Symmetry, Integrability and Geometry: Methods and Applications SIGMA **20** (2024), 024, 25 pages

Hodge Diamonds of the Landau–Ginzburg Orbifolds

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Received July 12, 2023, in final form March 06, 2024; Published online March 25, 2024
https://doi.org/10.3842/SIGMA.2024.024

Abstract. Consider the pairs (f, G) with $f = f(x_1, \dots, x_N)$ being a polynomial defining a quasihomogeneous singularity and G being a subgroup of $\mathrm{SL}(N, \mathbb{C})$, preserving f. In particular, G is not necessary abelian. Assume further that G contains the grading operator j_f and f satisfies the Calabi–Yau condition. We prove that the nonvanishing bigraded pieces of the B-model state space of (f, G) form a diamond. We identify its topmost, bottommost, leftmost and rightmost entries as one-dimensional and show that this diamond enjoys the essential horizontal and vertical isomorphisms.

Key words: singularity theory; Landau–Ginzburg orbifolds

2020 Mathematics Subject Classification: 32S05; 14J33

1 Introduction

Let a polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ be quasihomogeneous with respect to some positive integers $d_0, d_1, \dots, d_N$, i.e.,

$$f\left(\lambda^{d_1} x_1, \dots, \lambda^{d_N} x_N\right) = \lambda^{d_0} f(x_1, \dots, x_N), \qquad \forall \lambda \in \mathbb{C}^*.$$

Assume also that $x_1 = \dots = x_N = 0$ is the only critical point of f and $d_1, \dots, d_N$ have no common factor. Then the zero set $f(x_1, \dots, x_N) = 0$ defines a degree d_0 quasismooth hypersurface X_f in $\mathbb{P}(d_1, \dots, d_N)$. Such hypersurfaces became of great interest in the early 90's in the context of mirror symmetry (cf. [7, 8]). In particular, if the *Calabi–Yau condition* $d_0 = \sum_{k=1}^N d_k$ holds, then the first Chern class of X_f vanishes and, hence, its canonical bundle is trivial meaning that X_f is a Calabi–Yau variety.

The polynomials f above define the so-called *quasihomogeneous singularities* and can be studied from the point of view of singularity theory. The varieties X_f at the same time are the objects of Kähler geometry. To relate the singularity theory properties of f to the Kähler geometry properties of X_f is an important problem. This problem is in particular interesting in the context of mirror symmetry.

1.1 Hodge diamonds of Calabi–Yau manifolds

The state space of a Calabi–Yau manifolds X is the cohomology ring $H^*(X)$. This cohomology ring is naturally bigraded building up a Hodge diamond of size $D := \dim_{\mathbb{C}} X$. In particular the following properties hold:

(1) $H^*(X) = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}(X)$,

(2) $\dim H^{p,q}(X) = 0$ if $p < 0$ or $q < 0$ or $p > D$ or $q > D$,

(3) $\dim H^{0,0}(X) = \dim H^{D,D}(X) = 1$,

(4) $\dim H^{D,0}(X) = \dim H^{0,D}(X) = 1$,

(5) there is a "horizontal" vector space isomorphism $H^{p,q}(X) \cong H^{q,p}(X)$,

(6) there is a "vertical" vector space isomorphism $H^{p,q}(X) \cong \left(H^{D-q,D-p}(X)\right)^\vee$, where $(-)^\vee$ stands for the dual vector space

In physics, this bigrading is coming from considering Calabi–Yau B-model associated to X and A-model bigrading is obtained from it by the so-called *rotation of the diamond by* $90°$. On the level of state spaces the mirror symmetry map is an isomorphism of the cohomology vector spaces for a *dual* pair of Calabi–Yau manifolds switching A- and B-model bigradings.

More, generally if considering Calabi–Yau orbifolds in place of quasismooth varieties, one replaces the ordinary cohomology ring by the Chen–Ruan cohomology ring H^*_{orb}. This is an essential question if H^*_{orb} forms a Hodge diamond too. Some of the Hodge diamond properties above follow directly from the definitions or could be verified in the similar way to our main Theorem 1.1 below, while the others (like the property (4)) were not investigated in literature up to our knowledge and do not look to be straightforward.

1.2 Landau–Ginzburg orbifolds

Another facet of mirror symmetry is given by matching the so-called Landau–Ginzburg orbifolds in place of Calabi–Yau manifolds or orbifolds (cf. [17, 22, 29, 30, 31]). Mathematically, these are the pairs (f, G) with f being a quasihomogeneous polynomial with the only critical point $0 \in \mathbb{C}^N$ and G being a group of symmetries of f.

Consider the *maximal group of linear symmetries* of f

$$\mathrm{GL}_f := \{g \in \mathrm{GL}(N,\mathbb{C}) \mid f(g\cdot \mathbf{x}) = f(\mathbf{x})\}.$$

It is nontrivial because it contains a nontrivial subgroup J generated by j_f

$$j_f \cdot (x_1,\dots,x_N) := \left(\mathrm{e}^{2\pi\sqrt{-1}d_1/d_0}x_1,\dots,\mathrm{e}^{2\pi\sqrt{-1}d_N/d_0}x_N\right).$$

Also important is the group $\mathrm{SL}_f := \mathrm{GL}_f \cap \mathrm{SL}(N,\mathbb{C})$ consisting of GL_f elements preserving the volume form of $\mathbb{C}^N$.

For any $G \subseteq \mathrm{GL}_f$, the pair (f, G) is called a *Landau–Ginzburg orbifold*. One associates to it the *state space*, which is the G-equivariant generalization of a Jacobian ring of f, together with A- and B- model bigradings, which again differ by $90°$-rotation from one another provided G acts by transformations with determinant 1. Since these are interdependent, within this paper we will focus on just the B-model, as it has clearer geometric interpretation, much alike bigrading on the cohomology of Calabi–Yau manifold. For this reason, we will call this state space endowed with B-model bigrading by $\mathcal{B}(f, G)$.

Up till now, Landau–Ginzburg orbifolds were mostly investigated for the groups G acting diagonally on $\mathbb{C}^N$ and also for f belonging to a very special class of polynomials – the so-called invertible polynomials (cf. [4, 5, 6, 14, 19, 20, 21]). Also some work was done for the symmetry groups $G = S \ltimes G^d$ with G^d acting diagonally and $S \subseteq S_N$ – a subgroup of a symmetric group on N elements [2, 3, 10, 11, 12, 18, 24, 27, 32]. We relax both conditions in this paper.

Once again, the mirror symmetry attempts to match A- and B-state spaces for *dual* pairs of Landau–Ginzburg orbifolds. It should be mentioned, that the state space enjoys several other structures besides being just bigraded vector space, like multiplication or bilinear form on both A- and B-sides, which should also be compatible with the mirror map. They will not be considered in the present paper.

1.3 Calabi–Yau/Landau–Ginzburg correspondence

The final piece of matching comes from Calabi–Yau/Landau–Ginzburg correspondence, which relates respective A-models and their state spaces coming from Calabi–Yau and Landau–Ginzburg geometries of $(X_f, G/J)$ and (f, G) provided $J \subseteq G$. Mathematically, up till now this was proved for diagonal symmetry groups by Chiodo and Ruan [9, Theorem 14] and in some special cases with $N = 5$ in [25].

However, if this correspondence holds, the vector space $\mathcal{B}(f, G)$ is also expected to form a diamond. Namely, it should satisfy the properties (1)–(6) above. Formulated in terms of $\mathcal{B}(f, G)$ this becomes a purely singularity theoretic question. It is the main topic of our paper.

Theorem 1.1. *Let $f \in \mathbb{C}[x_1, \dots, x_N]$ be a quasihomogeneous polynomial satisfying Calabi–Yau condition and defining an isolated singularity. Then for any $G \subseteq \mathrm{SL}_f$, such that $J \subseteq G$ the state space $\mathcal{B}(f, G)$ forms a diamond of size $N-2$ in a sense that it satisfies conditions* (1)–(6) *above.*

Proof. The proof is summed up in Propositions 5.2, 5.3 and 5.4. The vertical and horizontal isomorphism are given in Section 4. ■

2 Preliminaries and notations

2.1 Quasihomogeneous singularities

The polynomial $f \in \mathbb{C}[x_1, \dots, x_N]$ is called *quasihomogeneous* if there are positive integers $d_0, d_1, \dots, d_N$, such that

$$f\big(\lambda^{d_1} x_1, \dots, \lambda^{d_N} x_N\big) = \lambda^{d_0} f(x_1, \dots, x_N), \qquad \forall \lambda \in \mathbb{C}. \tag{2.1}$$

In what follows we will say that f is quasihomogeneous with respect to the *weights* $d_0, d_1, \dots, d_N$ or the *reduced weights* $q_1 := d_1/d_0, \dots, q_N := d_N/d_0$.

We will say that f defines an *isolated singularity* at $0 \in \mathbb{C}^N$ if 0 is the only critical point of f. According to K. Saito [28, Satz 1.3] one may consider without changing the singularity only the quasihomogeneous polynomials, such that $0 < q_k \le 1/2$ for all $k = 1, \dots, N$. Moreover, we may assume that all variable x_k with $q_k = 1/2$ enter f only in monomial x_k^2, in particular, there are no monomials of the form $x_i x_j$ with $i \neq j$. Then the number of its monomials is not less than N, the number of variables.

Example 2.1. Fermat, chain and loop type polynomials are examples of quasihomogeneous singularities for any natural $a_i \ge 2$

$$\begin{aligned}
&f = x_1^{a_1} \qquad \text{Fermat type},\\
&f = x_1^{a_1} + x_1 x_2^{a_2} + \dots + x_{N-1} x_N^{a_N} \qquad \text{chain type},\\
&f = x_1^{a_1} x_2 + x_2^{a_2} x_3 + \dots + x_{N-1}^{a_{N-1}} x_N + x_N^{a_N} x_1 \qquad \text{loop type}.
\end{aligned}$$

Using the word ‘type’, we mean the certain structure of the monomials without specifying the exponents a_i.

It’s easy to see that for Fermat, chain and loop type polynomials, the reduced weights $q_1, \dots, q_N$ are defined in a unique way. This is also true for any quasihomogeneous singularity that we assume (cf. [28, Korollar 1.7]). We have

$$
\begin{aligned}
q_1 &= \frac{1}{a_1} && \text{Fermat type,}\\
q_i &= \sum_{j=i}^{N} \frac{(-1)^{j-i}}{a_1\cdots a_j} && \text{chain type,}\\
q_i &= (-1)^{N-1}\frac{1-a_i+a_i\sum_{k=2}^{N-1}(-1)^k\prod_{l=2}^{k}a_{i-l+1}}{\prod_{k=1}^{N}a_k-(-1)^N} && \text{loop type,}
\end{aligned}
$$

where one assumes $a_0 := a_N$, $a_{-1} := a_{N-1}$, $a_{-2} := a_{N-2}$ and so on.

Given $f \in \mathbb{C}[x_1,\dots,x_N]$ and $g \in \mathbb{C}[y_1,\dots,y_M]$ both defining quasihomogeneous singularities it follows immediately that $f+g \in \mathbb{C}[x_1,\dots,x_N,y_1,\dots,y_M]$ defines a quasihomogeneous singularity too. We will denote such a sum by $f \oplus g$.

Example 2.2 ([1, Section 13.1])**.** If $N \le 2$, then all quasihomogeneous isolated singularities are given by the $\oplus$-sums of the Fermat, chain or loop type polynomials.

