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Perfect Integrability and Gaudin Models

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Abstract. We suggest the notion of perfect integrability for quantum spin chains and conjecture that quantum spin chains are perfectly integrable. We show the perfect integrability for Gaudin models associated to simple Lie algebras of all finite types, with periodic and regular quasi-periodic boundary conditions.

Key words: Gaudin model; Bethe ansatz; Frobenius algebra

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1 Introduction

Quantum spin chains are important models in integrable system. These models have numerous deep connections with other areas of mathematics and physics. In this article, we would like to suggest the notion of perfect integrability for quantum spin chains.

Let us recall Gaudin models and XXX spin chains. Let $\mathfrak{g}$ be a simple (or reductive) Lie (super)algebra and G the corresponding Lie group. Let $\mathcal{A}_\mathfrak{g}$ be an affinization of $\mathfrak{g}$ where the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of $\mathfrak{g}$ can be identified as a Hopf subalgebra of $\mathcal{A}_\mathfrak{g}$. Here $\mathcal{A}_\mathfrak{g}$ is either the universal enveloping algebra of the current algebra $\mathrm{U}(\mathfrak{g}[t])$ which describes the symmetry for Gaudin models, or Yangian $\mathrm{Y}(\mathfrak{g})$ associated to $\mathfrak{g}$ for XXX spin chains. In both cases the algebra $\mathcal{A}_\mathfrak{g}$ has a remarkable commutative subalgebra called the *Bethe algebra*. We denote the Bethe algebra by $\mathcal{B}_\mathfrak{g}$. The Bethe algebra $\mathcal{B}_\mathfrak{g}$ commutes with $\mathrm{U}(\mathfrak{g})$. Take any finite-dimensional irreducible representation M of $\mathcal{A}_\mathfrak{g}$, then $\mathcal{B}_\mathfrak{g}$ acts naturally on the space of singular vectors $M^{\rm sing}$. Let $\mathcal{B}_\mathfrak{g}(M^{\rm sing})$ be the image of $\mathcal{B}_\mathfrak{g}$ in $\mathrm{End}(M^{\rm sing})$. The problem is to study the spectrum of $\mathcal{B}_\mathfrak{g}(M^{\rm sing})$ acting on $M^{\rm sing}$.[1]

With the agreement with the philosophy of geometric Langlands correspondence, it is important to understand and describe the finite-dimensional algebra $\mathcal{B}_\mathfrak{g}(M^{\rm sing})$ and the corresponding scheme $\mathsf{Spec}(\mathcal{B}_\mathfrak{g}(M^{\rm sing}))$. Or more generally, find a geometric object parameterizing the eigenspaces of $\mathcal{B}_\mathfrak{g}$ when M runs over all finite-dimensional irreducible representations (up to isomorphism). In Gaudin models, the underlying geometric objects are described by the sets of monodromy-free ${}^L\mathfrak{g}$-opers with regular singularities of prescribed residues at evaluation points, see [4, 21], where ${}^L\mathfrak{g}$ is the Langlands dual of $\mathfrak{g}$. Moreover, when $\mathfrak{g}=\mathfrak{gl}_N$, the algebra $\mathcal{B}_\mathfrak{g}(M^{\rm sing})$ is interpreted as the space of functions on the intersection of suitable Schubert cycles in a Grassmannian variety, see [16]. This interpretation gives a relation between representation theory and Schubert calculus useful in both directions which has important applications in real algebraic geometry, see [13, 16].

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[1]The reason these models are called spin chains is that M is usually a tensor product of evaluation modules where each factor corresponds to a particle of some spin.

Any finite-dimensional unital commutative algebra $\mathcal{B}$ is a module over itself induced by left multiplication. We call this module the *regular representation of* $\mathcal{B}$. The dual space $\mathcal{B}^*$ is naturally a $\mathcal{B}$-module which is called the *coregular representation*. A Frobenius algebra is a finite-dimensional unital commutative algebra whose regular and coregular representations are isomorphic, see Section 2.6.

Let V be a finite-dimensional $\mathcal{B}$-module. Let $\mathcal{B}(V)$ be the image of $\mathcal{B}$ in $\mathrm{End}(V)$. We say that the $\mathcal{B}$-module V is *perfectly integrable* if $\mathcal{B}$ acts on V cyclically and the algebra $\mathcal{B}(V)$ is a Frobenius algebra. Note that in this case, the $\mathcal{B}(V)$-module V is isomorphic to the regular and coregular representations of $\mathcal{B}(V)$.

Based on the extensive study of quantum spin chains, see the evidences from [2, 4, 10, 15, 16, 17, 21], the following conjecture is expected to hold.

