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A Generalization of Zwegers' μ-Function According to the q-Hermite–Weber Difference Equation

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Abstract. We introduce a one parameter deformation of the Zwegers' μ-function as the image of q-Borel and q-Laplace transformations of a fundamental solution for the q-Hermite–Weber equation. We further give some formulas for our generalized μ-function, for example, forward and backward shift, translation, symmetry, a difference equation for the new parameter, and bilateral q-hypergeometric expressions. From one point of view, the continuous q-Hermite polynomials are some special cases of our μ-function, and the Zwegers' μ-function is regarded as a continuous q-Hermite polynomial of "-1 degree".

Key words: Appell–Lerch series; q-Boerl transformation; q-Laplace transformation; q-hypergeometric series; continuous q-Hermite polynomial; mock theta functions

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1 Introduction

Throughout this paper, let $\mathbb{Z}$ denote the set of integers and let $\mathbb{C}$ denote the set of complex numbers, $\mathrm{i} := \sqrt{-1}$ be the imaginary unit, $\tau \in \mathbb{C}$ be a complex number $\mathrm{Im}(\tau) > 0$, and $q := \mathrm{e}^{2\pi\mathrm{i}\tau}$. We define the q-shifted factorials and Jacobi theta functions as follows:

$$(x)_\infty = (x;q)_\infty := \prod_{j=0}^{\infty}\big(1 - xq^j\big), \qquad (x)_n = (x;q)_n := \frac{(x;q)_\infty}{(q^n x;q)_\infty}, \qquad n \in \mathbb{Z},$$
$$\vartheta_{11}(u,\tau) = \vartheta_{11}(u) := \sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}(n+\frac{1}{2})(u+\frac{1}{2})+\pi\mathrm{i}(n+\frac{1}{2})^2\tau} = -\mathrm{i}q^{\frac{1}{8}}\mathrm{e}^{-\pi\mathrm{i}u}\big(q, \mathrm{e}^{2\pi\mathrm{i}u}, q\mathrm{e}^{-2\pi\mathrm{i}u}; q\big)_\infty,$$
$$\theta_q(x) := (q, -x, -q/x)_\infty = \sum_{n\in\mathbb{Z}} x^n q^{\frac{n(n-1)}{2}},$$

and for appropriate complex numbers $a_1,\dots,a_r,b_1,\dots,b_s,x$, we define the q-hypergeometric series as follows:

$${}_r\phi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,x\right) := \sum_{n=0}^{\infty}\frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s,q)_n}\Big((-1)^n q^{\frac{n(n-1)}{2}}\Big)^{s-r+1} x^n,$$
$${}_r\psi_s\left(\begin{matrix}a_1,\dots,a_r\\ b_1,\dots,b_s\end{matrix};q,x\right) := \sum_{n\in\mathbb{Z}}\frac{(a_1,\dots,a_r)_n}{(b_1,\dots,b_s)_n}\Big((-1)^n q^{\frac{n(n-1)}{2}}\Big)^{s-r} x^n,$$

where

$$(a_1,\dots,a_r)_n = (a_1,\dots,a_r;q)_n := (a_1;q)_n\cdots(a_r;q)_n, \qquad n \in \mathbb{Z}\cup\{\infty\}.$$

Mock theta functions first appeared in Ramanujan's last letter to Hardy in 1920. In this letter, Ramanujan told Hardy that he had discovered a new class of functions which he called mock theta functions. Mock theta functions are functions that have asymptotic behavior similar to "theta functions" (i.e., modular forms) at roots of unity but are not "theta functions", and Ramanujan gave 17 examples of mock theta functions. Some typical examples are as follows:

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-q;q)_n},\qquad \phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left(-q^2;q^2\right)_n},\qquad \psi(q)=\sum_{n=1}^{\infty}\frac{q^{n^2}}{\left(q;q^2\right)_n}.$$

Later, Andrews and Hickerson gave a detailed definition of the mock theta function [2]. For more background on mock theta functions, see, for example, [1, 10].

Ramanujan's mock theta functions are represented as linear combinations of specializations of the universal mock theta function:

$$g_3(x;q):=\sum_{n=1}^{\infty}\frac{q^{n(n-1)}}{(x)_n\left(x^{-1}q\right)_n}$$

and some q-infinite products, which have come to be known as the mock theta conjectures. For example [5, Appendix A],

$$f_0(q)=-2q^2g_3\left(q^2;q^{10}\right)+\frac{\left(q^5;q^5\right)_\infty\left(q^5;q^{10}\right)_\infty}{\left(q;q^5\right)_\infty\left(q^4;q^5\right)_\infty}.$$

Such identities were proved by Hickerson [11]. More fundamentally, the universal mock theta function $g_3(x;q)$ can be written in the form [12, Theorem 3.1]

$$q^{-\frac{1}{24}}x^{\frac{3}{2}}g_3(x;q)=\frac{q^{\frac{1}{3}}\left(q^3;q^3\right)_\infty^3}{(q;q)_\infty\vartheta_{11}(3u,3\tau)}+q^{-\frac{1}{6}}x\mu(3u,\tau,3\tau)+q^{-\frac{2}{3}}x^2\mu(3u,2\tau,3\tau),$$

where μ is the μ-function defined by Zwegers as follows [23]:

$$\mu(u,v;\tau):=\frac{\mathrm{e}^{\pi\mathrm{i}u}}{\vartheta_{11}(v)}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\mathrm{e}^{2\pi\mathrm{i}nv}q^{\frac{n(n+1)}{2}}}{1-\mathrm{e}^{2\pi\mathrm{i}u}q^n}.$$

For convenience, we also use the following multiplicative notation of the μ-function:

$$\mu(x,y;q)=\mu(x,y):=\frac{\mathrm{i}q^{-\frac{1}{8}}\sqrt{xy}}{\theta_q(-y)}\sum_{n\in\mathbb{Z}}\frac{(-1)^ny^nq^{\frac{n(n+1)}{2}}}{1-xq^n}.$$

If we substitute $x=\mathrm{e}^{2\pi\mathrm{i}u}$ and $y=\mathrm{e}^{2\pi\mathrm{i}v}$, then $\mu(x,y;q)=\mu(u,v;\tau)$.

Zwegers showed that the μ-function satisfies a transformation law like Jacobi forms by adding an appropriate non-holomorphic function to the μ-function [23]:

$$\tilde{\mu}(u,v;\tau+1)=\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\tilde{\mu}(u,v;\tau),$$
$$\tilde{\mu}\left(\frac{u}{\tau},\frac{v}{\tau};-\frac{1}{\tau}\right)=-\mathrm{i}\sqrt{-\mathrm{i}\tau}\,\mathrm{e}^{\pi\mathrm{i}\frac{(u-v)^2}{\tau}}\tilde{\mu}(u,v;\tau),$$

where

$$\tilde{\mu}(u,v;\tau):=\mu(u,v;\tau)+\frac{\mathrm{i}}{2}R(u-v;\tau),\qquad E(x):=2\int_0^x\mathrm{e}^{-\pi z^2}\mathrm{d}z,$$
$$R(u;\tau):=\sum_{\nu\in\mathbb{Z}+\frac{1}{2}}\left\{\operatorname{sgn}(\nu)-E\left((\nu+a)\sqrt{2t}\right)\right\}(-1)^{\nu-\frac{1}{2}}\mathrm{e}^{-\pi\mathrm{i}\nu^2\tau-2\pi\mathrm{i}\nu u},$$

and $t = \mathrm{Im}(\tau)$, $a = \frac{\mathrm{Im}(u)}{\mathrm{Im}(\tau)}$. This result was a pioneering work in the study of mock modular forms (see [5]). Thus, the μ-function is very important for the study of mock theta functions.

Further, Zwegers gave the following formulas:

$$\mu(u+1,v) = \mu(u,v+1) = -\mu(u,v), \tag{1.1}$$

$$\mu(u+\tau,v) = -\mathrm{e}^{2\pi\mathrm{i}(u-v)}q^{\frac{1}{2}}\mu(u,v) - \mathrm{i}\mathrm{e}^{\pi\mathrm{i}(u-v)}q^{\frac{3}{8}}, \tag{1.2}$$

$$\mu(u+z,v+z) = \mu(u,v) + \frac{\mathrm{i}q^{\frac{1}{8}}(q)_\infty^3\vartheta_{11}(z)\vartheta_{11}(u+v+z)}{\vartheta_{11}(u)\vartheta_{11}(v)\vartheta_{11}(u+z)\vartheta_{11}(v+z)}, \tag{1.3}$$

$$\begin{aligned} \mu(u,v) &= \mu(u+\tau,v+\tau) && (1.4)\\ &= \mu(v,u) && (1.5)\\ &= \mu(-u,-v). && (1.6) \end{aligned}$$

On the other hand, the μ-function is also a very interesting object from the viewpoint of q-hypergeometric functions. For example, let us recall the well-known Kronecker formula [20]:

$$k(x,y) := \frac{1}{1-x}\,{}_1\psi_1\left(\begin{matrix}x\\qx\end{matrix};q,y\right) = \frac{(q,q,xy,q/xy)_\infty}{(x,q/x,y,q/y)_\infty} = \frac{(q)_\infty^3\theta_q(-xy)}{\theta_q(-x)\theta_q(-y)}. \tag{1.7}$$

The second equality is the case of $a = x$, $b = xq$, $z = y$ in Ramanujan's summation formula:

$${}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q,z\right) = \sum_{n\in\mathbb{Z}}\frac{(a)_n}{(b)_n}z^n = \frac{(az,q/az,q,b/a)_\infty}{(z,b/az,b,q/a)_\infty}.$$

Note that from the most right-hand side of (1.7), one recognizes the symmetry $k(x,y) = k(y,x)$ explicitly.

Obviously, the μ-function is an analogue of the Kronecker summation:

$$k(x,y) = \sum_{n\in\mathbb{Z}}\frac{y^n}{1-xq^n}.$$

Hence the symmetric property $\mu(x,y) = \mu(y,x)$ holds in a similar way as $k(x,y) = k(y,x)$, that is μ-function has the following expressions:

$$\begin{aligned} \mu(x,y) &= \frac{\mathrm{i}q^{-\frac{1}{8}}\sqrt{xy}}{\theta_q(-x)}\frac{1}{1-y}\,{}_1\psi_2\left(\begin{matrix}y\\0,qy\end{matrix};q,qx\right) && (1.8)\\ &= \frac{\mathrm{i}q^{-\frac{1}{8}}\sqrt{xy}}{(x,y)_\infty}\,{}_2\psi_2\left(\begin{matrix}x,y\\0,0\end{matrix};q,q\right) && (1.9)\\ &= \frac{\mathrm{i}q^{-\frac{1}{8}}\sqrt{xy}}{(q/x,q/y)_\infty}\frac{1}{(1-x)(1-y)}\,{}_0\psi_2\left(\begin{matrix}-\\qx,qy\end{matrix};q,qxy\right). && (1.10) \end{aligned}$$

