

Tau Function and Moduli of Meromorphic Quadratic Differentials

Dmitry KOROTKIN^{ab} and Peter ZOGRAF^{bc}

a) *Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve West, Montreal, H3G 1M8 Quebec, Canada*
E-mail: dmitry.korotkin@concordia.ca

b) *Euler International Mathematical Institute, Pesochnaja nab. 10, Saint Petersburg, 197022 Russia*
E-mail: zograf@pdmi.ras.ru

c) *Chebyshev Laboratory, St. Petersburg State University, 14th Line V.O. 29, Saint Petersburg, 199178 Russia*

Received August 09, 2021, in final form December 28, 2021; Published online January 03, 2022

<https://doi.org/10.3842/SIGMA.2022.001>

Abstract. The Bergman tau functions are applied to the study of the Picard group of moduli spaces of quadratic differentials with at most n simple poles on genus g complex algebraic curves. This generalizes our previous results on moduli spaces of holomorphic quadratic differentials.

Key words: quadratic differentials; tau function; moduli spaces

2020 Mathematics Subject Classification: 14H15; 14H70; 14K20; 30F30

To Leon Takhtajan on occasion of his 70th birthday

1 Introduction and statement of results

The theory of the Bergman tau function was applied to the study of geometry of various moduli spaces in [2, 6, 8, 9, 10]; see [7] for a review of known results.

The present paper is a continuation of [10]. Here we use Bergman tau functions to study the geometry of the moduli space $\mathcal{Q}_{g,n}$ of quadratic differentials with n simple poles on genus g complex algebraic curves. More precisely, the space $\mathcal{Q}_{g,n}$ is defined as the set of isomorphism classes of pairs (C, q) , where C is a smooth genus g complex curve with n labeled distinct marked points, and q is a meromorphic quadratic differential on C with at most simple poles at the marked points and no other poles (throughout the paper we will assume that $2g + n > 3$). It is well known that $\mathcal{Q}_{g,n}$ is naturally isomorphic to $T^*\mathcal{M}_{g,n}$, the total space of the holomorphic cotangent bundle on the moduli space $\mathcal{M}_{g,n}$ of pointed complex algebraic curves.

The bundle $T^*\mathcal{M}_{g,n}$ can be extended to the Deligne–Mumford boundary of $\mathcal{M}_{g,n}$ in two natural ways: first, as the cotangent bundle $T^*\overline{\mathcal{M}}_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$, the Deligne–Mumford moduli space of stable curves, and second, as the total space of the direct image $\pi_*\omega_{g,n}^2$, where $\omega_{g,n}$ is the relative dualizing sheaf on the universal curve $\pi: \overline{\mathcal{C}}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$. These two extensions are rather close to each other: namely, $\det T^*\overline{\mathcal{M}}_{g,n} = \det \pi_*\omega_{g,n}^2 - \delta_{\text{DM}}$, where $\delta_{\text{DM}} = \overline{\mathcal{M}}_{g,n} \setminus \mathcal{M}_{g,n}$

This paper is a contribution to the Special Issue on Mathematics of Integrable Systems: Classical and Quantum in honor of Leon Takhtajan.

The full collection is available at <https://www.emis.de/journals/SIGMA/Takhtajan.html>

is the Deligne–Mumford boundary class (a detailed analytic treatment of these subjects can be found in [12]). In this paper we will use the second extension and put $\overline{\mathcal{Q}}_{g,n} = \pi_*\omega_{g,n}^2$.

The moduli space $\overline{\mathcal{Q}}_{g,n}$ has an open dense subset $\mathcal{Q}(1^{4g-4+n}, -1^n)$ that consists of isomorphism classes of pairs (C, q) , where C is a smooth curve, and q has exactly $4g - 4 + n$ simple zeros and n simple poles. We call $\mathcal{Q}(1^{4g-4+n}, -1^n)$ the *principal stratum*.

Denote by $P\overline{\mathcal{Q}}_{g,n} = \overline{\mathcal{Q}}_{g,n}/\mathbb{C}^*$ the projectivization of $\overline{\mathcal{Q}}_{g,n}$, where \mathbb{C}^* acts on quadratic differentials by multiplication. The complement $P\overline{\mathcal{Q}}_{g,n} \setminus P\mathcal{Q}(1^{4g-4+n}, -1^n)$ is a union of divisors that we denote by D_{deg}^0 , D_{deg}^∞ , and D_{DM} (the subscript deg stands for *degenerate*, as opposed to differentials in $\mathcal{Q}(1^{4g-4+n}, -1^n)$ that we call *generic*). The divisor D_{deg}^0 is the closure of the set of differentials q with multiple zeros. The divisor D_{deg}^∞ is the closure of n strata $P\mathcal{Q}(1^{4g-5+n}, 0, -1^{n-1})$ (one for each pole) parameterizing differentials with $4g - 5 + n$ simple zeros, $n - 1$ simple poles and one marked ordinary point (where a zero and a pole of a generic differential coalesce). Finally, D_{DM} is the pullback to $P\overline{\mathcal{Q}}_{g,n}$ of the Deligne–Mumford boundary of $\overline{\mathcal{M}}_{g,n}$. The latter consists of the divisor D_{irr} of irreducible nodal curves with n marked points, and divisors $D_{j,k}$ parameterizing reducible curves with components of genera j and $g - j$ having k and $n - k$ marked points respectively, where $j = 0, \dots, [g/2]$, $k = 0, \dots, n$ and $2 < 2j + k < 2g + n - 2$. (Note that while a quadratic differential may have only simple poles at the n marked points, it may have poles up to order n at the nodes.)