Conjecture 1.1. *The $\mathcal{B}_{\mathfrak{g}}$-module M^{sing} is perfectly integrable.*

In fact there is a family of commutative Bethe algebras $\mathcal{B}_{\mathfrak{g}}^{\mu}$ depending on an element $\mu \in \mathfrak{g}^*$ (resp. $\mu \in G$). From here to Conjecture 1.2, we use the parenthesis to indicate the modifications for XXX spin chains. If $\mu \in \mathfrak{g}^*$ (resp. $\mu \in G$) is a regular semi-simple element, we say that the corresponding spin chain has regular quasi-periodic boundary condition. Moreover, if $\mu = 0$ (resp. $\mu = \mathrm{Id}$), then the algebra $\mathcal{B}_{\mathfrak{g}}^{0}$ (resp. $\mathcal{B}_{\mathfrak{g}}^{\mathrm{Id}}$) coincides with the algebra $\mathcal{B}_{\mathfrak{g}}$ considered above. If $\mu = 0$ (resp. $\mu = \mathrm{Id}$), we say that the corresponding spin chain has periodic boundary condition.

For regular quasi-periodic spin chains the Bethe algebra does not commute with $\mathrm{U}(\mathfrak{g})$ and one replaces M^{sing} with M. Denote by $\mathcal{B}_{\mathfrak{g}}^{\mu}(M)$ the image of $\mathcal{B}_{\mathfrak{g}}^{\mu}$ in $\mathrm{End}(M)$. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and H the Cartan subgroup of G.

Conjecture 1.2. *If $\mu \in \mathfrak{h}^*$ (resp. $\mu \in H$) is regular, then the $\mathcal{B}_{\mathfrak{g}}^{\mu}$-module M is perfectly integrable.*

For more general $\mu \in \mathfrak{g}^*$, one has to replace M^{sing} or M with an appropriate subspace of M depending on μ, see Conjecture 2.9 in Section 2.8.

When Conjectures 1.1, 1.2, and 2.9 hold, we say that the corresponding quantum spin chains are *perfectly integrable*.

The perfect integrability was shown for

- Gaudin models of $\mathfrak{gl}_N$ in [15, 16] with periodic and regular quasi-periodic boundary conditions;
- XXX (resp. XXZ) spin chains of $\mathfrak{gl}_N$ associated to irreducible tensor products of vector representations in [17] (resp. [18]) with periodic and regular quasi-periodic boundary conditions;
- XXX spin chains of $\mathfrak{gl}_{1|1}$ associated to cyclic tensor products of polynomial representations in [10] with periodic and regular quasi-periodic boundary conditions;
- XXX spin chains of $\mathfrak{gl}_{m|n}$ associated to irreducible tensor products of vector representations in [2] with periodic boundary condition.

Our main result confirms Conjectures 1.1 and 1.2 for Gaudin models of all finite types, see Theorem 2.8. We deduce Theorem 2.8 from [4, Corollary 5], [21, Theorem 3.2], and [6, Theorem 8.1.5].

Our suggestion to call the situations in Conjectures 1.1 and 1.2 "perfect integrability" is motivated by Lemma 1.3 below.

Let $\mathcal{B}$ be a finite-dimensional unital commutative algebra. Let V be a finite-dimensional $\mathcal{B}$-module and $\mathcal{E}\colon \mathcal{B}\to\mathbb{C}$ a character, then the *$\mathcal{B}$-eigenspace* and *generalized $\mathcal{B}$-eigenspace* associated to $\mathcal{E}$ in V are defined by

$$\bigcap_{a\in\mathcal{B}}\ker(a|_V-\mathcal{E}(a))\qquad\text{and}\qquad\bigcap_{a\in\mathcal{B}}\Big(\bigcup_{m=1}^{\infty}\ker(a|_V-\mathcal{E}(a))^m\Big),$$

respectively. Let $\mathcal{B}(V)$ be the image of $\mathcal{B}$ in $\mathrm{End}(V)$.

Lemma 1.3. *If the $\mathcal{B}$-module V is perfectly integrable, then every $\mathcal{B}$-eigenspace in V has dimension one, and there exists a bijection between $\mathcal{B}$-eigenspaces in V and $\mathsf{Specm}(\mathcal{B}(V))$ – the subset of closed points in $\mathsf{Spec}(\mathcal{B}(V))$. Moreover, each generalized $\mathcal{B}$-eigenspace is a cyclic $\mathcal{B}$-module, and the algebra $\mathcal{B}(V)$ is a maximal commutative subalgebra in $\mathrm{End}(V)$ of dimension $\dim V$.*

This lemma easily follows from general well-known facts about regular and coregular representations of a finite-dimensional unital commutative algebra, see, e.g., [16, Section 3.3]. We provide a proof of Lemma 1.3 in Section 2.6.

Note that we expect that the dimensions of eigenspaces are one from the general philosophy of Bethe ansatz conjecture. The integrability in any sense always asserts that the algebra of Hamiltonians is maximal commutative. And we also expect that the Bethe algebra has geometric nature based on the geometric Langlands correspondence [6].