These expressions follow from some Bailey transformations for ${}_2\psi_2$ (see [8, p. 150, Exercise 5.20]):

$$\begin{aligned} {}_2\psi_2\left(\begin{matrix}a,b\\c,d\end{matrix};q,z\right) &= \frac{(az,c/a,d/b,qc/abz)_\infty}{(z,c,q/b,cd/abz)_\infty}\,{}_2\psi_2\left(\begin{matrix}a,abz/c\\az,d\end{matrix};q,\frac{c}{a}\right) && (1.11)\\ &= \frac{(bz,d/b,c/a,qd/abz)_\infty}{(z,d,q/a,cd/abz)_\infty}\,{}_2\psi_2\left(\begin{matrix}b,abz/d\\bz,c\end{matrix};q,\frac{d}{b}\right) && (1.12)\\ &= \frac{(az,d/a,c/b,qd/abz)_\infty}{(z,d,q/b,cd/abz)_\infty}\,{}_2\psi_2\left(\begin{matrix}a,abz/d\\az,c\end{matrix};q,\frac{d}{a}\right) && (1.13)\\ &= \frac{(bz,c/b,d/a,qc/abz)_\infty}{(z,c,q/a,cd/abz)_\infty}\,{}_2\psi_2\left(\begin{matrix}b,abz/c\\bz,d\end{matrix};q,\frac{c}{b}\right), && (1.14) \end{aligned}$$

which is a bilateral version of Heine's transformation formula:

$$ {}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,z\right)=\frac{(az,c/a)_\infty}{(z,c)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,abz/c\\az\end{matrix};q,\frac{c}{a}\right). $$

In fact, by taking the limit $z\to z/b$, $b\to\infty$, $c=0$ in (1.11)–(1.14), we derive

$$ {}_1\psi_2\left(\begin{matrix}a\\0,d\end{matrix};q,z\right)=\frac{(z,dq/az)_\infty}{(d,q/a)_\infty}\,{}_1\psi_2\left(\begin{matrix}az/d\\0,z\end{matrix};q,d\right) \tag{1.15} $$

$$ =\frac{(d/a,dq/az)_\infty}{(d)_\infty}\,{}_2\psi_2\left(\begin{matrix}a,az/d\\0,0\end{matrix};q,\frac{d}{a}\right) \tag{1.16} $$

$$ =\frac{(z,d/a)_\infty}{(q/a)_\infty}\,{}_0\psi_2\left(\begin{matrix}-\\z,d\end{matrix};q,az\right), \tag{1.17} $$

and we obtain (1.8), (1.9) and (1.10). In particular, the expression (1.10) was pointed out by Choi (see [6, Theorem 4] and [4, Theorem 5.1]).

Thus the μ-function is an interesting object deeply studied. But, it seems that there are many problems to be clarified further. For example, why the μ-function is a two-variable function, whereas the universal mock theta function $g_3(x,q)$ is a one-variable function (i.e., what is the origin of the extra variable of the μ-function?). Also, why the factors $\mathrm{e}^{\pi iu}$ in the numerator and $\vartheta_{11}(v)$ in the denominator are needed. Further, where the translation formula (1.3) comes from. Some explanation has been given for these questions. For example, Zwegers gives constructions of mock theta functions as indefinite theta series, of which μ-function is a special case, and these naturally have theta functions in denominators which are explainable from the geometry of the quadratic forms. Also Zwegers' two-variable $\tilde{\mu}$-function is decomposed into a sum of meromorphic Jacobi form and one-variable Maaß–Jacobi form [21] (see also [5, Theorem 8.18]). However, as we will show in this paper, one can give another answer from the view of q-special functions.

By using the pseudo-periodicity equation (1.2) and a bit of calculation, we obtain the following second-order q-difference equation for the μ-function:

$$ \left[T_x^2-q^{\frac12}\left(1-\frac{x}{y}q\right)T_x-\frac{x}{y}q\right]\mu(x,y)=0,\qquad T_xf(x):=f(qx). \tag{1.18} $$

This q-difference equation (1.18) coincides with the specialization $a=q$ and the transformation $x\mapsto\frac{x}{y}$ of the following equation essentially:

$$ \left[T_x^2-(1-xq)\sqrt{a}T_x-xq\right]f(x)=0, \tag{1.19} $$

which is the transformation $a\mapsto\frac{q}{a}$, $u(xq)=\theta_q\left(-\frac{\sqrt{a}}{xq}\right)f(x)$ of the q-Hermite–Weber equation:

$$ \left[axT_x^2+(1-x)T_x-1\right]u(x)=0. $$

The equation (1.19), which we also call the "q-Hermite–Weber equation", is a typical example of the second-order q-difference equation of the Laplace type:

$$ \left[(a_0+b_0x)T_x^2+(a_1+b_1x)T_x+(a_2+b_2x)\right]u(x)=0. $$

The q-difference equations of the Laplace type have long been studied. For example, the q-difference equation satisfied by Heine's hypergeometric function ${}_2\phi_1$:

$$ \left[(c-abqx)T_x^2-(c+q-(a+b)qx)T_x+q(1-x)\right]f(x)=0, \tag{1.20} $$

which is the most popular and master class in the following hierarchy of the second order q-difference equations, was discovered in the 19th century. The q-Hermite–Weber equation (1.19) is one of some equations in the hierarchy obtained by taking some limits of q-difference equation (1.20) (for example, see [16, Figure 2]).

However, a systematic study of the global analysis of degenerate Laplace types had to wait for J.P. Ramis, J. Sauloy and C. Zhang's work [18] in the 21st century. They introduced some q-Borel and q-Laplace transformations:

$$\mathcal{B}^+(f)(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n, \qquad \mathcal{L}^+(f)(x,\lambda) := \sum_{n\in\mathbb{Z}} \frac{f(\lambda q^n)}{\theta_q(\lambda q^n/x)}, \tag{1.21}$$

for the formal series

$$f(x) = \sum_{n\geq 0} a_n x^n \in \mathbb{C}[\![x]\!],$$

which play a fundamental role in the study of the Laplace type q-difference equations. In particular, C. Zhang [22] obtained some connection formulas for the q-convergent hypergeometric series ${}_2\phi_0(a,b;q,x)$ by applying these transformations. By considering the degeneration of Zhang's result, C. Zhang and Y. Ohyama [17] gave some connection formulas for the q-Hermite series ${}_1\phi_1(a;0;q,x)$.

In view of the story discussed above, we introduce the following generalization of the μ-function.

Definition 1.1. Let α, a be complex parameters such that $u-\alpha\tau, v \in \mathbb{C}\backslash(\mathbb{Z}+\mathbb{Z}\tau)$, and such that $xa^{-1}, y \in \mathbb{C}\backslash\{q^n\}_{n\in\mathbb{Z}}$. We define $\mu(u,v;\alpha)$ or $\mu(x,y;a)$ as the following series:

$$\mu(u,v;\alpha,\tau) = \mu(u,v;\alpha) := \frac{\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}}{\vartheta_{11}(v)} \sum_{n\in\mathbb{Z}} (-1)^n \mathrm{e}^{2\pi\mathrm{i}(n+\frac{1}{2})v} q^{\frac{n(n+1)}{2}} \frac{\left(\mathrm{e}^{2\pi\mathrm{i}u} q^{n+1}\right)_\infty}{\left(\mathrm{e}^{2\pi\mathrm{i}u} q^{n-\alpha+1}\right)_\infty},$$

$$\mu(x,y;a,q) = \mu(x,y;a) := -\mathrm{i}q^{-\frac{1}{8}} \frac{(x/y)^{\frac{\alpha}{2}}}{\theta_q(-y)} \frac{(x)_\infty}{(x/a)_\infty} {}_1\psi_2\left(\begin{matrix} x/a \\ 0, x \end{matrix}; q, y\right).$$

In particular, we have $\mu(u,v;0) = -\mathrm{i}q^{-\frac{1}{8}}$ and $\mu(u,v;1) = \mu(u,v)$.

If we substitute $x = \mathrm{e}^{2\pi\mathrm{i}u}$, $y = \mathrm{e}^{2\pi\mathrm{i}v}$ and $a = \mathrm{e}^{2\pi\mathrm{i}\alpha\tau} = q^\alpha$, then $\mu(x,y;a,q) = \mu(u,v;\alpha,\tau)$. The definition of $\mu(u,v;\alpha)$ is equal to the composition of the q-Borel and q-Laplace transformations of the formal solution

$$\widetilde{f_0}(x) = {}_2\phi_0\left(\begin{matrix} a, 0 \\ - \end{matrix}; q, \frac{x}{a}\right)$$

for the q-Hermite–Weber equation (1.19) around $x=0$ (see Section 2).

Theorem 1.2. *Let $f_0(x,\lambda)$ be the image of the q-Borel and q-Laplace transformations of the fundamental solution $\widetilde{f_0}$ of the q-Hermite–Weber equation at $x=0$:*

$$f_0(x) := x^{\frac{\alpha}{2}} \mathcal{L}^+ \circ \mathcal{B}^+\big(\widetilde{f_0}\big)(x,\lambda).$$

Then we have

$$f_0\big(\mathrm{e}^{2\pi\mathrm{i}(u-v)}, -\mathrm{e}^{2\pi\mathrm{i}u}\big) = \mu(u,v;\alpha). \tag{1.22}$$

As a corollary, we obtain an interpretation of $\mu(u,v)=\mu(u,v;1)$ that the original μ-function is regarded as the image of the q-Borel and q-Laplace transformations of the fundamental solution $\widetilde{f}_0$ of the q-Hermite–Weber equation (very recently S. Garoufalidis and C. Wheeler [7] pointed out this fact independently of us). This interpretation gives some answers to our questions mentioned before. For example, the μ-function is a two-variable function with $x=\mathrm{e}^{2\pi\mathrm{i}u}$ and $y=\mathrm{e}^{2\pi\mathrm{i}v}$ due to the extra parameter λ which arises from the q-Laplace transformation (1.21). The denominator $\vartheta_{11}(v)$ of the μ-function arises from the definition of q-Laplace transformation. The numerator $\mathrm{e}^{\pi\mathrm{i}u}$ in $\mu(u,v)$ corresponds to the characteristic index of the solution around the origin of (1.19).

For this function $\mu(u,v;\alpha)$, we give the following formulas similar to those of the original μ-function.

Theorem 1.3.

$$\mu(u+2\tau,v;\alpha)=\big(1-\mathrm{e}^{2\pi\mathrm{i}(u-v)}q\big)q^{\frac{\alpha}{2}}\mu(u+\tau,v;\alpha)+\mathrm{e}^{2\pi\mathrm{i}(u-v)}q\mu(u,v;\alpha), \tag{1.23}$$

$$\mu(u,v;\alpha)=\mathrm{e}^{-\pi\mathrm{i}\alpha}\mu(u+1,v;\alpha)=\mathrm{e}^{\pi\mathrm{i}\alpha}\mu(u,v+1;\alpha), \tag{1.24}$$

$$\mu(u+\tau,v;\alpha)=-\mathrm{e}^{2\pi\mathrm{i}(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha)+\mathrm{e}^{\pi\mathrm{i}(u-v)}q^{\frac{\alpha}{2}}\mu(u,v;\alpha-1), \tag{1.25}$$

$$\mu(u-\tau,v;\alpha)=q^{\frac{\alpha}{2}}\mu(u,v;\alpha)-2\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}(u-v)}\sin(\pi\alpha\tau)\mu(u,v;\alpha+1), \tag{1.26}$$

$$\begin{aligned}\mu(u+z,v+z;\alpha)&=\frac{\vartheta_{11}(u+z)\vartheta_{11}(v+z-\alpha\tau)}{\vartheta_{11}(u+z-\alpha\tau)\vartheta_{11}(v+z)}\mathrm{e}^{2\pi\mathrm{i}\alpha(u-v)}\mu(v,u;\alpha)\\&\quad-\frac{i(q^{\alpha})_\infty(q)_\infty^2q^{\frac{1-4\alpha}{8}}\vartheta_{11}(z)\vartheta_{11}(u+v+z-\alpha\tau)}{\vartheta_{11}(u)\vartheta_{11}(v-\alpha\tau)\vartheta_{11}(u+z-\alpha\tau)\vartheta_{11}(v+z)}\\&\quad\times\mathrm{e}^{\pi\mathrm{i}(\alpha-1)(u-v)}{}_1\phi_1\left(\begin{matrix}q^{1-\alpha}\\0\end{matrix};q,\mathrm{e}^{-2\pi\mathrm{i}(u-v)}q\right),\end{aligned} \tag{1.27}$$

$$\mu(u,v;\alpha)=\mu(u+\tau,v+\tau;\alpha) \tag{1.28}$$

$$=\frac{\vartheta_{11}(v-\alpha\tau)\vartheta_{11}(u)}{\vartheta_{11}(u-\alpha\tau)\vartheta_{11}(v)}\mathrm{e}^{2\pi\mathrm{i}\alpha(u-v)}\mu(v,u;\alpha) \tag{1.29}$$

$$=\frac{\vartheta_{11}(v-\alpha\tau)\vartheta_{11}(u)}{\vartheta_{11}(u-\alpha\tau)\vartheta_{11}(v)}\mathrm{e}^{2\pi\mathrm{i}\alpha(u-v)}\mu(-u+\alpha\tau,-v+\alpha\tau;\alpha), \tag{1.30}$$

$$2\cos\pi(u-v)\mu(u,v;\alpha)=(1-q^{-\alpha})\mu(u,v;\alpha+1)+\mu(u,v;\alpha-1). \tag{1.31}$$