Denote by $L \rightarrow P\overline{\mathcal{Q}}_{g,n}$ the tautological line bundle associated with the projectivization $\overline{\mathcal{Q}}_{g,n} \rightarrow P\overline{\mathcal{Q}}_{g,n}$, and put $\phi = c_1(L) \in \text{Pic}(P\overline{\mathcal{Q}}_{g,n}) \otimes \mathbb{Q}$. Denote by λ the pullback of the Hodge class $\lambda_1 = \det \pi_*\omega_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$. Furthermore, denote by δ_{deg}^0 , $\delta_{\text{deg}}^\infty$, δ_{irr} , $\delta_{j,k}$ the classes in $\text{Pic}(P\overline{\mathcal{Q}}_{g,n}) \otimes \mathbb{Q}$ of the corresponding divisors. We will also use the standard notation ψ_i , $i = 1, \dots, n$, for the tautological classes on $\overline{\mathcal{M}}_{g,n}$ as well as for their pullbacks to $P\overline{\mathcal{Q}}_{g,n}$.

The following fact is standard (cf., e.g., [4]):

Lemma 1.1. *The rational Picard group $\text{Pic}(P\overline{\mathcal{Q}}_{g,n}) \otimes \mathbb{Q}$ is freely generated over \mathbb{Q} by the classes ϕ , λ , ψ_i , δ_{irr} , $\delta_{j,k}$, where $i, k = 1, \dots, n$, $j = 0, \dots, [g/2]$, and $2 < 2j + k < 2g + n - 2$.*

To each pair (C, q) one can canonically associate a twofold branched cover $f: \widehat{C} \rightarrow C$ and an abelian differential v on \widehat{C} , where $\widehat{C} = \{(x, v(x)) \mid x \in C, v(x) \in T_x^*C, v(x)^2 = q(x)\}$. For a generic $(C, q) \in \mathcal{Q}_{g,n}$ the curve \widehat{C} is smooth of genus $\widehat{g} = 4g - 3 + n$ and v is holomorphic.

The covering f is invariant under the canonical involution $(x, v(x)) \mapsto (x, -v(x))$ on \widehat{C} that we denote by μ . Zeros and poles of q of odd order are branch points of the covering f . The abelian differential v has second order zeros at simple zeros of q , and at the simple poles of q the differential v is holomorphic and nonvanishing.

Consider the map $\widehat{p}: P\overline{\mathcal{Q}}_{g,n} \rightarrow \overline{\mathcal{M}}_{\widehat{g}}$, $(C, q) \mapsto \widehat{C}$. This map induces a vector bundle $\widehat{p}^*\Lambda_{\widehat{g}}^1 \rightarrow \overline{\mathcal{Q}}_{g,n}$ of dimension $\widehat{g} = 4g - 3 + n$, where $\Lambda_{\widehat{g}}^1 \rightarrow \overline{\mathcal{M}}_{\widehat{g}}$ is the Hodge vector bundle. The involution μ on \widehat{C} induces an involution μ^* on the vector bundle $\widehat{p}^*\Lambda_{\widehat{g}}^1$. Hence we have a decomposition

$$\widehat{p}^*\Lambda_{\widehat{g}}^1 = \Lambda_+ \oplus \Lambda_-,$$

where Λ_+ (resp. Λ_-) is the eigenbundle corresponding to the eigenvalue $+1$ (resp. -1) of μ^* . Clearly, $\Lambda_+ = p^*\pi_*\omega_g$ is the pullback of the Hodge bundle on $\overline{\mathcal{M}}_g$, where $p: P\overline{\mathcal{Q}}_{g,n} \rightarrow \overline{\mathcal{M}}_g$ is a natural projection (forgetful map). We call Λ_- the *Prym bundle*. Its fibers are the spaces of Prym differentials on \widehat{C} and have dimension $3g - 3 + n$. We call $\lambda_{\text{P}} = c_1(\Lambda_-) \in \text{Pic}(P\overline{\mathcal{Q}}_{g,n}) \otimes \mathbb{Q}$ the *Prym class* (slightly abusing the notation, we often denote the line bundles and their classes in the Picard group by the same symbols).