Example 2.3 ([1, Section 13.2])**.** If $N = 3$, then all quasihomogeneous isolated singularities are given by the following polynomials $f_{\mathrm{I}} = x_1^{a_1} + x_2^{a_2} + x_3^{a_3}$, $f_{\mathrm{II}} = x_1^{a_1} + x_2^{a_2}x_3 + x_3^{a_3}$, $f_{\mathrm{III}} = x_1^{a_1} + x_2^{a_2}x_1 + x_3^{a_3}x_1 + \varepsilon x_2^p x_3^q$, $f_{\mathrm{IV}} = x_1^{a_1} + x_2^{a_2}x_3 + x_3^{a_3}x_2$, $f_{\mathrm{V}} = x_1^{a_1} + x_2^{a_2}x_1 + x_3^{a_3}x_2$, $f_{\mathrm{VI}} = x_1^{a_1}x_2 + x_2^{a_2}x_1 + x_3^{a_3}x_1 + \varepsilon x_2^p x_3^q$, $f_{\mathrm{VII}} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + x_3^{a_3}x_1$ with some positive a_1, a_2, a_3. The numbers a_i are arbitrary for f_{I}, f_{II}, f_{IV}, f_{V}, f_{VII}, however the polynomials f_{III} and f_{VI} are only quasihomogeneous if $\varepsilon \neq 0$ and some additional combinatorial condition on a_1, a_2, a_3 holds. In particular the least common divisor of (a_2, a_3) should be divisible by $a_1 - 1$ for f_{III} to exist. Allowed values of ε depend quiet heavily on a_i. In particular $\varepsilon \notin \{0, \sqrt{-1}, -\sqrt{-1}\}$ for $f = x_1^3 + x_1x_2^2 + x_1x_3^2 + \varepsilon x_1 x_2^2$ to define an isolated singularity.

2.2 Graph of a quasihomogeneous singularity

Let $f \in \mathbb{C}[x_1,\dots,x_N]$ define an isolated singularity. Then for every index $j \le N$ the polynomial f has either the summand x_j^a or a summand $x_j^a x_k$ for some exponent $a \ge 2$ and index $k \le N$ (cf. [28, Korollar 1.6] and [16, Theorem 2.2]). Construct a map $\kappa\colon \{1,\dots,N\} \to \{1,\dots,N\}$. Set $\kappa(j) := j$ in the first case above and $\kappa(j) := k$ in the second.

Following [16, Section 3] associate to f the graph[1] Γ_f with N vertices labelled with the numbers $1,\dots,N$ and the oriented arrows $j \to \kappa(j)$ if $j \neq \kappa(j)$. In other words, the vertices correspond to the variables x_i and the arrows to the monomials $x_j^a x_k$.

Example 2.4. The graphs of the $N = 3$ quasihomogeneous singularities are all listed in Figure 1.

Call a tree oriented if its root has only incoming edges adjacent to it and any other vertex has exactly one outgoing edge and several incoming edges adjacent to it. The following proposition is immediate.

Proposition 2.5 (cf. [16, Lemma 3.1])**.** *Any graph Γ_f is a disjoint union of the graphs of the following two types*

(1) *oriented tree,*

(2) *oriented circle with the oriented trees having the roots on this oriented circle.*

In what follows we consider the root of the type (1) graph above as a cycle with one vertex. This merges the two types above.

[1] Such graphs were first considered by Arnold, however with the self-pointing arrows $j \to j$ too. We decide to remove such arrows to reduce complexity.

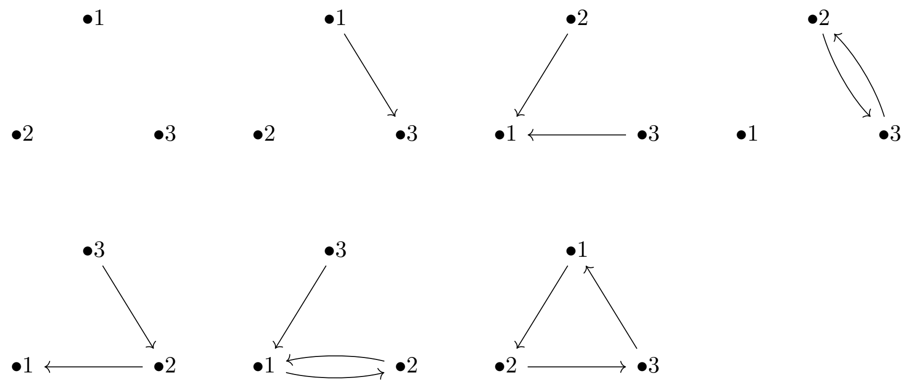


Figure 1. Graphs of $N = 3$ quasihomogenous singularities (see Example 2.3).

It's easy to see that $\Gamma_{f \oplus g} = \Gamma_f \sqcup \Gamma_g$, but it is not true that f decomposes into the $\oplus$-sum if Γ_f has more than one component. For example the graph of $f = x_1^3 + x_2^3 + x_3^3 + x_1 x_2 x_3$ is just the disjoint union of three vertices without any edges.

2.3 Graph decomposition of a polynomial

Assume we only know the graph Γ_f and not the polynomial f itself. The graph structure indicates some monomials that are the summands of f. Call these monomials *graph monomials*. In particular, f has only graph monomials if it is of Fermat, chain or loop type or a $\oplus$-sum of them.

Let f be such that Γ_f has only one connected component. Then Γ_f has one oriented circle, and p oriented trees with the roots on this circle. We have the decomposition

$$f = f_0 + f_1 + \cdots + f_p + f_{\text{add}}, \tag{2.2}$$

with

(1) $f_0, f_1, \ldots, f_p, f_{\text{add}} \in \mathbb{C}[x_1, \ldots, x_N]$,

(2) f_0 having as the summands only those graph monomials of f, that build up the oriented circle or the common root,

(3) f_k having as the summands only those graph monomials of f, that build up the k-th oriented tree,

(4) $f_{\text{add}} := f - f_0 - f_1 - \cdots - f_p$ having as the summands all the non-graph monomials of f.

This decomposition extends easily to the case of Γ_f having several components. Note that we could have had $p = 0$, but it should always hold that $f_0 \neq 0$.

Writing the decomposition above we had to order the trees by the index of f_i. This ordering is not important in what follows.

Example 2.6. The polynomial $f = x_1^3 + x_1\left(x_2^2 + x_3^2 + x_4^2\right) + \epsilon x_2 x_3 x_4$ with $\varepsilon \in \mathbb{C} \setminus \left\{0, \pm 2\sqrt{-1}\right\}$ defines an isolated singularity. It's also quasihomogeneous with $q_1 = \cdots = q_4 = 1/3$. We have $p = 3$,

$$f_0 = x_1^3, \qquad f_1 = x_1 x_2^2, \qquad f_2 = x_1 x_3^2, \qquad f_3 = x_1 x_4^2, \qquad f_{\text{add}} = \varepsilon x_2 x_3 x_4.$$

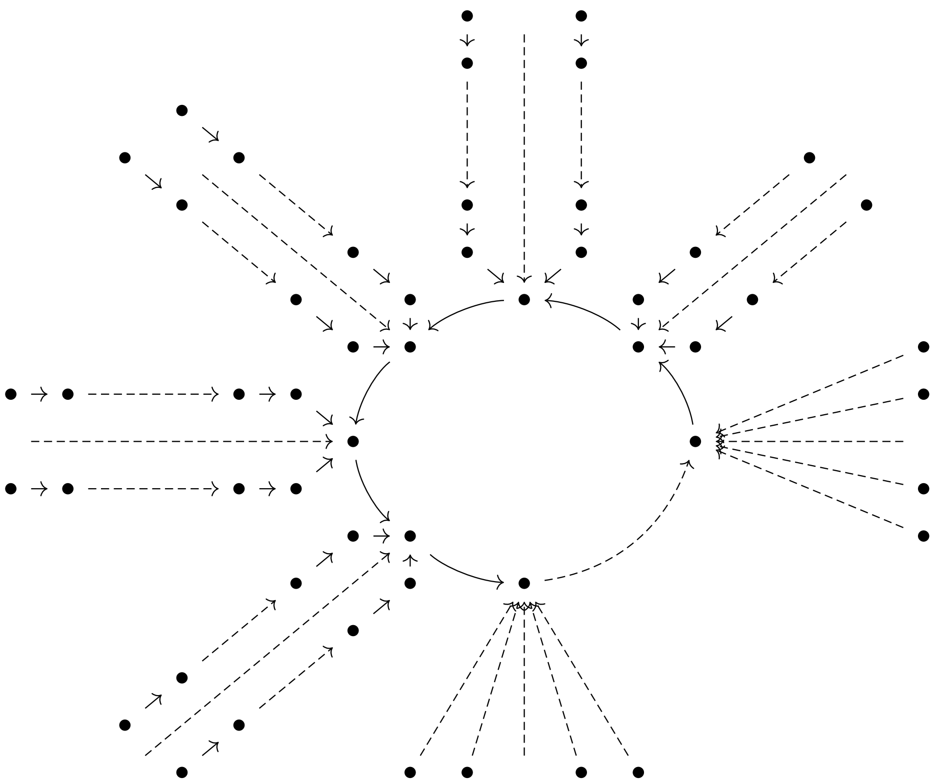


Figure 2. Connected component of Γ_f (see Proposition 2.5).

2.4 Graph exponents matrix

Let f define a quasihomogeneous singularity. We introduce the matrix E_f with the entries in $\mathbb{Z}_{\geq 0}$. It follows from Proposition 2.5 that f has exactly N graph monomials. Let every row of E_f correspond to a graph monomial. The components of this row will be $(\alpha_1, \dots, \alpha_N)$ if and only if the corresponding graph monomial is $\varepsilon \cdot x_1^{\alpha_1} \cdots x_N^{\alpha_N}$ for some $\varepsilon \in \mathbb{C}^*$.

The matrix E_f is only defined up to a permutation of the rows. We will call it *graph exponents matrix*.

Let E_{ij} denote the components of E_f. Then for some non-zero constants c_k we have

$$f - f_{\mathrm{add}} = \sum_{k=1}^{N} c_k x_1^{E_{k1}} \cdots x_N^{E_{kN}}. \tag{2.3}$$

Remark 2.7. Such a matrix was previously defined in the literature only for the invertible polynomials (see Section 2.5). We consider it here in a wider context.

Example 2.8. The matrices E_f of the Example 2.3 are

$$E_{f_{\mathrm{I}}} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad E_{f_{\mathrm{II}}} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 1 \\ 0 & 0 & a_3 \end{pmatrix}, \qquad E_{f_{\mathrm{III}}} = \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_2 & 0 \\ 1 & 0 & a_3 \end{pmatrix},$$

$$E_{f_{\mathrm{IV}}} = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 1 \\ 1 & 0 & a_3 \end{pmatrix}, \qquad E_{f_{\mathrm{V}}} = \begin{pmatrix} a_1 & 0 & 0 \\ 1 & a_2 & 0 \\ 0 & 1 & a_3 \end{pmatrix}, \qquad E_{f_{\mathrm{VI}}} = \begin{pmatrix} a_1 & 1 & 0 \\ 1 & a_2 & 0 \\ 1 & 0 & a_3 \end{pmatrix},$$

$$E_{f_{\rm VII}} = \begin{pmatrix} a_1 & 1 & 0 \\ 0 & a_2 & 1 \\ 1 & 0 & a_3 \end{pmatrix}.$$

The graph exponents matrices of loop and chain type polynomials read

$$E_{\rm loop} = \begin{pmatrix} a_1 & 1 & \dots & 0 & 0 \\ 0 & a_2 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & 0 & \dots & a_{N-1} & 1 \\ 1 & 0 & 0 & \dots & a_N \end{pmatrix}, \quad E_{\rm chain} = \begin{pmatrix} a_1 & 0 & \dots & 0 & 0 \\ 1 & a_2 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 1 & a_{N-1} & 0 \\ 0 & 0 & \dots & 1 & a_N \end{pmatrix}. \tag{2.4}$$

In general, by Proposition 2.5 if Γ_f has only one connected component, the matrix E_f after some renumbering of the variables and the rows has the block form. The diagonal blocks are several chain type exponent matrices and exactly one loop type exponents matrix as in equation (2.4), such that for every chain type block there is exactly one additional matrix entry 1 in the first row of this block and the column of a loop type block. All the other matrix entries except listed vanish,

$$E_f = \begin{pmatrix} A_0 & 0 & 0 & 0 \\ U_{i_1j_1} & A_1 & 0 & 0 \\ \vdots & & \ddots & 0 \\ U_{i_pj_p} & 0 & 0 & A_p \end{pmatrix}, \tag{2.5}$$

where A_0 is a loop type polynomial exponents matrix and $A_1, \dots, A_p$ are chain type polynomial exponent matrices, the matrix U_{ij} is the rectangular matrix with 1 at position (i, j) and all other entries 0.

