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Deformations of Instanton Metrics

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Abstract. We discuss a class of bow varieties which can be viewed as Taub-NUT deformations of moduli spaces of instantons on noncommutative $\mathbb{R}^4$. Via the generalized Legendre transform, we find the Kähler potential on each of these spaces.

Key words: instanton; bow variety; hyperkähler geometry; generalised Legendre transform

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To Nicholas Stephen Manton
on his 70th birthday

Bow varieties, introduced by the third author [6, 7], are a common generalisation of quiver varieties and of moduli spaces of solutions to Nahm's equations. A class of bow varieties describes, via an analog of the ADHM construction, moduli spaces of (framed) instantons on ALF-spaces. In the present paper, we are interested in a very particular type of bow varieties, which can be viewed as a moduli space of $U(r)$ instantons on the noncommutative Taub-NUT space (cf. Section 3). The case $r=1$ of these has been studied by Takayama [21]. Our approach is via spectral curves and line bundles. This allows us to give a formula for the Kähler potential of the hyperkähler metric via the generalised Legendre transform of Lindström and Roček [16]. We also derive the asymptotic metric in the region where the $U(r)$-instantons of charge k can be approximated by kr well-separated constituents (cf. [7, Section 9]), which we expect to be Euclidean $U(2)$-monopoles (cf. [10]).

1 Spectral curves, line bundles, and matrix polynomials

The complex manifold $\mathbb{T} = \mathrm{T}\mathbb{P}^1$ is equipped with the standard atlas (ζ,η), $(\tilde{\zeta},\tilde{\eta})$, where $\tilde{\zeta}=\zeta^{-1}$, $\tilde{\eta}=\eta\,\zeta^{-2}$. We recall [1, Proposition 2.2] that $H^1(\mathbb{T},\mathcal{O}_{\mathbb{T}})$ is generated by monomials of the form $\eta^i\zeta^{-j}$, $i>0$, $j<2i$. Of particular interest is the line bundle $\mathscr{L}^z$, $z\in\mathbb{C}$, with transition function $\exp(z\eta/\zeta)$.

A *spectral curve* (of degree k) is a compact 1-dimensional subscheme of $\mathrm{T}\mathbb{P}^1$ defined by the equation $P(\zeta,\eta)=0$, where $P(\zeta,\eta)=\eta^k+\sum_{i=1}^k p_i(\zeta)\eta^{k-i}$, $\deg p_i = 2i$. It can be reducible or nonreduced, and its arithmetic genus g is equal to $(k-1)^2$.

On a spectral curve S, we consider the Jacobian $\mathrm{Jac}^{g-1}(S)$ of line bundles L (i.e., invertible sheaves) of degree $g-1=k^2-2k$, i.e., satisfying $\chi(L)=0$. The line bundle $\mathcal{O}_S(k-2)$ has degree $g-1$, and therefore we have an isomorphism $\mathrm{Pic}^0(S)\to\mathrm{Jac}^{g-1}(S)$, $L\mapsto L(k-2)$. As shown in

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[1, Proposition 2.1], any line bundle on S of degree zero is a restriction of a line bundle on $\mathbb{T}$, and hence, the same holds for line bundles of degree $g-1$. The theta divisor $\Theta_S \subset \mathrm{Jac}^{g-1}(S)$ consists of line bundles with nontrivial cohomology. Beauville [2] has shown that any $L \in \mathrm{Jac}^{g-1}(S)\backslash\Theta_S$, viewed as a sheaf on $\mathbb{T}$, has a free resolution of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{T}}(-3)^{\oplus k} \xrightarrow{\eta - A(\zeta)} \mathcal{O}_{\mathbb{T}}(-1)^{\oplus k} \longrightarrow L \longrightarrow 0, \tag{1.1}$$

where $A(\zeta) = A_0 + A_1\zeta + A_2\zeta^2$, $A_i \in \mathrm{Mat}_{k,k}(\mathbb{C})$, is a quadratic matrix polynomial, the characteristic polynomial of which is $P(\zeta,\eta)$. The essential idea is that, since $\pi\colon S \to \mathbb{P}^1$, $(\zeta,\eta) \mapsto \zeta$, is a finite flat morphism, and L is locally free, the direct image π_*L is also locally free. Since $h^0(L) = h^1(L) = 0$, the same holds for π_*L, and so $\pi_*L \simeq \mathcal{O}(-1)^{\oplus k}$. Moreover, π_*L is a module over π_*S, i.e., it corresponds to a homomorphism $A\colon \pi_*L \to \pi_*L(2)$ satisfying $P(\zeta, A(\zeta)) = 0$. Since L is a line bundle, the matrix $A(\zeta)$ is regular for every ζ, and hence $P(\zeta,\eta)$ is the characteristic polynomial of $A(\zeta)$.

Remark 1.1. The Beauville correspondence described above can be also rephrased as follows. Consider the set Q of quadratic matrix polynomials $A(\zeta)$ such that $A(\zeta_0)$ is a regular matrix for every $\zeta_0 \in \mathbb{P}^1$. This is an open subset of $\mathbb{C}^{3k^2}$ and since $\mathrm{GL}_n(\mathbb{C})$ is reductive, there exists a good quotient $\mathscr{J}_k = Q/\mathrm{GL}_k(\mathbb{C})$. This quotient, with its scheme structure, can be viewed as the universal Jacobian of spectral curves, parametrising pairs (S,L), where S is a spectral curve and $L \in \mathrm{Jac}^{g-1}(S)\backslash\Theta_S$. It can also be viewed as an open subset of Simpson's moduli space of semistable 1-dimensional sheaves on the Hirzebruch surface $\mathbb{F}_2$ with Hilbert polynomial $h(m) = km$ [20].

1.1 Real structures

The manifold $\mathbb{T}$ is equipped with a real structure (i.e., an antiholomorphic involution) σ defined by

$$\sigma(\zeta,\eta) = \big(-1/\bar\zeta, -\bar\eta/\bar\zeta^2\big).$$

If a spectral curve S is real (i.e., σ-invariant), then we obtain an induced antiholomorphic involution σ on $\mathrm{Pic}(S)$. This involution corresponds to complex conjugation of the matrix polynomial in (1.1) [4, Section 1.2]. Since we are interested in Hermitian conjugation, we need to replace σ by the following antiholomorphic conjugation on $\mathrm{Jac}^{g-1}(S)$:

$$L \mapsto \sigma(L)^* \otimes \mathcal{O}_S(2k-4).$$

We denote the invariant subset of $\mathrm{Jac}^{g-1}(S)$ by $\mathrm{Jac}^{g-1}_{\mathbb{R}}(S)$ and the corresponding subset of $\mathscr{J}_k$ (cf. Remark 1.1) by $\mathscr{J}_k^{\mathbb{R}}$. A line bundle L belongs to $\mathrm{Jac}^{g-1}_{\mathbb{R}}(S)$ if and only if it is of the form $L_0(k-2)$, where L_0 is a degree 0 line bundle with transition function $\exp q(\zeta,\eta)$ satisfying $\overline{q(\zeta,\eta)} = q(\sigma(\zeta,\eta))$.