In this paper we prove the following generalization of Theorem 1 in [10] to the case $n > 0$:

Theorem 1.2. *The Hodge class λ and the Prym class λ_P can be expressed in terms of the tautological class ϕ and the classes δ_{deg}^0 , $\delta_{\text{deg}}^\infty$ and δ_{DM} by the formulas*

$$\begin{aligned}\lambda &= \left(\frac{5(g-1)}{36} - \frac{n}{36} \right) \phi + \frac{1}{72} \delta_{\text{deg}}^0 - \frac{1}{18} \delta_{\text{deg}}^\infty + \frac{1}{12} \delta_{\text{DM}}, \\ \lambda_P &= \left(\frac{11(g-1)}{36} + \frac{5n}{36} \right) \phi + \frac{13}{72} \delta_{\text{deg}}^0 + \frac{5}{18} \delta_{\text{deg}}^\infty + \frac{1}{12} \delta_{\text{DM}}.\end{aligned}$$

On the other hand, according to [10, formula (5.16) and explanations thereafter],

$$\lambda_P = \lambda_2 - \frac{1}{2}(3g - 3 + n)\phi, \quad (1.1)$$

where $\lambda_2 = \det \pi_* \omega_{g,n}^2$. This implies

Corollary 1.3. *The following relation holds in $\text{Pic}(P\overline{\mathcal{Q}}_{g,n}) \otimes \mathbb{Q}$:*

$$\lambda_2 - 13\lambda = n\phi + \delta_{\text{deg}}^\infty - \delta_{\text{DM}}. \quad (1.2)$$

Furthermore, combining (1.2) with Mumford's formula

$$\lambda_2 - 13\lambda = \sum_{i=1}^n \psi_i - \delta_{\text{DM}} \quad (1.3)$$

(cf., e.g., [1]), formula (7.8)), we get the following

Corollary 1.4. *The class $\delta_{\text{deg}}^\infty$ is expressed via the sum of ψ -classes and class ϕ as follows:*

$$\delta_{\text{deg}}^\infty = -n\phi + \sum_{i=1}^n \psi_i. \quad (1.4)$$

Remark 1.5. An analytic approach to the classes ψ_i was outlined, in particular, in [11].

The paper is organized as follows. In Section 2 we introduce a twofold canonical cover corresponding to a pair consisting of a complex algebraic curve and a quadratic differential on it with n simple poles. The main objective of this section is to discuss the action of the covering involution on (co)homology of the cover and the associated matrix of b -periods. In Section 3 we define two tau functions corresponding to the eigenvalues ± 1 of the covering map, discuss their basic properties and interpret them as holomorphic sections of line bundles on the moduli space of quadratic differentials with simple poles. In Section 4 we study the asymptotic behavior of the tau functions near the divisor D_{deg}^∞ and use it to express the Hodge and Prym classes via the classes of the boundary divisors and the tautological class (the asymptotics of the tau functions near the divisors D_{deg}^0 and $D_{j,k}$ were thoroughly studied in [10]).

2 Geometry of the double cover

Let $f: \widehat{C} \rightarrow C$ be the double cover defined by the meromorphic quadratic differential q with simple zeros $\{x_1, \dots, x_{4g-4+n}\}$ and simple poles $\{y_1, \dots, y_n\}$ on a smooth curve C , and let $\mu: \widehat{C} \rightarrow \widehat{C}$ be the corresponding involution. The covering map f is ramified over x_j, y_k and we put $\hat{x}_j = f^{-1}(x_j)$, $\hat{y}_k = f^{-1}(y_k)$; the points \hat{x}_j, \hat{y}_k are exactly the fixed points of μ . By μ_* (resp. μ^*) we denote the involution induced by μ in homology (resp. in cohomology) of \widehat{C} . The space $\Lambda_{\widehat{C}}^1$ of holomorphic abelian differentials on \widehat{C} splits into two eigenspaces Λ_+ and Λ_- of

complex dimension g and $3g - 3 + n$ respectively that correspond to the eigenvalues ± 1 of μ^* . We have a similar decomposition in the real homology of \widehat{C} : $H_1(\widehat{C}, \mathbb{R}) = H_+ \oplus H_-$, where $\dim H_+ = 2g$, $\dim H_- = 6g - 6 + 2n$. Following [3], we pick $8g - 6 + 2n$ smooth 1-cycles on \widehat{C}

$$\{a_j, a_j^*, \tilde{a}_k, b_j, b_j^*, \tilde{b}_k\}, \quad j = 1, \dots, g, \quad k = 1, \dots, 2g + n - 3, \quad (2.1)$$

in such a way that

$$\mu_* a_j = a_j^*, \quad \mu_* b_j = b_j^*, \quad \mu_* \tilde{a}_k + \tilde{a}_k = \mu_* \tilde{b}_k + \tilde{b}_k = 0,$$

and the intersection matrix is

$$\begin{pmatrix} 0 & I_{4g-3+n} \\ -I_{4g-3+n} & 0 \end{pmatrix}$$

(here I_k denotes the $k \times k$ identity matrix).