In the case of Gaudin models, it is proved in [21, Theorem 3.2] (resp. [4, Corollary 5]) that $\mathcal{B}_{\mathfrak{g}}$ (resp. $\mathcal{B}_{\mathfrak{g}}^{\mu}$ with regular μ) acts cyclically on $M^{\rm sing}$ (resp. M). For generic values of evaluation parameters (in the periodic case or in the case of generic regular $\mu\in\mathfrak{h}^*$) the action of Bethe algebra is diagonalizable and we immediately obtain that eigenspaces have dimension one. However, we cannot make such a conclusion for *arbitrary* parameters. Indeed, if a linear operator acts cyclically on a vector space then all its eigenspaces have dimension one. But the same result fails if we replace a single operator by a set of commuting linear operators, as the following simple example shows.

Example. Let $\mathcal{A}=\mathbb{C}[x_1,x_2]/\langle x_1^2,x_2^2,x_1x_2\rangle$. Consider the regular representation $\mathcal{A}$. Then the eigenspace corresponding to the trivial character is spanned by x_1 and x_2 which is two-dimensional.

We supplement the results of [4] and [21] with the nondegenerate symmetric bilinear form on M which makes $\mathcal{B}_{\mathfrak{g}}^{\mu}(M)$ Frobenius which allows us to use Lemma 1.3. The bilinear form comes from the tensor product of Shapovalov forms on M, we show that all elements of Bethe algebra $\mathcal{B}_{\mathfrak{g}}^{\mu}(M)$ with $\mu\in\mathfrak{h}^*$ are symmetric with respect to this form, see Lemma 2.6.

In the rest of the paper, we only deal with Gaudin models. We refer the readers to [7] for details about the Bethe algebra of Yangian $\mathrm{Y}(\mathfrak{g})$ (XXX spin chains). We expect the conjectures with proper modifications also hold for XXZ and XYZ spin chains.

2 Perfect integrability of Gaudin models

2.1 Feigin–Frenkel center

In this section, we recall the definition of Feigin–Frenkel center and its properties.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank r. Consider the affine Kac–Moody algebra $\widehat{\mathfrak{g}}$,

$$\widehat{\mathfrak{g}}=\mathfrak{g}\big[t,t^{-1}\big]\oplus\mathbb{C}K,\qquad \mathfrak{g}\big[t,t^{-1}\big]=\mathfrak{g}\otimes\mathbb{C}\big[t,t^{-1}\big],$$

where $\mathbb{C}\big[t,t^{-1}\big]$ is the algebra of Laurent polynomials in t. For $X\in\mathfrak{g}$ and $s\in\mathbb{Z}$, we simply write $X[s]$ for $X\otimes t^s$. Let $\mathfrak{g}_-=\mathfrak{g}\otimes t^{-1}\mathbb{C}\big[t^{-1}\big]$ and $\mathfrak{g}[t]=\mathfrak{g}\otimes\mathbb{C}[t]$.

Let $h^\vee$ be the *dual Coxeter number* of $\mathfrak{g}$. Define the module $V_{-h^\vee}(\mathfrak{g})$ as the quotient of $\mathrm{U}(\widehat{\mathfrak{g}})$ by the left ideal generated by $\mathfrak{g}[t]$ and $K+h^\vee$. We call the module $V_{-h^\vee}(\mathfrak{g})$ the *Vaccum module at the critical level over* $\widehat{\mathfrak{g}}$. The vacuum module $V_{-h^\vee}(\mathfrak{g})$ has a vertex algebra structure.

Define the subspace $\mathfrak{z}(\widehat{\mathfrak{g}})$ of $V_{-h^\vee}(\mathfrak{g})$ by

$$\mathfrak{z}(\widehat{\mathfrak{g}}) = \{v \in V_{-h^\vee}(\mathfrak{g}) \,|\, \mathfrak{g}[t]v = 0\}.$$

Using the PBW theorem, it is clear that $V_{-h^\vee}(\mathfrak{g})$ is isomorphic to $\mathrm{U}(\mathfrak{g}_-)$ as vector spaces. There is an injective homomorphism from $\mathfrak{z}(\widehat{\mathfrak{g}})$ to $\mathrm{U}(\mathfrak{g}_-)$. Hence $\mathfrak{z}(\widehat{\mathfrak{g}})$ is identified as a commutative subalgebra of $\mathrm{U}(\mathfrak{g}_-)$. We call $\mathfrak{z}(\widehat{\mathfrak{g}})$ the *Feigin–Frenkel center*.

There is a distinguished element $S_1 \in \mathfrak{z}(\widehat{\mathfrak{g}})$ given by

$$S_1 = \sum_{a=1}^{\dim \mathfrak{g}} X_a[-1]^2,$$

where $\{X_a\}$ is an orthonormal basis of $\mathfrak{g}$ with respect to the Killing form. The element S_1 is called the *Segal–Sugawara vector*.