Assuming the decomposition of equation (2.2), the matrix A_0 is exactly the exponent matrix of f_0 and the matrices $A_1, \dots, A_p$ are defined by $f_1, \dots, f_p$.

Proposition 2.9. *Let f define a quasihomogeneous singularity. Then*

(i) the matrix E_f is invertible,

(ii) there is a canonical choice of the integer weights $(d_0, d_1, \dots, d_N)$.

Proof. Let f be quasihomogeneous with the reduced weights $(q_1, \dots, q_N)$. We show first that these weights are defined uniquely by the graph monomials of f.

Let f be decomposed as in equation (2.2). Then f_0 is of Fermat or loop type and the weights of its variables are defined uniquely. Similarly for any f_k with $k = 1, \dots, p$ corresponding to the tree with the root on the oriented circle, the weight of the root's variable is defined by the quasihomogeneity of f_0, going up the tree of deduces uniquely the weight of every variable of f_k corresponding to the consequent vertex.

Introduce two $\mathbb{Z}^N$ vectors: $\overline{q} := (q_1, \dots, q_N)^{\mathsf{T}}$ and $\mathbf{1} := (1, \dots, 1)^{\mathsf{T}}$. Then the quasihomogeneity condition on f is equivalent to the $\mathbb{Q}^N$ vector equality $E_f \cdot \overline{q} = \mathbf{1}$. It follows now from Cramer rule that $\det(E_f) \neq 0$ because this equation has a unique solution. This completes (i).

The canonical weight set is obtained by taking $d_f := \det(E_f)$ and solving $E_f \cdot \overline{d} = d_f \mathbf{1}$ for $\overline{d} := (d_1, \dots, d_N)^{\mathsf{T}}$. ■

2.5 Invertible polynomials

The set of all quasihomogeneous singularities contains the following important class. The polynomial f defining an isolated quasihomogeneous singularity having no monomial of the form $x_i x_j$

with $i \neq j$ and as many monomials as the variables is called *invertible polynomial* and is said to define an *invertible singularity.*

The following statement can be assumed as a well-known (cf. [23]), we add up the proof for completeness.

Proposition 2.10. *Let f be an invertible polynomial. Then after some rescaling and renumbering of the variables we have $f = f^{(1)} \oplus \cdots \oplus f^{(n)}$ for $f^{(k)}$ being either of Fermat, chain or loop type.*

Proof. Assume Γ_f to contain a vertex with two incoming arrows. Then f is of the form

$$\alpha_1 x_i^a x_l^K + \alpha_2 x_j^b x_i + \alpha_3 x_k^c x_i + g(x),$$

where $K \in \{0,1\}$, $\alpha_1\alpha_2\alpha_3 \neq 0$, $b, c \geq 2$ and g does not depend on x_k, x_i, x_j. Computing

$$\frac{\partial f}{\partial x_i} = a\alpha_1 x_i^{a-1} x_l^K + \alpha_2 x_j^b + \alpha_3 x_k^c, \qquad \frac{\partial f}{\partial x_j} = b\alpha_2 x_j^{b-1} x_i, \qquad \frac{\partial f}{\partial x_k} = a\alpha_3 x_k^{c-1} x_i.$$

Setting $x_i = 0$, we see that vanishing $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_j} = \frac{\partial f}{\partial x_k} = 0$ is equivalent to $\alpha_2 x_j^b + \alpha_3 x_k^c = 0$ what shows that $x_i = x_j = x_k = 0$ is not an isolated critical point of f. ∎

The graphs of invertible singularities are disjoint unions of isolated vertices (Fermat types), oriented cycles (loop types) and one branch trees (chain types).

In the notation of Section 2.3, the graph decomposition equation (2.2) of Fermat, loop and chain type polynomials is the following. We have always $f_{\mathrm{add}} = 0$, $p = 0$ and $f_0 = f$ for Fermat and loop types, but $p = 1$ and $f_0 + f_1 = f$ for chain type with $f_0 = x_m^{a_m}$.

Example 2.11. The quasihomogeneous singularities with $N = 2$ are all invertible. The quasihomogeneous singularities with $N = 3$ are not all invertible. In the notation of Example 2.3, we have f_{I} – Fermat$\oplus$Fermat$\oplus$Fermat, f_{II} – Fermat$\oplus$chain, f_{III} – not invertible, f_{IV} – Fermat$\oplus$loop, f_{V} – chain, f_{VI} – not invertible, f_{VII} – loop.

3 Symmetries

Given a quasihomogeneous polynomial $f = f(x_1, \ldots, x_N)$ consider the *maximal group of linear symmetries* of f defined by

$$\mathrm{GL}_f := \{g \in \mathrm{GL}(N, \mathbb{C}) \mid f(g \cdot \mathbf{x}) = f(\mathbf{x})\}.$$

Lemma 3.1. *Under our assumptions on f any $g \in \mathrm{GL}_f$ necessarily preserves the weights of the variables, i.e., maps each x_i to a linear combination of x_j with the same weight.*

Proof. The action of g preserves the homogeneous components of f. In particular, the variables in the quadratic terms of f map to linear combinations of variables in the quadratic terms of f and hence weight $1/2$ subspace is preserved by f.

We may now assume that f has no quadratic terms. In this case, the argument of [26, Theorem 2.1] applies verbatim to the quasihomogeneous situation to prove that GL_f is finite. Let

$$\widetilde{\mathrm{GL}}_f = \{g \in \mathrm{GL}(N, \mathbb{C}) \mid f(g \cdot \mathbf{x}) = \chi(g) f(\mathbf{x}), \chi(g) \in \mathbb{C}^*\}.$$

The map $\chi\colon \widetilde{\mathrm{GL}}_f \to \mathbb{C}^*$ is a character and GL_f is precisely the kernel of χ, in particular, it is a normal subgroup. Moreover, the condition (2.1) provides an inclusion $t\colon \mathbb{C}^* \hookrightarrow \widetilde{\mathrm{GL}}_f$, such that $\chi \circ t$ is a degree $d_0 > 0$ map. The action by conjugation of connected subgroup $t(\mathbb{C}^*)$ on the finite subgroup GL_f is necessary trivial, which means that GL_f commute with $t(\mathbb{C}^*)$. This means that GL_f preserves the eigenspaces of $t(\mathbb{C}^*)$ as desired. ∎

Remark 3.2. Notably, the observation of this Lemma seems to be unknown to previous authors (for example, it was imposed as condition in [25] and some subsequent works).

Let $G_f^d \subseteq \mathrm{GL}_f$ be the *maximal group of diagonal symmetries* of f. This is the group of all diagonal elements of $\mathrm{GL}(N, \mathbb{C})$ belonging to GL_f.

We have

$$G_f^d \cong \big\{(\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \mid f(\lambda_1 x_1, \dots, \lambda_N x_N) = f(x_1, \dots, x_N)\big\}.$$

It's obvious that

$$G^d_{f' \oplus f''} \cong G^d_{f'} \times G^d_{f''}. \tag{3.1}$$

Note, however, that the same does not necessarily hold for $\mathrm{GL}_{f'\oplus f''}$.

In what follows we will use the notation

$$\mathbf{e}\,[\alpha] := \exp\big(2\pi\sqrt{-1}\alpha\big), \qquad \alpha \in \mathbb{R}.$$

Each element $g \in G_f^d$ has a unique expression of the form

$$g = \mathrm{diag}\left(\mathbf{e}\left[\frac{\alpha_1}{r}\right], \dots, \mathbf{e}\left[\frac{\alpha_N}{r}\right]\right) \qquad \text{with} \quad 0 \le \alpha_i < r, \quad \alpha_i \in \mathbb{N},$$

where r is the order of g. We adopt the additive notation

$$\overline{g} = (\alpha_1/r, \dots, \alpha_N/r) \qquad \text{or} \qquad \overline{g} = \frac{1}{r}(\alpha_1, \dots, \alpha_N)$$

for such an element g.

Example 3.3. For $f = x_1^{a_1}$ we have $\mathrm{GL}_f = G_f^d = \langle g \rangle$ with $g \in \mathbb{C}^*$ acting by $g(x_1) = \mathbf{e}\big[\frac{1}{a_1}\big] \cdot x_1$. Its order is a_1 and in the additive notation we have $\overline{g} = (1/a_1)$, giving us $\mathrm{GL}_f \cong \mathbb{Z}/a_1\mathbb{Z}$.

Example 3.4. For $f = x_1^{a_1}x_2 + x_2^{a_2}$ we have $G_f^d = \langle g_1, g_2 \rangle$ with $g_1 \cdot (x_1, x_2) = (\mathbf{e}[\frac{1}{a_1}]x_1, x_2)$ and $g_2 \cdot (x_1, x_2) = \big(\mathbf{e}[\frac{1}{a_2}(1 - \frac{1}{a_1})]x_1, \mathbf{e}\big[\frac{1}{a_2}\big]x_2\big)$. In the additive notation $\overline{g}_1 = (1/a_1, 0)$ and $\overline{g}_2 = ((a_1 - 1)/(a_1 a_2), 1/a_2)$. In this example $\mathrm{GL}_f = G_f^d$ because $q_1 \neq q_2$.

Let $(q_1, \dots, q_N)$ be the reduced weight set of f. Then we have

$$j_f := (\mathbf{e}[q_1], \dots, \mathbf{e}[q_N]) \in G_f^d.$$

In particular, it follows that G_f^d and GL_f are not empty whenever f is quasihomogeneous. Denote by J the group generated by j_f:

$$J := \langle j_f \rangle \subseteq G_f^d.$$

Since $g \in \mathrm{GL}_f$ preserves the weights we see that j_f commutes with g. In other words J is the central subgroup of GL_f.

3.1 Fixed loci of the GL_f elements

For each $g \in \mathrm{GL}_f$, denote by $\mathrm{Fix}(g)$ *the fixed locus of* g

$$\mathrm{Fix}(g) := \big\{(x_1, \dots, x_N) \in \mathbb{C}^N \mid g \cdot (x_1, \dots, x_N) = (x_1, \dots, x_N)\big\}.$$

This is an eigenvalue 1 subspace of $\mathbb{C}^N$ and therefore a linear subspace of $\mathbb{C}^N$. By $N_g := \dim_{\mathbb{C}} \mathrm{Fix}(g)$ denote its dimension and by $f^g := f|_{\mathrm{Fix}(g)}$ the restriction of f to the fixed locus of g. For $g \in G_f^d$, this linear subspace is furthermore a span of a collection of standard basis vectors.