The projections $f_* a_j = f_*(a_j^*)$ and $f_* b_j = f_*(b_j^*)$ on C form a canonical basis in $H_1(C)$, while the projections $f_* \tilde{a}_k$ and $f_* \tilde{b}_k$ are trivial in $H_1(C)$.

Remark 2.1. To avoid complicated notation, we will use the same symbols for the cycles (2.1), their homology classes in $H_1(\widehat{C})$, and their pushforwards in $H_1(C)$. In particular, the images of (2.1) give rise to a canonical basis in $H_1(C)$.

Denote by $\{u_j, u_j^*, \tilde{u}_k\}$ the basis of normalized abelian differentials on \widehat{C} associated with (2.1), so that the action of μ^* on $\Lambda_{\widehat{C}}^1$ is given by the matrix

$$M = \begin{pmatrix} 0 & I_g & 0 \\ I_g & 0 & 0 \\ 0 & 0 & -I_{2g-3+n} \end{pmatrix}.$$

The differentials $u_j^+ = u_j + u_j^*$, $j = 1, \dots, g$, provide a basis in the space Λ_+ , whereas a basis in Λ_- is given by $3g - 3 + n$ Prym differentials u_l^- , where

$$u_l^- = \begin{cases} u_l - u_l^*, & l = 1, \dots, g, \\ \tilde{u}_{l-g}, & l = g + 1, \dots, 3g - 3 + n. \end{cases}$$

We also introduce the bases in the spaces H_+ and H_- . The classes

$$\alpha_j^+ = \frac{1}{2}(a_j + a_j^*), \quad \beta_j^+ = b_j + b_j^*, \quad j = 1, \dots, g,$$

form a symplectic basis in H_+ , whereas the classes

$$\begin{aligned} \alpha_l^- &= \frac{1}{2}(a_l - a_l^*), & \beta_l^- &= b_l - b_l^*, & l &= 1, \dots, g, \\ \alpha_l^- &= \tilde{a}_{l-g}, & \beta_l^- &= \tilde{b}_{l-g}, & l &= g + 1, \dots, 3g - 3 + n \end{aligned}$$

form a symplectic basis in H_- . The basis $\{\alpha_j^+, \alpha_l^-, \beta_j^+, \beta_l^-\}$, $j = 1, \dots, g$, $l = 1, \dots, 3g - 3 + n$, is related to the canonical basis (2.1) by means of a (non-integer) symplectic matrix

$$S = \begin{pmatrix} T & 0 \\ 0 & (T^t)^{-1} \end{pmatrix}$$

with

$$T = \begin{pmatrix} I_g & I_g & 0 \\ I_g & -I_g & 0 \\ 0 & 0 & I_{2g-3+n} \end{pmatrix}. \quad (2.2)$$

The differentials u_j^+ , u_l^- are normalized relative to the classes α_j^+ , α_l^- respectively in the sense that

$$\int_{\alpha_i^+} u_j^+ = \delta_{ij}, \quad \int_{\alpha_k^-} u_l^- = \delta_{kl},$$

where $i, j = 1, \dots, g$ and $k, l = 1, \dots, 3g - 3 + n$. The corresponding matrices of β -periods Ω_+ and Ω_- are given by

$$\begin{aligned} (\Omega_+)_{ij} &= \int_{\beta_i^+} u_j^+, & i, j &= 1, \dots, g, \\ (\Omega_-)_{kl} &= \int_{\beta_k^-} u_l^-, & k, l &= 1, \dots, 3g - 3 + n. \end{aligned}$$

The matrix $\widehat{\Omega}$ of b -periods of $\{u_j, \tilde{u}_k, u_j^*\}$ with respect to the homology basis (2.1) on \widehat{C} is related to Ω^+ and Ω^- by the formula

$$\widehat{\Omega} = T^{-1} \begin{pmatrix} \Omega_+ & 0 \\ 0 & \Omega_- \end{pmatrix} (T^t)^{-1}.$$

We proceed with bidifferentials and projective connections on the double covers. Let $\widehat{B}(x, y)$ denote the canonical (Bergman) bidifferential on $\widehat{C} \times \widehat{C}$ associated with the homology basis (2.1); $\widehat{B}(x, y)$ is symmetric, has the second order pole on the diagonal $x = y$ with biresidue 1, and all a -periods of $\widehat{B}(x, y)$ on \widehat{C} vanish. We put

$$B_+(x, y) = \widehat{B}(x, y) + \mu_y^* \widehat{B}(x, y), \quad B_-(x, y) = \widehat{B}(x, y) - \mu_y^* \widehat{B}(x, y),$$

(the subscript y at μ^* means that we take the pullback with respect to the involution on the second factor in $\widehat{C} \times \widehat{C}$). The bidifferential $B_+(x, y)$ is the pullback of the canonical bidifferential $B(x, y)$ on $C \times C$ (normalized relative to the classes $f_* a_j$, where $f: \widehat{C} \rightarrow C$ is the covering map). The bidifferential $B_-(x, y)$ is called *Prym bidifferential* in [10].