Proposition 2.1 ([20])**.** *The subalgebra $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the centralizer of S_1 in $\mathrm{U}(\mathfrak{g}_-)$.*

Let $e_1,\dots,e_r$, $h_1,\dots,h_r$, $f_1,\dots,f_r$ be a set of Chevalley generators of $\mathfrak{g}$. Let $\varpi\colon \mathfrak{g}\to\mathfrak{g}$ be the *Cartan anti-involution* sending $e_1,\dots,e_r$, $h_1,\dots,h_r$, $f_1,\dots,f_r$ to $f_1,\dots,f_r$, $h_1,\dots,h_r$, $e_1,\dots,e_r$, respectively. Let $\widehat{\varpi}$ be the anti-involution on $\widehat{\mathfrak{g}}$ defined by

$$\widehat{\varpi}\colon\ \widehat{\mathfrak{g}}\to\widehat{\mathfrak{g}}, \qquad X[s]\mapsto \varpi(X)[s],$$

for all $X\in\mathfrak{g}$ and $s\in\mathbb{Z}$. We also call $\widehat{\varpi}$ the Cartan anti-involution.

Corollary 2.2. *The Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is invariant under $\widehat{\varpi}$.*

Proof. Since by Proposition 2.1, $\mathfrak{z}(\widehat{\mathfrak{g}})$ is the centralizer of S_1 in $\mathrm{U}(\mathfrak{g}_-)$, the statement follows from the fact that $\widehat{\varpi}(S_1)=S_1$. ■

2.2 Affine Harish-Chandra homomorphism

Let $\mathfrak{n}_+$ be the nilpotent Lie subalgebra generated by $e_1,\dots,e_r$. Let $\mathfrak{n}_-$ be the nilpotent Lie subalgebra generated by $f_1,\dots,f_r$. Let $\mathfrak{h}$ be the Cartan subalgebra generated by $h_1,\dots,h_r$. One has the triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$.

The Lie algebra $\mathfrak{g}$ is considered as a subalgebra of $\widehat{\mathfrak{g}}$ via identifying $X\in\mathfrak{g}$ with $X[0]\in\widehat{\mathfrak{g}}$. The Lie subalgebra $\mathfrak{h}$ acts on $\widehat{\mathfrak{g}}$ adjointly and hence acts on $\mathrm{U}(\mathfrak{g}_-)$. Let $\mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}}$ be the centralizer of $\mathfrak{h}$ in $\mathrm{U}(\mathfrak{g}_-)$.

Let J be the intersection of $\mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}}$ and the left ideal of $\mathrm{U}(\mathfrak{g}_-)$ generated by $t^{-1}\mathfrak{n}_-[t^{-1}]$. Then we have the direct sum of vector spaces,

$$\mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}} = \mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)\oplus J. \tag{2.1}$$

Hence we have the projection

$$\mathfrak{f}\colon\ \mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}}\to\mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big).$$

It is clear that $\mathfrak{f}$ is a homomorphism of algebras. We call $\mathfrak{f}$ the *affine Harish-Chandra homomorphism*. We use the same letter $\mathfrak{f}$ for the restriction map $\mathfrak{f}\colon \mathfrak{z}(\widehat{\mathfrak{g}})\to\mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$.

The following proposition is a part of [6, Theorem 8.1.5].

Proposition 2.3. *The homomorphism* $\mathfrak{f}\colon \mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$ *is injective.*

Using Proposition 2.3, we improve Corollary 2.2 to the following proposition.

Proposition 2.4. *For any element* $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$*, we have* $\widehat{\varpi}(S) = S$*.*

The proposition was proved in [14, Proposition 8.4] for type A and in [8, Proposition 6.1] for types B and C.

Proof. Now take $S \in \mathfrak{z}(\widehat{\mathfrak{g}})$ and write the decomposition of S as in (2.1), $S = S_{\mathfrak{h}} + S_{\mathfrak{j}}$, where $S_{\mathfrak{h}} \in \mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$ and $S_{\mathfrak{j}} \in J$. Then $\widehat{\varpi}(S) = \widehat{\varpi}(S_{\mathfrak{h}}) + \widehat{\varpi}(S_{\mathfrak{j}})$. Note that $\widehat{\varpi}$ fix elements in $\mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$ and $S_{\mathfrak{h}} \in \mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$ we have $\widehat{\varpi}(S_{\mathfrak{h}}) = S_{\mathfrak{h}}$. Note also that $\widehat{\varpi}$ maps $\mathrm{U}\big(t^{-1}\mathfrak{n}_+\big[t^{-1}\big]\big)$ to $\mathrm{U}\big(t^{-1}\mathfrak{n}_-\big[t^{-1}\big]\big)$ and $\mathrm{U}\big(t^{-1}\mathfrak{n}_-\big[t^{-1}\big]\big)$ to $\mathrm{U}\big(t^{-1}\mathfrak{n}_+\big[t^{-1}\big]\big)$, we have $\widehat{\varpi}(S_{\mathfrak{j}}) \in J$ since J is the intersection of the $\mathfrak{h}$-centralizer $\mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}}$ with the left ideal of $\mathrm{U}(\mathfrak{g}_-)$ generated by $t^{-1}\mathfrak{n}_-\big[t^{-1}\big]$ and also the intersection of $\mathrm{U}(\mathfrak{g}_-)^{\mathfrak{h}}$ with the right ideal of $\mathrm{U}(\mathfrak{g}_-)$ generated by $t^{-1}\mathfrak{n}_+\big[t^{-1}\big]$. It follows that

