} - 1\}$.

Lemma 5.5. *If $m \notin X$, then the base 3 representation of m contains no 1's.*

Proof. Suppose m had the digit 1 in the a-th position from the right. Then $m \pmod{3^a}$ has a leading digit of 1 if we choose $m \pmod{3^a}$ to be between 0 and $3^a - 1$ inclusive. However, this implies that $m \pmod{3^a}$ is in $\{3^{a-1}, 3^{a-1}+1, \ldots, 2 \cdot 3^{a-1} - 1\}$, so m is a counterexample. ■

Corollary 5.6. *If $m \notin X$, then m is even.*

Proof. From Lemma 5.5 m has no 1's in its base 3 representation, so

$$m = \sum_{j=0} c_j 3^j,$$

where c_j is 0 or 2. Thus m must be even. ■

Corollary 5.7. *For all $m \notin X$, we have $m + 1 \in X$.*

Proof. By Corollary 5.6, if $m \notin X$, m is even. Then $m + 1$ is odd, so by the contrapositive of Corollary 5.6, $m + 1 \in X$. ■

Now we begin describing the degrees of Ren–Xu counterexamples.

Lemma 5.8. *If $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a generator in degree $3m + 1$, then $m + 1 \in X$ and* $\deg R_{m+1} = 3m + 3$.

Proof. If $m \in X$, we must have $\deg R_m < 3m + 1$. This implies a generator in a degree less than $3m + 1$, violating Lemma 4.3. Thus $m \notin X$, implying that $m + 1 \in X$ by Corollary 5.7.

Because $\deg R_{m+1} < 3m + 4$ and $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}} \subset Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$, we have $3m + 1 \leq \deg R_{m+1} < 3m + 4$. By construction $3 | \deg R_{m+1}$, so $\deg R_{m+1} = 3m + 3$. ■

We now introduce a few useful lemmas.

Lemma 5.9. *Suppose $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a smallest degree generator L in degree $3m + 1$. Assume that for all $j < m$, if $j \notin X$, then $Q_j(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a degree $3j + 1$ generator. Then $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}}$ has no nonsymmetric degree $3m + 1$ or $3m + 2$ element.*

Proof. Any nonsymmetric $3m + 1$ degree element in $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}}$ must be a scalar multiple of L, so assume for contradiction L is in $Q_{m+1}(3, \mathbf{F}_3)$. By Lemma 5.8, $R_{m+1} = P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2b}$ is in degree $3m + 3$ for natural numbers a, b, k. By Lemma 5.4, $k \notin X$ implying P_k is a $3k + 1$ generator of $Q_k(3, \mathbf{F}_3)_{\text{sign-triv}}$ using our assumption. Moreover, with any other generator in a degree less than $3m + 3$ violating Lemma 4.3, R_{m+1} must be generated by L, so $P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2b} = SL$ for some degree 2 symmetric polynomial S. A degree 2 symmetric polynomial divisible by $(x_{i_1} - x_{i_2})$ is impossible, so $S | P_k^{3^a}$ which implies either $S | P_k$ or $(x_1 + x_2 + x_3) | P_k$. Since $\overline{P_k}$ is in the (-1)-eigenspace of s_{12}, P_k is as well and by Lemma 4.1 we have $P_k = P'_k (x_1 - x_2)^{2k+1}$. In both cases either $S | P'_k$ or $(x_1 + x_2 + x_3) | P'_k$. However, by our assumption P_k is a generator, so P_k' is not divisible by any nonconstant symmetric polynomial by Corollary 4.2.

Similarly, suppose for contradiction that K is a nonsymmetric element of $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}}$ of degree $3m+2$. Since $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}}$ has no nonsymmetric $3m+1$ degree element, K must be a generator. By Lemma 4.3, K is the only generator in degree less than $3m+3$, so $P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2b}$ is a symmetric polynomial multiple of K. However, the only symmetric polynomials of degree 1 are multiples of $x_1 + x_2 + x_3$, implying $(x_1 + x_2 + x_3)|P_k$ which is impossible by Corollary 4.2. ■

Note that by [9], $Q_m(3, \mathbf{Q})_{\text{std}}$ has generators in degree $3m+1$ and $3m+2$, and by [13], such generators with even degree are divisible by $x_1 + x_2 - 2x_3$. Let π be the canonical mapping from characteristic 0 to characteristic 3. We then have the following lemma.

Lemma 5.10. *Suppose $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a generator L in degree $3m+1$. We can choose the generators of $Q_m(3, \mathbf{Q})_{\text{std}}$ to be integer polynomials L' and $(x_1 + x_2 - 2x_3)K'$ with $\pi(K') = \pi(L') = L$. Moreover, if*

$$G = (x_1 + x_2 + x_3)\left(\frac{K' - L'}{3}\right) - x_3 K',$$

then

$$\pi(G) = (x_1 + x_2 + x_3)\pi\left(\frac{K' - L'}{3}\right) - x_3 L$$

is a degree $3m+2$ generator for $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$.

Proof. Let L' be an arbitrary $3m+1$ degree generator of $Q_m(3, \mathbf{Q})_{\text{std}}$ with coprime integer coefficients in the (-1)-eigenspace of s_{12}. By Lemma 4.1, $\pi(L')$ is an element of the (-1)-eigenspace of s_{12} in $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ and if $\pi(L')$ is not a scalar multiple of L then there must exist some other generator of $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ with degree less than or equal to $3m+1$. That generator and L would violate Lemma 4.3, so we may set $\pi(L') = L$.

A higher degree generator of $Q_m(3, \mathbf{Q})_{\text{std}}$ has degree $3m+2$. With $\deg L = 3m+1$ implying $m \notin X$, $3m+2$ is even by Corollary 5.6. Using [13], we let $(x_1 + x_2 - 2x_3)K'$ be an arbitrary degree $3m+2$ generator for $Q_m(3, \mathbf{Q})_{\text{std}}$ with coprime integer coefficients. Similarly, $\pi((x_1 + x_2 - 2x_3)K') = (x_1 + x_2 + x_3)\pi(K')$ is an element of $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$, so $\pi(K')$ is a non-symmetric polynomial of degree $3m+1$ in $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$. Thus it must be a scalar multiple of L, and we may set $\pi(K') = L$.

Let $G = (x_1 + x_2 + x_3)\left(\frac{K' - L'}{3}\right) - x_3 K'$. Since

$$(x_1 + x_2 - 2x_3)K' - (x_1 + x_2 + x_3)L' = (x_1 + x_2 + x_3)\left(K' - L'\right) - 3x_3 K'$$

and $\pi(K' - L') = L - L = 0$, we have $G \in Q_m(3, \mathbf{Q}) \cap \mathbf{Z}[x_1, x_2, x_3]$. Then

$$\pi(G) = (x_1 + x_2 + x_3)\pi\left(\frac{K' - L'}{3}\right) - x_3 L.$$

If $\pi(G)$ generated by L, we must have $\pi(G) = c(x_1 + x_2 + x_3)L$ for some $c \in \mathbf{F}_3$ since $\deg(\pi(G)) = \deg(L) + 1$. However, $x_1 + x_2 + x_3$ does not divide $x_3 L$ since L is a generator, so $x_1 + x_2 + x_3 \nmid \pi(G)$. Then if $\pi(G)$ was not a generator, there must be some generator other than L for $Q_m(3, \mathbf{F}_3)$ in degree less than $3m+2$ which violates Lemma 4.3. Thus, $\pi(G)$ is a generator. ■

We aim to prove that minimum Ren–Xu counterexamples are generators and represent the only cases, where the Hilbert series of the quasi-invariants differs between characteristics 0 and 3. To this end, we describe the degree of Ren–Xu counterexamples.