It has been shown in [4, Proposition 1.7] that $\mathscr{J}_k^{\mathbb{R}}$ decomposes into disjoint subsets $\mathscr{J}_k^p$, $p = 0,\dots,[k/2]$, corresponding to standard Hermitian forms $q = -\sum_{i=1}^p |z_i|^2 + \sum_{i=p+1}^k |z_i|^2$ of signature $(p,k-p)$ on $\mathbb{C}^k$. Denoting by q also the diagonal matrix defining the quadratic form, $\mathscr{J}_k^p$ consists of $\mathrm{SU}(p,k-p)$-conjugacy classes of quadratic matrix polynomials $A(\zeta)$ which satisfy

$$qA_0q^{-1} = -A_2^*, \qquad qA_1q^{-1} = A_1^*, \qquad qA_2q^{-1} = -A_0^*.$$

Remark 1.2. Equivalently, the component $\mathscr{J}_k^p$ to which a real (S,L) belongs is determined by the signature of Hitchin's metric on $H^0(S,L(1))$ [11, Section 6].

Remark 1.3. It is perhaps worth pointing out that, for any real spectral curve S and any p, $\mathrm{Jac}^{g-1}_{\mathbb{R}}(S)\backslash\Theta_S$ has a nonempty intersection with $\mathscr{J}^p_k$. Indeed, for small $s \in \mathbb{R}$, the line bundle $\mathscr{L}^s(k-2)|_S$ belongs to $\mathscr{J}^0_k$ (cf. [11, paragraph after formula (6.11)]). Thus the map associating to $L \in \mathscr{J}^0_k$ its support S is surjective. Each $\mathscr{J}^p_k$ is, however isomorphic to $\mathscr{J}^0_k$, e.g., via $A(\zeta) \mapsto D_1 A(\zeta) D_2$ for an appropriately chosen pair of diagonal matrices.

We shall be interested only in the component $\mathscr{J}^0_k$. The sheaves in this component are represented by matrix polynomials of the form

$$T(\zeta) = (T_2 + \mathrm{i}T_3) + 2\mathrm{i}T_1\zeta + (T_2 - \mathrm{i}T_3)\zeta^2, \qquad T_i \in \mathfrak{u}(k), \tag{1.2}$$

modulo conjugation by $U(k)$. As in [4], we shall call sheaves belonging to $\mathscr{J}^0_k$ *definite.*

1.2 Nahm's equations

$\mathrm{Jac}^{g-1}(S)$ is a torsor for $\mathrm{Pic}^0(S)$. Therefore the tangent bundle of $\mathrm{Jac}^{g-1}(S)$ is parallelisable and canonically isomorphic to $\mathrm{Jac}^{g-1}(S) \times H^1(S, \mathcal{O}_S)$. If we choose an element of $H^1(S, \mathcal{O}_S)$, we obtain a linear flow on $\mathrm{Jac}^{g-1}(S)$. Restricting this flow to the complement of the theta divisor, and choosing an appropriate connection (cf. [11] and [1]) yields a flow of quadratic matrix polynomials corresponding to elements of $\mathrm{Jac}^{g-1}(S)\backslash\Theta_S$. In particular, for the flow given by $[\eta/\zeta] \in H^1(S, \mathcal{O}_S)$, i.e., $L \mapsto L \otimes \mathscr{L}^z$, there is a connection such that the restriction of the flow to $z \in \mathbb{R}$ and to the definite line bundles (i.e., to matrix polynomials of form (1.2)) is given by

$$\frac{\partial T(\zeta)}{\partial z} = \frac{1}{2}\left[T(\zeta), \frac{\partial T(\zeta)}{\partial \zeta}\right],$$

which is equivalent to Nahm's equations

$$\dot{T}_i + \frac{1}{2}\sum_{j,k} \epsilon_{ijk}[T_j.T_k] = 0, \qquad i = 1, 2, 3. \tag{1.3}$$

2 Factorisation of matrix polynomials

We consider the flat hyperkähler manifold $T^* \mathrm{Mat}_{k,k}(\mathbb{C})$, which we identify with $\mathrm{Mat}_{k,k}(\mathbb{C}) \oplus \mathrm{Mat}_{k,k}(\mathbb{C})$. It has a natural tri-Hamiltonian $U(k) \times U(k)$-action given by

$$(g, h).(A, B) = \big(gAh^{-1}, hBg^{-1}\big),$$

and the corresponding hyperkähler moment maps are:

$$(\mu_2 + \mathrm{i}\mu_3)(A, B) = AB, \qquad 2\mathrm{i}\mu_1(A, B) = AA^* - B^*B,$$
$$(\nu_2 + \mathrm{i}\nu_3)(A, B) = -BA, \qquad 2\mathrm{i}\nu_1(A, B) = BB^* - A^*A.$$

We can view these moment maps as sections of $\mathcal{O}(2) \otimes \mathfrak{gl}_k(\mathbb{C})$ over the $\mathbb{P}^1$ parametrising complex structure, and write them as quadratic matrix polynomials:

$$\mu(\zeta) = (A - B^*\zeta)(B + A^*\zeta), \tag{2.1}$$
$$\nu(\zeta) = -(B + A^*\zeta)(A - B^*\zeta). \tag{2.2}$$

As explained in the previous section $\mu(\zeta)$ and $-\nu(\zeta)$ define 1-dimensional sheaves $\mathcal{F}$, $\mathcal{F}'$ in $\mathscr{J}^0_k$ (i.e., real, acyclic, and definite). Moreover, $\mathcal{F}$ and $\mathcal{F}'$ are supported on the same spectral curve S. Our first goal is to relate $\mathcal{F}'$ to $\mathcal{F}$. Since we do not need the reality conditions for this, let us consider arbitrary linear matrix polynomials $A(\zeta)$, $B(\zeta)$, such that the roots of $\det A(\zeta)$ are disjoint from the roots of $\det B(\zeta)$. Let $\mathcal{F} \in \mathscr{J}_k$ (resp. $\mathcal{F}' \in \mathscr{J}_k$) be the sheaf determined by $A(\zeta)B(\zeta)$ (resp. by $B(\zeta)A(\zeta)$). Let S be the common support of $\mathcal{F}$ and $\mathcal{F}'$, and let Δ_A (resp. Δ_B) be the Cartier divisor on S given by $\eta = 0$ on the open subset $\det B(\zeta) \neq 0$ (resp. on the open subset $\det A(\zeta) \neq 0$).