$$\mathfrak{f}(S) = S_{\mathfrak{h}} = \mathfrak{f} \circ \widehat{\varpi}(S).$$

Note that by Corollary 2.2 both S and $\widehat{\varpi}(S)$ are elements in $\mathfrak{z}(\widehat{\mathfrak{g}})$. Since by Proposition 2.3 the homomorphism $\mathfrak{f}\colon \mathfrak{z}(\widehat{\mathfrak{g}}) \to \mathrm{U}\big(t^{-1}\mathfrak{h}\big[t^{-1}\big]\big)$ is injective, we conclude that $S = \widehat{\varpi}(S)$, completing the proof. ■

2.3 Gaudin models

We recall the construction of Gaudin models from, e.g., [19, 21]. The coproduct of $\mathrm{U}(\mathfrak{g}_-)$ is given by

$$\Delta\colon\ X[s] \mapsto X[s] \otimes 1 + 1 \otimes X[s], \qquad X \in \mathfrak{g}, \qquad s < 0.$$

Let ℓ be a positive integer. Using the iterated coproduct, one has the homomorphism

$$\mathrm{U}(\mathfrak{g}_-) \to \mathrm{U}(\mathfrak{g}_-)^{\otimes(\ell+1)}.$$

For any $z \in \mathbb{C}^\times$, one gets the *evaluation map* at z

$$\varphi_z\colon\ \mathrm{U}(\mathfrak{g}_-) \to \mathrm{U}(\mathfrak{g}), \qquad X[s] \mapsto z^s X.$$

For any $\mu \in \mathfrak{g}^*$, one obtains the character

$$\psi_\mu\colon\ \mathrm{U}(\mathfrak{g}_-) \to \mathbb{C}, \qquad X[s] \mapsto \delta_{s,-1}\mu(X),$$

Fix a sequence of pairwise distinct nonzero complex numbers $\boldsymbol{z} = (z_1, \dots, z_\ell)$. Then using these three homomorphisms, one obtains a new homomorphism

$$\varphi_{\boldsymbol{z},\mu}\colon\ \mathrm{U}(\mathfrak{g}_-) \to \mathrm{U}(\mathfrak{g})^{\otimes\ell}, \qquad \varphi_{\boldsymbol{z},\mu}(X[s]) = \sum_{a=1}^{\ell} z_a^s (X)_a + \delta_{s,-1}\mu(X),$$

where $(X)_a = 1^{\otimes(a-1)} \otimes X \otimes 1^{\otimes(\ell-a)}$.

Set $u - \boldsymbol{z} = (u - z_1, \dots, u - z_\ell)$. Define the *Gaudin algebra* as a subalgebra generated by elements in $\varphi_{u-\boldsymbol{z},\mu}(\mathfrak{z}(\widehat{\mathfrak{g}})) \subset \mathrm{U}(\mathfrak{g})^{\otimes\ell}$ for all $u \in \mathbb{C} \setminus \{z_1, \dots, z_\ell\}$. The Gaudin algebra is commutative and it is denoted by $\mathcal{A}_{\boldsymbol{z},\mu}$. When $\mu = 0$, the Gaudin algebra commutes with the diagonal action of $\mathrm{U}(\mathfrak{g})$, see, e.g., [19, Proposition 3].

Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_\ell)$ be a sequence of dominant integral weights. Denote by V_{λ_i} the finite-dimensional irreducible $\mathfrak{g}$-module of highest weight λ_i. We set $V_{\boldsymbol{\lambda}} = \otimes_{i=1}^{\ell} V_{\lambda_i}$ and

$$(V_{\boldsymbol{\lambda}})^{\mathrm{sing}} = \{v \in V_{\boldsymbol{\lambda}} \mid \mathfrak{n}_+ v = 0\}, \qquad \mathcal{M}_{\boldsymbol{\lambda},\mu} = \begin{cases} (V_{\boldsymbol{\lambda}})^{\mathrm{sing}}, & \text{if } \mu = 0, \\ V_{\boldsymbol{\lambda}}, & \text{if } \mu \in \mathfrak{h}^* \text{ is regular.} \end{cases}$$

Here we identify $\mathfrak{h}^*$ with the subspace of $\mathfrak{g}^*$ consisting of all elements annihilating $\mathfrak{n}_+ \oplus \mathfrak{n}_-$. By the construction of $\mathcal{A}_{\boldsymbol{z},\mu}$, $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ is an $\mathcal{A}_{\boldsymbol{z},\mu}$-module.