Example 5.11. We notice a "staircase" pattern for Ren–Xu counterexamples. The following are counterexamples for $m = 3, 4, 5$:

$$(x_1 - x_2)^9, \qquad (x_1 - x_2)^9, \qquad (x_1 - x_2)^9 \prod (x_{i_1} - x_{i_2})^2.$$

We note that since $(x_1 - x_2)^9 \in Q_4(3, \mathbf{F}_3)$, $(x_1 - x_2)^9$ is the Ren–Xu counterexample for both $m = 3$ and $m = 4$. Moreover, the counterexample in $Q_5(3, \mathbf{F}_3)$ is the previous counterexample $(x_1 - x_2)^9$ multiplied by $\prod (x_{i_1} - x_{i_2})^2$ to add the extra factor of $(x_1 - x_2)^2$. In this way the degree of counterexample stays constant for the first half of the "staircase" and climbs by 6 per each increase in m thereafter. Moreover, we note that $m = 2, 6 \notin X$, so our "staircase" is surrounded by non-counterexamples. One can also compute another generator for $m = 3, 4, 5$ in degree 12, 18, and 18 respectively. Since $9+12 = 6\cdot 3+3$, $9+18 = 6\cdot 4+3$, and $15+18 = 6\cdot 5+3$, $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ is freely generated by each of these generators and 1 by Lemma 4.4. This way we see that the upper degree generators form a complement to the lower degree ones, climbing by 6 degrees initially and staying constant for the second half of the staircase.

Visually, the following figure shows the degree of the generators for $Q_m(3, \mathbf{F}_3)$ with respect to m were the staircase pattern and Theorem 5.3 to hold.

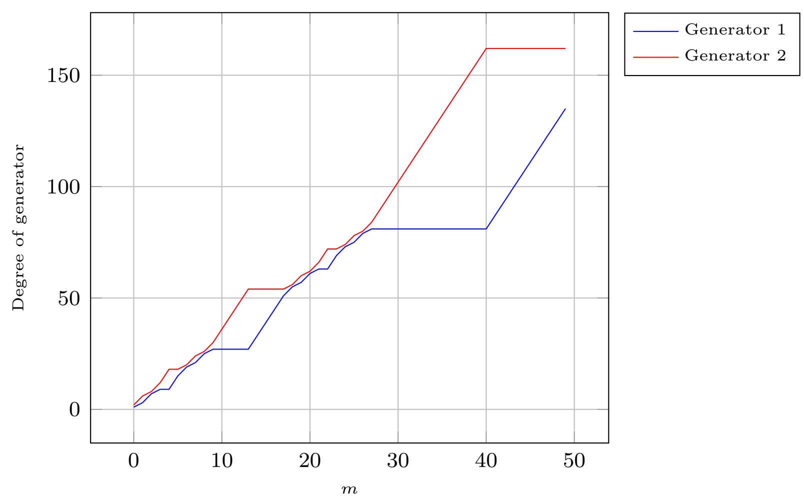

Figure 1. Degrees of generators in characteristic 3 with respect to m.

We prove that Ren–Xu counterexamples follow this staircase pattern.

Lemma 5.12. *Let m be a natural number not in X and let d be the largest integer such that R_{m+1} lies in $Q_{m+d}(3, \mathbf{F}_3)$. Suppose that for all $k \leq m$, if $k \notin X$, then $Q_k(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has a generator in degree $3k + 1$. Then $R_{m+j} = R_{m+1}$ in degree $3m + 3$ for $1 \leq j \leq d$ and $R_{m+j} = R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(j-d)}$ in degree $3m + 3 + 6(j - d)$ for $d < j < 2d$.*

Proof. Let

$$R_{m+1} = P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2b},$$

where k is a nonnegative integer, a is a positive integer, and $b = \max\{0, \frac{2m+3-3^a(2k+1)}{2}\}$. If b is positive, the polynomial $P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2(b-1)}$ has degree less than $3m - 2$ and is at least m-quasi-invariant since $P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2b}$ has degree less than $3m+4$. Thus $P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2(b-1)}$ is a Ren–Xu counterexample for $Q_m(3, \mathbf{F}_3)$, a contradiction.

In this way, we have $R_{m+1} = P_k^{3^a}$. Moreover, $Q_k(3, \mathbf{F}_3)$ must be a non-counterexample by Lemma 5.4, so by our assumption P_k is a generator. By Lemma 5.9, P_k is not in $Q_{k+1}(3, \mathbf{F}_3)$, so the largest power of (x_1-x_2) dividing into R_{m+1} must be $(x_1-x_2)^{3^a(2k+1)}$ and $m+d = \frac{3^a(2k+1)-1}{2}$ by Lemma 4.1. Then for all $1 \leq j \leq d$,

$$\frac{2(m+j)+1-3^a(2k+1)}{2} \leq \frac{2(m+d)+1-3^a(2k+1)}{2} = 0.$$

Thus $R_{m+j} = P_k^{3^a} = R_{m+1}$ which is indeed in degree $3m+3$ by Lemma 5.8.

We claim that for $d < j < 2d$, $m+j \in X$. Let I be the set of integers h such that a Ren–Xu counterexample for $Q_h(3, \mathbf{F}_3)$ is $P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2b}$ for some $b \in \mathbf{Z}_{\geq 0}$. By [11], $m \in I$ if and only if

$$k + \frac{1}{3} \leq \frac{m}{3^a} \leq k + \frac{2}{3} - \frac{1}{3^a},$$

which implies I is $\left\{s, s+1, s+2, \ldots, s+3^{a-1}-1\right\}$ for some $s \equiv 3^{a-1} \pmod{3^a}$. Then note that $m+1 \in I$, yet $m \notin I$ since $m \notin X$. Thus $m \equiv 3^{a-1}-1 \pmod{3^a}$. Since $m+d = \frac{3^a(2k+1)-1}{2} \in I$ as well, we have $s = 3^a k + 3^{a-1}$, $m = 3^a k + 3^{a-1} - 1$, and $d = \frac{3^{a-1}+1}{2}$. Then

$$\frac{3^a(2k+1)-1}{2} < m+j < \frac{3^{a-1}+1}{2} + \frac{3^a(2k+1)-1}{2} = 3^a k + 2 \cdot 3^{a-1},$$

so $m+j$ is in I and thus in X.

If $R_{m+j} = P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2b}$, where $b = \frac{2(m+j)+1-3^a(2k+1)}{2}$ for $d < j < 2d$, then $m+d = \frac{3^a(2k+1)-1}{2}$ implies $b = j-d$. Thus $R_{m+j} = P_k^{3^a} \prod(x_{i_1} - x_{i_2})^{2(j-d)}$ has degree $3m+3+6(j-d)$ as desired. ■

In [13], the first author proved that generators of $Q_m(3, \mathbf{F}_p)_{\text{std}}$ for $p > 3$ lie in $\mathbf{F}_p[x_1 - x_3, x_2 - x_3]$ using that $\mathbf{F}_p[x_1 - x_3, x_2 - x_3, x_1 + x_2 + x_3] = \mathbf{F}_p[x_1, x_2, x_3]$. However, this is not true for $p = 3$ since $x_1 - x_3 + x_2 - x_3 = x_1 + x_2 + x_3$ in characteristic 3, so we instead consider the space $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]$. From now on, we say a polynomial's degree in x_3 is with respect to the basis $\{x_1 - x_3, x_2 - x_3, x_3\}$. Moreover, in [13] the first author defined the polynomial

$$M_d = (x_1 + x_2 - 2x_3)^{2\{\frac{d}{2}\}}(x_1 - x_3)^{\lfloor \frac{d}{2} \rfloor}(x_2 - x_3)^{\lfloor \frac{d}{2} \rfloor}$$

for natural numbers d and proved that homogeneous s_{12}-invariant elements of $\mathbf{F}_p[x_1 - x_3, x_2 - x_3]/(x_1 - x_2)^2$ are equal to constant multiples of M_d. Extending this gives that elements of $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$ are polynomials in x_3 with coefficients that are constant multiples of M_d. Some further nice properties of M_d are the following.

Lemma 5.13. *For any $j, j' \in \mathbf{Z}_{\geq 0}$,*

1. $(x_1 + x_2 + x_3)M_j = M_{j+1}$ *in* $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$.
2. $M_j M_{j'} = M_{j+j'}$ *in* $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$.