Proposition 2.1. *$\mathcal{F}' \simeq \mathcal{F}(1)[-\Delta_A]$.*

Proof. We have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{T}}(-3)^{\oplus k} & \xrightarrow{\eta - A(\zeta)B(\zeta)} & \mathcal{O}_{\mathbb{T}}(-1)^{\oplus k} & \longrightarrow & \mathcal{F} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle B(\zeta)} & & \downarrow{\scriptstyle B(\zeta)} & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{\mathbb{T}}(-2)^{\oplus k} & \xrightarrow{\eta - B(\zeta)A(\zeta)} & \mathcal{O}_{\mathbb{T}}^{\oplus k} & \longrightarrow & \mathcal{F}'(1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C & \xrightarrow{\eta} & C(2) & \longrightarrow & \mathcal{O}_{\Delta_B} & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where C is the cokernel of $B(\zeta)$. Therefore, $\mathcal{F}'(1) \simeq \mathcal{F}[\Delta_B]$. Since $[\Delta_A + \Delta_B] \simeq \mathcal{O}_S(2)$, the claim follows. ■

We now ask whether a given quadratic polynomial $T(\zeta)$, corresponding to a sheaf in $\mathscr{J}_k^0$, can be factorised as in formula (2.1). Generically, the answer is yes.

Proposition 2.2. *Let $T(\zeta)$ be of form (1.2) and suppose that*

(i) the polynomial $\det T(\zeta)$ has $2n$ distinct zeros $\zeta_1, \dots, \zeta_{2n}$,

(ii) the corresponding eigenvectors $v_i \in \operatorname{Ker} T(\zeta_i)$, $i = 1, \dots, 2n$, are in general position, i.e., for any choice $i_1 < \dots < i_n \in \{1, \dots, 2n\}$, $v_{i_1}, \dots, v_{i_n}$ are linearly independent.

Then $T(\zeta)$ can be factorised as $(A - B^*\zeta)(B + A^*\zeta)$.

Proof. After rotating $\mathbb{P}^1$, we can assume that $\zeta = \infty$ is not a root of $\det T(\zeta)$. Let $\Delta \cup \sigma(\Delta)$ be a decomposition of the set of zeros of $\det T(\zeta)$. Theorem 1 in [17] implies that there is a decomposition $T(\zeta) = (C_1 + D_1\zeta)(C_2 + D_2\zeta)$ such that Δ is the set of roots of $\det(C_2 + D_2\zeta)$. Applying the real structure shows that $(D_2^* - C_2^*\zeta)(-D_1^* + C_1^*\zeta)$ is also a factorisation of $T(\zeta)$. We can rewrite these factorisations as

$$T(\zeta) = \big(C_1 D_1^{-1} + \zeta\big)(D_1 C_2 + D_1 D_2 \zeta) = \big(-D_2^*(C_2^*)^{-1} + \zeta\big)(C_2^* D_1^* - C_2^* C_1^* \zeta).$$

Theorem 2 in [17] implies now that $C_1 D_1^{-1} = -D_2^*(C_2^*)^{-1}$, i.e., $D_1^{-1}C_2^* = -C_1^{-1}D_2^*$. In addition, comparing the constant coefficients of the two factorisations, we have $C_1 C_2 = -D_2^* D_1^*$. Hence

$$\big(D_1^{-1}C_2^*\big)^* = C_2(D_1^*)^{-1} = -C_1^{-1}D_2^*D_1^*(D_1^*)^{-1} = -C_1^{-1}D_2^* = D_1^{-1}C_2^*.$$

Therefore, $D_1^{-1}C_2^*$ is hermitian (and invertible). We can write it as $-gdg^*$, where g is invertible and d is diagonal with diagonal entries equal ± 1. Then

$$T(\zeta) = (C_1 + D_1\zeta)gg^{-1}(C_2 + D_2\zeta) = (C_1 g + D_1 g\zeta)d(-g^*D_1^* + g^*C_1^*\zeta). \tag{2.3}$$

The uniqueness of monic factors of $T(\zeta)$ implies that the map $\Delta \mapsto d$ is injective. Since both sets have the same cardinality (equal to 2^k), this map is surjective, i.e., there is a choice of Δ such that the corresponding d is the identity matrix, and formula (2.3) becomes the desired factorisation. ■

3 Deformed instanton moduli spaces

We consider a bow variety $\mathcal{M}$ corresponding to the bow representation diagram in Figure 1: with r λ-points and the rank of all bundles equal to k. In other words, $\mathcal{M}$ is the moduli space

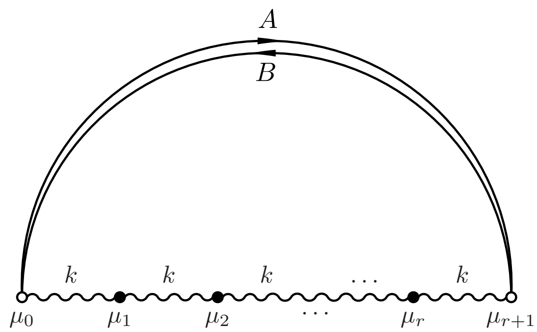

Figure 1. Bow representation diagram with r λ-points $\mu_1, \dots, \mu_r$ and constant rank k.

of $\mathfrak{u}(k)$-valued solutions to Nahm's equations on $[\mu_0, \mu_{r+1}]$ which have rank 1 discontinuity in $(T_2 + \mathrm{i}T_3) + 2\mathrm{i}T_1\zeta + (T_2 - \mathrm{i}T_3)\zeta^2$ at each μ_i, $i = 1, \dots, r$, and $(T_2 + \mathrm{i}T_3) + 2\mathrm{i}T_1\zeta + (T_2 - \mathrm{i}T_3)\zeta^2$ is equal to $(B + A^*\zeta)(A - B^*\zeta) + c_L(\zeta)\mathrm{Id}$ at μ_0 and to $(A - B^*\zeta)(B + A^*\zeta) + c_R(\zeta)\mathrm{Id}$ at μ_{r+1}, where $A, B \in \mathrm{Mat}_{k,k}(\mathbb{C})$ and c_L, c_R are quadratic polynomials satisfying the reality condition.