Let $\mathcal{A}_{\boldsymbol{z},\mu}$ be the algebra of Hamiltonians and $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ the spin chain. We call the corresponding integrable system the *Gaudin model*. We say that the Gaudin model has *periodic boundary condition* if $\mu = 0$ and *regular quasi-periodic boundary condition* if $\mu \in \mathfrak{h}^*$ is regular. We would like to study the spectrum of $\mathcal{A}_{\boldsymbol{z},\mu}$ acting on $\mathcal{M}_{\boldsymbol{\lambda},\mu}$.

The following theorem is obtained in [4, Corollary 5] for any regular $\mu \in \mathfrak{g}^*$ and in [21, Theorem 3.2] for $\mu = 0$.

Theorem 2.5. *If $\mu \in \mathfrak{h}^*$ is regular or if $\mu = 0$, then the space $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ is cyclic as an $\mathcal{A}_{\boldsymbol{z},\mu}$-module.*

2.4 Bethe algebra

Note that our definition of Gaudin models is slightly different from that in Introduction. In this section, we define the Bethe algebra in $\mathrm{U}(\mathfrak{g}[t])$ and clarify this point.

Following, e.g., [12, Section 5], we recall the definition of Bethe algebra using Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$. Note that the Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$ is in $\mathrm{U}(\mathfrak{g}_-)$ while the Bethe algebra is in $\mathrm{U}(\mathfrak{g}[t])$.

For $\mathcal{X} \in \mathfrak{g}$ and $\mu \in \mathfrak{g}^*$, define the current $\mathcal{X}^\mu(u)$ by

$$\mathcal{X}^\mu(u) = \mu(\mathcal{X}) + \sum_{r \geqslant 0} \mathcal{X}[r] u^{-r-1} \in \mathrm{U}(\mathfrak{g}[t])\big[\big[u^{-1}\big]\big].$$

For any element a of the form

$$a = \sum \mathcal{X}_1[-s_1 - 1] \mathcal{X}_2[-s_2 - 1] \cdots \mathcal{X}_k[-s_k - 1] \in \mathfrak{z}(\widehat{\mathfrak{g}})$$

for some $k \in \mathbb{Z}_{>0}$, $\mathcal{X}_i \in \mathfrak{g}$, $s_i \in \mathbb{Z}_{\geqslant 0}$, define a series in u^{-1} whose coefficients are in $\mathrm{U}(\mathfrak{g}[t])$ by

$$\sum \frac{(-1)^k}{s_1! s_2! \cdots s_k!} \partial_u^{s_1} \mathcal{X}_1^\mu(u) \partial_u^{s_2} \mathcal{X}_2^\mu(u) \cdots \partial_u^{s_k} \mathcal{X}_k^\mu(u) \in \mathrm{U}(\mathfrak{g}[t])\big[\big[u^{-1}\big]\big]. \tag{2.2}$$

The *Bethe algebra* $\mathcal{B}_{\mathfrak{g}}^\mu$ is the subalgebra of $\mathrm{U}(\mathfrak{g}[t])$ generated by the coefficients of all such series of form (2.2) as a runs over $\mathfrak{z}(\widehat{\mathfrak{g}})$.

The Bethe algebra $\mathcal{B}_{\mathfrak{g}}^\mu$ (or Feigin–Frenkel center $\mathfrak{z}(\widehat{\mathfrak{g}})$) is considered as the universal Gaudin algebra, see, e.g., [7]. The Gaudin algebra $\mathcal{A}_{\boldsymbol{z},\mu}$ in $\mathrm{U}(\mathfrak{g})^{\otimes \ell}$ can also be obtained from $\mathcal{B}_{\mathfrak{g}}^\mu$ by taking the $(\ell-1)$-th fold coproduct and then applying to the i-th factor the evaluation map at z_i for every $1 \leqslant i \leqslant \ell$. In particular, the image of the Gaudin algebra $\mathcal{A}_{\boldsymbol{z},\mu}$ acting on $V_{\boldsymbol{\lambda}}$ coincides with that of Bethe algebra $\mathcal{B}_{\mathfrak{g}}^\mu$ acting on tensor product of evaluation modules $V_{\boldsymbol{\lambda}}$ with evaluation points at $\boldsymbol{z} = (z_1, \dots, z_\ell)$, see [21, Propositions 2.3 and 2.5] or [3, 5, 19]. Note that in this case, all finite-dimensional irreducible $\mathrm{U}(\mathfrak{g}[t])$-modules are tensor products of evaluation modules with pairwise distinct evaluation parameters.