For each $h \in G_f^d$, let $I_h := \{i_1, \dots, i_{N_h}\}$ be a subset of $\{1, \dots, N\}$ such that

$$\mathrm{Fix}(h) = \big\{(x_1, \dots, x_N) \in \mathbb{C}^N \mid x_j = 0, j \notin I_h\big\}.$$

In the other words, $\mathrm{Fix}(h)$ is indexed by I_h. In particular, $I_{\mathrm{id}} = \{1, \dots, N\}$.

More generally, for $g \in \mathrm{GL}_f$, since g preserves the weight subspaces of $\mathbb{C}^N$, the weights of the subspace $\mathrm{Fix}(g)$ are well-defined and are the subset of $\{q_1, \dots, q_N\}$. Fix a subset $I_g \subset \{1, \dots, N\}$ such that q_k with $k \in I_g$ are exactly all the weights of $\mathrm{Fix}(g)$, so that, in particular, we have $|I_g| = N_g$. Note that if $g \notin G_f^d$ there is no canonical choice for I_g, but the choice made at this step will not impact our results.

Denote by I_h^c the complement of I_h in I_{id} and set $d_h := N - N_h$, the codimension of $\mathrm{Fix}(h)$.

Proposition 3.5. *For any diagonalizable $g \in \mathrm{GL}_f$ with $N_g > 0$ there is a choice of coordinates on $\mathrm{Fix}(g)$ linear in x_i, such that the polynomial f^g also defines a quasihomogeneous singularity.*

Proof. Let $\widetilde{x}_1, \dots, \widetilde{x}_N$ be the coordinates of $\mathbb{C}^N$ dual to the basis diagonalizing g. In this coordinates the polynomial f^g is obtained by setting some of $\widetilde{x}_\bullet$ to zero. The proof follows now by the same argument as in [13, Proposition 5]. ■

Example 3.6. If f is of Fermat or loop type, then any $g \in \mathrm{GL}_f^d$, such that $g \neq \mathrm{id}$ satisfies $\mathrm{Fix}(g) = 0$. If $f = x_1^{a_1} + x_1 x_2^{a_2} + \dots + x_{N-1} x_N^{a_N}$ is of chain type, then any $g \in \mathrm{GL}_f^d$, such that $g \neq \mathrm{id}$ satisfies $\mathrm{Fix}(g) = \big\{(x_1, \dots, x_p, 0, \dots, 0) \in \mathbb{C}^N \mid x_k \in \mathbb{C}\big\}$ for some p depending on g. The polynomial f^g is of chain type again: $f^g = x_1^{a_1} + x_1 x_2^{a_2} + \dots + x_{p-1} x_p^{a_p}$.

Denote also

$$\mathrm{SL}_f := \mathrm{GL}_f \cap \mathrm{SL}(N, \mathbb{C}).$$

This group will be important later on because it preserves the volume form of $\mathbb{C}^N$.

3.2 Age of a GL_f element

For $g \in \mathrm{GL}_f$ let $\lambda_1, \dots, \lambda_N$ be the collection of its eigenvalues. Let $0 \le \alpha_i < 1$ be such that $\lambda_i = \mathbf{e}[\alpha_i]$, then *age* of g is defined as the number

$$\mathrm{age}(g) := \sum_{k=1}^{N} \alpha_k.$$

The following properties are clear but will be important in what follows.

Proposition 3.7.

(1) *For any $g \in \mathrm{GL}_f$ we have*

$$\mathrm{age}(g) + \mathrm{age}\big(g^{-1}\big) = N - N_g = d_g. \tag{3.2}$$

(2) *For a diagonalizable $g \in \mathrm{GL}_f$ we have $\mathrm{age}(g) = 0$ if and only if $g = \mathrm{id}$.*

(3) *We have $g \in \mathrm{SL}_f$ if and only if $\mathrm{age}(g) \in \mathbb{N}$.*

3.3 Diagonal symmetries and a graph Γ_f

Proposition 3.8. *Let Γ_f be a graph of a quasihomogeneous singularity f and $g \in \mathrm{GL}_f$. Then if g acts nontrivially on x_k, then it acts nontrivially on all x_i, such that there is an oriented path from i-th to the k-th vertex.*

Proof. We first show the statement for the arrows pointing at k. Having an arrow $j \to k$ means that f has a monomial $x_j^{a_j} x_k$ as a summand with a nonzero coefficient. We have $g \cdot x_k \neq x_k$ and therefore the summand can only be preserved under the action of g if $g \cdot x_j \neq x_j$. Having an oriented path $i \to j_1 \to \cdots \to j_n \to k$ we have by using the previous step that $g \cdot x_{j_n} \neq x_{j_n}$ and then $g \cdot x_{j_a} \neq x_{j_a}$ for all a. Hence, for x_i too. ∎

Let E_f be the graph exponents matrix of f. Consider

$$G_f^{\mathrm{gr}} := \left\{ (\lambda_1, \dots, \lambda_N) \in (\mathbb{C}^*)^N \,\middle|\, \prod_{j=1}^N \lambda_j^{E_{1j}} = \cdots = \prod_{j=1}^N \lambda_j^{E_{Nj}} = 1 \right\}$$

with E_{ij} being the components of E_f. The group G_f^{gr} is exactly the maximal group of diagonal symmetries of the difference $f - f_{\mathrm{add}}$. In particular, every element of G_f^{gr} preserves all graph monomials of f.

We have $G_f^d \subseteq G_f^{\mathrm{gr}}$ and hence G_f^d is a finite group. An element $\overline{g} = \frac{1}{r}(\alpha_1, \dots, \alpha_N)$ belonging to G_f^{gr} satisfies $E_f \cdot \overline{g} \in \mathbb{Z}^N$. This gives yet another characterization of the group G_f^{gr}

$$G_f^{\mathrm{gr}} \cong \left\{ \overline{g} \in (\mathbb{Q}/\mathbb{Z})^N \mid E_f \cdot \overline{g} \in \mathbb{Z}^N \right\} = E_f^{-1} \mathbb{Z}^N / \mathbb{Z}^N.$$

It follows that every vector $\overline{g}$ giving a G_f^{gr}-element is a linear combination with integer coefficients of the columns of E_f^{-1}. Following the notation of Krawitz [21] define $\overline{\rho}_i$ as the i-th column of E_f^{-1}

$$E_f^{-1} = (\overline{\rho}_1 | \dots | \overline{\rho}_N).$$

Denote also $\rho_i := \mathbf{e}[\overline{\rho}_i] \in G_f^d$.

The elements ρ_k generate G_f^{gr} and $j_f = \rho_1 \cdots \rho_N$. The columns of E_f generate all relations on $\rho_1, \dots, \rho_N$.

In particular, for $(E_{1k}, \dots, E_{Nk})^\mathsf{T}$ being a k-th column of E_f we have in G_f^{gr}

$$\rho_1^{E_{1k}} \cdots \rho_N^{E_{Nk}} = \mathrm{id},$$

and all other relations among $\{\rho_k\}_{k=1}^N$ follow from those written above.

3.4 Diagonal symmetries of an invertible singularity

In [14] for an invertible f, the authors gave the set $\mathcal{S}_f$ of all N-tuples $(s_1, \dots, s_N)$ such that every $g \in G_f^d \setminus \{\mathrm{id}\}$ is written uniquely by

$$g = \prod_{k \in I_g^c} \rho_k^{s_k},$$

and $s_k = 0$ if and only if $k \in I_g$. Due to equation (3.1) and Proposition 2.10, it is enough to construct such set for Fermat, loop or chain type polynomials.

Proposition 3.9. *For f being of Fermat, chain or loop type the set $\mathcal{S}_f$ consists of all $s = (s_1, \dots, s_N)$, such that*

- (*Fermat type*): $1 \le s_1 \le a_1 - 1$
- (*loop type*): $1 \le s_k \le a_k$ *and*

$$s \neq (a_1, 1, a_3, 1, \dots, a_{N-1}, 1), \qquad s \neq (1, a_2, 1, a_4, 1, \dots, a_N)$$

if N is even.

- (*chain type*): *s is of the form*

$$(0, \dots, 0, s_p, s_{p+1}, \dots, s_N), \qquad \text{with} \quad \{1, \dots, p-1\} = I_g,$$

with $1 \le s_p \le a_p - 1$, $1 \le s_k \le a_k$ for $k > p$.

Corollary 3.10. *In the additive notation for the column $s^{\mathsf{T}} = (s_1, \dots, s_N)^{\mathsf{T}}$, we have $\overline{g} = E_f^{-1} s^{\mathsf{T}}$.*

3.5 Diagonal symmetries of a quasihomogeneous singularity

For any quasihomogeneous singularity f, consider its graph decomposition as in equation (2.2). Up to the renumbering and rescaling of the variables, we have

$$f_0 = x_1^{a_1} \qquad \text{or} \qquad f_0 = x_1^{a_1} x_2 + \dots + x_K^{a_K} x_1,$$
$$f_1 = x_1 x_{K+1}^{b_1} + x_{K+1} x_{K+2}^{b_2} + \dots + x_{K+L-1} x_{K+L}^{b_L}$$

with the similar expression for $f_2, \dots, f_p$.

Any nontrivial $g \in G_{f_0}^d$ extends to an element $\widetilde{g} \in G_f^{\mathrm{gr}}$. Moreover it follows that $\mathrm{Fix}(g) = 0$ and also $\mathrm{Fix}(\widetilde{g}) = 0$ as long as $g \neq \mathrm{id}$. Similarly any element $h \in G_f^{\mathrm{gr}}$ with $\mathrm{Fix}(h) = 0$ acts nontrivially on $x_1, \dots, x_K$ preserving f_0. Hence it defines $h_0 \in G_{f_0}^d$ by the restriction.

At the same time any $h \in (\mathbb{C}^*)^L$ acting diagonally on $(x_{K+1}, \dots, x_{K+L})$ preserving f_1 extends to an element of G_f^{gr} assuming it to act trivially on f_0 and all other $f_2, \dots, f_p$. One notes immediately that such elements h are the elements of chain type polynomial symmetry group. Denote the group of all such elements by $G_{f_1}^{\circ}$.

We construct the groups $G_{f_2}^{\circ}, \dots, G_{f_p}^{\circ}$ in a similar way.

For a nontrivial element $g \in G_f^{\mathrm{gr}}$ and its restriction $g_0 \in G_{f_0}^d$, the extension $\widetilde{g}_0$ is not unique. However, having it fixed, we have by Proposition 3.8 that there is a unique set of $g_k \in G_{f_k}^{\circ}$ for $k = 1, \dots, p$, s.t.

$$g = \widetilde{g}_0 \cdot g_1 \cdots g_p,$$

We have that every g_k acts non-trivially only on the variables of f_k preserving all the variables of f_0 identically.

We have

$$|G_f^{\mathrm{gr}}| = |G_{f_0}^d| \cdot |G_{f_1}^{\mathrm{gr}}| \cdots |G_{f_p}^{\mathrm{gr}}|.$$

Associate to every $g_0, g_1, \dots, g_p$ an element $s_0, s_1, \dots, s_p$ as in Proposition 3.9. Composing them in one column s, we have

$$\overline{g} = E_f^{-1} s^{\mathsf{T}}.$$

Note that for $s_0 \neq 0$ and $s_1 = 0, \dots, s_p = 0$, all components of $\overline{g}$ are nonzero. We follow the convention $s_1 \neq 0, \dots, s_p \neq 0$ if g is such that $g_0 \neq \mathrm{id}$.

The following proposition is very important in what follows.