Let us consider two limiting cases.

First, is the case when we let the lengths of all intervals go to zero, then $\mathcal{M}$ is the quotient by $U(k)$ of the set of solutions to the following matrix equations:

$$[A - B^*\zeta, B + A^*\zeta] = \sum_{i=1}^{r} (v_i - \bar{w}_i\zeta)(w_i + \bar{v}_i\zeta)^{\mathrm{T}} + (c_L(\zeta) - c_R(\zeta)),$$

where $v_i, w_i \in \mathbb{C}^k$. In particular, if $c_L(\zeta) - c_R(\zeta) = a\zeta$, then $\mathcal{M}$ with the complex structure corresponding to $\zeta = 0$ is biholomorphic to the moduli space of framed torsion-free sheaves on $\mathbb{P}^2$ with rank r and $c_2 = k$ [18, Theorem 2.1]. For an arbitrary nonzero $(c_L(\zeta) - c_R(\zeta))$, $\mathcal{M}$ (with $\mu_0 = \dots = \mu_{r+1}$) has been interpreted by Nekrasov and Schwarz as a moduli space of instantons on a noncommutative $\mathbb{R}^4$ [19]. We can, therefore, view $\mathcal{M}$ with arbitrary μ_i as a *deformation* of the moduli space of instantons on noncommutative $\mathbb{R}^4$ with the noncommutativity parameter $c_L(\zeta) - c_R(\zeta)$. It changes the space geometry from a higher-dimensional ALE to ALF kind, as we explain in the beginning of Section 4. For $r = 1$, these moduli spaces have been investigated in detail by Takayama [21].

We remark that the hyperkähler metric on our $\mathcal{M}$ has a T^r-symmetry, compared to a $U(r)$-symmetry of the moduli space of instantons on the noncommutative $\mathbb{R}^4$.

Second, in the case with $c_L(\zeta) = c_R(\zeta)$, $\mathcal{M}$ is isometric to the moduli space of instantons on the Taub-NUT space [8]. Notably, while the deformation to nonzero $c_L(\zeta) - c_R(\zeta)$ appears rather benign from the moduli space point of view, it is nearly fatal to the ADHM-type transform from the bow to the instanton, since the corresponding *small bow representation* moduli space becomes empty, instead of being the Taub-NUT space. This is completely analogous to the situation with the original ADHM construction and its noncommutative deformation of Nekrasov and Schwarz.

3.1 Complex structures

We shall now show that the complex-symplectic structures of $\mathcal{M}$ do not depend on the μ_i (this has been shown by Takayama for $r = 1$). First of all, $\mathcal{M}$ is isomorphic to a hyperkähler quotient of $\tilde{\mathcal{M}} \times T^* \mathrm{Mat}_{k,k}(\mathbb{C})$ by $U(k) \times U(k)$, where $\tilde{\mathcal{M}}$ is the moduli space of solutions to

Nahm's equations on $r+1$ intervals as above, without the bifundamental representation, i.e., without the half-circles labelled by A and B. We discuss first the complex-symplectic structures of $\tilde{\mathcal{M}}$. Let us consider the complex structure I corresponding to $0 \in \mathbb{P}^1$ (all complex structures of $\tilde{\mathcal{M}}$ are isomorphic). We can, following Donaldson [9], separate the data given by Nahm's equations and boundary conditions, into a complex and a real part. The complex part is given by solutions of the Lax equation $\dot{\beta} = [\beta, \alpha]$ on each interval $[\mu_i, \mu_{i+1}]$, where $\beta(t) = T_2(t) + \mathrm{i}T_3(t)$, $\alpha(t) = \mathrm{i}T_1(t)$ with rank 1 discontinuity at $\mu_1, \dots, \mu_r$. It follows from results of Donaldson [9] and Hurtubise [13] that $\tilde{\mathcal{M}}$ is biholomorphic to the quotient of this space by $\mathrm{GL}(k,\mathbb{C})$-valued gauge transformations which are identity at μ_0 and μ_{r+1} and match at the remaining μ_i. This biholomorphism preserves also the complex-symplectic form. On each interval one can apply a complex gauge transformation to make α identically zero and β constant. If we do this beginning with the left-most interval and such a gauge transformation with $g(\mu_0) = 1$, we can make $\beta(t)$ equal to a constant β_i on each $[\mu_{i-1}, \mu_i]$, $i = 1, \dots, r+1$, with $\beta_{i+1} - \beta_i = I_i J_i$ for a vector I_i and a covector J_i. The map associating to $(\beta(\mu_0), g(\mu_{r+1}), I, J)$, where $I = [I_1, \dots, I_r]$ and $J = [J_1, \dots, J_r]^{\mathrm{T}}$ to a point of $\tilde{\mathcal{M}}$ is a complex-symplectic isomorphism between $\tilde{\mathcal{M}}$ and $T^*\mathrm{GL}(k,\mathbb{C}) \times T^*\,\mathrm{Mat}_{k,r}$.

The complex-symplectic quotient of the product of $T^*\mathrm{GL}(k,\mathbb{C}) \times T^*\,\mathrm{Mat}_{k,r}$ and $T^*\,\mathrm{Mat}_{k,k}(\mathbb{C})$ by $\mathrm{GL}(k,\mathbb{C}) \times \mathrm{GL}(k,\mathbb{C})$ (which is the remaining gauge freedom at μ_0 and μ_{r+1}) can be performed in two stages: the quotient by the left copy of $\mathrm{GL}(k,\mathbb{C})$ (the one which acts trivially on I and J) is $T^*\,\mathrm{Mat}_{k,r} \times T^*\,\mathrm{Mat}_{k,k}(\mathbb{C})$. The remaining symplectic quotient is the same one as in the case with $\mu_0 = \dots = \mu_{r+1}$. This shows that, as long as $c_L(\zeta) - c_R(\zeta) \neq 0$, $\mathcal{M}$ is isomorphic, as a complex-symplectic manifold, to the corresponding space of noncommutative instantons.