2.5 Shapovalov form

For a dominant integral weight λ, there is a unique nondegenerate symmetric bilinear form S_λ on V_λ such that

$$\mathcal{S}_\lambda(v_\lambda, v_\lambda) = 1, \qquad \mathcal{S}_\lambda(Xv, w) = \mathcal{S}_\lambda(v, \varpi(X)w),$$

where v_λ is a highest weight vector of V_λ and $v, w \in V_\lambda$. We call $\mathcal{S}_\lambda$ the *Shapovalov form on* V_λ. The Shapovalov form $\mathcal{S}_\lambda$ is positive definite on the real part of V_λ.

The Shapovalov forms $\mathcal{S}_{\lambda_i}$ induce a nondegenerate symmetric bilinear form $\mathcal{S}_{\boldsymbol{\lambda}} = \otimes_{i=1}^{\ell} \mathcal{S}_{\lambda_i}$ on $V_{\boldsymbol{\lambda}}$. The restriction of $\mathcal{S}_{\boldsymbol{\lambda}}$ on the singular subspace $(V_{\boldsymbol{\lambda}})^{\mathrm{sing}}$ is also nondegenerate.

Suppose $\mu \in \mathfrak{h}^*$, then it is clear that

$$\mathcal{S}_{\boldsymbol{\lambda}}(\varphi_{\boldsymbol{z},\mu}(X[s])v, w) = \mathcal{S}_{\boldsymbol{\lambda}}(v, \varphi_{\boldsymbol{z},\mu}(\varpi(X)[s])w) = \mathcal{S}_{\boldsymbol{\lambda}}(v, \varphi_{\boldsymbol{z},\mu} \circ \widehat{\varpi}(X[s])w), \tag{2.3}$$

for all $v, w \in V_{\boldsymbol{\lambda}}$ and $X \in \mathfrak{g}$.

Let $\rho_{\boldsymbol{\lambda},\boldsymbol{z},\mu} \colon \mathcal{A}_{\boldsymbol{z},\mu} \to \mathrm{End}(\mathcal{M}_{\boldsymbol{\lambda},\mu})$ be the representation of the natural action of $\mathcal{A}_{\boldsymbol{z},\mu}$ on $\mathcal{M}_{\boldsymbol{\lambda},\mu}$. Let $\mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$ be the image of $\mathcal{A}_{\boldsymbol{z},\mu}$ under $\rho_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$.

Lemma 2.6. *Let $a \in \mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$ and $v, w \in \mathcal{M}_{\boldsymbol{\lambda},\mu}$. If $\mu \in \mathfrak{h}^*$, then we have $\mathcal{S}_{\boldsymbol{\lambda}}(av, w) = \mathcal{S}_{\boldsymbol{\lambda}}(v, aw)$.*

Proof. The statement follows from (2.3) and Proposition 2.4. ■

2.6 Frobenius algebra

Let A be a finite-dimensional commutative unital algebra. If there exists a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on A such that

$$(ab, c) = (a, bc) \qquad \text{for all } a, b, c \in A,$$

then it is clear that the regular and coregular representations of A are isomorphic. Thus A is a Frobenius algebra.

We prepare the following lemma for the proof of the main theorem. Suppose A is a unital commutative algebra acting on a finite-dimensional space V, $\rho\colon A \to \mathrm{End}(V)$. Let $\mathfrak{A}$ be the image of A under ρ in $\mathrm{End}(V)$. Clearly, $\mathfrak{A}$ is a finite-dimensional unital commutative algebra.

Lemma 2.7. *Suppose A acts on V cyclically. If there is a nondegenerate symmetric bilinear form $(\cdot|\cdot)$ on V such that*

$$(av|w) = (v|aw), \qquad \textit{for all } a \in \mathfrak{A}, \quad v, w \in V,$$

then the algebra $\mathfrak{A}$ is a Frobenius algebra. In particular, the A-module V is perfectly integrable.

Proof. Let v^+ be a cyclic vector of the action of $\mathfrak{A}$ on V. Define a linear map ξ by

$$\xi\colon\ \mathfrak{A} \to V, \qquad a \mapsto av^+.$$

Clearly, ξ is surjective.

We claim that ξ is injective. Indeed, suppose that $a \in \ker\xi$, then $a \in \mathrm{End}(V)$ and $av^+ = 0$. Hence $aa'v^+ = a'av^+ = 0$ for all $a' \in \mathfrak{A}$, namely $a\,\xi(\mathfrak{A}) = 0$. Since $\xi(\mathfrak{A}) = V$, we conclude that $aV = 0$. Therefore $a = 0$, which implies ξ is injective and hence a bijection. Then it is clear that ξ defines an $\mathfrak{A}$-module isomorphism between the regular representation of $\mathfrak{A}$ and the $\mathfrak{A}$-module V.