Properties of the differentials B_{\pm} are summarized in [10, Lemma 2]. Near the diagonal $x = y$ in $\widehat{C} \times \widehat{C}$ we have

$$B_{\pm}(x, y) = \frac{d\zeta(x)d\zeta(y)}{(\zeta(x) - \zeta(y))^2} + \frac{1}{6} S_{B_{\pm}}(\zeta(x)) + \dots,$$

where $\zeta(x)$ is a local parameter at $x \in \widehat{C}$, and two projective connections S_{B_+} and S_{B_-} that are related by

$$S_{B_{\pm}}(x) = S_{\widehat{B}}(x) \pm 6\mu_y^* \widehat{B}(x, y)|_{y=x}. \quad (2.3)$$

Here S_{B_-} is the *Prym projective connection*.

Now we describe the dependence of bidifferentials and projective connections on the choice of homology basis. Let σ be a symplectic (i.e., preserving the intersection form) transformation of $H_1(\widehat{C}, \mathbb{Z})$. In the canonical basis (2.1), σ is represented by an $\mathrm{Sp}(8g - 6 + 2n, \mathbb{Z})$ -matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

with square blocks of size $4g - 3 + n$. The canonical bidifferential $\widehat{B}^\sigma(x, y)$ on $\widehat{C} \times \widehat{C}$ associated with the new basis is

$$\widehat{B}^\sigma(x, y) = \widehat{B}(x, y) - 2\pi\sqrt{-1}\mathbf{u}(x)^t(C\widehat{\Omega} + D)^{-1}C\mathbf{u}(y),$$

where $\mathbf{u} = \{u_j, \tilde{u}_k, u_j^*\}^t$ is the vector of holomorphic abelian differentials on \widehat{C} normalized with respect to $\{a_j, \tilde{a}_k, a_j^*\}$, $j = 1, \dots, g$, $k = 1, \dots, 2g - 3 + n$.

If σ commutes with the involution μ_* , then $\sigma = \text{diag}(\sigma_+, \sigma_-)$, where σ_+ (resp. σ_-) is a symplectic transformation of H_+ (resp. H_-) with half-integer coefficients (note that the intersection form on $H_1(\widehat{C}, \mathbb{R})$ is invariant with respect to μ_*). In the bases $\{\alpha_j^+, \beta_j^+\}$, $j = 1, \dots, g$, and $\{\alpha_l^-, \beta_l^-\}$, $l = 1, \dots, 3g - 3 + n$, these transformations can be written as

$$\sigma_\pm := \begin{pmatrix} A_\pm & B_\pm \\ C_\pm & D_\pm \end{pmatrix}. \quad (2.4)$$

Transformation properties of basic holomorphic differentials, Prym matrix, differentials B_\pm and projective connections S_{B_\pm} are described in [10, Lemma 3].

3 Tau functions

Here we define the necessary tau functions and study their basic properties. Only slight modifications are needed comparing to the case $n = 0$ [10].

3.1 Definition of τ_\pm

Consider two meromorphic quadratic differentials $S_{B_\pm} - S_v$ on \widehat{C} , where S_{B_\pm} are the Bergman projective connections (2.3) and S_v is given by

$$S_v = \left(\frac{v'}{v}\right)' - \frac{1}{2} \left(\frac{v'}{v}\right)^2.$$

where the prime means the derivative with respect to a local parameter $\zeta(x)$ (in other words, S_v is the Schwarzian derivative of the abelian integral $\int_{x_0}^x v$). Take the trivial line bundle on the principal stratum $\mathcal{Q}(1^{4g-4+n}, -1^n)$ and consider two connections

$$d_\pm = d + \xi_\pm$$

with

$$\xi_\pm = - \sum_{i=1}^{6g-6+2n} \left(\int_{s_i^*} \varphi_\pm \right) d \int_{s_i} v,$$

where $\{s_i\} = \{\alpha_i, \beta_i\}$ and $\{s_i^*\} = \{\beta_i, -\alpha_i\}$, $i = 1, \dots, 3g - 3 + n$, being the dual bases in H_- as above, and the meromorphic abelian differentials φ_\pm are given by

$$\varphi_\pm = - \frac{2}{\pi\sqrt{-1}} \frac{S_{B_\pm} - S_v}{v}.$$

The connections d_\pm are flat, see [5, 7, 10] for details.

Definition 3.1. The tau functions τ_\pm are (locally) covariant constant sections of the trivial line bundle on the principal stratum $\mathcal{Q}(1^{4g-4+n}, -1^n)$ with respect to the connections d_\pm , that is,

$$d_\pm \tau_\pm = 0. \quad (3.1)$$

Explicit formulas for the tau functions τ_\pm can be derived similar to [5, 10], but we are not going to use them here.

Remark 3.2. The definition (3.1) follows the convention of [10]. The tau functions defined by (3.1) are equal to the 48th power of the tau functions defined in [5, 7].