3.2 Spectral curves

We shall now describe the moduli space $\mathcal{M}$ using the language of spectral curves and line bundles. We denote by S_i the spectral curve on the interval $[\mu_i, \mu_{i+1}]$. Due to the matching conditions, S_r is equal to S_0 shifted by $\eta \mapsto \eta + c(\zeta)$, where $c(\zeta) = c_L(\zeta) - c_R(\zeta)$.

Hurtubise and Murray [14] analysed what happens to spectral curves and line bundles at rank 1 discontinuity of solutions to Nahm's equations. Namely, for $i = 0, \dots, r-1$, we have $S_i \cap S_{i+1} = D_{i,i+1} \cup D_{i+1,i}$ with $\sigma(D_{i,i+1}) = D_{i+1,i}$ and the line bundles at μ_{i+1} equal to $\mathcal{O}_{S_i}(2k)[-D_{i,i+1}] \in \mathrm{Jac}^{g-1}(S_i)$, $\mathcal{O}_{S_{i+1}}(2k)[-D_{i,i+1}] \in \mathrm{Jac}^{g-1}(S_{i+1})$. It follows that $S_1, \dots, S_{r-1}$ satisfy the following condition

$$\mathscr{L}_{S_i}^{\mu_{i+1}-\mu_i}[D_{i,i+1} - D_{i-1,i}] \simeq \mathcal{O}_{S_i}. \tag{3.1}$$

It remains to identify the condition satisfied by S_0 and S_r. The line bundles at μ_0 and at μ_{r+1} are $\mathscr{L}_{S_0}^{\mu_0-\mu_1}(2k)[-D_{0,1}]$ and μ_{r+1} is $\mathscr{L}_{S_r}^{\mu_{r+1}-\mu_r}(2k)[-D_{r-1,r}]$, respectively. For any quadratic polynomial $c = c(\zeta)$ denote by ϕ_c the automorphism of $\mathbb{T} = \mathrm{T}\mathbb{P}^1$ given by $\eta \mapsto \eta + c(\zeta)$. The induced map on $H^1(\mathbb{T}, \mathcal{O}_{\mathbb{T}})$ is trivial. Let us denote by S_c the image of S_0 under ϕ_{c_L} (equivalently, the image of S_r under ϕ_{c_R}). It follows that $B(\zeta)A(\zeta)$ represents the line bundle $\mathscr{L}_{S_c}^{\mu_0-\mu_1}(2k)[-\phi_{c_L}(D_{0,1})]$ and $A(\zeta)B(\zeta)$ represents the line bundle $\mathscr{L}_{S_c}^{\mu_{r+1}-\mu_r}(2k)[-\phi_{c_R}(D_{r-1,r})$. Proposition 2.1 implies that

$$\mathscr{L}_{S_c}^{\mu_0-\mu_1}(2k)\big[-\phi_{c_L}(D_{0,1})\big] \simeq \mathscr{L}_{S_c}^{\mu_{r+1}-\mu_r}(2k)\big[-\phi_{c_R}(D_{r-1,r})\big] \otimes \mathcal{O}_{S_c}(1)[-\Delta_A],$$

that is,

$$\mathscr{L}_{S_c}^{\mu_{r+1}-\mu_r+\mu_1-\mu_0}(1)\big[\phi_{c_L}(D_{0,1}) - \phi_{c_R}(D_{r-1,r}) - \Delta_A\big] \simeq \mathcal{O}_{S_c}, \tag{3.2}$$

where Δ_A is be the divisor on S_c cut out by $\eta = 0$ on the open subset $\det B(\zeta) \neq 0$ (thus $\det A(\zeta) = 0$ on Δ_A). In addition, the spectral curves $S_c, S_1, \dots, S_{r-1}$ satisfy appropriate nondegeneracy conditions, which simply mean that the flow of line bundles on each S_i does not meet the theta divisor. Conversely, given generic curves $S_c, S_1, \ldots, S_{r-1}$ satisfying these conditions together with trivialisations in the formulas (3.1) and (3.2), we obtain, using the results of [14] and Proposition 2.2, a unique gauge equivalence class of solutions to Nahm's equations in $\mathcal{M}$. Here "generic" means that $S_i \cap S_{i+1}$ for $i = 0, \ldots, r-1$ as well as $S_c \cap \{\eta = 0\}$ consist of distinct points.

3.3 Generalised Legendre transform

The complex symplectic quotient described in Section 3.1 can be performed for each complex structure, i.e., on the fibres of the twistor space of $\tilde{\mathcal{M}} \times T^* \operatorname{Mat}_{k,k}(\mathbb{C})$. The spectral curves and (real) trivialisations of line bundles (3.1) and (3.2) provide twistor lines corresponding to an open dense subset of $\mathcal{M}$. In particular, for each complex structure, the roots of polynomials defining spectral curves and values of trivialising sections of line bundles (3.1)–(3.2) define Darboux coordinates for the corresponding complex-symplectic form. This picture is a particular case of the generalised Legendre transform construction of Lindström and Roček [12, 16], which we now recall.

The generalised Legendre transform describes $4n$-dimensional hyperkähler metrics, the twistor space Z^{2n+1} of which admits a projection to the total space of a vector bundle $E = \bigoplus_{i=1}^n \mathcal{O}(2k_i)$ over $\mathbb{P}^1$, $k_i \geq 1$, $i = 1, \ldots, n$. The projection is required to commute with real structures and its fibres for each $\zeta \in \mathbb{P}^1$ are Lagrangian for the fibre-wise complex symplectic form on Z^{2n+1}. The hyperkähler structure is then defined on a subset M of real sections of E consisting of those $\alpha_i(\zeta) = \sum_{a=0}^{2k_i} w_{ia}\zeta^a$, $i = 1, \ldots, n$, which satisfy

$$F_{w_{ia}} := \frac{\partial F}{\partial w_{ia}} = 0 \qquad \text{for} \quad a = 2, \ldots, 2k_i - 2, \tag{3.3}$$

for a function F defined as a contour integral

$$F(w_{ia}) = \oint_c G\big(\zeta, \alpha_1(\zeta), \ldots, \alpha_n(\zeta)\big) \frac{\mathrm{d}\zeta}{\zeta^2}. \tag{3.4}$$

Complex coordinates on $\mathcal{M}$ with respect to the complex structure corresponding to $\zeta = 0$ are given by $z_i = w_{i0}$, $i = 1, \ldots, n$, and by u_i, where $u_i = F_{w_{i1}}$ if $k_i \geq 2$ and $u_i + \overline{u}_i = F_{w_{i1}}$ if $k_i = 1$. The other coefficients w_{ia} with $a > 0$ are understood to be functions of $\{z_i, u_i\}$ determined by equations (3.3). The Kähler potential is given by $K = F - 2\sum_{i=1}^n \operatorname{Re} u_i w_{i1}$.