We see that these formulas correspond to that of the original μ-function as the periodicity: (1.24) $\leftrightarrow$ (1.1), forward shift: (1.25) $\leftrightarrow$ (1.2), translation: (1.3) $\leftrightarrow$ (1.3), τ-periodicity: (1.28) $\leftrightarrow$ (1.4), symmetry: (1.29) $\leftrightarrow$ (1.5), pseudo periodicity: (1.30) $\leftrightarrow$ (1.6). The equation (1.23) is a rewriting that $\mu(u,v;\alpha)$ satisfies the q-Hermite–Weber equation (1.19). Also, the property (1.31) is one of the specific properties of $\mu(u,v;\alpha)$, which coincides essentially with the q-Bessel equation:

$$\left[T_x-\big(q^{\frac{\nu}{2}}+q^{-\frac{\nu}{2}}\big)T_x^{\frac{1}{2}}+\left(1+\frac{x^2}{4}\right)\right]f(x)=0,\qquad T_x^{\frac{1}{2}}f(x):=f\big(q^{\frac{1}{2}}x\big).$$

Also, the second term on the right-hand side of (1.3) is written by the q-Bessel function (see Corollary 3.1). Further, we prove the translation formula (1.3) as a connection formula for the q-Hermite–Weber equation, that means the mysterious translation (1.3) is regarded as a variation of a connection formula (see Theorem 2.3 and (2.4)).

We also immediately obtain the q-hypergeometric expressions for $\mu(u,v;\alpha)$ corresponding to (1.8), (1.9) and (1.10) from (1.15), (1.16) and (1.17).

Theorem 1.4.

$$\begin{aligned}\mu(x,y;a) &= -\mathrm{i}q^{-\frac{1}{8}}\frac{(x/y)^{\frac{\alpha}{2}}}{\theta_q(-x/a)}\frac{(aq/y)_\infty}{(q/y)_\infty}{}_1\psi_2\left(\begin{matrix}y/a\\0,y\end{matrix};q,x\right)\\ &= -\mathrm{i}q^{-\frac{1}{8}}\left(\frac{x}{y}\right)^{\frac{\alpha}{2}}\frac{(a,q,aq/x,aq/y)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_2\psi_2\left(\begin{matrix}x/a,y/a\\0,0\end{matrix};q,a\right)\\ &= -\mathrm{i}q^{-\frac{1}{8}}\left(\frac{x}{y}\right)^{\frac{\alpha}{2}}\frac{(a,q,x,y)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_0\psi_2\left(\begin{matrix}-\\x,y\end{matrix};q,\frac{xy}{a}\right).\end{aligned}$$

Note that the symmetry (1.29) is proved by a specialization of the translation (1.3), but we also get another proof from these q-hypergeometric expressions.

The above results are for general complex parameters with α, but by restricting α to the integer k, we obtain the following simplified formulation.

Corollary 1.5. *Let k be an integer, we have*

$$\begin{aligned}&\mu(u+1,v;k) = \mu(u,v+1;k) = (-1)^k\mu(u,v;k),\\ &\mu(u+\tau,v;k) = -\mathrm{e}^{2\pi\mathrm{i}(u-v)}q^{\frac{k}{2}}\mu(u,v;k)+\mathrm{e}^{\pi\mathrm{i}(u-v)}q^{\frac{k}{2}}\mu(u,v;k-1),\\ &\mu(u-\tau,v;k) = q^{\frac{k}{2}}\mu(u,v;k)-2\mathrm{i}\mathrm{e}^{-\pi\mathrm{i}(u-v)}\sin(\pi k\tau)\mu(u,v;k+1),\\ &\mu(u+z,v+z;k+1) = \mu(u,v;k+1)+\frac{\mathrm{i}q^{\frac{1}{8}}(q)_\infty^3\vartheta_{11}(z)\vartheta_{11}(u+v+z)}{\vartheta_{11}(u)\vartheta_{11}(v)\vartheta_{11}(u+z)\vartheta_{11}(v+z)}\\ &\qquad\times\frac{\mathrm{e}^{-\pi\mathrm{i}k(u-v)}}{(q^{-k})_k}{}_1\phi_1\left(\begin{matrix}q^{-k}\\0\end{matrix};q,\mathrm{e}^{2\pi\mathrm{i}(u-v)}q\right),\\ &\mu(u,v;k) = \mu(u+\tau,v+\tau;k)\\ &\qquad= \mu(v,u;k)\\ &\qquad= \mu(-u,-v;k),\\ &2\cos\pi(u-v)\mu(u,v;k) = \left(1-q^{-k}\right)\mu(u,v;k+1)+\mu(u,v;k-1). \qquad (1.32)\end{aligned}$$

In particular, if k is a positive integer, we have

$$\begin{aligned}\mu(u,v;k) &= \frac{\mathrm{e}^{\pi\mathrm{i}k(u-v)}}{\vartheta_{11}(v)}\sum_{n\in\mathbb{Z}}(-1)^n\mathrm{e}^{2\pi\mathrm{i}\left(n+\frac{1}{2}\right)v}q^{\frac{n(n+1)}{2}}\prod_{l=0}^{k-1}\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}u}q^{n-l}}\\ &= \mathrm{e}^{-\pi\mathrm{i}(k-1)(u-v-\tau)}\sum_{j=0}^{k-1}\frac{(-1)^{k-1-j}}{(q)_j(q)_{k-1-j}}q^{\frac{(k-1-j)^2}{2}}\mu(u-j\tau,v). \qquad (1.33)\end{aligned}$$

Also from (1.32), we obtain the following result which is non-trivial from the definition of $\mu(u,v;\alpha)$.

Theorem 1.6. *Let k be a non-negative integer, we have*

$$\mu(u,v;-k) = -\mathrm{i}q^{-\frac{1}{8}}H_k(\cos\pi(u-v)\mid q),$$

where $H_k(\cos\pi(u-v)\mid q)$ is a continuous q-Hermite polynomial of degree k [13, p. 543]:

$$H_k(\cos\pi(u-v)\mid q) := \sum_{l=0}^{k}\frac{(q)_k}{(q)_l(q)_{k-l}}\mathrm{e}^{\pi\mathrm{i}(k-2l)(u-v)}.$$

From this theorem, $\{\mu(u,v;k+1)\}_{k\geq 0}$ is a family of “minus degree” continuous q-Hermite polynomials, and in particular, the original μ-function is regarded as a “-1 degree” continuous q-Hermite polynomial.

To give some relations of $\{\mu(u,v;k)\}_{k\in\mathbb{Z}}$, we introduce the following generating function of $\{\mu(u,v;k+1)\}_{k\geq 0}$:

$$S(u,v,r)=S(r):=\sum_{k=0}^{\infty}\mu(u,v;k+1)r^k,$$

which is a variation (i.e., minus degree version) of the well-known generating function of the continuous q-Hermite polynomials (for example, see [13, p. 542]):

$$\sum_{n\geq 0}\frac{H_n(\cos\theta\mid q)}{(q)_n}r^n=\frac{1}{\left(r\mathrm{e}^{\mathrm{i}\theta},r\mathrm{e}^{-\mathrm{i}\theta}\right)_\infty}. \tag{1.34}$$

For the generating function $S(r)$, we obtain the following q-difference relations and expressions.

Theorem 1.7.

(1) *The generating function $S(r)$ satisfies the following q-difference equations:*

$$S(r)=\left(1-r\mathrm{e}^{\pi\mathrm{i}(u-v)}q\right)\left(1-r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)S(rq)-\mathrm{i}rq^{\frac{7}{8}}, \tag{1.35}$$

$$\begin{aligned}&\left[\left(1-r\mathrm{e}^{\pi\mathrm{i}(u-v)}q^2\right)\left(1-r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q^2\right)T_r^2\right.\\&\quad\left.-\left(1+q\left(1-r\mathrm{e}^{\pi\mathrm{i}(u-v)}q\right)\left(1-r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)\right)T_r+q\right]S(r)=0.\end{aligned} \tag{1.36}$$

(2) *The generating function $S(r)$ has the following closed forms:*

$$\begin{aligned}S(r)&=\left(r\mathrm{e}^{\pi\mathrm{i}(u-v)}q,r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)_\infty\mu(u,v)-\mathrm{i}rq^{\frac{7}{8}}\,{}_3\phi_2\left(\begin{matrix}q,r\mathrm{e}^{\pi\mathrm{i}(u-v)}q,r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\\0,0\end{matrix};q,q\right)\\&=\left(r\mathrm{e}^{\pi\mathrm{i}(u-v)}q,r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)_\infty\\&\quad\times\left\{\mu(u,v)-\frac{\mathrm{i}rq^{\frac{7}{8}}}{1-q}\Phi^{(1)}\left(\begin{matrix}a;0,0\\q^2\end{matrix};q;r\mathrm{e}^{\pi\mathrm{i}(u-v)}q,r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)\right\}\end{aligned} \tag{1.37}$$

$$=\left(r\mathrm{e}^{\pi\mathrm{i}(u-v)}q,r\mathrm{e}^{-\pi\mathrm{i}(u-v)}q\right)_\infty\sum_{m\geq 0}\frac{\mu(u,v;1-m)}{(q)_m}q^mr^m, \tag{1.38}$$

where $\Phi^{(1)}$ is the q-Appell hypergeometric function:

$$\Phi^{(1)}\left(\begin{matrix}a;b_1,b_2\\c\end{matrix};q;x,y\right):=\sum_{m,n\geq 0}\frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n}(q)_m(q)_n}x^my^n. \tag{1.39}$$

In particular, note that the q-difference equation (1.36) is a degeneration of the q-Heun equation:

$$\left[\left(a_0+b_0x+c_0x^2\right)T_x^2+\left(a_1+b_1x+c_1x^2\right)T_x+\left(a_2+b_2x+c_2x^2\right)\right]u(x)=0,$$

and $S(r)$ is its solution. Furthermore, by comparing the coefficients in the expansion of the relation (1.38) at $r=0$, we also obtain some relations for $\mu(u,v;k+1)$, see Corollary 3.6.