Proof. 1. In $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$, for $j \in Z_{\geq 0}$,

$$(x_1 + x_2 + x_3)M_{2j} = (x_1 + x_2 + x_3)(x_1 - x_3)^j(x_2 - x_3)^j = M_{2j+1}$$

and

$$\begin{aligned}(x_1 + x_2 + x_3)M_{2j+1} &= (x_1 + x_2 + x_3)^2(x_1 - x_3)^j(x_2 - x_3)^j \\ &= (x_1 - x_3)(x_2 - x_3)M_{2j} = M_{2j+2}.\end{aligned}$$

2. From (1), we have $M_j = (x_1 + x_2 + x_3)^j$ and $M_{j'} = (x_1 + x_2 + x_3)^{j'}$ in $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$, and our equality follows. ■

This gives us intuition for the following lemmas.

Lemma 5.14. *Let e_1, e_2, and e_3 be the elementary symmetric polynomials for $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}$ in degree 1, 2, and 3 respectively. If n is a natural number such that $n \not\equiv 0 \pmod 3$, for all natural numbers $j < n$ there exists a monomial P in e_1, e_2, e_3 such that P has degree n and degree j in x_3. If n is a natural number such that $n \equiv 0 \pmod 3$, for all natural numbers $j < n-1$ there exists a monomial P in e_1, e_2, e_3 such that P has degree n and degree j in x_3.*

Proof. We choose e_1, e_2, and e_3 to be

$$\begin{aligned} e_1 &= x_1 + x_2 + x_3 = (x_1 - x_3) + (x_2 - x_3), \\ e_2 &= x_1x_2 + x_1x_3 + x_2x_3 = (x_1 - x_3)(x_2 - x_3) + 2((x_1 - x_3) + (x_2 - x_3))x_3, \\ e_3 &= x_1x_2x_3 = (x_1 - x_3)(x_2 - x_3)x_3 + ((x_1 - x_3) + (x_2 - x_3))x_3^2 + x_3^3. \end{aligned}$$

We prove the lemma by decreasing induction on j.

The base case for n where $3 \nmid n$ is $j = n-1$. If $j = n-1$ and $n \equiv 1 \pmod 3$, we can let $P = e_3^{(n-1)/3}e_1$. If $n \equiv 2 \pmod 3$, we let $P = e_3^{(n-2)/3}e_2$. The base case when $3|n$ is $j = n-2$, so we can let $P = e_1e_2e_3^{(n/3)-1}$.

Suppose that, when $3 \nmid n$, for all j' such that $n > j' > j$ where $j \in \mathbf{N}$ and $0 \le j < n-1$ there exists a monomial in e_1, e_2, e_3 with degree n and degree j' in x_3. Suppose the same for when $3|n$ but with $n-1 > j' > j$ and $j < n-2$. Then there exists a monomial $m = e_1^a e_2^b e_3^c$ with degree $j+1$ in x_3 in $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$. If $b \neq 0$ we can take the monomial $e_1^{a+2}e_2^{b-1}e_3^c$ to be P since it has degree n and degree j in x_3. If $b = 0$ and $a, c > 0$, then we take $P = e_1^{a-1}e_2^{b+2}e_3^{c-1}$. Finally, we are left with the cases $a, b = 0$ or $b, c = 0$. The former would imply $m = e_3^{\frac{n}{3}}$ is our monomial, but $3 \nmid n$ would imply m is not a polynomial and $3|n$ implies m has degree $j+1 = n$ in x_3 and $j = n-1 \not< n-2$. For the latter case, we have that $a = n$, so $m = e_1^n$ implies that $j+1 = 0$ which is below our range for j. ■

Lemma 5.15. *For all $f_j \in \mathbf{F}_3$ and $n \not\equiv 0 \pmod 3$, there exists a $P \in \mathbf{F}_3[x_1, x_2, x_3]^{S_3}$ such that*

$$P = f_0M_nx_3^0 + f_1M_{n-1}x_3^1 + \cdots + f_{n-2}M_2x_3^{n-2} + f_{n-1}M_1x_3^{n-1}$$

in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$. If $n \equiv 0 \pmod 3$, for all $f_j \in \mathbf{F}_3$ there exists a $P \in \mathbf{F}_3[x_1, x_2, x_3]^{S_3}$ such that

$$P = f_0M_nx_3^0 + f_1M_{n-1}x_3^1 + \cdots + f_{n-2}M_2x_3^{n-2}$$

in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$. Moreover, P also satisfies the property that if it has degree k in x_3 in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$, then it has degree k in x_3 in $\mathbf{F}_3[x_1 - x_3, x_2 - x_3, x_3]$.

Proof. A weaker statement is that there exists some fixed $c_0, c_1, \ldots, c_j \in \mathbf{F}_3$ such that for all $f_{j+1}, f_{j+2}, \ldots, f_{n-1} \in \mathbf{F}_3$, there exists a symmetric polynomial

$$\begin{aligned} P \equiv\ & c_0M_nx_3^0 + c_1M_{n-1}x_3^1 + \cdots + c_jM_{n-j}x_3^j \\ & + f_{j+1}M_2x_3^{n-j-1} + f_{j+2}M_1x_3^{n-j-2} + \cdots + f_{n-1}M_1x_3^{n-1} \pmod{(x_1 - x_2)^2}, \end{aligned}$$

when $n \not\equiv 3 \pmod 3$ and $j \in \mathbf{Z}_{\ge 0}$. A similar weaker statement can be made for the $n \equiv 0 \pmod 3$ case. We prove the statement in the lemma by induction on this j.

For the base case when $n \not\equiv 0 \pmod 3$, we claim there exists coefficients $c_j \in \mathbf{F}_3$ such that the polynomial $c_0M_nx_3^0 + c_1M_{n-1}x_3^1 + \cdots + c_{n-2}M_2x_3^{n-2} + c_{n-1}M_1x_3^{n-1}$ is in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$. The symmetric polynomial 0 satisfies these conditions and has degree 0 in x_3. For the base case when $n \equiv 0 \pmod 3$, we claim there exists coefficients $c_0, \ldots, c_{n-2}$ such that the polynomial $c_0M_nx_3^0 + c_1M_{n-1}x_3^1 + \cdots + c_{n-2}M_2x_3^{n-2}$ is in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$. The symmetric polynomial 0 satisfies this.

We consider the case where $n \not\equiv 0 \pmod 3$. Suppose that for all $n \geq j' > j$ there exists coefficients $c_0, \ldots, c_{j'-1}$ such that for all $f_{j'}, f_{j'+1}, \ldots, f_{n-1}$ there exists a symmetric polynomial P such that

$$P = c_0 M_n x_3^0 + c_1 M_{n-1} x_3^1 + \cdots + c_{j'-1} M_{n-j'+1} x_3^{j'-1} + f_{j'} M_{n-j'} x_3^{j'} + \cdots + f_{n-1} M_1 x_3^{n-1}$$

lies in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$, where $j \in \mathbf{N}$, $0 \leq j \leq n-1$. Moreover, suppose the polynomial P exists such that it has degree in x_3 equal to the degree in x_3 in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$.

Consider arbitrary coefficients $f_j, f_{j+1}, \ldots, f_{n-1}$. If they are each 0, then we can take 0 to be our polynomial just like our base case. Otherwise, let l be the greatest natural number $l \geq j$ such that $f_l \neq 0$. If $l = j$, by Lemma 5.14 there exists a monomial m in e_1, e_2, e_3 with degree j in x_3 and we may take $f_j m$ to be our symmetric polynomial.

If $l > j$, by assumption there exists coefficients $c_0, c_1, \ldots, c_j$ such that

$$S = c_0 M_n x_3^0 + c_1 M_{n-1} x_3^1 + \cdots + c_j M_{n-j} x_3^j + f_{j+1} M_{n-j-1} x_3^{j+1} + \cdots + f_{n-1} M_1 x_3^{n-1}$$

lies in $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$. By assumption, S has degree l in x_3.