3.2 Transformation properties of τ_{\pm}

For our purposes we need to analyze transformation properties of τ_{\pm} . The group \mathbb{C}^* of nonzero complex numbers acts by multiplication on quadratic differential q on C and, therefore, on the tau functions τ_{\pm} . Under the action of $\epsilon \in \mathbb{C}^*$ the tau functions τ_{\pm} transform like $\tau_{\pm}(C, \epsilon q) = \epsilon^{\kappa_{\pm}} \tau_{\pm}(C, q)$, where

$$\kappa_+ = \frac{20(g-1) - 4n}{3}, \quad \kappa_- = \frac{44(g-1) + 20n}{3}. \quad (3.2)$$

The formulas (3.2) follow from formulas (6.54) and (6.55) of [7] taking into account an extra factor of 48 due to a different definition of τ_{\pm} adopted here, see also (4.4).

Let us now describe the behavior of the tau functions under a change of the canonical homology basis that commutes with the action of the involution μ . Note that τ_+ (resp. τ_-) is uniquely determined by the subspace in H_+ (resp. H_-) generated by the classes α_i^+ (resp. α_j^-).

Let σ be a symplectic transformation of $H_1(\widehat{C}, \mathbb{R})$ commuting with μ_* that is given by the matrices σ_{\pm} in the basis $\{\alpha_i^+, \beta_i^+, \alpha_j^-, \beta_j^-\}$, $i = 1, \dots, g$, $j = 1, \dots, 3g - 3 + n$, as in (2.4). Then the tau functions τ_{\pm} transform under the action of σ by the formula

$$\frac{\tau_{\pm}^{\sigma}}{\tau_{\pm}} = \gamma_{\pm}(\sigma_{\pm}) \det(C_{\pm} \Omega_{\pm} + D_{\pm})^{48}, \quad (3.3)$$

where $\gamma_{\pm}(\sigma_{\pm})^3 = 1$.

The proof of this statement for $n > 0$ is similar to the case $n = 0$ considered in [10, Theorem 3]. Namely, the transformation property (3.3) (with some constants γ_{\pm}) follows from the differential equations (3.1) and the transformation rule of the Bergman projective connections $S_{B_{\pm}}$ under the change of canonical bases in H_{\pm} . The proof that $\gamma_{\pm}^3 = 1$ uses explicit formulas for τ_{\pm} which use the distinguished local parameters near the ramification points of the cover $C \rightarrow \widehat{C}$. The details can be found in [7], see formulas (6.46) and (6.50) therein.

The following statement is an immediate consequence of (3.2) and (3.3):

Theorem 3.3. *For the tau function τ_+ (resp. τ_-) its 3rd power τ_+^3 (resp. τ_-^3) is a nowhere vanishing holomorphic section of the line bundle $\lambda^{144} \otimes L^{20(g-1)-4n}$ (resp. $\lambda_{\mathbb{P}}^{144} \otimes L^{44(g-1)+20n}$) on the (projectivized) principal stratum $P\mathcal{Q}(1^{4g-4+n}, -1^n)$, where L denotes the tautological line bundle on $P\mathcal{Q}(1^{4g-4+n}, -1^n)$.*

Corollary 3.4. *The Hodge and Prym classes in the rational Picard group of $P\mathcal{Q}(1^{4g-4+n}, -1^n)$ satisfy the relations*

$$\lambda = \frac{5(g-1) - n}{36} \phi, \quad \lambda_{\mathbb{P}} = \frac{11(g-1) + 5n}{36} \phi,$$

where $\phi = c_1(L)$.

4 The Hodge and Prym classes on $P\overline{\mathcal{Q}}_{g,n}$

Here we compute the divisor classes of the tau functions τ_{\pm} viewed as holomorphic sections of line bundles on the compactification $P\overline{\mathcal{Q}}_{g,n}$. The boundary $P\overline{\mathcal{Q}}_{g,n} \setminus P\mathcal{Q}(1^{4g-4+n}, -1^n)$ consists of divisors D_{deg}^0 , D_{deg}^{∞} and D_{DM} . The asymptotic behaviour of τ_{\pm} near D_{deg}^0 and D_{DM} for $n > 0$ is the same as for $n = 0$, the case considered in detail in [10].

Namely, in terms of transversal local coordinate t near D_{DM} we have

$$\tau_{\pm} = t^4(\text{const} + o(1)) \quad \text{as } t \rightarrow 0, \quad (4.1)$$

cf. formulas (5.10) and (5.12) in [10].

In terms of a local transversal coordinate t near D_{deg}^0

$$\tau_+ = t^{2/3}(\text{const} + o(1)), \quad \tau_- = t^{26/3}(\text{const} + o(1)) \quad (4.2)$$

as $t \rightarrow 0$, cf. Lemmas 8 and 9 in [10].

Therefore, it remains to analyze the asymptotics of τ_{\pm} near the divisor D_{deg}^{∞} .