This paper is organized as follows. First, in Section 2, we present some preliminary results on the basic solutions and connection formulas for some q-difference equations that are degenerations of the q-difference equation (1.20). In particular, for the q-Hermite–Weber equation (1.19), we give a proof of its connection formula according to Ohyama's private note [17]. Furthermore, we show that the images of the q-Borel and q-Laplace transformations for the divergent solution $\widetilde{f}_0(x)$ is essentially equivalent to the generalized μ-function $\mu(u,v;\alpha)$. Then, we prove the connection formula for the case where the parameters λ and λ' are different, and state that it is equivalent to the translation formula (1.3) of $\mu(u,v;\alpha)$. Next, in Section 3, we prove the above main results concerning $\mu(u,v;\alpha)$. In Section 4, we mention that $\mu(u,v;k,\tau)$ satisfies the modular transformations (4.1) and (4.2). Finally, in Section 5, we discuss a possible direction to future works.

During the preparation of this paper, we have been informed by Hikami and Matsusaka of a recent very interesting paper [7]. Some of the examples of the paper [7] overlap with our results on the μ-function, though the one parameter generalization $\mu(u,v;\alpha)$ is not considered there (e.g., see equation (1.18) and [7, Section 4.2]).

2 Fundamental solutions and connection formulas of some q-difference equations

In this section, based on [16, 18, 22], we present some preliminary results on the basic solutions and connection formulas for some q-difference equations.

Lemma 2.1 ([22, p. 18, Theorem 2.2.1])**.** *We have the fundamental solutions of the q-difference equation*

$$\left[(1-abxq)T_x^2-(1-(a+b)xq)T_x-xq\right]f(x)=0 \tag{2.1}$$

around $x=0$ and $x=\infty$:

$$\mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda),\qquad F_2(x)=\frac{(abx)_\infty}{\theta_q(-xq)}\,{}_2\phi_1\left(\begin{matrix}q/a,q/b\\0\end{matrix};q,abx\right),$$

$$G_1(x)=\frac{\theta_q(-axq)}{\theta_q(-xq)}\,{}_2\phi_1\left(\begin{matrix}a,0\\aq/b\end{matrix};q,\frac{q}{abx}\right),\qquad G_2(x)=\frac{\theta_q(-bxq)}{\theta_q(-xq)}\,{}_2\phi_1\left(\begin{matrix}b,0\\bq/a\end{matrix};q,\frac{q}{abx}\right),$$

where $F_1(x)$ is the formal solution around $x=0$:

$$F_1(x):={}_2\phi_0\left(\begin{matrix}a,b\\-\end{matrix}q,x\right).$$

In this case, the connection formulas for $\mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda)$, $F_2(x)$ and $G_1(x)$, $G_2(x)$ are as follows:

$$\begin{aligned}\mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda)&=\frac{(b)_\infty\theta_q(a\lambda)\theta_q(axq/\lambda)\theta_q(-xq)}{(b/a)_\infty\theta_q(\lambda,xq/\lambda,-axq)}G_1(x)\\&\quad+\frac{(a)_\infty\theta_q(b\lambda)\theta_q(bxq/\lambda)\theta_q(-xq)}{(a/b)_\infty\theta_q(\lambda)\theta_q(xq/\lambda)\theta_q(-bxq)}G_2(x),\\F_2(x)&=\frac{(q/a)_\infty}{(b/a,q)_\infty}G_1(x)+\frac{(q/b)_\infty}{(a/b,q)_\infty}G_2(x).\end{aligned}$$

Putting $x\mapsto a^{-1}x$, $b=0$ in the q-difference equation (2.1), we have

$$\left[T_x^2-(1-xq)T_x-\frac{x}{a}q\right]f\left(a^{-1}x\right)=0.$$

Furthermore, we put $a=q^\alpha$, $x^{\frac{\alpha}{2}}f\left(a^{-1}x\right)=g(x)$, then $g(x)$ is the solution of the q-Hermite–Weber equation (1.19).

Lemma 2.2. *The fundamental solutions of the q-difference equation (1.19) around $x=0$ are*

$$f_0(x) := x^{\frac{\alpha}{2}}\mathcal{L}^+\circ\mathcal{B}^+\big(\widetilde{f_0}\big)(x,\lambda), \qquad g_0(x) := \frac{x^{1-\frac{\alpha}{2}}}{\theta_q(-x)}\,{}_1\phi_1\left(\begin{matrix}q/a\\0\end{matrix};q,xq\right),$$

where $\widetilde{f_0}(x)$ is the formal solution around $x=0$:

$$\widetilde{f_0}(x) = {}_2\phi_0\left(\begin{matrix}a,0\\-\end{matrix};q,\frac{x}{a}\right).$$

And that of around $x=\infty$ are

$$f_\infty(x) := f_0\big(x^{-1}\big), \qquad g_\infty(x) = g_0\big(x^{-1}\big).$$

In this case, the connection formulas for $f_0(x,\lambda)$, $f_\infty(x,\lambda)$ and $g_0(x)$, $g_\infty(x)$ are as follows:

$$\begin{pmatrix}f_0(x,\lambda)\\f_\infty(x,\lambda)\end{pmatrix} = -\frac{(q)_\infty}{(q/a)_\infty}\begin{pmatrix}\dfrac{\theta_q(\lambda)\theta_q(ax/\lambda)x^\alpha}{\theta_q(\lambda/a)\theta_q(x/\lambda)a} & 1\\ 1 & \dfrac{\theta_q(\lambda)\theta_q(x\lambda/a)}{\theta_q(\lambda/a)\theta_q(x\lambda)}x^{-\alpha}\end{pmatrix}\begin{pmatrix}g_0(x)\\g_\infty(x)\end{pmatrix}. \tag{2.2}$$

For this Lemma 2.2, we give a proof by following Ohyama [17].

Proof. It is sufficient to prove the case of $x=0$, since the q-difference equation (1.19) is symmetric under the transformation $x\leftrightarrow x^{-1}$.

The image of $\mathcal{L}^+\circ\mathcal{B}^+$ for $\widetilde{f_0}(x)$ gives the convergent series $f_0(x)$, and it gives a fundamental solution of (1.19). This fact follows from the property of the q-Borel and q-Laplace transformations:

$$\mathcal{B}^+\big(x^mT_x^nf(x)\big)(\xi) = q^{\frac{m(m-1)}{2}}\xi^mT_\xi^{m+n}\mathcal{B}^+(f)(\xi),$$
$$\mathcal{L}^+\big(\xi^mT_\xi^nf(\xi)\big)(x,\lambda) = q^{-\frac{m(m-1)}{2}}x^mT_x^{n-m}\mathcal{L}^+(f)(x,\lambda),$$

or

$$\mathcal{L}^+\circ\mathcal{B}^+\big(x^mT_x^nf\big)(x,\lambda) = x^mT_x^n\mathcal{L}^+\circ\mathcal{B}^+(f)(x,\lambda).$$

Since the limit of $F_2(x)$ as $x\to a^{-1}x$, $b\to 0$ degenerates to $g_0(x)$, we prove $g_0(x)$ is another fundamental solution of (1.19).

The proof of the connection formula is obtained from the degeneration limit of the connection formula in Lemma 2.1. We take the limit of the Heine's transformation formula

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};q,x\right) = \frac{(abx/c)_\infty}{(x)_\infty}\,{}_2\phi_1\left(\begin{matrix}c/b,c/a\\c\end{matrix};q,\frac{abx}{c}\right),$$

as $a\mapsto 0$, then

$${}_2\phi_1\left(\begin{matrix}0,b\\c\end{matrix};q,x\right) = \frac{1}{(x)_\infty}\,{}_1\phi_1\left(\begin{matrix}q/b\\c\end{matrix};q,bx\right).$$

Then we rewrite the solution $G_2(x)$ in Lemma 2.1 as

$$\begin{aligned}G_2(x) &= \frac{\theta_q(-bxq)}{\theta_q(-xq)}\,{}_2\phi_1\left(\begin{matrix}b,0\\bq/a\end{matrix};q,\frac{q}{abx}\right) = \frac{\theta_q(-bxq)}{(q/abx)_\infty\theta_q(-xq)}\,{}_1\phi_1\left(\begin{matrix}q/a\\bq/a\end{matrix};q,\frac{q}{ax}\right)\\ &= (q,abx)_\infty\frac{\theta_q(-bxq)}{\theta_q(-abx)\theta_q(-xq)}\,{}_1\phi_1\left(\begin{matrix}q/a\\bq/a\end{matrix};q,\frac{q}{ax}\right).\end{aligned}$$

Therefore the connection formula of $\mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda)$ is given by

$$\mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda)=C_1(x)F_2(x)+C_2(x)\frac{(abx)_\infty}{\theta_q(-ax)}\,{}_1\phi_1\left(\begin{matrix}q/a\\ bq/a\end{matrix};q,\frac{q}{ax}\right), \tag{2.3}$$

where

$$C_1(x)=\frac{(b,q)_\infty}{(q/a)_\infty}\frac{\theta_q(a\lambda)\theta_q(axq/\lambda)\theta_q(-xq)}{\theta_q(\lambda)\theta_q(xq/\lambda)\theta_q(-axq)},$$
$$C_2(x)=\frac{(q)_\infty\theta_q(-ax)}{\theta_q(\lambda)\theta_q(\frac{xq}{\lambda})}\left\{(a,bq/a,q)_\infty\frac{\theta_q(b\lambda)\theta_q(bxq/\lambda)}{\theta_q(-bq/a)\theta_q(-abx)}\right.$$
$$\left.-\frac{(bq/a)_\infty}{(q/a)_\infty}\frac{\theta_q(-b)\theta_q(a\lambda)\theta_q(axq/\lambda)\theta_q(-bxq)}{\theta_q(-bq/a)\theta_q(-axq)\theta_q(-abx)}\right\}.$$

By replacing $x\mapsto a^{-1}x$, $\lambda\mapsto a^{-1}\lambda$, $b\mapsto-\lambda^{-1}q^n$ in (2.3) and taking the limit of each term of (2.3) as $n\to\infty$, we have

$$\frac{(abx)_\infty}{\theta_q(-ax)}\,{}_1\phi_1\left(\begin{matrix}q/a\\ bq/a\end{matrix};q,\frac{q}{ax}\right)\to -x^{-\frac{\alpha}{2}}g_\infty(x),$$
$$F_2(x)\to\frac{\theta_q(-x)}{\theta_q(-xq/a)}x^{\frac{\alpha}{2}-1}g_0(x),\qquad \mathcal{L}^+\circ\mathcal{B}^+(F_1)(x,\lambda)\to\mathcal{L}^+\circ\mathcal{B}^+\big(\widetilde{f}_0\big)(x,\lambda).$$

Hence, we obtain the conclusion. ■

Next, we prove Theorem 1.2 which means that the image of the composition of the q-Borel and q-Laplace transformations of the divergent series $\widetilde{f}_0(x)$ is essentially equivalent to our $\mu(u,v;\alpha)$.