Without loss of generality let the leading coefficient of m be M_{n-j}, so

$$\begin{aligned} S + (f_j - c_j)m = {} & c'_0 M_n x_3^0 + c'_1 M_{n-1} x_3^1 + \cdots + c'_{j-1} M_{n-j+1} x_3^{j-1} \\ & + f_j M_{n-j} x_3^j + \cdots + f_{n-1} M_1 x_3^{n-1} \end{aligned}$$

for some coefficients $c'_0, c'_1, \ldots, c'_{j-1}$. Moreover, $S + (f_j - c_j)m$ is still a symmetric polynomial and m has degree j in x_3 while S has degree l, so $S + (f_j - c_j)m$ has degree l as desired.

An identical argument holds for $n \equiv 0 \pmod 3$. ∎

Now we have the tools to prove $m \notin X$ implies $m + 1$ begins our staircase.

Lemma 5.16. *Suppose that for all $k \leq m$, if $k \notin X$ then $Q_k(3, \mathbf{F}_3)$ has a $3k+1$ degree generator, where m is a natural number. Then if $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a generator in degree $3m + 1$, $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign-triv}}$ has a generator in degree $3m + 6$.*

Proof. By Lemma 5.10, the generators for $Q_m(3, \mathbf{F}_3)_{\text{sign-triv}}$ are

$$\left((x_1 + x_2 + x_3)\pi\left(\frac{A' - B'}{3}\right) - x_3 B\right)(x_1 - x_2)^{2m+1}$$

in degree $3m + 2$, and

$$B(x_1 - x_2)^{2m+1}$$

in degree $3m + 1$, where $(x_1 - x_2)^{2m+1}(x_1 + x_2 - 2x_3)A'$ and $(x_1 - x_2)^{2m+1}B'$ are the generators of $Q_m(3, \mathbf{Q})_{\text{std}}$, B is an s_{12}-invariant polynomial, and $\pi(A') = \pi(B') = B$.

For the greater degree generator, let $C = \big((x_1 + x_2 + x_3)\pi\big(\frac{A'-B'}{3}\big) - x_3 B\big)$. We would like to show there exists symmetric polynomials P and Q in degree 4 and 5 respectively such that

$$PC + QB \equiv 0 \pmod{(x_1 - x_2)^2}.$$

Since $\frac{PC+BQ}{(x_1-x_2)^2}$ is still s_{12}-invariant, this would then imply $(PC+QB)(x_1-x_2)^{2m+1} \in Q_{m+1}(3, \mathbf{F}_3)$ by Lemma 4.1. Consider writing

$$P = f_0 M_4 x_3^0 + f_1 M_3 x_3^1 + f_2 M_2 x_3^2 + f_3 M_1 x_3^3$$

and

$$Q = h_0 M_5 x_3^0 + h_1 M_4 x_3^1 + h_2 M_3 x_3^2 + h_3 M_2 x_3^3 + h_4 M_1 x_3^4$$

for arbitrary f_j and h_j. By Lemma 5.15, we know that for any choice of f_j and h_j, we have $P, Q \in \mathbf{F}_3[x_1, x_2, x_3]^{S_3}/(x_1 - x_2)^2$.

We claim that $B|\pi\left(\frac{A'-B'}{3}\right)$ in $\mathbf{F}_3[x_1, x_2, x_3]/(x_1 - x_2)^2$. By [13], A' and B' are both polynomials in the variables $(x_1 - x_2)^2$ and $(x_1 - x_3)(x_2 - x_3)$. Moreover, by Lemma 5.9, $(x_1 - x_2)^2 \nmid B$ so $B \equiv cM_m \pmod{(x_1 - x_2)^2}$ for some $c \in \mathbf{F}_3$ such that $c \neq 0$. Similarly, we know $\pi\left(\frac{A'-B'}{3}\right) \equiv c'M_m \pmod{(x_1 - x_2)^2}$ for some $c' \in \mathbf{F}_3$. Thus we have $\pi\left(\frac{A'-B'}{3}\right) = dB$, where $d = \frac{c'}{c}$.

We use Lemma 5.13 to expand $PC + BQ$ in $\mathbf{F}_3[x_1, x_2, x_3]/(x_1 - x_2)^2$,

$$\begin{aligned}
PC + QB &= \left(h_0 M_5 B + f_0(x_1 + x_2 + x_3) M_4 \pi\left(\frac{A' - B'}{3}\right) x_3^0\right) \\
&\quad + \sum_{j=1}^{3} \Bigg(h_j M_{5-j} B x_3^j + f_j(x_1 + x_2 + x_3) M_{4-j} \pi\left(\frac{A' - B'}{3}\right) x_3^j \\
&\qquad - f_{j-1} M_{5-j} B x_3^j\Bigg) + h_4 M_1 B x_3^4 - f_3 M_1 B x_3^4 \\
&= \left(h_0 B + f_0 \pi\left(\frac{A' - B'}{3}\right)\right) M_5 \\
&\quad + \sum_{j=1}^{3} \left(\left((h_j - f_{j-1}) B + f_j \pi\left(\frac{A' - B'}{3}\right)\right) M_{5-j} x_3^j\right) + (h_4 - f_3) M_1 B x_3^4. \\
&= (h_0 + f_0 d) B M_5 + \sum_{j=1}^{3} \Big((h_j - f_{j-1}) + f_j d\Big) B M_{5-j} x_3^j + (h_4 - f_3) M_1 B x_3^4.
\end{aligned}$$

Letting h_j be arbitrary for $j > 0$, set $f_3 = h_4$, $f_{j-1} = h_j + f_j d$ for $0 < j < 3$ and set $h_0 = -f_0 d$. This makes the expression $PC + QB = 0$.

We claim $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has a degree $3m+3$ generator, namely R_{m+1}. From Lemma 5.9, $Q_{m+1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has no degree $3m+1$ or $3m+2$ generator, so it has no generators in degree less than $3m+3$. By Lemma 5.8, R_{m+1} is in degree $3m+3$ so it must be a generator. Without loss of generality, we let

$$R_{m+1} = \left((x_1 + x_2 + x_3)C + SB\right)(x_1 - x_2)^{2m+1},$$

where S is a degree 2 symmetric polynomial.

If $(PC + QB)(x_1 - x_2)^{2m+1}$ were generated by R_{m+1}, there would exist a symmetric polynomial I such that $IR_{m+1} = (PC + QB)(x_1 - x_2)^{2m+1}$. This implies $(I(x_1 + x_2 + x_3) - P)C + (IS - Q)B = 0$. If $I(x_1 + x_2 + x_3) - P \neq 0$ or $IS - Q \neq 0$, there is a relation on C and B over $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}$, but $C(x_1 - x_2)^{2m+1}$ and $B(x_1 - x_2)^{2m+1}$ are generators of $Q_m(3, \mathbf{F}_3)$. Thus we must have $P = I(x_1 + x_2 + x_3)$, so $(x_1 + x_2 + x_3)|P$. Now we consider the symmetric polynomials $P' = P + e_2^2 + e_2 e_1^2 + e_1^4$ and $Q' = Q + e_3 e_1^2 + (-d - 1)e_2^2 e_1 - d e_1^3 e_2 + (-d + 1)e_1^5$. In $F_3[x_1 - x_3, x_2 - x_3, x_3]/(x_1 - x_2)^2$, we get that

$$P' = f_0 M_4 x_3^0 + f_1 M_3 x_3^1 + (f_2 + 1) M_2 x_3^2 + f_3 M_1 x_3^3$$

and

$$Q' = h_0 M_5 x_3^0 + h_1 M_4 x_3^1 + (h_2 - d) M_3 x_3^2 + (h_3 + 1) M_2 x_3^3 + h_4 M_1 x_3^4.$$

Then $f_2 + 1 = (h_3 + f_3 d) + 1 = (h_3 + 1) + f_3 d$, $f_1 = h_2 + f_2 d = (h_2 - d) + (f_2 + 1)d$, and the rest of the equations necessary for $P'C + Q'B \equiv 0 \pmod{(x_1 - x_2)^2}$ are the same as $PC + QB \equiv 0 \pmod{(x_1 - x_2)^2}$. Thus $P'C + Q'B \equiv 0 \pmod{(x_1 - x_2)^2}$. Moreover, $(x_1 + x_2 + x_3)$ divides into $P + e_2 e_1^2 + e_1^4$ but not e_2^2, so $(x_1 + x_2 + x_3) \nmid P'$. We have shown that if $(PC + QB)(x_1 - x_2)^{2m+1}$

is generated by R_{m+1}, then $(x_1+x_2+x_3)|P$, implying $(P'C+Q'B)(x_1-x_2)^{2m+1}$ is not generated by R_{m+1}. If $(P'C+Q'B)(x_1-x_2)^{2m+1}$ is not a generator, then whatever generates it violates Lemma 4.3, so $(P'C+Q'B)(x_1-x_2)^{2m+1}$ is indeed a degree $3m+6$ generator of $Q_{m+1}(3,\mathbf{F}_3)$. ■

Now we prove that if R_{m+1} begins our staircase, then it is the lower degree generator for the first half of the staircase.