Proposition 3.11. *For any $g \in G_f^d$ such that $\overline{g} = E_f^{-1} s$, we have*

$$\mathrm{age}(g) = (1, \dots, 1) E_f^{-1} s^{\mathsf{T}}.$$

Proof. We need to show that the components of $\overline{g}$ belong to $[0, 1)$. This follows immediately from the equality $E_f \overline{g} = s$, the bounds on s and the special form of the matrix E_f (see equation (2.5)). ∎

3.6 Symmetries and the Calabi–Yau condition

Let the reduced weight set $q_1, \dots, q_N$ of f satisfy $\sum_{k=1}^N q_k = 1$. This equality is called the *Calabi–Yau* condition and we will say that f *satisfies the CY condition.* We show in this section that it puts significant restrictions on the symmetries of f.

Let the matrix $E_f^\top$ define a polynomial $f^\top$. Namely, if for f we have (2.3), then

$$f^\top := \sum_{k=1}^N c_k x_1^{E_{1k}} \cdots x_N^{E_{Nk}}.$$

This polynomial does not necessarily define an isolated singularity. However, it is quasihomogeneous again with some weights $q_1^\top, \dots, q_N^\top$ by the same argument as in Proposition 2.9.

We call f star-shaped if its graph Γ_f consists of $N-1$ vertices all adjacent to one vertex. Namely, $f_0 = x_1^{a_1}$, $p = N-1$ and $f_i = x_1 x_{i+1}^{b_{i+1}}$. Such a polynomial satisfies the CY condition if and only if

$$\sum_{i=2}^N \frac{1}{b_i} = 1.$$

Example of such a polynomial is given by

$$f = x_1^{a_1} + x_1\left(x_2^3 + x_3^3 + x_4^3\right) + x_2^2 x_3^2 + x_2^2 x_4^2 + x_3^2 x_4^2$$

with the Milnor number 81.

We will treat the star-shaped polynomials separately.

Proposition 3.12. *Let f not being a star-shaped polynomial, satisfy the CY condition. Then the weights $q_1^\top, \dots, q_N^\top$ are all positive.*

Proof. This lemma is obvious for invertible polynomial f and we assume only noninvertible cases in the proof.

Let E_f be written as in equation (2.5) and A_0 be a $K \times K$ loop type matrix as in equation (2.4). It is immediate that $q_{K+1}^\top, \dots, q_N^\top$ are positive.

For $i = 1, \dots, K$, denote by $\mathcal{A}_i$ the sum of all $q_j^\top$ with $j > K$, s.t. j-vertex of Γ_f is adjacent to the i-th vertex.

Lemma 3.13. *We have $0 \le \mathcal{A}_i < 1$ for any $1 \le i \le K$.*

Proof. $\mathcal{A}_i$ is non-negative as the sum of the positive weights. However this sum can be empty.

Let $\mathcal{A}_i \ge 1$ for some i. Let the vertices adjacent to the i-th vertex be labelled by $K+1, \dots, K+m$ contributing to f with the monomials $x_i x_{K+1}^{b_{K+1}}, \dots, x_i x_{K+m}^{b_{K+m}}$. Then

$$q_{K+j} = \frac{1}{b_{K+j}}(1 - q_i) \qquad \text{and} \qquad q_{K+j}^\top \le \frac{1}{b_{K+j}}.$$

Denote $S := \sum_{j=1}^m \dfrac{1}{b_{K+j}}$. If the CY condition holds, then

$$q_i + S - q_i S \le 1 \;\Leftrightarrow\; (S-1) \le q_i(S-1).$$

If $S > 1$, this gives $q_i \ge 1$ which contradicts the quasihomogeneity condition of f. If $S = 1$, then $q_{K+1} + \cdots + q_{K+m} = 1 - q_i$ and the CY condition can only hold if f is a star-shaped CY polynomial. ■

Let us show that q_1^T is positive. The proof for $q_2^\mathsf{T},\dots,q_N^\mathsf{T}$ is similar.

Let c_{ij} stand for the components of the $K\times K$ matrix A_0^{-1}. Note that up to a sign these are just the products of a_i divided by $\det A = a_1\cdots a_K + (-1)^{K-1}$. In particular, we have

$$
\begin{aligned}
&c_{i,i} = \frac{a_1\cdots a_K}{a_i \det A}, \qquad c_{i,i+r} = (-1)^r \frac{a_1\cdots a_K}{a_i\cdots a_{i+r}\det A}, \qquad 1\le r\le K-i,\\
&c_{i,i-1} = (-1)^{K-1}\frac{1}{\det A}, \qquad c_{i,i-r} = (-1)^{K-r}\frac{a_{i-r+1}\cdots a_{i-1}}{\det A}, \qquad 1\le r\le i-1.
\end{aligned}
$$

Then

$$
q_i^\mathsf{T} = c_{1i} + \cdots + c_{Ki} - c_{1i}\mathcal{A}_1 - \cdots - c_{Ki}\mathcal{A}_K
$$

for $1\le i\le K$ and

$$
\sum_{i=1}^N q_i \ge \sum_{i=1}^K q_i + \sum_{i=1}^K \mathcal{A}_i(1-q_i) = \sum_{i,j=1}^K c_{ij} + \sum_{i=1}^K \mathcal{A}_i(1-c_{i1}-\cdots-c_{iK}).
$$

Under the CY condition we have

$$
\sum_{i,j=1}^K c_{ij} + \sum_{i=2}^K \mathcal{A}_i(1-c_{i1}-\cdots-c_{iK}) - 1 \le -\mathcal{A}_1(1-c_{11}-\cdots-c_{1K}).
$$

The bracket on the right hand side is positive because $q_1<1$. This gives the estimate

$$
-c_{11}\mathcal{A}_1 \ge \frac{c_{11}}{1-c_{11}-\cdots-c_{1K}}\left(\sum_{i,j=1}^K c_{ij} - 1 + \sum_{i=2}^K \mathcal{A}_i(1-c_{i1}-\cdots-c_{iK})\right)
$$

because c_{11} is positive. We get then the estimate

$$
\begin{aligned}
q_1^\mathsf{T} &\ge c_{11}+\cdots+c_{K1} + \frac{c_{11}}{1-c_{11}-\cdots-c_{1K}}\left(\sum_{i,j=1}^K c_{ij} - 1\right)\\
&\quad + \sum_{i=2}^K \mathcal{A}_i\left(-c_{i1} + c_{11}\frac{1-c_{i1}-\cdots-c_{iK}}{1-c_{11}-\cdots-c_{1K}}\right)\\
&= \frac{(c_{11}+\cdots+c_{K1})(1-c_{11}-\cdots-c_{1K}) + c_{11}\left(\sum_{ij=1}^K c_{ij}-1\right)}{1-c_{11}-\cdots-c_{1K}}\\
&\quad + \sum_{i=2}^K \mathcal{A}_i \frac{c_{11}(1-c_{i1}-\cdots-c_{iK}) - c_{i1}(1-c_{11}-\cdots-c_{1K})}{1-c_{11}-\cdots-c_{1K}}.
\end{aligned}
$$

Introduce the positive numbers T_r and P_r by

$$
\begin{aligned}
&T_r = a_{K-1}(\cdots a_{r-2}(a_{r-1}(a_r-1)+1)-1)\cdots + (-1)^{r-1},\\
&P_{K-1} = a_{K-1}, \qquad P_r = a_r(T_{r+2} + P_{r+1} + (-1)^r), \qquad r\le K-2.
\end{aligned}
$$

Some computations give us

$$
(c_{11}+\cdots+c_{K1})(1-c_{11}-\cdots-c_{1K}) + c_{11}\left(\sum_{ij=1}^K c_{ij} - 1\right) = \frac{a_K}{\det A}(T_3+P_2-1)
$$

and

$$(1 - c_{21} - \cdots - c_{2K}) - c_{21}(1 - c_{11} - \cdots - c_{1K}) = \frac{a_K}{\det A}(T_2 + 1),$$
$$(1 - c_{i1} - \cdots - c_{iK}) - c_{i1}(1 - c_{11} - \cdots - c_{1K}) = \frac{a_K}{\det A}\left(T_i + (-1)^i\right)\prod_{r=2}^{i-1} a_i, \qquad i \geq 3.$$

These are positive numbers for $a_i \geq 2$, what gives the proof after applying the lemma above. ■

Proposition 3.14. *Let f satisfy the CY condition. Then for any diagonalizable $g \in \mathrm{GL}_f$ such that $N_g = 0$, we have*

$$\mathrm{age}(g) \geq \sum_{k=1}^{N} q_k.$$

The equality is only reached if $g = j_f$.

Proof. Rewrite f in the coordinates $\widetilde{x}_1, \dots, \widetilde{x}_N$ dual to the basis diagonalizing g. Then each $\widetilde{x}_k$ is a linear combination of $x_1, \dots, x_N$. Moreover, one can renumber the new variables such that the weight of $\widetilde{x}_k$ is the same as the weight of x_k, namely q_k.

The element j_f is represented in the old and the new basis by the same diagonal matrix. The given element g acts of $\widetilde{x}_k$ just by a rescaling. Therefore it is enough to show the proposition for g belonging to the maximal group of diagonal symmetries.

To prove the propositions for $g \in G_f^d$ it is enough to prove the inequality for any $g \in G_f^{\mathrm{gr}}$ with $N_g = 0$ and f, such that the graph Γ_f has only one connected component.

We have

$$\sum_{k=1}^{N} q_k = (1, \dots, 1)E_f^{-1}\mathbf{1} = (1, \dots, 1)\left(E_f^{\mathsf{T}}\right)^{-1}\mathbf{1} = \sum_{k=1}^{N} q_k^{\mathsf{T}}. \tag{3.3}$$

For a given g assume s, such that $\overline{g} = E_f^{-1}s$ as in Proposition 3.11. None of $s_k = 0$ because $N_g = 0$. We have

$$\mathrm{age}(g) = ((1, \dots, 1)E_f^{-1}s)^{\mathsf{T}} = s^{\mathsf{T}}\left(E_f^{\mathsf{T}}\right)^{-1}\mathbf{1} = \sum_{k=1}^{N} s_k q_k^{\mathsf{T}}.$$

First assume f is not star-shaped. Then

$$\sum_{k=1}^{N} s_k q_k^{\mathsf{T}} \geq \sum_{k=1}^{N} q_k^{\mathsf{T}}$$

because every $s_k \geq 1$ and q_i^{T} are all positive. Combining with equation (3.3) we get the inequality claimed. Moreover it is obvious that the equality is only reached if $s_k = 1$ for all k. This is equivalent to the fact that $g = j_f$.

Now let f be star-shaped. We have $q_1^{\mathsf{T}} = 0$ and $q_k^{\mathsf{T}} > 0$ for $k = 2, \dots, N$. By the same reasoning as above it is enough to consider $\overline{g} = E_f^{-1}s$ with $s_2 = \cdots = s_N = 1$. Then

$$\overline{g} = \left(\frac{s_1}{a_1}, \frac{1}{b_2}\frac{a_1 - s_1}{a_1}, \dots, \frac{1}{b_N}\frac{a_1 - s_1}{a_1}\right)$$

for some $s_1 = 1, \dots, a_1 - 1$. If f defines an isolated singularity, it should have at least one summand $x_2^{r_2} \cdots x_N^{r_N}$ for some nonnegative $r_2, \dots, r_N$. The quasihomogeneity and the g-invariance conditions on this summand give

$$\sum_{k=2}^{N} \frac{r_k}{b_k} \left(1 - \frac{1}{a_1}\right) = 1, \qquad \sum_{k=2}^{N} \frac{r_k}{b_k} \frac{a_1 - s_1}{a_1} \in \mathbb{Z}_{\geq 1}.$$

These two conditions can only hold when $s_1 = 1$. ■

Remark 3.15. The proposition above holds for any invertible polynomial without the Calabi–Yau condition too. However for noninvertible polynomial without Calabi–Yau condition the proposition does not hold in general. If particular for $f = x_1^{10} + x_1\left(x_2^2 + x_3^2 + x_4^2\right)$ and $\overline{g} = \left(\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right)^{\mathsf{T}}$ we have $\overline{q} = \left(\frac{1}{10}, \frac{9}{20}, \frac{9}{20}, \frac{9}{20}\right)$ and $\operatorname{age}(g) - \sum_{k=1}^{4} q_k = -\frac{1}{20}$.