Proof of Theorem 1.2. From the definition of the q-Borel transformation and the q-binomial theorem (see [8, p. 8, equation (1.3.2)]),

$$\mathcal{B}^+\left(\widetilde{f}_0\right)(\xi)=\sum_{n\geq0}\frac{(a)_n}{(q)_n}(-1)^n\left(\frac{\xi}{a}\right)^n=\frac{(-\xi)_\infty}{(-\xi/a)_\infty}.$$

By a simple calculation, we have

$$\mathcal{L}^+\circ\mathcal{B}^+\big(\widetilde{f}_0\big)(x,-\lambda)=\sum_{n\in\mathbb{Z}}\frac{1}{\theta_q(-\lambda q^n/x)}\frac{(\lambda q^n)_\infty}{(\lambda q^n/a)_\infty}$$
$$=\frac{1}{\theta_q(-\lambda/x)}\sum_{n\in\mathbb{Z}}\left(-\frac{\lambda}{x}\right)^{n+1}q^{\frac{n(n+1)}{2}}\frac{\left(\lambda q^{n+1}\right)_\infty}{\left(\lambda q^{n-\alpha+1}\right)_\infty}.$$

The second equality follows from the well-known q-difference relation of the theta function:

$$\theta_q(q^nx)=q^{-\frac{n(n-1)}{2}}x^{-n}\theta_q(x),\qquad n\in\mathbb{Z}.$$

By substituting $x=\mathrm{e}^{2\pi\mathrm{i}(u-v)}$ and $\lambda=\mathrm{e}^{2\pi\mathrm{i}u}$, we have

$$f_0\big(\mathrm{e}^{2\pi\mathrm{i}(u-v)},-\mathrm{e}^{2\pi\mathrm{i}u}\big)=\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}\mathcal{L}^+\circ\mathcal{B}^+\big(\widetilde{f}_0\big)\big(\mathrm{e}^{2\pi\mathrm{i}(u-v)},-\mathrm{e}^{2\pi\mathrm{i}v}\big)$$
$$=-\frac{\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}}{\theta_q\big(-\mathrm{e}^{2\pi\mathrm{i}v}\big)}\sum_{n\in\mathbb{Z}}(-1)^n\mathrm{e}^{2\pi\mathrm{i}(n+1)v}q^{\frac{n(n+1)}{2}}\frac{\big(\mathrm{e}^{2\pi\mathrm{i}u}q^{n+1}\big)_\infty}{\big(\mathrm{e}^{2\pi\mathrm{i}u}q^{n-\alpha+1}\big)_\infty}$$
$$=\frac{\mathrm{i}\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}}{\vartheta_{11}(v)}\sum_{n\in\mathbb{Z}}(-1)^n\mathrm{e}^{2\pi\mathrm{i}\left(n+\frac12\right)v}q^{\frac{\left(n+\frac12\right)^2}{2}}\frac{\big(\mathrm{e}^{2\pi\mathrm{i}u}q^{n+1}\big)_\infty}{\big(\mathrm{e}^{2\pi\mathrm{i}u}q^{n-\alpha+1}\big)_\infty}$$
$$=\mathrm{i}q^{\frac18}\mu(u,v;\alpha).$$
■

Finally, we give a connection formula which derives the translation formula (1.3).

Theorem 2.3. *The following equation holds:*

$$f_0(x,\lambda')=\frac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a}f_\infty(x,\lambda)\\ -(a)_\infty(q)_\infty^2\frac{\theta_q(-\lambda'/x\lambda)\theta_q(-x)\theta_q(-\lambda\lambda'/a)}{\theta_q(x\lambda)\theta_q(\lambda'/x)\theta_q(\lambda'/a)\theta_q(a/\lambda)}g_\infty(x). \tag{2.4}$$

Proof. We replace the variable λ by λ' in the connection formula (2.2) of the case of $f_0(x,\lambda)$:

$$\begin{pmatrix} f_0(x,\lambda') \\ f_\infty(x,\lambda)\end{pmatrix}=-\frac{(q)_\infty}{(q/a)_\infty}\begin{pmatrix}\dfrac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a} & 1 \\ 1 & \dfrac{\theta_q(\lambda)\theta_q(x\lambda/a)}{\theta_q(\lambda/a)\theta_q(x\lambda)}x^{-\alpha}\end{pmatrix}\begin{pmatrix} g_0(x) \\ g_\infty(x)\end{pmatrix}. \tag{2.5}$$

By erasing the function $g_0(x)$ in the above (2.5), we have

$$\begin{aligned} f_0(x,\lambda')&-\frac{\theta_q(\lambda')\theta_q(ax/\lambda')x^\alpha}{\theta_q(\lambda'/a)\theta_q(x/\lambda')a}f_\infty(x,\lambda)\\ &=\frac{(q)_\infty}{(q/a)_\infty}\left\{\frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(x\lambda/a)\theta_q(ax/\lambda')}{a\theta_q(\lambda/a)\theta_q(\lambda'/a)\theta_q(x\lambda)\theta_q(x/\lambda')}-1\right\}g_\infty(x)\\ &=\frac{(a)_\infty(q)_\infty^2}{\theta_q(-a)}\frac{\theta_q(-\lambda'/x\lambda)\theta_q(-x)}{\theta_q(x\lambda)\theta_q(\lambda'/x)}C(x)g_\infty(x),\end{aligned}$$

where

$$C(x)=\frac{\lambda'}{ax}\frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(x\lambda/a)\theta_q(ax/\lambda')}{\theta_q(-\lambda'/x\lambda)\theta_q(-x)\theta_q(\lambda/a)\theta_q(\lambda'/a)}-\frac{\theta_q(x\lambda)\theta_q(\lambda'/x)}{\theta_q(-\lambda'/x\lambda)\theta_q(-x)}.$$

The function $C(x)$ has a simple pole in the points $x=q^j,\frac{\lambda'}{\lambda}q^j$ for some $j\in\mathbb{Z}$, and satisfies $C(xq)=C(x)$. We calculate the residue at the pole $x=1$,

$$\begin{aligned}\lim_{x\to1}\theta_q(-x)C(x)&=\frac{\lambda'}{a}\frac{\theta_q(\lambda)\theta_q(\lambda')\theta_q(\lambda/a)\theta_q(a/\lambda')}{\theta_q(-\lambda'/\lambda)\theta_q(\lambda/a)\theta_q(\lambda'/a)}-\frac{\theta_q(\lambda)\theta_q(\lambda')}{\theta_q(-\lambda'/\lambda)}\\ &=\frac{\theta_q(\lambda)\theta_q(\lambda')}{\theta_q(-\lambda'/\lambda)}\left\{\frac{\lambda'}{a}\frac{\theta_q(a/\lambda')}{\theta_q(\lambda'/a)}-1\right\}=0.\end{aligned}$$

It follows that the doubly periodic function $C(x)$ is constant in x. Hence, we have

$$C(x)=C(-a/\lambda)=-\frac{\theta_q(-a)\theta_q(-\lambda\lambda'/a)}{\theta_q(\lambda'/a)\theta_q(a/\lambda)}.$$ ■

3 Proofs of the main results

In this section, we prove the main theorems and corollaries. First, we prove Theorem 1.3.

Proof of Theorem 1.3. The equation (1.23) is obvious from Theorem 1.2 and Lemma 2.2.

The pseudo periodicity (1.24) follows directly from the definition of $\mu(u,v;\alpha)$.

The forward shift (1.25) is proved as follows:

$$\mu(u+\tau,v;\alpha)=\frac{\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}}{\vartheta_{11}(v)}q^{\frac{\alpha}{2}}\sum_{n\in\mathbb{Z}}(-1)^n\mathrm{e}^{2\pi\mathrm{i}(n+\frac12)v}q^{\frac{n(n+1)}{2}}$$

$$\times \frac{(e^{2\pi i u} q^{n+2})_\infty}{(e^{2\pi i u} q^{n-\alpha+2})_\infty} \left(e^{2\pi i u} q^{n+1} + 1 - e^{2\pi i u} q^{n+1}\right)$$
$$= \frac{e^{\pi i\alpha(u-v)}}{\vartheta_{11}(v)} q^{\frac{\alpha}{2}} \left\{ \sum_{n\in\mathbb{Z}} e^{2\pi i(u-v)} (-1)^{n-1} e^{2\pi i(n+\frac{1}{2})v} q^{\frac{n(n+1)}{2}} \frac{(e^{2\pi i u} q^{n+1})_\infty}{(e^{2\pi i u} q^{n-\alpha+1})_\infty} \right.$$
$$\left. + \sum_{n\in\mathbb{Z}} (-1)^n e^{2\pi i(n+\frac{1}{2})v} q^{\frac{n(n+1)}{2}} \frac{\left(e^{2\pi i u} q^{n+1}\right)_\infty}{\left(e^{2\pi i u} q^{n-(\alpha-1)+1}\right)_\infty} \right\}$$
$$= -e^{2\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(u, v; \alpha) + e^{\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(u, v; \alpha - 1).$$

The translation formula (1.3) is proved by putting $x = e^{2\pi i(u-v)}$, $\lambda = -e^{2\pi i v}$, and $\lambda' = -e^{2\pi i(u+z)}$ in the connection formula (2.4) and using (1.22). The τ-periodicity (1.28), symmetry (1.29) and pseudo-periodicity (1.30) are the case of $z = 0$, τ and $-u - v + \alpha\tau$ in the translation formula (1.3), respectively,

$$\mu(u, v; \alpha) = \frac{\vartheta_{11}(v - \alpha\tau)\vartheta_{11}(u)}{\vartheta_{11}(u - \alpha\tau)\vartheta_{11}(v)} e^{2\pi i\alpha(u-v)} \mu(v, u; \alpha),$$
$$\mu(u, v; \alpha) = \mu(u + \tau, v + \tau; \alpha),$$
$$\mu(u, v; \alpha) = \frac{\vartheta_{11}(v - \alpha\tau)\vartheta_{11}(u)}{\vartheta_{11}(u - \alpha\tau)\vartheta_{11}(v)} e^{2\pi i\alpha(u-v)} \mu(-u + \alpha\tau, -v + \alpha\tau; \alpha).$$

Finally to prove the equation (1.31), let us put

$$\Phi(u, v; \alpha) := \frac{\vartheta_{11}(v - \alpha\tau)\vartheta_{11}(u)}{\vartheta_{11}(u - \alpha\tau)\vartheta_{11}(v)} e^{2\pi i\alpha(u-v)},$$

which satisfies

$$\Phi(u, v; \alpha) = \Phi(u, v; \alpha - 1) = \Phi(u, v + \tau; \alpha) = \Phi(v, u; \alpha)^{-1}.$$

Then, from (1.3) and (1.25), we have the desired result

$$\begin{aligned}\mu(u, v + \tau; \alpha) &= \Phi(u, v + \tau; \alpha)\mu(v + \tau, u; \alpha) \\ &= \Phi(u, v; \alpha)\left(-e^{-2\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(v, u; \alpha) + e^{-\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(v, u; \alpha - 1)\right) \qquad (3.1) \\ &= -e^{-2\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(u, v; \alpha) + e^{-\pi i(u-v)} q^{\frac{\alpha}{2}} \mu(u, v; \alpha - 1).\end{aligned}$$ ■

We define the function $J_\nu^{(2)}$, one of Jackson q-Bessel functions, as follows [13, p. 23]:

$$J_\nu^{(2)}(x; q) := \frac{\left(q^{\nu+1}\right)_\infty}{(q)_\infty} \left(\frac{x}{2}\right)^\nu {}_0\phi_1\left(\begin{matrix} - \\ q^{\nu+1} \end{matrix}; q, -\frac{x^2 q^{\nu+1}}{4}\right).$$

It is known that this function has the following expression [14, equation (3.2)]:

$$J_\nu^{(2)}(x; q) = \frac{1}{(q)_\infty} \left(\frac{x}{2}\right)^\nu {}_1\phi_1\left(\begin{matrix} -x^2/4 \\ 0 \end{matrix}; q, q^{\nu+1}\right). \qquad (3.2)$$

From the equations (1.3), (2.2) and (3.2), we obtain some relations between the q-Bessel function and $\mu(u, v; \alpha)$.