Lemma 5.17. *Let $m \notin X$ for some natural number m. Suppose R_{m+1} is a degree $3m+3$ generator of $Q_{m+1}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ and L is another generator in degree $3m+6$. Further, let R_{m+1} lie in $Q_{m+d}(3,\mathbf{F}_3)$, where d is maximal. Then $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$, R_{m+1}, and 1 freely generate $Q_{m+j}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ for $1 \le j \le d$.*

Proof. As a generator, L lies in a copy of $\text{sign}-\text{triv}$ and is divisible by $(x_1-x_2)^{2(m+1)+1}$ by Lemma 4.1. Since $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$ is divisible by $(x_1-x_2)^{2(m+j)+1}$, by the second part of Lemma 4.1, $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$ is in $Q_{m+j}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$. If $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$ is not a generator, R_{m+1} must generate $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$, implying a relation between R_{m+1} and L. Thus $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$ is indeed a generator.

Moreover, $3m+3+3m+6j=6(m+j)+3$ so by Lemma 4.4, $L\prod(x_{i_1}-x_{i_2})^{2(j-1)}$ and R_{m+1} generate $Q_{m+j}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$. ■

Next, we prove that, for all consecutive spaces of quasi-invariants in the second half of the staircase, the lower degree generator is $\prod(x_{i_1}-x_{i_2})^2$ times the previous lower degree generator.

Lemma 5.18. *Let $m \notin X$ for some natural number m. Suppose R_{m+1} is a degree $3m+3$ generator of $Q_m(3,\mathbf{F}_3)$ and L is another generator in degree $3m+6$. Let R_{m+1} lie in $Q_{m+d}(3,\mathbf{F}_3)$, where d is maximal. Further, let L have degree at most 5 in x_3. Then for all $d \le j < 2d$, $Q_{m+j}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ is freely generated by a generator in degree $3m+6d$, $R_{m+1}\prod(x_{i_1}-x_{i_2})^{2(j-d)}$ in degree $3m+6(j-d)+3$, and 1.*

Proof. We proceed with induction.

The generator R_{m+1} of $Q_{m+d}(3,\mathbf{F}_3)$ is in degree $3m+3=3m+6(d-d)+3$, and from Lemma 5.17 a second generator is $L\prod(x_{i_1}-x_{i_2})^{2(d-1)}$ in degree $3m+6d$. Moreover, these are the only generators so the claim is true for $j=d$.

Let k be a natural number with $d<k<2d$ and suppose $Q_{m+j}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ has a generator in degree $3m+6d$ and degree $3m+6(j-d)+3$ for all $d \le j < k$, where this upper degree generator is a polynomial of degree at most 5 in x_3 and is not generated by R_{m+1}. Consider $Q_{m+k}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$. We know $R_{m+1}\prod(x_{i_1}-x_{i_2})^{2(k-d-1)}$ is an element of $Q_{m+k-1}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ of degree $3m+6(k-d-1)+3$ by Lemma 4.1. Since $k-1<k$, our inductive hypothesis implies $R_{m+1}\prod(x_{i_1}-x_{i_2})^{2(k-d-1)}$ is a generator for $Q_{m+k-1}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$.

Let T be the degree $3m+6d$ generator for $Q_{m+k-1}(3,\mathbf{F}_3)_{\text{sign}-\text{triv}}$ with degree 5 in x_3. We write $R_{m+1}\prod(x_{i_1}-x_{i_2})^{2(k-d-1)} = R'_{m+1}(x_1-x_2)^{2(m+k-1)+1}$ and $T=T'(x_1-x_2)^{2(m+k-1)+1}$ for s_{12} invariant polynomials R'_{m+1} and T'. If $o=m+4k-6d-2$ and $r=m+6d-2k+1$, then $\deg R'_{m+1}=o$ and $\deg T'=r$. We want to find a degree $r-o$ symmetric polynomial P such that

$$-PR'_{m+1}+T' \equiv 0 \pmod{(x_1-x_2)^2}.$$

We claim that R'_{m+1} has degree 0 in x_3. This is because $R_{m+1}\prod(x_{i_1}-x_{i_2})^{2(k-d-1)} = P_l^{3^a}\prod(x_{i_1}-x_{i_2})^{2(k-d-1)}$ as we proved in Lemma 5.12. Since P_l is the map of the generator of $Q_l(3,\mathbf{Q})$ into characteristic 3, P_l must be constant in the variable x_3. We can see $\prod(x_{i_1}-x_{i_2})^{2(k-d-1)}$ is also constant in x_3, so R_{m+1} and R'_{m+1} are constant in x_3.

Having assumed that T' is at most degree 5 in x_3,

$$T' = t_0 M_r x_3^0 + t_1 M_{r-1} x_3^1 + t_2 M_{r-2} x_3^2 + t_3 M_{r-3} x_3^3 + t_4 M_{r-4} x_3^4 + t_5 M_{r-5} x_3^5$$

and

$$R'_{m+1} = a M_o$$

for coefficients t_j and a in $\mathbf{F}_3$. Since R_{m+1} is not in $Q_{m+d+1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, we have $a \neq 0$. We let

$$\begin{aligned} P = &\frac{t_0}{a} M_{r-o} x_3^0 + \frac{t_1}{a} M_{r-o-1} x_3^1 + \frac{t_2}{a} M_{r-o-2} x_3^2 + \frac{t_3}{a} M_{r-o-3} x_3^3 \\ &+ \frac{t_4}{a} M_{r-o-4} x_3^4 + \frac{t_5}{a} M_{r-o-5} x_3^5, \end{aligned}$$

so that $T' - PR'_{m+1} \equiv 0 \pmod{(x_1 - x_2)^2}$ by Lemma 5.13. Since $\deg(P) = r - o = 12d - 6k + 3 \geq 9 > 7$, by Lemma 5.15 such a symmetric polynomial P is attainable with P having degree at most degree 5 in x_3. Since T' also has at most degree 5 in x_3 and R'_{m+1} has degree 0, $(-PR'_{m+1} + T')$ has at most degree 5 in x_3. Letting $U = \left(-PR'_{m+1} + T'\right)(x_1 - x_2)^{2(m+k-1)+1}$, we have U is in $Q_{m+k}(3, \mathbf{F}_3)$ with degree $3m + 6d$ and since $\left(-PR'_{m+1}\right)(x_1 - x_2)^{2(m+k-1)+1}$ is generated by R_{m+1} and T is not, U is not generated by R_{m+1}. Finally, we also have $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ is in $Q_{m+k}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ with degree $3m + 6(k - d) + 3$. Thus what is left to prove is $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ and $\left(-PR'_{m+1} + T'\right)(x_1 - x_2)^{2(m+k-1)+1}$ are generators for $Q_{m+k}(3, \mathbf{F}_3)$.

Assume for sake of contradiction that U and $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ are not both generators. If $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ is a generator, then any other generator must be of at least degree $3m + 6d$ by Lemma 4.3. Yet U is not generated by $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ since it is not generated by R_{m+1}. Thus U must be a generator.