In the case of our bow variety $\mathcal{M}$, $E = \bigoplus_{i=1}^k \mathcal{O}(2i)^{\oplus r}$ with the summands corresponding to coefficients of powers of η in the polynomials defining the spectral curves $S_c, S_1, \ldots, S_{r-1}$. It has been shown in [5] that conditions such as (3.1) and (3.2) on spectral curves correspond to a particular choice of the function G and the contour c in formula (3.4). In fact, one can replace the usually multi-valued function G with a single-valued function on a branched cover of $\mathbb{P}^1$. This cover is precisely the union of spectral curves $S_c \cup S_1 \cup \cdots \cup S_{r-1}$. Although it is not necessary (as long as we allow integration over chains rather than contours), it is better to enlarge this cover by the fixed projective line $\eta = 0$ (the integration contour will enter this line from S_c at points of Δ_B and leave it at points of Δ_A).

In order to have trivialising sections satisfying assumptions of [5, Theorem 7.5] (cf. Example 8.2 there), we need to replace a nonvanishing section s_i of the left-hand side in formula (3.1) by $s_i/\overline{\sigma^* s_i}$, which is a section of

$$\mathscr{L}_{S_i}^{2(\mu_{i+1}-\mu_i)}[D_{i,i+1} + D_{i,i-1} - D_{i+1,i} - D_{i-1,i}].$$

Similarly, we obtain from formula (3.2) a section of

$$\mathscr{L}_{S_c}^{2(\mu_{r+1}-\mu_r+\mu_1-\mu_0)}\big[\phi_{c_L}(D_{0,1}) + \phi_{c_R}(D_{r,r-1}) + \Delta_B - \phi_{c_L}(D_{1,0}) - \phi_{c_R}(D_{r-1,r}) - \Delta_A\big].$$

The assumptions of [5, Theorem 7.5] are now satisfied, and we can conclude from it that the hyperkähler metric on $\mathcal{M}$ is given by the generalised Legendre transform applied to the function $F(w_{ia})$ given by

$$\oint_\gamma \frac{\eta}{2\zeta^2}\,\mathrm{d}\zeta - \frac{1}{2\pi\mathrm{i}}\sum_{i=1}^{r-1}(\mu_{i+1}-\mu_i)\oint_{\tilde{0}_i}\frac{\eta^2}{2\zeta^3}\,\mathrm{d}\zeta - \frac{1}{2\pi\mathrm{i}}(\mu_{r+1}-\mu_r+\mu_1-\mu_0)\oint_{\tilde{0}_c}\frac{\eta^2}{2\zeta^3}\,\mathrm{d}\zeta,$$

where $\tilde{0}_i$ (resp. $\tilde{0}_c$) is the sum of simple contours around points in S_i (resp. in S_c) lying over $0\in\mathbb{P}^1$, while γ is a contour which enters (resp. leaves) each S_i, $i=2,\dots,r$, at points of $D_{i+1,i}+D_{i-1,i}$ (resp. $D_{i,i+1}+D_{i,i-1}$), and it enters (resp. leaves) S_c at points of $\phi_{c_L}(D_{0,1})+\phi_{c_R}(D_{r-1,r})+\Delta_A$ (resp. $\phi_{c_L}(D_{1,0})+\phi_{c_R}(D_{r,r-1})+\Delta_B$).

4 Asymptotic metrics

In the case $\mu_0=\dots=\mu_{r+1}$, the hyperkähler metric on $\mathcal{M}$ has Euclidean volume growth (i.e., proportional to R^{4kr}) and it is asymptotic to a Riemannian cone on a singular 3-Sasakian manifold. Allowing the length of m of the r intervals $[\mu_i,\mu_{i+1}]$ to be positive, reduces the volume growth power exponent by mk. In particular, if $\mu_{i+1}-\mu_i>0$ for every $i=0,\dots,r$, then the volume growth is proportional to R^{3kr}. In this section, we shall show that, on an open dense subset, the metric is asymptotic to the Lee–Weinberg–Yi metric [15].

The basic idea is the same as in [3]: the functions $\hat{T}_i(t)=\epsilon T_i(\epsilon t)$ satisfy the same Nahm equations (1.3) as the original $T_i(t)$. Thus, exploring infinity of $\mathcal{M}$ is equivalent to studying finite $\hat{T}_i$ on rescaled long intervals. Under such rescaling, the lengths of the intervals go to infinity and we can consider a hyperkähler manifold "glued together" from r moduli spaces of solutions to Nahm's equations on $\mathbb{R}$ with a rank 1 discontinuity at $t=0$, plus diagonal matrices A, B. The resulting hyperkähler metric will be the asymptotic metric in the region of $\mathcal{M}$ where spectral curves degenerate to unions of lines. Let us recall from [3] the precise definition of these building blocks.

4.1 Building blocks

Let a_-, a_+ be positive real numbers. We shall denote[1] by $\mathcal{N}_k(a_-,a_+)$ the moduli space of $\mathfrak{u}(k)$-valued solutions $(T_0(t),T_1(t),T_2(t),T_3(t))$ to Nahm's equations on $\mathbb{R}$ satisfying the following conditions:

- The solutions are analytic on $(-\infty,0]$ and on $[0,\infty)$. At $t=0$, there is a rank one discontinuity, i.e., there exist vectors $I,J^*\in\mathbb{C}^k$ such that $(T_2+iT_3)(0_+)-(T_2+iT_3)(0_-)=IJ$ and $T_1(0_+)-T_1(0_-)=\frac{1}{2}(II^*-J^*J)$.
- The $\hat{T}_i$ approach exponentially fast a diagonal limit as $t\to\pm\infty$ with $(T_1(-\infty),T_2(-\infty),T_3(-\infty))$ and $(T_1(+\infty),T_2(+\infty),T_3(+\infty))$ regular triples, i.e., the centraliser of the triple consists of diagonal matrices.
- The gauge group has a Lie algebra consisting of functions $\rho\colon\mathbb{R}\to\mathfrak{u}(k)$ such that:

  (1) $\rho(0)=0$ and $\dot{\rho}$ has a diagonal limit at $t\to\pm\infty$,

  (2) $(\dot{\rho}-\dot{\rho}(+\infty))$ and $[\rho,\tau]$ decay exponentially fast for any regular diagonal matrix $\tau\in\mathfrak{u}(k)$, and similarly at $t=-\infty$,

  (3) $a_+\dot{\rho}(+\infty)+\lim_{t\to+\infty}(\rho(t)-t\dot{\rho}(+\infty))=0$, and similarly at $t=-\infty$.