4 The total space

Consider the quotient ring

$$\operatorname{Jac}(f) := \mathbb{C}[x_1, \dots, x_N] \Big/ \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_N}\right).$$

It is a finite-dimensional $\mathbb{C}$-vector space whenever f defines an isolated singularity. Call it *Jacobian algebra* of f and set $\mu_f := \dim_{\mathbb{C}} \operatorname{Jac}(f)$ – the Milnor number of f.

We will assume an additional convention: for the constant function $f = 0$ set $\operatorname{Jac}(f) := \mathbb{C}$, $\mu_f := 1$.

4.1 Grading

The reduced weights $q_1, \dots, q_N$ of f define the $\mathbb{Q}$-grading on $\mathbb{C}[x_1, \dots, x_N]$. Introduce the $\mathbb{Q}$-grading on $\operatorname{Jac}(f)$ by setting

$$\deg\left(\left[x_1^{\alpha_1} \cdots x_N^{\alpha_N}\right]\right) := \alpha_1 q_1 + \cdots + \alpha_N q_N.$$

Let $\phi_1, \dots, \phi_\mu$ be the classes of monomials, generating $\operatorname{Jac}(f)$ as a $\mathbb{C}$-vector space. We say that $X \in \operatorname{Jac}(f)$ is of degree κ if it is expressed as a $\mathbb{C}$-linear combination of degree κ elements $\phi_\bullet$.

Denote by $\operatorname{Jac}(f)_\kappa$ the linear subspace of $\operatorname{Jac}(f)$ spanned by the degree κ elements. Let the *Hessian* of f be defined as the following determinant:

$$\operatorname{hess}(f) := \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j=1,\dots,N}.$$

Its class is nonzero in $\operatorname{Jac}(f)$.

Proposition 4.1. *The maximal degree of a* $\operatorname{Jac}(f)$*-element is* $\widehat{c} = \widehat{c}(f) := \sum_{k=1}^{N} (1 - 2q_k)$. *Moreover we have*

$$\operatorname{Jac}(f)_{\widehat{c}} = \mathbb{C}\langle[\operatorname{hess}(f)]\rangle.$$

Proof. See [1, Section II]. ■

4.2 Pairing

The algebra $\mathrm{Jac}(f)$ can be endowed with the $\mathbb{C}$-bilinear nondegenerate pairing η_f called *residue pairing* (see [15, Chapter 5], [1, Section 5.11]). The value $\eta_f([u],[v])$ is taken as the projection of $[u][v]$ to the top graded component $\mathrm{Jac}(f)_{\widehat{c}}$ divided by its generator $[\mathrm{hess}(f)]$. In particular, we have $\eta_f([1],[\mathrm{hess}(f)]) = 1$.

Proposition 4.2. *For any β, such that $0 \le \beta \le \widehat{c}$ the perfect pairing η_f induces an equivalence*

$$\phi_{f,\beta}\colon\ \mathrm{Jac}(f)_\beta \cong (\mathrm{Jac}(f)_{\widehat{c}-\beta})^\vee, \qquad [p] \mapsto \eta_f([p],-),$$

where $(-)^\vee$ stands for the dual vector space.

Proof. See [1, Section II]. ■

4.3 The total space

For each $g \in \mathrm{GL}_f$, fix a generator of a one-dimensional vector space $\Lambda(g) := \bigwedge^{d_g}\big(\mathbb{C}^N/\mathrm{Fix}(g)\big)$. Denote it by ξ_g.

For $g \in G_f^d$, it is standard to choose the generator to be the wedge product of x_k with $k \in I_g^c$ taken in increasing order.

Define $\mathcal{B}_{\mathrm{tot}}(f)$ as the $\mathbb{C}$-vector spaces of dimension $\sum_{g\in\mathrm{GL}_f} \dim \mathrm{Jac}(f^g)$

$$\mathcal{B}_{\mathrm{tot}}(f) := \bigoplus_{g\in\mathrm{GL}_f} \mathrm{Jac}(f^g)\xi_g. \tag{4.1}$$

Each direct summand $\mathrm{Jac}(f^g)\xi_g$ will be called the *g-th sector*. We will write just $\mathcal{B}_{\mathrm{tot}}$ when the polynomial is clear from the context.

Remark 4.3. Note that for $g, h \in G$, such that $\mathrm{Fix}(g) = \mathrm{Fix}(h)$, we have $f^g = f^h$. Then $\mathrm{Jac}(f^g) = \mathrm{Jac}(f^h)$, but the formal letters $\xi_g \neq \xi_h$ help to distinguish $\mathrm{Jac}(f^g)\xi_g$ and $\mathrm{Jac}(f^h)\xi_h$, such that $\mathrm{Jac}(f^g)\xi_g \oplus \mathrm{Jac}(f^h)\xi_h$ is indeed a direct sum of dimension $\dim \mathrm{Jac}(f^g) + \dim \mathrm{Jac}(f^h)$.

4.4 B-model group action

Note that an element $h \in \mathrm{GL}_f$ induces a map

$$h\colon\ \mathrm{Fix}(g) \to \mathrm{Fix}\big(hgh^{-1}\big) \qquad \text{and hence} \qquad h\colon\ \Lambda(g) \to \Lambda\big(hgh^{-1}\big).$$

Since we have fixed the generators $\xi_\bullet$, the latter map provides a constant $\rho_{h,g} \in \mathbb{C}^*$ such that $h(\xi_g) = \rho_{h,g}\xi_{hgh^{-1}}$. We have

$$\rho_{h_2,h_1gh_1^{-1}}\rho_{h_1,g} = \rho_{h_2h_1,g}. \tag{4.2}$$

Note, that if $g, h \in G_f^d$ or, more generally, if g and h commute $\rho_{h,g}$ is independent of the choice of the generators since $g = hgh^{-1}$. More precisely, in this case it could be computed as follows. Let λ_k, λ'_k be the eigenvalues of h and g in their common eigenbasis, then

$$\rho_{h,g} = \prod_{\substack{k=1,\dots,N\\ \lambda'_k\neq 1}} \lambda_k.$$

We define the action of GL_f on $\mathcal{B}_{\mathrm{tot}}$ by

$$h^*([p(\mathbf{x})]\xi_g) = \rho_{h,g}\big[p\big(h^{-1}\cdot\mathbf{x}\big)\big]\xi_{hgh^{-1}}.$$

This is indeed a group action, i.e., $(h_2h_1)^* = h_2^* \cdot h_1^*$. Indeed, using equation (4.2) we get

$$\begin{aligned}(h_2h_1)^*[p(\mathbf{x})]\xi_g &= \rho_{h_2h_1,g}\big[p\big((h_2h_1)^{-1}\cdot\mathbf{x}\big)\big]\xi_{(h_2h_1)g(h_2h_1)^{-1}} \\ &= \rho_{h_2,h_1gh_1^{-1}}\rho_{h_1,g}\big[p\big(h_1^{-1}h_2^{-1}\cdot\mathbf{x}\big)\big]\xi_{h_2h_1gh_1^{-1}h_2^{-1}} \\ &= h_2^*\big(\rho_{h_1,g}\big[p\big(h_1^{-1}\cdot\mathbf{x}\big)\big]\xi_{h_1gh_1^{-1}}\big) = h_2^*h_1^*([p(\mathbf{x})]\xi_g).\end{aligned}$$

Note that, in particular, if $g, h \in G_f^d$ then h acts on ξ_g by

$$h\colon\ \xi_g \mapsto h^*(\xi_g) := \prod_{k\in I_g^c} h_k\ \cdot \xi_g.$$

Example 4.4. Because $I_{\mathrm{id}}^c = I_{j_f} = \varnothing$, we have

$$h^*(\xi_{\mathrm{id}}) = \xi_{\mathrm{id}} \qquad \text{and} \qquad h^*(\xi_{j_f}) = \det(h)\xi_{j_f}$$

for any $h \in \mathrm{GL}_f$. Similarly for any $[p]\xi_g$ with a homogeneous $p \in \mathbb{C}[x_1, \dots, x_N]$ and $g \in \mathrm{GL}_f$ we have

$$(j_f)^*([p]\xi_g) = \mathbf{e}\Bigg[-\deg(p) + \sum_{k\in I_g^c} q_k\Bigg] \cdot [p]\xi_g.$$

For a finite $G \subseteq \mathrm{GL}_f$ put

$$\mathcal{B}_{\mathrm{tot},G} := \bigoplus_{g\in G} \mathrm{Jac}(f^g)\xi_g \subset \mathcal{B}_{\mathrm{tot}}$$

and define the *B-model state space* $\mathcal{B}(f, G)$ by $\mathcal{B}(f, G) := (\mathcal{B}_{\mathrm{tot},G})^G$. Namely, the linear span of the $\mathcal{B}_{\mathrm{tot}}$ vectors that are invariant with respect to the action of all elements of G.

Remark 4.5. In the literature (see, for example, [25]) a different definition could be found where the sum is taken over the representatives of the conjugacy classes of G and the invariants in each sector are taken with respect to the centralizer of the corresponding g. The two definitions are in fact equivalent in the same way as in [3, Proposition 42].

Example 4.6. Let $f = x_1^{a_1}$ – the Fermat type polynomial. Assume $a_1 = rm$ and consider G to be generated by $g = (1/r)$. We have

$$\mathcal{B}_{\mathrm{tot}} = \mathbb{C}\big\langle[1]\xi_{\mathrm{id}}, [x_1]\xi_{\mathrm{id}}, \dots, \big[x_1^{rm-2}\big]\xi_{\mathrm{id}}\big\rangle \oplus \mathbb{C}\langle[1]\xi_g, \dots, [1]\xi_{g^{r-1}}\rangle.$$

Because $I_g^c = I_{g^k}^c = \{1\}$, we have

$$\big(g^k\big)^*(\xi_{g^l}) = \exp\left(2\pi\mathrm{i}\cdot\frac{k}{r}\right)\xi_{g^l}.$$

However $\big(g^k\big)^*\big(\big[x_1^l\big]\big) = \exp\big(-2\pi\mathrm{i}\cdot\frac{kl}{r}\big)\big[x_1^l\big]$ and the G-invariant monomials are x_1^{rn} with $n \in \mathbb{Z}$. This gives

$$\mathcal{B}(f, G) = \mathbb{C}\big\langle[1]\xi_{\mathrm{id}}, [x_1^r]\xi_{\mathrm{id}}, \dots, \big[x_1^{r(m-1)}\big]\xi_{\mathrm{id}}\big\rangle.$$

4.5 Bigrading

The following operators $q_l, q_r \colon \mathcal{B}_{\mathrm{tot}} \to \mathbb{Q}$ were first introduced in [17] giving the bigrading we use.