Next, we consider if $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ is not a generator. For $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ to not be a generator there must be a generator in a degree less than $3m + 6(k - d) + 3$. Let it be G, and by Lemma 4.3, any other generator must have degree greater than $3m + 6d$. Thus U is not a generator, so U and $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ are both generated by G and specifically $U = QG$ and $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)} = SG$ for symmetric polynomials P and Q. Moreover, $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d-1)}$ is the lowest degree generator for $Q_{m+k-1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, so $G = CR_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d-1)}$ for a symmetric polynomial C. This implies $C | \prod (x_{i_1} - x_{i_2})^2$, and G is not a scalar multiple of $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$, so C is a constant. We then have U is a constant multiple of $QR_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d-1)}$, so U is generated by R_{m+1} which is a contradiction.

Thus U and $R_{m+1} \prod (x_{i_1} - x_{i_2})^{2(k-d)}$ are each generators and together with 1 they freely generate $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ by Lemma 4.4. ■

Finally, we show that after the staircase completes, the next space of quasi-invariants has no counterexamples.

Lemma 5.19. *Let $Q_{m-1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ have generators K in degree $3m - 3$ and T in degree $3m$ such that K is not in $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$. If m is even, then $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ is freely generated by a generator in degree $3m + 1$, $3m + 2$, and 1.*

Proof. Suppose for the sake of contradiction that $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has a generator U in degree $3m - 1$ or $3m - 2$. Then since U is also in the -1 s_{12} eigenspace of $Q_{m-1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, U must be generated by K over $\mathbf{F}[x_1, x_2, x_3]^{S_3}$. Yet K being divisible by a symmetric polynomial violates Corollary 4.2.

Suppose for the sake of contradiction that $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has a generator in degree $3m$. Without loss of generality let that generator be T. From [9], we can let L' be a degree $3m + 1$

generator of $Q_m(3,\mathbf{Q})_{\text{std}}$ with coprime integer coefficients. Then $\pi(L') \in Q_m(3,\mathbf{F}_3)_{\text{sign-triv}}$, so $\pi(L')$ must be generated by T since any other generator in degree less than degree $3m+1$ would violate Lemma 4.3. Moreover, the only degree 1 symmetric polynomials are constant multiples of $x_1+x_2+x_3$, so we can assume without loss of generality that

$$\pi(L') = (x_1+x_2+x_3)T.$$

Note that from [13] all generators of $Q_m(3,\mathbf{Q})_{\text{std}}$ must lie in $\mathbf{Q}[x_1-x_3, x_2-x_3]$. Thus $(x_1+x_2+x_3)T \in \mathbf{F}_3[x_1-x_3, x_2-x_3]$ and so $T \in \mathbf{F}_3[x_1-x_3, x_2-x_3]$.

We also have $T = (x_1-x_2)^{2m+1}T'$ for some s_{12}-invariant polynomial T'. Thus by the fundamental theorem of symmetric polynomials $T' \in \mathbf{F}_3[(x_1-x_3)(x_2-x_3), x_1+x_2+x_3]$. Note that $\deg T' = 3m-2m-1 = m-1$ and m is even, so T' has an odd degree. However, since it is generated by $(x_1-x_3)(x_2-x_3)$ and $x_1+x_2+x_3$, we must have $(x_1+x_2+x_3)|T'$. This gives a contradiction because T is a generator. ■

Finally, we have the lemmas to prove Theorem 5.3.

Proof of Theorem 5.3. We prove this using induction on m.

The generators for $Q_0(3,\mathbf{F}_3)_{\text{sign-triv}}$ are x_1-x_2 and $x_3(x_1-x_2)$. These generators are in degree $3\cdot 0+1$ and $3\cdot 0+2$ so the theorem is true for the base case.

Assume the claim is true when $m<j$ for some $j\in\mathbf{N}$. Consider the space $Q_j(3,\mathbf{F}_3)_{\text{sign-triv}}$. Let t be the largest natural number less than j such that $t\notin X$. By the inductive hypothesis, $Q_t(3,\mathbf{F}_3)$ has a generator in degree $3t+1$ and $3t+2$. By Lemma 5.10, we may let the generators be

$$\left(x_1+x_2+x_3)\pi\left(\frac{A'-B'}{3}\right) - x_3B\right)(x_1-x_2)^{2t+1} \qquad \text{and} \qquad B(x_1-x_2)^{2t+1},$$

where $(x_1-x_2)^{2t+1}(x_1+x_2-2x_3)A'$ and $(x_1-x_2)^{2t+1}B'$ are generators for $Q_t(3,\mathbf{Q})_{\text{std}}$ and $\pi(A')=\pi(B')=B$. From Lemma 5.16, $Q_{t+1}(3,\mathbf{F}_3)_{\text{sign-triv}}$ is generated by a generator in degree $3t+6$ and $3t+3$. Moreover, R_{t+1} is the $3t+3$ degree generator by Lemma 5.8. Let L be the degree $3t+6$ generator. Suppose R_{t+1} lies in $Q_{t+d}(3,\mathbf{F}_3)$, but not $Q_{t+d+1}(3,\mathbf{F}_3)$, where d is a natural number.

First, we consider when $t+d\geq j\geq t+1$. By Lemma 5.17, $Q_j(3,\mathbf{F}_3)_{\text{sign-triv}}$ has generators R_{t+1} and $L\prod(x_{i_1}-x_{i_2})^{2(j-t-1)}$. Note that $R_{t+1}=R_j$ by Lemma 5.12, and further $\deg\left(L\prod(x_{i_1}-x_{i_2})^{2(j-t-1)}\right)+\deg(R_{t+1}) = (6(j-t-1)+3t+6)+3t+3 = 6j+3$. By Lemma 4.4, we then have that R_{t+1} and $L\prod(x_{i_1}-x_{i_2})^{2(j-t-1)}$ generate $Q_j(3,\mathbf{F}_3)_{\text{sign-triv}}$.

Next, we consider the case where $t+2d-1\geq j\geq t+d+1$. Notice that by our construction in Lemma 5.16, we can choose L such that it has at most degree 5 in x_3. Thus we can apply Lemma 5.18, which gives us that $Q_j(3,\mathbf{F}_3)_{\text{sign-triv}}$ is generated by $R_{t+1}\prod(x_{i_1}-x_{i_2})^{2(j-t-d)}$ and a generator in degree $3t+6d$. Note that $R_{t+1}\prod(x_{i_1}-x_{i_2})^{2(j-t-d)}$ is a constant multiple of R_j by Lemma 5.12. Moreover, the sum of their degrees is $3t+6(j-t-d)+3+3t+6d = 6j+3$ as desired.

Finally, we consider if $j=t+2d$. Note that by Lemma 5.18, $Q_{t+2d-1}(3,\mathbf{F}_3)$ has a generator in degree $3t+6d$ and $3t+6(d-1)+3$. The degree $3t+6(d-1)+3$ generator is $R_{t+1}\prod(x_{i_1}-x_{i_2})^{2(d-1)}$, and R_{t+1} is divisible by $(x_1-x_2)^{2(t+d)+1}$, where d is maximal, so $R_{t+1}\prod(x_{i_1}-x_{i_2})^{2(d-1)}$ does not lie in $Q_{t+2d}(3,\mathbf{F}_3)$. Moreover, $Q_t(3,\mathbf{F}_3)$ is a non Ren–Xu counterexample, so t must be even by Lemma 5.6. Then $t+2d$ is even as well, so by Lemma 5.19, $Q_{t+2d}(3,\mathbf{F}_3)$ has a generator in degree $3(t+2d)+1$ and $3(t+2d)+2$.

Now we claim we have exhausted all cases. If we had $j>t+2d$, since we just showed $t+2d\notin X$, we would not have chosen t to be the largest natural number less than j not in X. ■

Remark 5.20. We can compute the degrees of generators of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ explicitly. If m has no digits 1 in its base 3 representation, then the generators have degree $3m+1$ and $3m+2$. Otherwise, the lower degree generator is R_m. We can deduce the minimal degree of the Ren–Xu counterexamples in $Q_m(3, \mathbf{F}_3)$: Let a be the greatest natural number such that the a-th term from the right in the base 3 representation of m is 1. Then if $\lceil \frac{\lceil \frac{m}{3^a} \rceil - 1}{2} \rceil = k$, a minimal degree Ren–Xu counterexample is $P_k^{3^a} \prod (x_{i_1} - x_{i_2})^{2b}$, where

$$b = \max\left\{\frac{2m+1-3^a(2k+1)}{2}, 0\right\}.$$

The degrees of the generators are then $3^a(2k+1)+6b$ and $6m+3-3^a(2k+1)-6b$.