Corollary 3.1. *We have*

$$i q^{\frac{1}{8}} \mu(u, v; \alpha) = \Phi(u, v; \alpha) j(u - v; \alpha) + j(v - u; \alpha), \qquad (3.3)$$

and

$$\frac{\vartheta_{11}(\alpha\tau)\vartheta_{11}(u-v)\vartheta_{11}(z)\vartheta_{11}(u+v+z-\alpha\tau)}{\vartheta_{11}(u-\alpha\tau)\vartheta_{11}(v)\vartheta_{11}(u+z-\alpha\tau)\vartheta_{11}(v+z)}\mathrm{e}^{2\pi\mathrm{i}\alpha(u-v)}j(u-v;\alpha+1)$$
$$= \mathrm{i}q^{\frac{1}{8}}\mu(u+z,v+z;\alpha+1) - \mathrm{i}q^{\frac{1}{8}}\mu(u,v;\alpha+1),$$

where

$$j(w;\alpha) := \mathrm{i}q^{\frac{1}{8}}\frac{(q)_\infty^2\mathrm{e}^{-\frac{\pi\mathrm{i}w}{2\tau}}}{\vartheta_{11}(w)(q^{1-\alpha})_\infty}J^{(2)}_{\frac{w}{\tau}}\left(2\mathrm{i}\mathrm{e}^{\pi\mathrm{i}(1-\alpha)\tau};q\right)$$
$$= \mathrm{i}q^{\frac{1}{8}}\frac{(q)_\infty}{(q^{1-\alpha})_\infty}\frac{\mathrm{e}^{\pi\mathrm{i}(1-\alpha)w}}{\vartheta_{11}(w)}\,{}_1\phi_1\left(\begin{matrix}q^{1-\alpha}\\0\end{matrix};q,\mathrm{e}^{2\pi\mathrm{i}w}q\right).$$

Remark 3.2. If α is a positive integer k in the equation (3.3), the function $j(w;\alpha)$ diverges. So, taking the limit $\alpha\to k$ of (3.3) is not straightforward. For example, if we take the limit $\alpha\to 1$ in an appropriate setting, we obtain

$$\vartheta_{11}(u-v)\mu(u,v) = \frac{1}{2\pi\mathrm{i}}\left(\frac{\vartheta'_{11}(u)}{\vartheta_{11}(u)} - \frac{\vartheta'_{11}(v)}{\vartheta_{11}(v)}\right) + \sum_{n\in\mathbb{Z}\setminus\{0\}}\frac{(-1)^n q^{\frac{n(n+1)}{2}}}{1-q^n}\mathrm{e}^{2\pi\mathrm{i}n(u-v)}. \tag{3.4}$$

This equation (3.4) is also obtained by the specialization of (1.3)

$$\mu(u,v+\epsilon) = \mu(\epsilon,u-v) + \frac{\mathrm{i}q^{\frac{1}{8}}(q)_\infty^3\vartheta_{11}(v)\vartheta_{11}(u+\epsilon)}{\vartheta_{11}(u)\vartheta_{11}(v+\epsilon)\vartheta_{11}(\epsilon)\vartheta_{11}(u-v)},$$

and taking the limit $\epsilon\to 0$.

Next, we prove Corollary 1.5. Since all other equations except equation (1.33) are obtained immediately from Theorem 1.3, we prove only (1.33).

Proof of Corollary 1.5. The first equation is obvious by the Definition 1.1. Next, using the partial fraction decomposition

$$\prod_{l=0}^{k-1}\frac{1}{1-\mathrm{e}^{2\pi\mathrm{i}u}q^{n-l}} = \sum_{j=0}^{k-1}\frac{(-1)^{k-j-1}}{(q)_j(q)_{k-1-j}}\frac{q^{\frac{(k-j)(k-j-1)}{2}}}{1-\mathrm{e}^{2\pi\mathrm{i}u}q^{n-j}},$$

we have

$$\mu(u,v;k) = \frac{\mathrm{e}^{\pi\mathrm{i}k(u-v)}}{\vartheta_{11}(v)}\sum_{j=0}^{k-1}\sum_{n\in\mathbb{Z}}\frac{(-1)^n\mathrm{e}^{2\pi\mathrm{i}(n+\frac{1}{2})v}q^{\frac{n(n+1)}{2}}}{1-\mathrm{e}^{2\pi\mathrm{i}u}q^{n-j}}\frac{(-1)^{k-1-j}q^{\frac{(k-j)(k-j-1)}{2}}}{(q)_j(q)_{k-1-j}}$$
$$= \mathrm{e}^{\pi\mathrm{i}(k-1)(u-v+\tau)}\sum_{j=0}^{k-1}\frac{(-1)^{k-1-j}}{(q)_j(q)_{k-1-j}}q^{\frac{(k-1-j)^2}{2}}\mu(u-j\tau,v). \qquad\blacksquare$$

As an important application of the recursion formula (1.31), we also obtain the following relation (reducing formula) for $\mu(u,v;\alpha)$.

Corollary 3.3. *For $\alpha\in\mathbb{C}$, we have*

$$\mu\left(u,u+\frac{1}{2};\alpha-1\right) = \left(q^{-\alpha}-1\right)\mu\left(u,u+\frac{1}{2};\alpha+1\right). \tag{3.5}$$

In particular, for any non-negative integer k,

$$\mu\left(u, u+\frac{1}{2}; 2k\right) = -\mathrm{i}q^{-\frac{1}{8}}\frac{q^{k^2}}{\left(q;q^2\right)_k},$$
$$\mu\left(u, u+\frac{1}{2}; 2k+1\right) = \frac{q^{k(k+1)}}{\left(q^2;q^2\right)_k}\mu\left(u, u+\frac{1}{2}\right) = \frac{q^{k(k+1)}}{\left(q^2;q^2\right)_k}\frac{\mathrm{i}q^{\frac{1}{4}}(q)_\infty^4(-q)_\infty^2\vartheta_{11}(2u)}{\vartheta_{11}(u)^2\vartheta_{11}\left(u+\frac{1}{2}\right)^2}.$$

Proof of Corollary 3.3. It is enough to show that

$$\mu\left(u, u+\frac{1}{2}\right) = \frac{\mathrm{i}q^{\frac{1}{4}}(q)_\infty^4(-q)_\infty^2\vartheta_{11}(2u)}{\vartheta_{11}(u)^2\vartheta_{11}\left(u+\frac{1}{2}\right)^2}.$$

If we put $v = u - \frac{1}{2}$ and $z = \frac{1}{2}$ in (1.3), then

$$\mu\left(u, u+\frac{1}{2}\right) = \mu\left(u-\frac{1}{2}, u\right) + \frac{\mathrm{i}q^{\frac{1}{8}}(q)_\infty^3\vartheta_{11}\left(\frac{1}{2}\right)\vartheta_{11}(2u)}{\vartheta_{11}\left(u-\frac{1}{2}\right)\vartheta_{11}(u)^2\vartheta_{11}\left(u+\frac{1}{2}\right)}.$$

Since

$$\mu\left(u-\frac{1}{2}, u\right) = -\mu\left(u+\frac{1}{2}, u\right) = -\mu\left(u, u+\frac{1}{2}\right),$$
$$\vartheta_{11}\left(u-\frac{1}{2}\right) = -\vartheta_{11}\left(u+\frac{1}{2}\right),$$

and

$$\vartheta_{11}\left(\frac{1}{2}\right) = -2q^{\frac{1}{8}}(q,-q,-q)_\infty,$$

we obtain the conclusion. ■

Remark 3.4. (1) Corollary 3.3 is a generalization of a classical evaluation [9]. In fact, Gauss proved the case of negative integers in our formula (3.5), which is the special value of continuous q-Hermite polynomial $H_n(x \mid q)$ at $x = 0$:

$$\mathrm{i}^{-n}H_n(0 \mid q) = \frac{\left(q^{n-1};q^{-2}\right)_\infty}{\left(q^{-1};q^{-2}\right)_\infty} = \begin{cases} 0 & \text{if } n = 2N-1, \\ (q;q^2)_N & \text{if } n = 2N. \end{cases} \tag{3.6}$$

This formula (3.6) is a key lemma to derive the product formula of the quadratic Gauss sum:

$$\sum_{k=1}^{2N} \mathrm{e}^{\frac{2\pi\mathrm{i}k^2}{2N+1}} = \prod_{j=1}^{N}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{2N+1}(2j-1)} - \mathrm{e}^{-\frac{2\pi\mathrm{i}}{2N+1}(2j-1)}\right). \tag{3.7}$$

In fact, by using this product formula (3.7), Gauss determined the sign of the quadratic Gauss sum, and gave a proof (the 4th proof) of the quadratic reciprocity law.

(2) The function $F(u) := \mu\left(u, u+\frac{1}{2}\right)$ also has the following expressions:

$$F(u) = -\frac{1}{2\pi\mathrm{i}}\frac{2K}{\vartheta_{11}\left(\frac{1}{2}\right)}\frac{1}{\mathrm{sn}\,(2Ku,k)\mathrm{sn}\,\left(2K\left(u+\frac{1}{2}\right),k\right)} \tag{3.8}$$
$$= -\frac{1}{2\pi\mathrm{i}}\frac{2K}{\vartheta_{11}\left(\frac{1}{2}\right)}\frac{\mathrm{dn}\,(2Ku,k)}{\mathrm{sn}\,(2Ku,k)\mathrm{cn}\,(2Ku,k)}. \tag{3.9}$$

Here, $k = k(\tau)$ is the elliptic modulus and $K = K(\tau)$ is the elliptic period:

$$K := \frac{\pi}{2}(q)_\infty^2\left(-q^{\frac{1}{2}}\right)_\infty^4 = \int_0^1 \frac{\mathrm{d}t}{\sqrt{\left(1-t^2\right)\left(1-k^2t^2\right)}},$$

and sn, cn and dn are Jacobian elliptic functions (see, for example, [3, Chapter 11]).

Although this fact is an exercise of the μ-function and elliptic functions, we have not been able to find any appropriate references about this result. Hence, we give a sketch of the proof of (3.8) and (3.9).

From the equations (1.2) and (1.6), the function $F(u)$ is odd:

$$F(-u) = \mu\left(-u, -u+\frac{1}{2}\right) = \mu\left(u, u-\frac{1}{2}\right) = -\mu\left(u, u+\frac{1}{2}\right) = -F(u).$$

The double periodicity of $F(u)$ follows from (1.2) and (1.4):

$$F(u+1) = \mu\left(u+1, u+1+\frac{1}{2}\right) = (-1)^2\mu\left(u, u+\frac{1}{2}\right) = F(u),$$
$$F(u+\tau) = \mu\left(u+\tau, u+\tau+\frac{1}{2}\right) = \mu\left(u, u+\frac{1}{2}\right) = F(u).$$

By the definition of Zwegers' μ-function, the function $F(u)$ has simple poles at each lattice point on $\mathbb{Z}+\mathbb{Z}\tau$ and $\mathbb{Z}+\frac{1}{2}+\mathbb{Z}\tau$ and no other singularities, and the residues of $F(u)$ at each pole are the following:

$$\operatorname*{Res}_{u=0} F(u) = \lim_{u\to 0} \frac{\mathrm{e}^{\pi\mathrm{i}u}}{\vartheta_{11}\left(u+\frac{1}{2}\right)}\frac{u}{1-\mathrm{e}^{2\pi\mathrm{i}u}} = -\frac{1}{2\pi\mathrm{i}}\frac{1}{\vartheta_{11}\left(\frac{1}{2}\right)},$$
$$\operatorname*{Res}_{u=-\frac{1}{2}} F(u) = \lim_{u\to -\frac{1}{2}} \frac{\mathrm{e}^{\pi\mathrm{i}\left(u+\frac{1}{2}\right)}}{\vartheta_{11}(u)}\frac{u}{1-\mathrm{e}^{2\pi\mathrm{i}\left(u+\frac{1}{2}\right)}} = \frac{1}{2\pi\mathrm{i}}\frac{1}{\vartheta_{11}\left(\frac{1}{2}\right)}.$$

On the other hand, by a simple calculation of elliptic functions, the right-hand side of (3.8) has exactly the same properties as $F(u)$. Since the function

$$G(u) := F(u) + \frac{1}{2\pi\mathrm{i}}\frac{2K}{\vartheta_{11}\left(\frac{1}{2}\right)}\frac{1}{\operatorname{sn}(2Ku,k)\operatorname{sn}\left(2K\left(u+\frac{1}{2}\right),k\right)}$$

is odd and a constant with respect to u, $G(u)$ is identically zero.