4.1 Local behavior of tau functions near D_{deg}^{∞}

Recall that the boundary component D_{deg}^{∞} is the closure in $P\overline{\mathcal{Q}}_{g,n}$ of the union of n copies of the set $\mathcal{Q}(1^{4g-5+n}, 0, -1^{n-1})$ of equivalence classes of pairs (C, q) , where C is a smooth curve with a marked point and q is a quadratic differential with $n-1$ labeled simple poles and $4g-5+n$ simple zeros (the marked point is neither zero nor pole of q and distinguish the point where a zero and a pole of q coalesce).

Consider a one parameter family (C_t, q_t) of generic quadratic differentials transversal to D_{deg}^{∞} that converges to $(C_0, q_0) \in D_{\text{deg}}^{\infty}$ as $t \rightarrow 0$ (we can actually keep C_t fixed and ignore the rational tail of C_0). Without loss of generality we can assume that under such a degeneration a simple zero x_1^t and a simple pole y_1^t of q_t on C_t coalesce as $t \rightarrow 0$. Denote by $s_t \in H_-$ the vanishing cycle on \widehat{C} going around the path connecting the preimages $\widehat{x}_1^t, \widehat{y}_1^t \in \widehat{C}$, and denote by $t = \int_{s_t} v_t$ the corresponding homological coordinate.

Assuming that the zero x_1^t and the pole y_1^t are close to each other, consider a small disk $U_t \subset C_t$ that contains x_1^t, y_1^t and no other singularities of q_t .

Lemma 4.1. *The local coordinate $t = \int_{s_t} v_t$ is transversal to D_{deg}^{∞} in a tubular neighborhood of $D_{\text{deg}}^{\infty} \subset \overline{\mathcal{Q}}_{g,n}$.*

Proof. The proof is similar to that of Lemma 8 in [10]. Namely, one can choose a coordinate ζ in U_t such that

$$q(\zeta) = \frac{\zeta - \zeta(x_1^t)}{\zeta - \zeta(y_1^t)} d\zeta^2.$$

Then

$$t = \int_{s_t} v_t = 2 \int_{\zeta(x_1^t)}^{\zeta(y_1^t)} \left(\frac{\zeta - \zeta(x_1^t)}{\zeta - \zeta(y_1^t)} \right)^{1/2} d\zeta = \text{const}(\zeta(x_1^t) - \zeta(y_1^t)).$$

Therefore, $t = \int_{s_t} v_t$ is transversal to D_{deg}^{∞} . ■

Lemma 4.2. *The tau functions $\tau_{\pm}(C_t, q_t)$ have the following asymptotic behaviour as $t \rightarrow 0$:*

$$\tau_+(C_t, q_t) \sim t^{-8/3} \tau_+(C_0, q_0), \quad \tau_-(C_t, q_t) \sim t^{40/3} \tau_-(C_0, q_0). \quad (4.3)$$

Proof. First, let us notice that for an arbitrary stratum $\mathcal{Q}(d_1, \dots, d_k)$, where d_1, \dots, d_k are integers not less than -1 and $\sum_{i=1}^k d_i = 4g-4$, the tau functions τ_{\pm} have the following homogeneity property: $\tau_{\pm}(C, \epsilon q) = \epsilon^{\kappa_{\pm}} \tau_{\pm}(C, q)$ with

$$\kappa_+ = \sum_i \frac{d_i(d_i+4)}{d_i+2}, \quad \kappa_- = \kappa_+ + 6 \sum_{d_i \text{ odd}} \frac{1}{d_i+2}, \quad (4.4)$$

(see equations (6.54) and (6.55) of [7], as well as Remark 3.2). Now, in the limit $t \rightarrow 0$, we have for some γ_{\pm}

$$\tau_{\pm} \sim t^{\gamma_{\pm}} \tau_{\pm}(C_0, q_0).$$

To compute γ_{\pm} we evaluate the differences of the homogeneity coefficients (4.4) on the strata $\mathcal{Q}(1^{4g-4+n}, -1^n)$ and $\mathcal{Q}(1^{4g-5+n}, 0, -1^n)$, i.e., when a pole and a zero annihilate each other (all other contributions remain the same):

$$\kappa_+(q) - \kappa_+(q_0) = \frac{1 \cdot 5}{3} + \frac{(-1) \cdot 3}{1} = -\frac{4}{3},$$

$$\kappa_-(q) - \kappa_-(q_0) = \kappa_+(q) - \kappa_+(q_0) + 6 \left(\frac{1}{3} + 1 \right) = \frac{20}{3}.$$

Since the homogeneity coefficient of the coordinate t is $1/2$, we have $\gamma_+ = -8/3$ and $\gamma_- = 40/3$. \blacksquare

4.2 Proof of Theorem 1.2 and its consequences

Combining Theorem 3.3 with the asymptotics (4.1), (4.2) and (4.3) of τ_{\pm} near D_{DM} , D_{deg}^0 and D_{deg}^{∞} , we get the formulas

$$48\lambda - \frac{20(g-1) - 4n}{3}\phi = \frac{2}{3}\delta_{\text{deg}}^0 - \frac{8}{3}\delta_{\text{deg}}^{\infty} + 4\delta_{\text{DM}}, \quad (4.5)$$

$$48\lambda_{\text{P}} - \frac{44(g-1) + 20n}{3}\phi = \frac{26}{3}\delta_{\text{deg}}^0 + \frac{40}{3}\delta_{\text{deg}}^{\infty} + 4\delta_{\text{DM}}, \quad (4.6)$$

which immediately imply Theorem 1.2.