6 Representations of S_3 in $Q_m(3, \mathrm{F}_3)$

Now that we have a complete picture of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, we consider generators that generate the other indecomposable modules of S_3. We start with triv − sign − triv, which behaves very similarly to sign − triv.

Proposition 6.1. *Suppose that for all $j \leq m$, $Q_j(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ has generators in degree d and $6j+3-d$ respectively for some d. If K, L are distinct generators of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ then there are two other homogeneous generators K_1, L_1 of $Q_m(3, \mathbf{F}_3)$ in the same degrees as K, L, respectively such that as a representation of S_3, K_1 generates a copy of* triv − sign − triv *containing K and L_1 generates a copy of* triv − sign − triv *containing L. Moreover, there are no relations between K_1, L_1 over the symmetric polynomials, and there are no other generators of $Q_m(3, \mathbf{F}_3)$ that generate a copy of* triv − sign − triv.

Proof. We prove this by induction on m. For the base case $m = 0$, note that by Example 2.9, for $K = x_1 - x_2$ we have that $K_1 = x_1$ satisfies the desired conditions. Similarly, for $L = (x_1 - x_2)x_3$, we have that $L_1 = x_1(x_2 + x_3)$ satisfies the desired conditions. These two are independent over the symmetric polynomials, as a relation between them would imply a relation between 1 and $x_2 + x_3$.

For the inductive step, let K', L' be the generators of $Q_{m-1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$ and let K_1', L_1' be the corresponding generators of $Q_{m-1}(3, \mathbf{F}_3)$. Without loss of generality, we can choose K_1', L_1' to be s_{23}-invariant with $(1 - s_{12})K_1' = K'$, $(1 - s_{12})L_1' = L'$ (similar to in the base case). Let K, L be generators of $Q_m(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$. Then since $K, L \in Q_{m-1}(3, \mathbf{F}_3)_{\text{sign}-\text{triv}}$, we can write $K = P_1K' + Q_1L'$, $L = P_2K' + Q_2L'$ for symmetric polynomials P_1, P_2, Q_1, Q_2. Then it follows that $K_1 := P_1K_1' + Q_1L_1'$, $L_1 := P_2K_1' + Q_2L_1'$ each generate a copy of triv − sign − triv that contains K, L, respectively. Moreover, if there is some relation $P_3K_1 + Q_3L_1 = 0$ for symmetric polynomials P_3, Q_3, then applying $1 - s_{12}$ to this equation would yield $P_3K + Q_3L = 0$, which violates Lemma 4.4.

Next, we show that K_1, L_1 are m-quasi-invariants. As the computations are the same for both polynomials, we give the proof only for K_1. First, note that $(1 - s_{23})K_1 = 0$ since both K_1', L_1' are s_{23}-invariant. Next, note that $(1 - s_{12})K_1 = K$ is divisible by $(x_1 - x_2)^{2m+1}$ by Lemma 4.1. Finally, note that since K_1 is s_{23}-invariant, we have

$$(1 - s_{13})K_1 = s_{23}(s_{23} - s_{23}s_{13})K_1 = s_{23}(1 - s_{23}s_{13}s_{23})K_1 = s_{23}(1 - s_{12})K_1$$

is divisible by $s_{23}(x_1 - x_2)^{2m+1} = (x_1 - x_3)^{2m+1}$.

Note that K_1, L_1 are the minimal degree polynomials such that $(1 - s_{12})K_1$, $(1 - s_{12})L_1$ are symmetric polynomial multiples of K, L, respectively, so they cannot be generated by any other generators and thus must be generators themselves. Then assume for contradiction that there is some other generator T of $Q_m(3, \mathbf{F}_3)$ that generates a copy of triv − sign − triv.

Then $(1 - s_{12})T$ is contained in a copy of sign − triv and is s_{12}-antiinvariant, so we can write $(1 - s_{12})T = S_1K + S_2L$ for symmetric polynomials S_1, S_2. Then T, $S_1K_1 + S_2L_1$ generate copies of triv − sign − triv with the same sign − triv submodule, so they generate a copy of

$$(\text{triv} - \text{sign} - \text{triv} \oplus \text{triv} - \text{sign} - \text{triv})/\text{sign} - \text{triv} \cong \text{triv} - \text{sign} - \text{triv} \oplus \text{triv}.$$

Thus T is generated by K_1, L_1, 1, and is not a generator itself. ■

Corollary 6.2. *The generators* 1*,* K*,* K_1*,* L*,* L_1 *of* $Q_m(3, \mathbf{F}_3)$ *defined in Proposition 2.11, Theorem 5.3 and Proposition 6.1 have no relations between them over the symmetric polynomials.*

Proof. Let

$$P_1 + P_2K + P_3L + P_4K_1 + P_5L_1 = 0$$

for symmetric polynomials $P_1, \ldots, P_5$. Then apply $1 + s_{12}$ to the equation to yield

$$2P_1 + P_4(2K_1 - K) + P_5(2L_1 - L) = 0$$

since K, L are s_{12}-antiinvariant. Next, apply $1 - s_{23}$ to this equation to yield

$$P_4(s_{23} - 1)K + P_5(s_{23} - 1)L = 0.$$

Note that $(s_{23} - 1)K$ generates the same copy of sign − triv as K, since $s_{23} - 1$ acts bijectively on sign (and similarly for L). So a relation between $(s_{23} - 1)K$, $(s_{23} - 1)L$ is equivalent to a relation between K, L, which cannot exist by Lemma 4.4. So we have $P_4 = P_5 = 0$.

Now, the result follows from Lemma 4.4. ■

Remark 6.3. In the non-modular case, one has that the polynomial $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1}$ is a generator of $Q_m(n, \Bbbk)$, as it is the lowest degree quasi-invariant in the sign module. However, from Lemma 4.4 we have that in characteristic 3,

$$(L + s_{23}L)K - (K + s_{23}K)L = c \prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1},$$

so $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1}$ is not a generator. We can take this calculation further, and note that $(L + s_{23}L)K_1 - (K + s_{23}K)L_1$ would then generate a copy of triv − sign, as the quotient of this module by the space generated by $(L + s_{23}L)K - (K + s_{23}K)L$ must be a trivial module.

It remains to consider the modules triv − sign, sign − triv − sign. To motivate the results that follow, we start by considering 0-quasi-invariants.

Example 6.4. Note that from Corollary 6.2 we know that $Q_0(3, \mathbf{F}_3)$ has 5 generators 1, $x_1 - x_2$, $(x_1 - x_2)x_3$, x_1, $x_1(x_2 + x_3)$ with no relations between them. By examining the dimension of the space of all homogeneous degree 3 polynomials, we have that $Q_0(3, \mathbf{F}_3)[3]$ is 10-dimensional. Since $\mathbf{F}_3[x_1, x_2, x_3]^{S_3}$ is 3-dimensional in degree 3, 2-dimensional in degree 2, and 1-dimensional in degree 1, so far we have accounted for only $3 + 2 + 2 + 1 + 1 = 9$ dimensions. Moreover, every irreducible representation is accounted for, so this extra dimension must be an extension of an existing indecomposable representation. The only indecomposable representations that have nontrivial extensions are the triv generated by $x_1x_2x_3$ and the triv − sign generated by

$$\begin{aligned} E &:= (x_1x_2 + x_1x_3 + x_2x_3)x_1 + (x_1 + x_2 + x_3)(x_1(x_2 + x_3)) \\ &= -x_1^2x_2 - x_1^2x_3 + x_1x_2^2 + x_1x_3^2. \end{aligned}$$

Indeed, the triv − sign generated by E extends to a sign − triv − sign generated by

$$F := (x_1 - x_2)x_1x_2.$$

We will later see that the polynomials E, F defined above are key to understanding $\mathrm{triv}-\mathrm{sign}$ and $\mathrm{sign}-\mathrm{triv}-\mathrm{sign}$ in the quasi-invariants.