[1] These were denoted by $\tilde{F}_{k,k}(a,a')$ in [3].

Let us denote by $\mathbf{x}_i^+$ (resp. $\mathbf{x}_i^-$) the i-th diagonal entry of the triple $(T_1(+\infty), T_2(+\infty), T_3(+\infty))$ (resp. $(T_1(-\infty), T_2(-\infty), T_3(-\infty))$). The collection $\{\mathbf{x}_i^+\}_{i=1}^k$ of k triplets (as well as $\{\mathbf{x}_i^-\}_{i=1}^k$) might be viewed as k points of $\mathbb{R}^3$. As shown in [3], $\mathcal{N}_k(a_-, a_+)$ is a hyperkähler[2] manifold, which topologically is a torus bundle over $\tilde{C}_k(\mathbb{R}^3) \times \tilde{C}_k(\mathbb{R}^3)$, where $\tilde{C}_k(\mathbb{R}^3)$ denotes the configuration space of k distinct and distinguishable points in $\mathbb{R}^3$. The action of the torus $T^k \times T^k$ is tri-Hamiltonian and the hyperkähler moment map is given by $\mathbf{x}_i^-$, $\mathbf{x}_i^+$, $i = 1, \dots, k$. Let us write $\mathbf{x}^{-i}$ for $\mathbf{x}_i^-$, $\mathbf{x}^i$ for $\mathbf{x}_i^+$, and $\mathbf{x}_\nu \in \mathbb{R}^{2k}$, $\nu = 1, 2, 3$, for the vector of ν-coordinates of the $\mathbf{x}^i$, $|i| = 1, \dots, k$. The metric is given by the Gibbons–Hawking ansatz, i.e., it is of the form

$$\sum_{\nu=1}^{3} d\mathbf{x}_\nu^{\mathrm{T}} \Phi d\mathbf{x}_\nu + (dt + A)^{\mathrm{T}} \Phi^{-1} (dt + A), \tag{4.1}$$

where dt is the diagonal matrix of 1-forms dual to Killing fields, A is a connection 1-form, and the matrix Φ (which determines the metric up to gauge equivalence) is given explicitly by

$$\Phi_{ij} = \begin{cases} a_{\operatorname{sgn}(i)} + \displaystyle\sum_{k \neq i} \frac{s_{ik}}{\|\mathbf{x}^i - \mathbf{x}^k\|} & \text{if } i = j, \\ -\dfrac{s_{ij}}{\|\mathbf{x}^i - \mathbf{x}^j\|} & \text{if } i \neq j, \end{cases}$$

where $s_{ij} = -\operatorname{sgn}(i)\operatorname{sgn}(j)$.

There is one more building block, corresponding to matrices A, B. In our asymptotic region, these will become almost diagonal, so that this building block is $\mathbb{H}^k$ with its standard flat metric and the diagonal torus action.

4.2 Asymptotic coordinates and metric

We now obtain the asymptotic metric, analogously to [3], by gluing together these building blocks, i.e., performing the hyperkähler quotient with respect to the torus.

We start with the product $\prod_{i=1}^r \mathcal{N}_k(a_-^i, a_+^i) \times \mathbb{H}^k$ with $a_+^i + a_-^{i+1} = \mu_{i+1} - \mu_i$ for $i = 0, \dots, r$, where $a_+^0 = a_-^{r+1} = 0$. This hyperkähler manifold has, as explained above, a tri-Hamiltonian action of $T^k \times T^k$ on each of the first r factors and of T^k on the last factor. Let us denote the torus $T^k \times T^k$ acting on $\mathcal{N}_k(a_-^i, a_+^i)$ by $T_i^- \times T_i^+$, where T_i^- (resp. T_i^+) is given by gauge transformations asymptotic to $\exp(a_\pm^i h - th)$ as $t \to -\infty$ (resp. $t \to +\infty$), with $h \in \mathfrak{u}(k)$. Let us also write T_0^+ for the standard torus action $(t, q) \mapsto \phi(t, q)$ on $\mathbb{H}^k$, and T_{r+1}^- for the action $(t, q) \mapsto \phi(t^{-1}, q)$. We now perform the hyperkähler quotient with respect to $(T^k)^{r+1}$, the i-th factor of which is embedded diagonally into $T_i^+ \times T_{i+1}^-$, $i = 0, \dots, r$. The level set of the hyperkähler moment map is $(c_L, \dots, c_L)$ for the first copy of T^k, by $(c_R, \dots, c_R)$ for the last copy, and is equal to 0 for all others (where c_L, c_R are points in $\mathbb{R}^3$ determined by the quadratic polynomials $c_L(\zeta), c_R(\zeta)$ used to define the bow variety $\mathcal{M}$).

The resulting metric is again of the form (4.1), where this time we have kr points $\mathbf{x}^i \in \mathbb{R}^3$: k for each of the middle $r-1$ intervals and k given by the moment map on each copy of $\mathbb{H}$. Let us denote by $\mathbf{x}_{ij}$, $j = 1, \dots, k$ the points corresponding to the interval $[\mu_i, \mu_{i+1}]$, $i = 1, \dots, r-1$. Each $\mathbf{x}_{ij}$ is equal to $\mathbf{x}_j^+$ for $\mathcal{N}_k(a_-^i, a_+^i)$ and also to $\mathbf{x}_j^-$ for $\mathcal{N}_k(a_-^{i+1}, a_+^{i+1})$. Let us also write $\mathbf{y}_1, \dots, \mathbf{y_k} \in \mathbb{R}^3$ for the coordinates on each $\mathbb{H}\backslash\{0\}$ given by the hyperkähler moment map. The metric on $\mathbb{H}$ can be also written in the form (4.1) with $\Phi = \|\mathbf{y}\|^{-1}$. Observe that $\mathbf{x}_j^-$ for $\mathcal{N}_k(a_-^1, a_+^1)$ (resp. $\mathbf{x}_j^+$ for $\mathcal{N}_k(a_-^r, a_+^r)$) satisfy $\mathbf{x}_j^- = \mathbf{y}_j + c_L$ (resp. $\mathbf{x}_j^+ = \mathbf{y}_j + c_R$).