For any homogeneous $p \in \mathbb{C}[x_1, \dots, x_N]$ define for $[p]\xi_g$ its *left charge* q_l and *right charge* q_r to be

$$(q_l, q_r) = \left(\deg p - \sum_{k \in I_g^c} q_k + \mathrm{age}(g), \deg p - \sum_{k \in I_g^c} q_k + \mathrm{age}\big(g^{-1}\big)\right). \tag{4.3}$$

This definition endows $\mathcal{B}_{\mathrm{tot}}$ with the structure of a $\mathbb{Q}$-bigraded vector space. For $u, v \in G_f^d$ it follows immediately that $q_\bullet(\xi_u) + q_\bullet(\xi_v) = q_\bullet(\xi_{uv})$ for $u, v \in G$, such that $I_u^c \cap I_v^c = \varnothing$.

This bigrading restricts to $\mathcal{B}(f, G)$ because q_l, q_r commute with the action of h^* for any $h \in \mathrm{GL}_f$, h preserves the weights and $\mathrm{age}(g) = \mathrm{age}\big(hgh^{-1}\big)$.

5 Hodge diamond of LG orbifolds

Assume $N \geq 3$ and the reduced weight set of f to satisfy the CY condition $\sum_{k=1}^N q_k = 1$ (see also Section 3.6).

Proposition 5.1. *For f satisfying CY condition and G, such that $J \subseteq G \subseteq \mathrm{SL}_f$ both left and right charges q_l and q_r of any $Y \in \mathcal{B}(f, G)$ are integer.*

Proof. Note that $q_r([p]\xi_g) = q_l([p]\xi_g) + (N - N_g) - 2\mathrm{age}(g)$. Due to $\mathrm{age}(g) \in \mathbb{Z}$ the right charge $q_r([p]\xi_g)$ is integral if and only if $q_l([p]\xi_g)$ is integral. It remains to recall that

$$(j_f)^*([p]\xi_g) = \mathbf{e}\left[-\deg(p) + \sum_{k \in I_g^c} q_k\right] \cdot [p]\xi_g$$

by Example 4.4. Hence for a class in $\mathcal{B}(f, G)$ we have $\mathbf{e}[-q_l + \mathrm{age}(g)] = 1$ and so q_l is integer. ■

The following two propositions state that the graded pieces of $\mathcal{B}(f, G)$ are organized into a diamond when CY condition holds.

Proposition 5.2. *Let f be a quasihomogeneous polynomial satisfying the CY condition, let $G \subseteq \mathrm{SL}_f$ be a finite subgroup, and let $V^{a,b}$ stand for the bidegree (a, b)-subspace of $\mathcal{B}(f, G)$. We have*

(i) $V^{a,b} = 0$ *for* $a < 0$ *or* $b < 0$;

(ii) $V^{0,0} \cong \mathbb{C}$, *generated by* $[1]\xi_{\mathrm{id}}$;

(iii) $V^{a,b} = 0$ *for* $a > N - 2$ *or* $b > N - 2$;

(iv) $V^{N-2,N-2} \cong \mathbb{C}$, *generated by* $[\mathrm{hess}(f)]\xi_{\mathrm{id}}$.

Proof. Assume $X = [p]\xi_g$ for p being a polynomial fixed by g.

(i) If $g = \mathrm{id}$ we have $q_l(X) = q_r(X) = \deg p \geq 0$. For $g \neq \mathrm{id}$ we have $\mathrm{age}(g) \in \mathbb{N}_{\geq 1}$. Rewriting

$$q_l(X) = \deg p - \sum_{k \in I_g^c} q_k + \mathrm{age}(g),$$

we see that $q_l(X) \geq 0$ because $\sum_{k \in I_g^c} q_k \leq \sum_{k=1}^N q_k = 1$. Similarly for $q_r(X)$ by the same argument applied to $\mathrm{age}\big(g^{-1}\big)$.

(ii) If $g = \mathrm{id}$ we have that $q_l(X) = q_r(X) = 0$ if and only if $\deg p = 0$. By Propositions 4.1 and 4.2, we have $[p] = \alpha[1]$ in $\mathrm{Jac}(f)$ for some constant $\alpha \in \mathbb{C}$.

For $g \neq \mathrm{id}$, we just saw that $\mathrm{age}(g) \geq 1$ and $\sum_{k\in I_g^c} q_k \leq 1$, so $q_l(X) = q_r(X) = 0$ is achieved only if $\deg p = 0$, $N_g = 0$ and $\mathrm{age}(g) = \mathrm{age}(g^{-1}) = 1$, which implies

$$N = \mathrm{age}(g) + \mathrm{age}\big(g^{-1}\big) + N_g = 2.$$

(iii) If $g = \mathrm{id}$, the statement follows from Proposition 4.1 as given the CY condition we have $\hat{c} = N - 2\sum_k q_k = N - 2$.

For $g \neq \mathrm{id}$, apply the same proposition again to estimate $\deg p$ in $\mathrm{Jac}(f^g)$. Namely, it gives

$$q_l(X) \leq N_g - 2\sum_{k\in I_g} q_k - \sum_{k\in I_g^c} q_k + \mathrm{age}(g) = N_g - \sum_{k\in I_g} q_k - 1 + \mathrm{age}(g).$$

At the same time, we have $N_g + \mathrm{age}(g) = N - \mathrm{age}\big(g^{-1}\big) \leq N - 1$ because $\mathrm{age}\big(g^{-1}\big) \in \mathbb{N}_{\geq 1}$. Combining this with the inequality above we get

$$q_l(X) \leq N - 2 - \sum_{k\in I_g} q_k \leq N - 2. \tag{5.1}$$

One gets in the similar way that $q_r(X) \leq N - 2$.

(iv) If $g = \mathrm{id}$, by Proposition 4.1, we see that $q_l([\mathrm{hess}(f)]\xi_{\mathrm{id}}) = q_r([\mathrm{hess}(f)]\xi_{\mathrm{id}}) = N - 2$. If $g \neq \mathrm{id}$, $q_l(X) = q_r(X) = N - 2$, then equation (5.1) implies that $\sum_{k\in I_g} q_k = 0$ and, hence, $N_g = 0$ and $\mathrm{age}(g) = \mathrm{age}\big(g^{-1}\big) = 1$, which altogether implies

$$N = \mathrm{age}(g) + \mathrm{age}\big(g^{-1}\big) + N_g = 2. \qquad \blacksquare$$

We now construct two symmetries of $\mathcal{B}_{\mathrm{tot}}$. The *horizontal* morphism Ψ and the *vertical* morphism Φ.

Consider the direct sum decomposition of $\mathcal{B}_{\mathrm{tot}}$ as in equation (4.1). We first extend the $\phi_{f,\beta}$ isomorphism of Proposition 4.2 to $\mathcal{B}_{\mathrm{tot}}$ in the following way. The hessian matrix of f^g viewed coordinate free is a bilinear form on the tangent bundle of $\mathrm{Fix}(g)$. Therefore, its determinant $\mathrm{hess}(f^g)$ is canonically an element of $\big(\Lambda^{N_g}\mathrm{Fix}(g)^\vee\big)^{\otimes 2} \otimes \mathbb{C}[\mathrm{Fix}(g)]$. Fix a generator

$$\xi_g^\vee := \iota_{\xi_g}\mathrm{d}x_1 \wedge \dots \wedge \mathrm{d}x_N \in \Lambda^{N_g}\mathrm{Fix}(g)^\vee,$$

where ι is the interior product operator and let the generator of $\big(\Lambda^{N_g}\mathrm{Fix}(g)^\vee\big)^{\otimes 2}$ to be $(\xi_g^\vee)^{\otimes 2}$. This choice allows us to fix $\mathrm{hess}(f^g)$ as a function on $\mathrm{Fix}(g)$ and, hence fix a pairing η_{f^g} for the g-sector. As in Proposition 4.2, this in turn defines an isomorphism $\phi_{f^g,\beta}$ on each sector. Now the vertical morphism Φ is the direct sum of these isomorphisms acting on each sector of $\mathcal{B}_{\mathrm{tot}}$

$$\Phi := \bigoplus_{g\in \mathrm{GL}_f, \beta\in\mathbb{Q}} \phi_{f^g,\beta}\colon\ \mathcal{B}_{\mathrm{tot}} \to \mathcal{B}_{\mathrm{tot}}^\vee.$$

It is an isomorphism restricted to $\mathcal{B}_{\mathrm{tot},G}$ for any finite G because each of $\phi_{f^g,\beta}$ is an isomorphism.

Define the horizontal morphism Ψ to act on the g-th sector by $\Psi([p]\xi_g) := [p]\xi_{g^{-1}}$. Extend it by linearity to all $\mathcal{B}_{\mathrm{tot}}$. This is an isomorphism because $f^g = f^{g^{-1}}$ and $\mathrm{Jac}(f^g) = \mathrm{Jac}\big(f^{g^{-1}}\big)$.

Proposition 5.3.

(1) *The maps Φ and Ψ are well defined on $\mathcal{B}(f,G)$ for any finite $G \subseteq \mathrm{SL}_f$.*

(2) *For f satisfying CY condition and a finite $G \subseteq \mathrm{SL}_f$, let $V^{a,b}$ stand for the bidegree (a,b)-subspace of $\mathcal{B}(f,G)$. Then the maps Ψ and Φ induce the $\mathbb{C}$-vector spaces isomorphisms*

$$V^{a,b} \cong V^{b,a} \qquad \textit{and} \qquad V^{a,b} \cong \big(V^{N-2-b,N-2-a}\big)^\vee.$$

Proof. 1) The map Ψ commutes with the G-action since $\mathrm{Fix}(g) = \mathrm{Fix}\big(g^{-1}\big)$. Hence Ψ preserves the invariants.

To see that Φ commute with the G-action, recall first that, $f\big(h^{-1} \cdot \mathbf{x}\big) = f(\mathbf{x})$ and, hence, $f^g\big(h^{-1} \cdot \mathbf{x}\big) = f^{hgh^{-1}}(\mathbf{x})$. Furthermore, since $\det(h) = 1$ we have $\rho_{h,g} h^*\big(\xi^\vee_{hgh^{-1}}\big) = \xi^\vee_g$ and we can conclude that

$$\rho^2_{h,g} \eta_{f^{hgh^{-1}}}\big(\big[p_1\big(h^{-1} \cdot \mathbf{x}\big)\big]\xi_{hgh^{-1}}, \big[p_2\big(h^{-1} \cdot \mathbf{x}\big)\big]\xi_{hgh^{-1}}\big) = \eta_{f^g}([p_1(\mathbf{x})]\xi_g, [p_2(\mathbf{x})]\xi_g).$$

Now, by the definition of G-action we get

$$\eta_{f^{hgh^{-1}}}(h^*([p_1(\mathbf{x})]\xi_g), h^*([p_2(\mathbf{x})]\xi_g)) = \eta_{f^g}([p_1(\mathbf{x})]\xi_g, [p_2(\mathbf{x})]\xi_g).$$

This implies the statement.

2) We have directly by the definition that $q_l(\Psi(X)) = q_r(X)$ and $q_r(\Psi(X)) = q_l(X)$. The first isomorphism follows.

To verify compatibility of Φ with the grading, note first, that $\widehat{c}(f^g) = \sum_{k \in I_g}(1 - 2q_k)$. Thus, by Proposition 4.2 the left charge of $[\phi_{f^g, \deg p}(p)]\xi_g$ is given by

$$\begin{aligned} q_l([\phi_{f^g,\deg p}(p)]\xi_g) &= \sum_{k \in I_g}(1 - 2q_k) - \deg p - \sum_{k \in I_g^c} q_k + \mathrm{age}(g) \\ &= N_g - 2\sum_{k \in I_g} q_k - \deg p - \sum_{k \in I_g^c} q_k + \big(N - N_g - \mathrm{age}\big(g^{-1}\big)\big) \\ &= -2 + \sum_{k \in I_g^c} q_k - \deg p + N - \mathrm{age}\big(g^{-1}\big) = N - 2 - q_r([p]\xi_g). \end{aligned}$$

The computation for the right charge is identical. ∎

Consider now two more special graded pieces of $\mathcal{B}(f, G)$.