Finally, by (3.8) and the translation formula

$$\operatorname{sn}(z+K,k) = \frac{\operatorname{cn}(z,k)}{\operatorname{dn}(z,k)},$$

we obtain the equation (3.9).

Next, we prove Theorem 1.6 for the relation between our μ-functions $\mu(u,v;k)$ and the continuous q-Hermite polynomials.

Proof of Theorem 1.6. For $n = 0, 1, 2, \dots$, the continuous q-Hermite polynomials satisfy the following recursion equation (see [13, p. 541, equation (14.26.3)]):

$$2xH_n(x \mid q) = H_{n+1}(x \mid q) + \left(1-q^n\right)H_{n-1}(x \mid q), \tag{3.10}$$
$$H_0(x \mid q) = 1, \qquad H_1(x \mid q) = 2x. \tag{3.11}$$

Let us now set

$$\widehat{H}_k(x \mid q) := \mathrm{i}q^{\frac{1}{8}}\mu(u,v;-k), \qquad x = \cos\pi(u-v).$$

To prove $\widehat{H}_k(x \mid q) = H_k(x \mid q)$, it is enough to show that the $\widehat{H}_k(x \mid q)$ satisfy the recurrence relation (3.10) and the initial condition (3.11). The recurrence relation (3.10) is equal to the formula (1.32) exactly. For the initial condition (3.11), by the definition and the triple product identity, we have

$$\widehat{H}_0(x \mid q) = \mathrm{i}q^{\frac{1}{8}}\mu(u,v;0) = \frac{1}{\vartheta_{11}(v)}\sum_{n\in\mathbb{Z}} \mathrm{e}^{2\pi\mathrm{i}(n+\frac{1}{2})(v+\frac{1}{2})+\pi\mathrm{i}(n+\frac{1}{2})^2\tau} = 1. \tag{3.12}$$

The equation $\widehat{H}_1(x \mid q) = 2x$ follows from (3.12) and the case of $k=0$ in (1.32). ■

Remark 3.5. In the equation (3.3), let α be a negative integer $-n$. In this case, the equation (3.3) is

$$\begin{aligned} H_n(x \mid q) = \frac{\mathrm{i}q^{\frac{1}{8}}(q)_n}{\vartheta_{11}(u-v)}\Bigg\{&\mathrm{e}^{\pi\mathrm{i}(1+n)(u-v)}{}_1\phi_1\left(\begin{matrix} q^{1+n} \\ 0\end{matrix}; q, \mathrm{e}^{2\pi\mathrm{i}(u-v)}q\right) \\ &- \mathrm{e}^{-\pi\mathrm{i}(1+n)(u-v)}{}_1\phi_1\left(\begin{matrix} q^{1+n} \\ 0\end{matrix}; q, \mathrm{e}^{-2\pi\mathrm{i}(u-v)}q\right)\Bigg\}. \end{aligned} \tag{3.13}$$

This is a non-trivial result that the right-hand side of (3.13) is the sum of two infinite sums, but cancels out with the theta function to become a finite sum.

This equation is also regarded as the $a,b,c,d \to 0$ limit case of an Askey–Wilson polynomial's expression by the sum of two ${}_4\phi_3$ functions [19, (2.11)] (but this reference has small typos. The denominator terms of ${}_4\phi_3$ (${}_4\varphi_3$) $q^{1-\nu+z}$ and $q^{1-\nu-z}$ should be $q^{1-\nu+z}/d$ and $q^{1-\nu-z}/d$, respectively).

Next, we prove Theorem 1.7 for the generating function $S(r)$.

Proof of Theorem 1.7. (1) Since the equation (1.36) follows from (1.35) immediately, it is enough to show (1.35). We find by (1.32) that

$$\begin{aligned} S(r) - S\left(\frac{r}{q}\right) &= \sum_{k=0}^{\infty}\left(1-q^{-k}\right)\mu(u,v;k+1)r^k \\ &= \sum_{k=0}^{\infty}(2\cos\pi(u-v)\mu(u,v;k) - \mu(u,v;k-1))r^k \\ &= 2r\cos\pi(u-v)\sum_{k=-1}^{\infty}\mu(u,v;k+1)r^k - r^2\sum_{k=-2}^{\infty}\mu(u,v;k+1)r^k \\ &= \left(2\cos\pi(u-v) - r^2\right)S(r) - r\mu(u,v;0) \\ &\quad + 2\cos\pi(u-v)\mu(u,v;0) - \mu(u,v;-1). \end{aligned}$$

By the case of $k=0$ in recursion equation (1.32):

$$2\cos\pi(u-v)\mu(u,v;0) - \mu(u,v;-1) = 0$$

and $\mu(u,v;0) = -\mathrm{i}q^{-\frac{1}{8}}$, we have the conclusion.

(2) Let $N = 0,1,2,\dots$. Using the equation (1.32), we find that

$$S(r) = \left(1 - r\mathrm{e}^{\pi\mathrm{i}(u-v)}q\right)\left(1 - r\mathrm{e}^{-\pi\mathrm{i}(u-v)q}\right)S(rq) - \mathrm{i}rq^{\frac{7}{8}}$$

$$= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_N S\left(rq^N\right) - irq^{\frac{7}{8}} \sum_{n=0}^{N-1} q^n \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_n. \tag{3.14}$$

Taking the limit $N \to \infty$ in (3.14), we have

$$\begin{aligned} S(r) &= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_\infty \mu(u,v) - irq^{\frac{7}{8}} \sum_{n=0}^{\infty} q^n \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_n \\ &= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_\infty \mu(u,v) - irq^{\frac{7}{8}} {}_3\phi_2\left(\begin{matrix} q, re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q \\ 0, 0 \end{matrix}; q, q\right). \end{aligned} \tag{3.15}$$

The conclusion follows from (3.15) and Andrews' formula [8, p. 298, Exercise 10.8]:

$${}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, x\right) = \frac{(ax, b, c)_\infty}{(x, d, e)_\infty} \Phi^{(1)}\left(\begin{matrix} x; d/b, e/c \\ ax \end{matrix}; q; b, c\right). \tag{3.16}$$

Finally, we prove (1.38). From (1.37), we have

$$\begin{aligned} S(r) &= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_\infty \\ &\quad\times \left\{\mu(u,v) - \frac{irq^{\frac{7}{8}}}{1-q} \Phi^{(1)}\left(\begin{matrix} q; 0, 0 \\ q^2 \end{matrix}; q; re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)\right\} \\ &= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_\infty \\ &\quad\times \left\{\mu(u,v) - \frac{iq^{\frac{7}{8}}}{1-q} \sum_{n\geq 0} \sum_{k=0}^{n} \frac{(q)_n}{(q)_k (q)_{n-k} (q^2)_n} e^{\pi i(2k-n)(u-v)} q^n r^{n+1}\right\} \\ &= \left(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q\right)_\infty \left\{\mu(u,v) - \sum_{n\geq 0} \frac{iq^{\frac{7}{8}} H_n(\cos\pi(u-v) \mid q)}{(q)_{n+1}} q^n r^{n+1}\right\}. \end{aligned}$$

From Theorem 1.6, $H_n(x \mid q) := iq^{\frac{1}{8}} \mu(u,v;-n)$, so (1.38) holds. ■

By re-expanding the generating function $S(r)$, we obtain the following relationship between the function $\mu(u,v;k+1)$ and the continuous q-Hermite polynomials.

Corollary 3.6. *For non-negative integers k, we have*

$$\mu(u,v;k+1) = \sum_{l=0}^{k} \frac{q^l}{(q)_l} F_{k-l+1}(\cos\pi(u-v) \mid q) \mu(u,v;1-l), \tag{3.17}$$

where

$$\begin{aligned} F_{n+1}(\cos\pi w \mid q) &:= e^{-\pi i n w} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, e^{2\pi i w} q\right) \frac{(-1)^n q^{\frac{n(n+1)}{2}}}{(q)_n} \\ &= e^{-\frac{\pi i w}{2\tau}} \frac{(q)_\infty}{(q^{-n})_n} J^{(2)}_{\frac{w}{\tau}}\left(2ie^{-\pi i n\tau}; q\right). \end{aligned}$$

For positive integers m, we have

$$\sum_{k=0}^{m} \mu(u,v;k+1) \frac{H_{m-k}(\cos\pi(u-v) \mid q)}{(q)_{m-k}} q^{m-k} = -\frac{iq^{\frac{7}{8}} H_{m-1}(\cos\pi(u-v) \mid q)}{(q)_m}. \tag{3.18}$$

Proof. From the Taylor expansion of the q-exponential function [8, equation (II.2)]

$$(x)_\infty = \sum_{n\geq 0} \frac{1}{(q)_n}(-1)^n q^{\frac{n(n-1)}{2}} x^n,$$

we have

$$\begin{aligned}\left(e^{i\theta}rq, e^{-i\theta}rq\right)_\infty &= \sum_{n\geq 0}(-qr)^n \sum_{k=0}^{n} \frac{q^{\binom{k}{2}}q^{\binom{n-k}{2}}}{(q)_k(q)_{n-k}} e^{i(n-2k)\theta} \\ &= \sum_{n\geq 0} {}_1\phi_1\left(\begin{matrix} q^{-n} \\ 0 \end{matrix}; q, e^{2i\theta}q\right) \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q)_n}\left(e^{-i\theta}rq\right)^n \\ &= \sum_{n\geq 0} F_{n+1}(\cos\pi(u-v) \mid q) r^n.\end{aligned}$$

By expanding (1.38) and comparing the coefficient in (1.38), we obtain (3.17). To prove (3.18), we divide by both sides of (1.38) by $\left(e^{i\theta}rq, e^{-i\theta}rq\right)_\infty$;

$$\frac{1}{(re^{\pi i(u-v)}q, re^{-\pi i(u-v)}q)_\infty} S(r) = \sum_{m\geq 0} \frac{\mu(u,v;1-m)}{(q)_m} q^m r^m. \tag{3.19}$$

By the generating function of continuous q-Hermite polynomial (1.34), the left-hand side of (3.19) is equal to

$$\sum_{m\geq 0}\sum_{k=0}^{m} \mu(u,v;k+1) \frac{H_{m-k}(\cos\pi(u-v) \mid q)}{(q)_{m-k}} q^{m-k} r^m.$$

Then we obtain the conclusion (3.18). ∎

4 Modular transformations related to $\mu(u, v; k, \tau)$

In this section, again let k be a positive integer. We state that $\mu(u,v;k,\tau)$ is essentially equivalent to the original μ-function with respect to the properties of a real-analytic Jacobi form.