The $kr \times kr$ matrix Φ defining the asymptotic metric is described as follows. Let Φ^i, $i = 1, \dots, r-1$, be the $2k \times 2k$ matrix describing the metric on $\mathcal{N}_k(a_-^i, a_+^i)$. We decompose each Φ^i

[2] Strictly speaking the metric is positive-definite only in an asymptotic region.

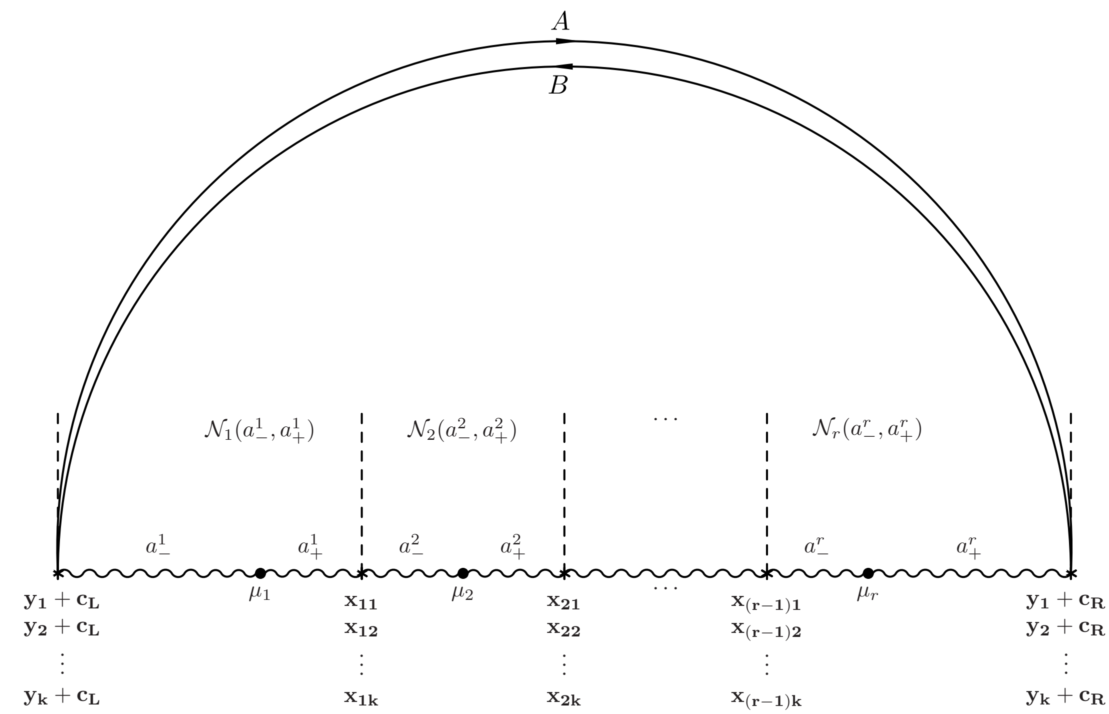

Figure 2. Bow asymptotic as hyperkähler reduction of the approximation blocks. The bow interval is cut at crosses into subintervals, each containing a single λ-point μ_i with length a_i^- to the left of μ_i and length a_i^+ to its right. The corresponding approximation space is $\mathcal{N}_k(a_-^i, a_+^i)$.

into $k \times k$ blocks (corresponding to the positive and negative superscripts labelling coordinates) as

$$\begin{pmatrix} \Phi_{11}^i & \Phi_{12}^i \\ \Phi_{21}^i & \Phi_{22}^i \end{pmatrix}.$$

Next, we form an $rk \times rk$-matrix Ψ^i as follows: the matrix Ψ^i has k^2 $r \times r$ blocks labelled by $\Psi^i_{(m,n)}$, where, for $i \le r-1$,

$$\Psi^i_{(m,n)} = \begin{cases} \Phi^i_{st} & \text{if } m = i+s-2 \text{ and } n = i+t-2, \\ 0 & \text{otherwise.} \end{cases}$$

For $i = r$, set $\Psi^r_{(r,r)} = \Phi^r_{11}$, $\Psi^r_{(r,1)} = \Phi^r_{12}$, $\Psi^r_{(1,r)} = \Phi^r_{21}$, $\Psi^r_{(1,1)} = \Phi^r_{22}$, and the remaining blocks equal to 0. Finally, let Ψ^0 have the $(1,1)$-block equal to $\operatorname{diag}\bigl(\|\mathbf{y_1}\|^{-1}, \dots, \|\mathbf{y_k}\|^{-1}\bigr)$, and all other blocks equal to 0. Then the matrix Φ for the asymptotic metric is the sum $\sum_{j=0}^r \Psi^j$ with $\mathbf{x}_j^-$ for $\mathcal{N}_k(a_-^1, a_+^1)$ and $\mathbf{x}_j^+$ for $\mathcal{N}_k(a_-^r, a_+^r)$ replaced by, respectively, $\mathbf{y}_j + c_L$ and $\mathbf{y}_j + c_R$.

To recapitulate: the asymptotic metric is given by formula (4.1) for the just defined $rk \times rk$ matrix Φ in coordinates $\mathbf{y}_1, \dots, \mathbf{y_k}$, $\mathbf{x}_{ij}$, $i = 1, \dots, r-1$, $j = 1, \dots, k$.

Remark 4.1. The asymptotic metric appears already, albeit in a different form, in [7, Section 9]. The setup we have just presented allows to prove easily that it is, indeed, the asymptotic metric on $\mathcal{M}$.

Let now

$$R = \min\bigl\{\|\mathbf{y}_m - \mathbf{y}_n\|, \|\mathbf{x}_{im} - \mathbf{x}_{in}\|;\ i = 1, \dots, r-1,\ m, n = 1, \dots, k,\ m \ne n\bigr\}.$$

If $R \to \infty$, then the spectral curves become close to unions of lines. The proof that this metric is exponentially (in the parameter R) close to the metric on $\mathcal{M}$ proceeds as in [3, Theorem 9.1], with minor modifications (the main one being that we can solve the real Nahm equation with boundary conditions of $\mathcal{M}$ since $R > 0$ guarantees that the stability condition for the complex-symplectic quotient of $\tilde{\mathcal{M}} \times T^* \operatorname{Mat}_{k,k}(\mathbb{C})$ (cf. Section 3) is satisfied).

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