Proposition 5.4. *For f satisfying CY condition and a finite G, such that $J \subseteq G \subseteq \mathrm{SL}_f$, let $V^{a,b}$ stand for the bidegree (a, b)-subspace of $\mathcal{B}(f, G)$. Then*

(1) $V^{N-2,0} \cong \mathbb{C}$, *generated by* $[1]\xi_{j_f^{-1}}$,

(2) $V^{0,N-2} \cong \mathbb{C}$, *generated by* $[1]\xi_{j_f}$.

Proof. One notes immediately that $[1]\xi_{j_f}$ and $[1]\xi_{j_f^{-1}}$ are non-zero in $\mathcal{B}(f, G)$ and belong to $V^{0,N-2}$ and $V^{N-2,0}$ respectively. By Proposition 5.3 it is enough to show one of the statements.

Lemma 5.5. $\dim V^{0,N-2} = \{g \in G \mid \mathrm{age}(g) = 1,\ N_g = 0\}$.

Proof. Let $[p]\xi_g \in V^{0,N-2}$. It follows from equations (3.2) and (4.3) that $\mathrm{age}(g) = 1 - N_g/2$. The statement follows by Proposition 3.7. ∎

Under the CY condition for $g \in \mathrm{GL}_f \backslash \{\mathrm{id}\}$ of finite order and with integral $\mathrm{age}(g)$ we have by Proposition 3.14 that $\mathrm{age}(g) \geq 1$ with the equality being reached only for $g = j_f$. This completes the proof. ∎

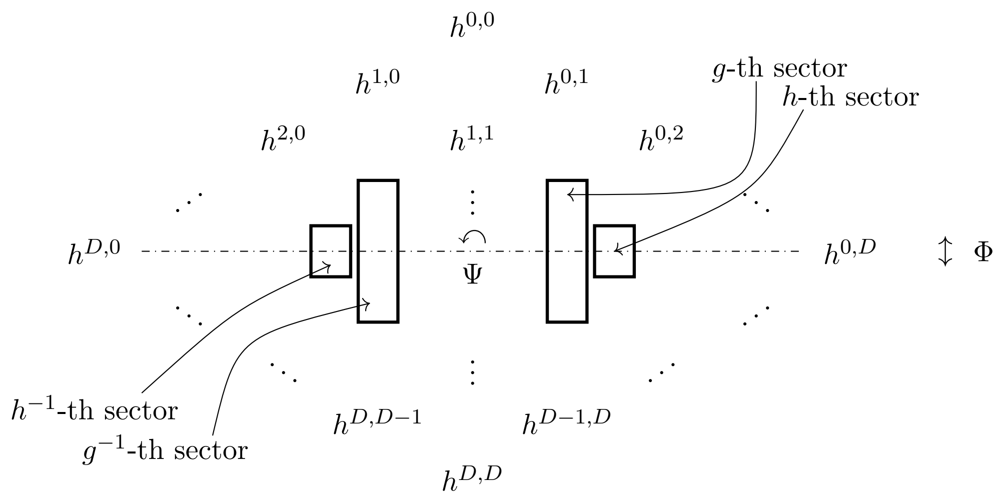
This completes the proof of Theorem 1.1.

For a fixed pair (f, G) set

$$h^{a,b} := \dim_{\mathbb{C}} \{X \in \mathcal{B}(f, G) \mid (q_l(X), q_r(X)) = (a, b)\}$$

and denote $D := N - 2$.

It follows from the propositions above that for f satisfying CY condition and G such that $J \subseteq G \subseteq \mathrm{SL}_f$, the numbers $h^{a,b}$ form a diamond,



Let us call the line $\left\{h^{a,b} \mid a+b=D\right\}$ the horizontal line and the line $\left\{h^{a,b} \mid a-b=0\right\}$ the vertical line. The Hodge diamond $\left\{h^{a,b}\right\}_{a,b=0}^{D}$ has the following special properties

(1) The g-th sector of $\mathcal{B}(f,G)$ contributes as a line symmetric with respect to the horizontal line.

(2) Every g-th sector of $\mathcal{B}(f,G)$ contributes together with a g^{-1}-th sector of $\mathcal{B}(f,G)$, located symmetrically with respect to the vertical line.

(3) All the elements of the form $\xi_{j_f^k}$ contribute to the horizontal line. In particular, $h^{a,D-a} \geq 1$ for all $a = 0,\dots,D$.

(4) All the elements of the form $[p]\xi_{\mathrm{id}}$ contribute to the vertical line.

Example 5.6. Consider $f = x_1^2x_2 + x_2^2 + x_2x_3^6 + x_4^6 + x_1x_3^9$ and $G = \mathrm{SL}_f$. Then $G = J = \langle j_f \rangle$ with

$$\bar{j}_f = \left(\frac{1}{4}, \frac{1}{2}, \frac{1}{12}, \frac{1}{6}\right).$$

The basis of $\mathcal{B}(f,G)$ is given by the elements

$$\xi_{j_f^3},\quad \xi_{j_f^5},\quad \xi_{j_f^7},\quad \xi_{j_f^9},\quad [x_1]\xi_{j_f^4},\quad [x_1]\xi_{j_f^8},\quad \left[x_4^2\right]\xi_{j_f^6},\quad \left[x_3^4x_4^4\right]\xi_{\mathrm{id}},$$
$$\left[x_1x_3x_4^4\right]\xi_{\mathrm{id}},\quad \left[x_1x_3^3x_4^3\right]\xi_{\mathrm{id}},\quad \left[x_2x_4^3\right]\xi_{\mathrm{id}},\quad \left[x_1^2x_4^3\right]\xi_{\mathrm{id}},\quad \left[x_1x_3^5x_4^2\right]\xi_{\mathrm{id}},$$
$$\left[x_2x_3^2x_4^2\right]\xi_{\mathrm{id}},\quad \left[x_1^2x_3^2x_4^2\right]\xi_{\mathrm{id}},\quad \left[x_2x_3^4x_4\right]\xi_{\mathrm{id}},\quad \left[x_1x_2x_3x_4\right]\xi_{\mathrm{id}},\quad \left[x_1^3x_3x_4\right]\xi_{\mathrm{id}},$$
$$\left[x_1^2\right]\xi_{\mathrm{id}},\quad \left[x_2^2\right]\xi_{\mathrm{id}}.$$

all having the bigrading $(1,1)$, and the elements

$$\xi_{j_f},\quad \xi_{j_f^{11}},\quad [1]\xi_{\mathrm{id}},\quad \left[x_1x_2^2x_3x_4^4\right]\xi_{\mathrm{id}},$$

having the bigrading $(0,2)$, $(2,0)$, $(0,0)$ and $(2,2)$, respectively.

One gets the following diamond:

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

Example 5.7. Let $f = x_1^5+x_2^5+x_3^5+x_4^5+x_5^5$ and $G = S \ltimes J$, where $S = \langle (1,2),(2,3)\rangle \subset S_5$ is the subgroup permuting first 3 variables and preserving the last two. Pick $\xi_{(1,2,3)}$ and $\xi_{(1,3,2)}$ in such a way that $(1,2)(\xi_{(1,2,3)}) = \xi_{(1,3,2)}$; $\xi_{(1,2,3)j_f}$ and $\xi_{(1,3,2)j_f}$ in such a way that $(1,2)(\xi_{(1,2,3)j_f}) = \xi_{(1,3,2)j_f}$ and so on. Then the basis of $\mathcal{B}(f,G)$ is

$$\xi_{\mathrm{id}} \qquad \text{and} \qquad \left[x_1^3x_2^3x_3^3x_4^3x_5^3\right]\xi_{\mathrm{id}}$$

in bidegrees $(0,0)$ and $(3,3)$, respectively;

$$\begin{gathered}
[x_1x_2x_3x_4x_5]\xi_{\mathrm{id}}, \quad \left[x_4^3x_5^2\right]\xi_{\mathrm{id}}, \quad \left[x_4^2x_5^3\right]\xi_{\mathrm{id}}, \quad \left[x_1x_2x_3x_4^2\right]\xi_{\mathrm{id}}, \quad \left[x_1x_2x_3x_5^2\right]\xi_{\mathrm{id}},\\
\left[(x_1+x_2+x_3)^2\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}), \quad \left[x_4^2\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}),\\
\left[x_5^2\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}), \quad [(x_1+x_2+x_3)x_4](\xi_{(1,2,3)}+\xi_{(1,3,2)}),\\
[(x_1+x_2+x_3)x_5](\xi_{(1,2,3)}+\xi_{(1,3,2)}), \quad [x_4x_5](\xi_{(1,2,3)}+\xi_{(1,3,2)})
\end{gathered}$$

in bidegree $(1,1)$;

$$\xi_{j_f} \qquad \text{and} \qquad \xi_{j_f^4}$$

in bidegrees $(3,0)$ and $(0,3)$, respectively;

$$\xi_{j_f^2}, \quad \xi_{(1,2,3)j_f}+\xi_{(1,3,2)j_f}, \quad \xi_{(1,2,3)j_f^2}+\xi_{(1,3,2)j_f^2}$$

in bidegree $(2,1)$;

$$\xi_{j_f^3}, \quad \xi_{(1,2,3)j_f^3}+\xi_{(1,3,2)j_f^3}, \quad \xi_{(1,2,3)j_f^4}+\xi_{(1,3,2)j_f^4}$$

in bidegree $(1,2)$;

$$\begin{gathered}
\left[x_1^2x_2^2x_3^2x_4^2x_5^2\right]\xi_{\mathrm{id}}, \quad \left[x_1^2x_2^2x_3^2x_4^3x_5\right]\xi_{\mathrm{id}}, \quad \left[x_1^2x_2^2x_3^2x_4x_5^3\right]\xi_{\mathrm{id}}, \quad \left[x_1^3x_2^3x_3^3x_4\right]\xi_{\mathrm{id}},\\
\left[x_1^3x_2^3x_3^3x_5\right]\xi_{\mathrm{id}}, \quad \left[(x_1+x_2+x_3)^3x_4^3x_5\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}),\\
\left[(x_1+x_2+x_3)^3x_4x_5^3\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}), \quad \left[(x_1+x_2+x_3)x_4^3x_5^3\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}),\\
\left[(x_1+x_2+x_3)^3x_4^2x_5^2\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}), \quad \left[(x_1+x_2+x_3)^2x_4^3x_5^2\right](\xi_{(1,2,3)}+\xi_{(1,3,2)}),\\
\left[(x_1+x_2+x_3)^2x_4^2x_5^3\right](\xi_{(1,2,3)}+\xi_{(1,3,2)})
\end{gathered}$$

in bidegree $(2,2)$.

This gives the following diamond:

$$\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 11 & & 0 & \\
1 & & 3 & & 3 & & 1\\
 & 0 & & 11 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}$$

Acknowledgements

The work of Alexey Basalaev was supported by International Laboratory of Cluster Geometry NRU HSE, RF Government grant, ag. no. 075-15-2021-608 dated 08.06.2021. The authors are grateful to Anton Rarovsky for sharing the pictures from his bachelor thesis. The authors are very grateful to the anonymous referees for many valuable comments.

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