Proposition 6.5. *$Q_0(3,\mathbf{F}_3)$ is freely generated by 1, x_1-x_2, $(x_1-x_2)x_3$, x_1, $x_1(x_2+x_3)$, F as a $\mathbf{F}_3[x_1,x_2,x_3]^{S_3}$-module.*

Proof. We already know that the first 5 polynomials are independent. Now, let

$$P_1+P_2(x_1-x_2)+P_3(x_1-x_2)x_3+P_4x_1+P_5(x_2+x_3)x_1+P_6F=0$$

for symmetric polynomials P_j. Apply $1-s_{12}$ to this equation to get

$$(P_4-P_2)(x_1-x_2)+(P_5-P_3)(x_1-x_2)x_3-P_6F=0.$$

Next, apply $1+s_{23}$ to get

$$(P_2-P_4)(x_1+x_2+x_3)+(P_5-P_3)(x_1x_2+x_1x_3+x_2x_3)+P_6E=0.$$

Finally, note that as E can be written in terms of symmetric polynomial multiples of x_1, $(x_2+x_3)x_1$, this equation would be a relation between the first 5 generators of $Q_0(3,\mathbf{F}_3)$. We have seen this is impossible, so we have $P_6=0$, and hence all of the P_j must be 0.

Let Q_0' be the submodule of $Q_0(3,\mathbf{F}_3)$ generated by these 6 polynomials. Then as the polynomials freely generate Q_0' as a $\mathbf{F}_3[x_1,x_2,x_3]^{S_3}$-module, we have that the Hilbert series of Q_0' is

$$\mathcal{H}(Q_0')=\left(1+2t+2t^2+t^3\right)\mathcal{H}\left(\mathbf{F}_3[x_1,x_2,x_3]^{S_3}\right)=\frac{1+2t+2t^2+t^3}{(1-t)(1-t^2)(1-t^3)}=\frac{1}{(1-t)^3}$$

by the fundamental theorem of symmetric polynomials. This is exactly the Hilbert series of $Q_0(3,\mathbf{F}_3)$, so $Q_0'=Q_0(3,\mathbf{F}_3)$ and there are no more generators of $Q_0(3,\mathbf{F}_3)$. ■

Similar to how we only considered polynomials in the (-1)-eigenspace of s_{12} for $\mathrm{sign}-\mathrm{triv}$, we only consider generators in the (-1)-eigenspace of s_{12} for $\mathrm{sign}-\mathrm{triv}-\mathrm{sign}$ and polynomials in the 1-eigenspace of s_{23} for $\mathrm{triv}-\mathrm{sign}$. Note that this is sufficient to describe the roles of $\mathrm{sign}-\mathrm{triv}-\mathrm{sign}, \mathrm{triv}-\mathrm{sign}$, as both modules are generated by an element satisfying their respective constraints.

Lemma 6.6.

1. *Let $T\in Q_m(3,\mathbf{F}_3)$ generate a copy of* $\mathrm{triv}-\mathrm{sign}$*. Then T is the sum of a symmetric polynomial multiple of $E\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m}$ and a symmetric polynomial. Conversely, any symmetric polynomial multiple of $E\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m}$ generates a copy of* $\mathrm{triv}-\mathrm{sign}$ *in $Q_m(3,\mathbf{F}_3)$.*
2. *Let $T_1\in Q_m(3,\mathbf{F}_3)$ generate a copy of* $\mathrm{sign}-\mathrm{triv}-\mathrm{sign}$*. Then T_1 is the sum of a symmetric polynomial multiple of $F\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m}$ and a symmetric polynomial multiple of $\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m+1}$. Conversely, any symmetric polynomial multiple of $F\prod_{i_1<i_2}(x_{i_1}-x_{i_2})^{2m}$ generates a copy of* $\mathrm{sign}-\mathrm{triv}-\mathrm{sign}$ *in $Q_m(3,\mathbf{F}_3)$.*

Proof. 1. We first prove the lemma for $m=0$. Consider some T as above, and note that $(1-s_{12})T$ is contained in the sign representation, so by Proposition 2.11 we have $(1-s_{12})T=P(x_1-x_2)(x_1-x_3)(x_2-x_3)$ for some symmetric polynomial P. Then note that PE, T generate two copies of $\mathrm{triv}-\mathrm{sign}$ with the same sign subrepresentation, so they generate a copy of

$$(\mathrm{triv}-\mathrm{sign}\oplus\mathrm{triv}-\mathrm{sign})/\mathrm{sign}\cong\mathrm{triv}-\mathrm{sign}\oplus\mathrm{triv}.$$

So T is the sum of PE and a symmetric polynomial, as claimed.

Now, consider general m. By the above we have that any T must be of the form $T = PE + Q$ for symmetric polynomials P, Q. Then since T is m-quasi-invariant, we have $(1 - s_{12})T = P(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ is divisible by $(x_1 - x_2)^{2m+1}$. So P is divisible by $(x_1 - x_2)^{2m}$, and it must also be divisible by $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ since it is symmetric.

The converse is clear.

2. This proof is similar to part (1). For $m = 0$, any T_1 must have that $(1 + s_{23})T_1$ is in a triv $-$ sign representation, so $(1 + s_{23})T_1 = PE$ for some $P \in \mathbf{F}_3[x_1, x_2, x_3]^{S_3}$. Then T_1, PF generate a copy of

$$(\text{sign} - \text{triv} - \text{sign} \oplus \text{sign} - \text{triv} - \text{sign})/\text{triv} - \text{sign} \cong \text{sign} - \text{triv} - \text{sign} \oplus \text{sign},$$

which implies the result for $m = 0$. Then the extension to general m is the same as in part (1). The converse is clear, as before. ∎

Finally, we can prove Theorem 1.3 for $p = 3$.

Theorem 6.7. *$Q_m(3, \mathbf{F}_3)$ is freely generated by 1, the two generators K, L from Theorem 5.3, the two generators K_1, L_1 from Proposition 6.1, and the generator $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ from Lemma 6.6.*

Proof. Let us first show that there are no other generators of $Q_m(3, \mathbf{F}_3)$. Assume for contradiction that there is some other generator T of $Q_m(3, \mathbf{F}_3)$. Then T cannot generate a copy of triv by Proposition 2.11 and it cannot generate a copy of sign $-$ triv or triv $-$ sign $-$ triv by Theorem 5.3 and Proposition 6.1. If it generates a copy of sign, then by Proposition 2.11 it must be $\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m+1}$, but this polynomial is generated by K, L by Lemma 4.4, so it cannot be a generator. If it generates a copy of triv $-$ sign, then it is $E\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ by Lemma 6.6. But this is generated by K_1, L_1 by Remark 6.3. Finally, by Lemma 6.6 the only generator that generates a copy of sign $-$ triv $-$ sign is $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$.

Finally, we show there are no relations between the 6 generators. Note that this also implies $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ is a generator, since it is not generated by the other 5 generators. But this is clear: we already know there are no relations between the first 5 generators by Corollary 6.2. If there was a relation involving $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$, then note that since every generator is generated by the generators of $Q_0(3, \mathbf{F}_3) = \mathbf{F}_3[x_1, x_2, x_3]$, this would induce a relation on those generators. Moreover, the generators other than $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ each generate a copy of an indecomposable representation that is not sign $-$ triv $-$ sign, so they are each generated by the first 5 generators of $Q_0(3, \mathbf{F}_3)$. Meanwhile, $F\prod_{i_1<i_2}(x_{i_1} - x_{i_2})^{2m}$ is the only generator not generated by the first 5 generators, so the induced relation would be nontrivial. But there is no such relation by Proposition 6.5. ∎

Note that these generators imply a Hilbert series that agrees with Theorem 1.3 since K is either a minimal degree Ren–Xu counterexample or has degree $3m + 1$ if one does not exist. In this way, the Hilbert series of $Q_m(3, \mathbf{F}_3)$ agrees with that of $Q_m(3, \mathbf{Q})$ if and only if there does not exist a Ren–Xu counterexample. Ren–Xu counterexamples only exist when the conditions of Conjecture 5.2 are satisfied, so Conjecture 5.2 is also implied.

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