Proposition 4.1. *We define a modified* $\mu(u,v;k,\tau)$ *by*

$$\tilde{\nu}(u,v;k,\tau) := \frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v) \mid q)} + \frac{1}{2i} R_{k+1}(u-v;\tau),$$

where

$$R_{k+1}(u;\tau) := R(u;\tau) - 2q^{-\frac{1}{8}} \sum_{l=1}^{k} \frac{q^l}{(q)_l} \frac{F_{k-l+1}(\cos\pi u \mid q)}{F_{k+1}(\cos\pi u \mid q)} H_{l-1}(\cos\pi u \mid q).$$

We have

$$\tilde{\mu}(u,v;\tau) = \tilde{\nu}(u,v;k,\tau).$$

Therefore, the following transformations hold;

$$\tilde{\nu}(u,v;k,\tau+1) = e^{-\frac{\pi i}{4}} \tilde{\nu}(u,v;k,\tau), \tag{4.1}$$

$$\tilde{\nu}\left(\frac{u}{\tau}, \frac{v}{\tau}; k, -\frac{1}{\tau}\right) = -i\sqrt{-i\tau}\, e^{\pi i \frac{(u-v)^2}{\tau}} \tilde{\nu}(u,v;k,\tau). \tag{4.2}$$

Proof. From equation (3.17) of Corollary 3.6,

$$\mu(u,v;\tau)-\frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v)\mid q)}=\mathrm{i}q^{-\frac{1}{8}}\sum_{l=1}^{k}\frac{q^l}{(q)_l}\frac{F_{k-l+1}(\cos\pi(u-v)\mid q)}{F_{k+1}(\cos\pi(u-v)\mid q)}H_{l-1}(\cos\pi(u-v)\mid q).$$

Therefore,

$$\begin{aligned}\tilde{\mu}(u,v;\tau)&=\mu(u,v;\tau)+\frac{1}{2\mathrm{i}}R(u-v;\tau)\\&=\frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v)\mid q)}+\frac{1}{2\mathrm{i}}R(u-v;\tau)\\&\quad+\mathrm{i}q^{-\frac{1}{8}}\sum_{l=1}^{k}\frac{q^l}{(q)_l}\frac{F_{k-l+1}(\cos\pi(u-v)\mid q)}{F_{k+1}(\cos\pi(u-v)\mid q)}H_{l-1}(\cos\pi(u-v)\mid q)\\&=\frac{\mu(u,v;k+1,\tau)}{F_{k+1}(\cos\pi(u-v)\mid q)}+\frac{1}{2\mathrm{i}}R_{k+1}(u-v;\tau).\end{aligned}$$ ∎

Remark 4.2. T. Matsusaka also mentions this proposition [15].

5 Remarks for further studies

In this paper, from the point of view of the analysis of one-variable linear q-difference equations of the Laplace type, we have studied a generalization of Zwegers' μ-function $\mu(x,y;a)$ which satisfies the q-Hermite–Weber equation (1.19).

On the other hand, the generalized μ-function $\mu(x,y;a)$ is a two-variable function originally. More precisely, $\mu(x,y;a)$ is closely related to the q-Appell hypergeometric function $\Phi^{(1)}$ (1.39) and its q-difference system which we call q-Appell difference system:

$$\begin{aligned}&[(1-T_x)(1-c/qT_xT_y)-x(1-aT_xT_y)(1-b_1T_x)]\Phi(x,y)=0,\\&[(1-T_y)(1-c/qT_xT_y)-y(1-aT_xT_y)(1-b_2T_y)]\Phi(x,y)=0,\\&[x(1-T_y)(1-b_1T_x)-y(1-T_x)(1-b_2T_y)]\Phi(x,y)=0.\end{aligned}\tag{5.1}$$

The first and second equations are essentially equivalent to Gasper–Rahman [8, Exercises 10.12(i) and (ii)] which have some small typos (the terms $(c/q-a)f(qx,qy)$ in (i) and (ii) should be $(c/q-ax)f(qx,qy)$ and $(c/q-ay)f(qx,qy)$). The third equation (5.1) is a reducible factor of the following q-difference equation:

$$\begin{aligned}&(1-T_x)[(1-T_y)(1-c/qT_xT_y)-y(1-aT_xT_y)(1-b_2T_y)]\Phi(x,y)\\&\qquad-(1-T_y)[(1-T_x)(1-c/qT_xT_y)-x(1-aT_xT_y)(1-b_1T_x)]\Phi(x,y)\\&\quad=(1-aT_xT_y)[x(1-T_y)(1-b_1T_x)-y(1-T_x)(1-b_2T_y)]\Phi(x,y)=0\end{aligned}$$

and we easily show that $\Phi^{(1)}$ satisfies (5.1).

First, $\Phi^{(1)}$ appears in the expression of $\mu(x,y;a)$. In Theorem 1.4, by dividing the bilateral sum of q-hypergeometric series ${}_2\psi_2$ and ${}_0\psi_2$ into two parts, positive and negative;

$$\begin{aligned}{}_2\psi_2\left(\begin{matrix}x/a,y/a\\0,0\end{matrix};q,a\right)&=\frac{xy}{a}\left(1-\frac{a}{x}\right)\left(1-\frac{a}{y}\right){}_3\phi_2\left(\begin{matrix}xq/a,yq/a,q\\0,0\end{matrix};q,a\right)\\&\quad+{}_2\phi_2\left(\begin{matrix}0,q\\aq/x,aq/y\end{matrix};q,-\frac{aq^2}{xy}\right),\end{aligned}$$

$$
{}_0\psi_2\left(\begin{matrix}-\\x,y\end{matrix};q,\frac{xy}{a}\right)=\frac{1}{(1-x)(1-y)}{}_1\phi_2\left(\begin{matrix}q\\xq,yq\end{matrix};q,\frac{xyq^2}{a}\right)+{}_3\phi_2\left(\begin{matrix}q/x,q/y,q\\0,0\end{matrix};q,a\right),
$$

we rewrite Theorem 1.4 as follows:

$$
\begin{aligned}
\mathrm{i}q^{\frac{1}{8}}\mu(x,y;a)&=\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}\left\{\frac{xy}{a}\frac{(a,q,a/x,a/y)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_3\phi_2\left(\begin{matrix}xq/a,yq/a,q\\0,0\end{matrix};q,a\right)\right.\\
&\qquad\left.+\frac{(a,q,aq/x,aq/y)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_2\phi_2\left(\begin{matrix}0,q\\aq/x,aq/y\end{matrix};q,-\frac{aq^2}{xy}\right)\right\}\\
&=\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}\left\{\frac{(a,q,xq,yq)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_1\phi_2\left(\begin{matrix}q\\xq,yq\end{matrix};q,\frac{xyq^2}{a}\right)\right.\\
&\qquad\left.+\frac{(a,q,x,y)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_3\phi_2\left(\begin{matrix}q/x,q/y,q\\0,0\end{matrix};q,a\right)\right\}.
\end{aligned}
$$

By Andrews' formula (3.16), we obtain

$$
\begin{aligned}
\mathrm{i}q^{\frac{1}{8}}\mu(x,y;a)&=\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}\left\{-\frac{(aq)_\infty\theta_q(-yq/a)}{(q)_\infty\theta_q(-yq)}\Phi^{(1)}\left(\begin{matrix}a;0,0\\aq\end{matrix};q;\frac{xq}{a},\frac{yq}{a}\right)\right.\\
&\qquad\left.+\frac{(a,q,aq/x,aq/y;q)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_2\phi_2\left(\begin{matrix}0,q\\aq/x,aq/y\end{matrix};q,-\frac{aq^2}{xy}\right)\right\}\\
&=\mathrm{e}^{\pi\mathrm{i}\alpha(u-v)}\left\{\frac{(a,q,xq,yq)_\infty}{\theta_q(-y)\theta_q(-x/a)}{}_1\phi_2\left(\begin{matrix}q\\xq,yq\end{matrix};q,\frac{xyq^2}{a}\right)\right.\\
&\qquad\left.+\frac{(aq)_\infty\theta_q(-x)}{(q)_\infty\theta_q(-x/a)}\Phi^{(1)}\left(\begin{matrix}a;0,0\\aq\end{matrix};q;\frac{q}{x},\frac{q}{y}\right)\right\}.
\end{aligned}
$$

Namely, $\mu(x,y;a)$ is regarded as a bilateral version of q-Appell hypergeometric functions

$$
\Phi^{(1)}\left(\begin{matrix}a;0,0\\aq\end{matrix};q;\frac{xq}{a},\frac{yq}{a}\right)\qquad\text{or}\qquad\Phi^{(1)}\left(\begin{matrix}a;0,0\\aq\end{matrix};q;\frac{q}{x},\frac{q}{y}\right).
$$

Further, $\mu(x,y;a)$ essentially satisfies the q-Appell difference system in the case of $b_1=b_2=0$, $c=aq$:

$$
\begin{aligned}
&[(1-x-T_x)(1-aT_xT_y)]\Phi(x,y)=0,\\
&[(1-y-T_y)(1-aT_xT_y)]\Phi(x,y)=0,\\
&[x(1-T_y)-y(1-T_x)]\Phi(x,y)=0.
\end{aligned}\tag{5.2}
$$

Theorem 5.1. *The function*

$$
\nu(x,y;a):=\mathrm{e}^{-\pi\mathrm{i}\alpha(u-v)}\frac{\theta_q(-ay)}{\theta_q(-y)}\mu(ax,ay;a)
$$

satisfies the multivariate q-difference equation (5.2). More precisely, we have

$$
(1-aT_xT_y)\nu(x,y;a)=0,\tag{5.3}
$$

$$
[x(1-T_y)-y(1-T_x)]\nu(x,y;a)=0.\tag{5.4}
$$

Proof. To prove the first and second equations of (5.2), it is enough to show (5.3). By simple calculation, we have

$$
aT_xT_y\nu(x,y;a)=a\frac{\theta_q(-ayq)}{\theta_q(-yq)}\mu(axq,ayq;a)=a\frac{-a^{-1}y^{-1}}{-y^{-1}}\frac{\theta_q(-ay)}{\theta_q(-y)}\mu(x,y;a)=\nu(x,y;a).
$$

Here, the second equality follows from τ-periodicity (1.28).

Another equation (5.4) is proved as follows:

$$\begin{aligned}(xT_y - yT_x)\nu(x,y;a) &= x\mathrm{e}^{-\pi\mathrm{i}\alpha(u-v-\tau)}\frac{\theta_q(-ayq)}{\theta_q(-yq)}\mu(ax,ayq;a)\\ &\quad - y\mathrm{e}^{-\pi\mathrm{i}\alpha(u-v+\tau)}\frac{\theta_q(-ay)}{\theta_q(-y)}\mu(axq,ay;a)\\ &= \mathrm{e}^{-\pi\mathrm{i}\alpha(u-v)}\frac{\theta_q(-ay)}{\theta_q(-y)}\big\{(-y\mu(ax,ay;a)+\sqrt{xy}\mu(ax,ay;a/q))\\ &\quad - (-x\mu(ax,ay;a)+\sqrt{xy}\mu(ax,ay;a/q))\big\}\\ &= (x-y)\nu(x,y;a).\end{aligned}$$

The first and second equalities in the above follow from the forward shift (1.25) and (3.1), respectively. ■

In particular, Zwegers' μ-function $\nu(x,y;q) = -\mathrm{e}^{-\pi\mathrm{i}(u+v)}\mu(x,y)$ also is a solution of the case of $a = q$ in the two-variate q-difference system (5.2):

$$\begin{aligned}&[1 - qT_xT_y]\nu(x,y;q) = 0,\\ &[x(1-T_y) - y(1-T_x)]\nu(x,y;q) = 0.\end{aligned} \tag{5.5}$$

Based on this fact, it would be desirable to study other generalizations and their global analysis of the μ-function $\mu(u,v)$ from the view of analysis of the q-Appell difference system (5.5).

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