Excluding δ_{deg}^0 from (4.5) and (4.6), we obtain

Corollary 4.3. *The Prym class λ_{P} on $P\overline{\mathcal{Q}}_{g,n}$ is expressed in terms of the Hodge class λ , the tautological class ϕ and the boundary classes D_{DM} , D_{deg}^{∞} by the formula*

$$\lambda_{\text{P}} - 13\lambda = \delta_{\text{deg}}^{\infty} - \delta_{\text{DM}} - \frac{1}{2}(3g - 3 - n)\phi.$$

Furthermore, using the formula $\lambda_{\text{P}} = \lambda_2 - \frac{1}{2}(3g - 3 + n)\phi$ that relates the Prym class to the class $\lambda_2 = \pi_*\omega_{g,n}^2$ (see (1.1)), we get

Corollary 4.4. *The following relation holds:*

$$\lambda_2 - 13\lambda = n\phi + \delta_{\text{deg}}^{\infty} - \delta_{\text{DM}}.$$

Combining this relation with the pullback to $P\overline{\mathcal{Q}}_{g,n}$ of Mumford's formula (1.3), we obtain the equation (1.4)

$$\delta_{\text{deg}}^{\infty} = -n\phi + \sum_{i=1}^n \psi_i.$$

On the other hand, excluding $\delta_{\text{deg}}^{\infty}$ from (4.5) and (4.6), we get the formula

$$\delta_{\text{deg}}^0 = 72\lambda + 4 \sum_{i=1}^n \psi_i - (10(g-1) + 2n)\phi - 6\delta_{\text{DM}}$$

that coincides with the formula (1.2) in [10] for $n = 0$.

Acknowledgements

This work was supported by the Ministry of Science and Higher Education of the Russian Federation, agreement no. 075-15-2019-1620. We thank A. Zorich for useful discussions. We are grateful to anonymous referees for carefully reading the manuscript and pointing out several typos and other inconsistencies.

References

- [1] Arbarello E., Cornalba M., Griffiths P.A., Geometry of algebraic curves. Vol. II, *Grundlehren der mathematischen Wissenschaften*, Vol. 268, Springer, Heidelberg, 2011.
- [2] Bertola M., Korotkin D., Hodge and Prym tau functions, Strebel differentials and combinatorial model of $\mathcal{M}_{g,n}$, *Comm. Math. Phys.* **378** (2020), 1279–1341, [arXiv:1804.02495](#).
- [3] Fay J.D., Theta functions on Riemann surfaces, *Lecture Notes in Math.*, Vol. 352, Springer-Verlag, Berlin – New York, 1973.
- [4] Fulton W., Intersection theory, 2nd ed., *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics*, Vol. 2, Springer-Verlag, Berlin, 1998.
- [5] Kokotov A., Korotkin D., Tau-functions on spaces of abelian differentials and higher genus generalizations of Ray–Singer formula, *J. Differential Geom.* **82** (2009), 35–100, [arXiv:math.SP/0405042](#).
- [6] Kokotov A., Korotkin D., Zograf P., Isomonodromic tau function on the space of admissible covers, *Adv. Math.* **227** (2011), 586–600, [arXiv:0912.3909](#).
- [7] Korotkin D., Bergman tau-function: from Einstein equations and Dubrovin–Frobenius manifolds to geometry of moduli spaces, in *Integrable Systems and Algebraic Geometry*, Editors R. Donagi, T. Shaska, *LMS Lecture Note Series*, Cambridge University Press, Cambridge, 2019, 215–287, [arXiv:1812.03514](#).
- [8] Korotkin D., Sauvaget A., Zograf P., Tau functions, Prym–Tyurin classes and loci of degenerate differentials, *Math. Ann.* **375** (2019), 213–246, [arXiv:1710.01239](#).
- [9] Korotkin D., Zograf P., Tau function and moduli of differentials, *Math. Res. Lett.* **18** (2011), 447–458, [arXiv:1003.2173](#).
- [10] Korotkin D., Zograf P., Tau function and the Prym class, in *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices*, *Contemp. Math.*, Vol. 593, Amer. Math. Soc., Providence, RI, 2013, 241–261, [arXiv:1302.0577](#).
- [11] Takhtajan L.A., Zograf P.G., A local index theorem for families of $\bar{\partial}$ -operators on punctured Riemann surfaces and a new Kähler metric on their moduli spaces, *Comm. Math. Phys.* **137** (1991), 399–426.
- [12] Wolpert S.A., Infinitesimal deformations of nodal stable curves, *Adv. Math.* **244** (2013), 413–440, [arXiv:1204.3680](#).