Define a bilinear form $(\cdot,\cdot)$ on $\mathfrak{A}$ as follows,

$$(a, b) = (av^+|bv^+), \qquad \text{for all } a, b \in \mathfrak{A}.$$

Since $(\cdot|\cdot)$ is symmetric, so is $(\cdot,\cdot)$. Because $(\cdot|\cdot)$ is nondegenerate and ξ is bijective, the form $(\cdot,\cdot)$ is nondegenerate as well. For $a,b,c\in\mathfrak{A}$, we also have

$$(ab,c)=(abv^+|cv^+)=(bav^+|cv^+)=(av^+|bcv^+)=(a,bc).$$

Hence $\mathfrak{A}$ is a Frobenius algebra. ■

Since Lemma 1.3 is central to the results of the present paper, we also provide a detailed proof for it.

Proof of Lemma 1.3. To simplify the notation, we write $\mathfrak{B}$ for $\mathcal{B}(V)$. If the $\mathcal{B}$-module V is perfectly integrable, then the $\mathfrak{B}$-module V is isomorphic to the regular representation $\mathfrak{B}$ and coregular representation $\mathfrak{B}^*$. Note that the $\mathcal{B}$-eigenspaces are essentially the same as $\mathfrak{B}$-eigenspaces, thus we shall use $\mathfrak{B}$-eigenspaces instead.

Let $\psi\in\mathfrak{B}^*$ be a $\mathfrak{B}$-eigenvector for the coregular representation $\mathfrak{B}^*$ with the eigenvalue $\xi_\psi\in\mathfrak{B}^*$, namely $a\psi=\xi_\psi(a)\psi$ for any $a\in\mathfrak{B}$. It is clear that ξ_ψ is a character of $\mathfrak{B}$.

On one hand, by definition of coregular representation, we have $(a\psi)(1)=\psi(a\cdot 1)=\psi(a)$ for any $a\in\mathfrak{B}$. On the other hand, since ψ is a $\mathfrak{B}$-eigenvector, we have

$$(a\psi)(1)=\big(\xi_\psi(a)\psi\big)(1)=\xi_\psi(a)\psi(1)$$

for any $a\in\mathfrak{B}$. Therefore $\psi(a)=\xi_\psi(a)\psi(1)$ for any $a\in\mathfrak{B}$, which means the $\mathfrak{B}$-eigenvector ψ is proportional to the corresponding eigenvalue ξ_ψ. This shows that every $\mathfrak{B}$-eigenspace in V has dimension one.

For any character $\xi\in\mathfrak{B}^*$ and any $a,b\in\mathfrak{B}$, we have

$$(a\xi)(b)=\xi(ab)=\xi(a)\xi(b).$$

Therefore, any character $\xi\in\mathfrak{B}^*$ is a $\mathfrak{B}$-eigenvector with the eigenvalue ξ.

Note that the characters of $\mathfrak{B}$ are parameterized by their kernels, that is the maximal ideals of $\mathfrak{B}$. Combining the facts above, we conclude that there is a bijection between $\mathfrak{B}$-eigenspaces and $\mathsf{Specm}(\mathfrak{B})$, namely the maximal ideals of $\mathfrak{B}$.

We then show that each generalized $\mathfrak{B}$-eigenspace is a cyclic $\mathfrak{B}$-module. We call a finite-dimensional commutative algebra A *local* if it has a unique maximal ideal $\mathfrak{m}$. Hence it has a unique character $\zeta\in A^*$. Moreover, $\mathfrak{m}$ is nilpotent, see [1, Proposition 8.6]. Therefore, the local algebra A itself as the regular representation is the generalized A-eigenspace corresponding to the eigenvalue ζ as $a-\zeta(a)\in\ker\zeta=\mathfrak{m}$. Note that every finite-dimensional commutative algebra is a direct sum of local algebras, see [1, Theorem 8.7]. In addition, each local summand is a generalized A-eigenspace with the corresponding eigenvalue uniquely determined by the summand, see the first paragraph of the proof of [1, Theorem 8.7]. This part now follows from the fact that V is isomorphic to the regular representation of $\mathfrak{B}$.

It is clearly that the algebra $\mathfrak{B}$ is maximal commutative in $\mathrm{End}(V)$. The last statement follows from the first half of the proof of Lemma 2.7. ■

2.7 Perfect integrability of Gaudin models

The following is our main theorem which asserts Gaudin models with periodic and regular quasi-periodic boundary conditions are perfectly integrable.

Theorem 2.8. *If $\mu\in\mathfrak{h}^*$ is regular or if $\mu=0$, then the $\mathcal{A}_{\boldsymbol{z},\mu}$-module $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ is perfectly integrable.*

Proof. By Theorem 2.5, Gaudin algebra $\mathcal{A}_{\boldsymbol{z},\mu}$ acts on $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ cyclically. Recall that $\rho_{\boldsymbol{\lambda},\boldsymbol{z},\mu}\colon\mathcal{A}_{\boldsymbol{z},\mu}\to\mathrm{End}(\mathcal{M}_{\boldsymbol{\lambda},\mu})$ and $\mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}=\rho_{\boldsymbol{\lambda},\boldsymbol{z},\mu}(\mathcal{A}_{\boldsymbol{z},\mu})$. It remains to show that $\mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$ is Frobenius.

By Lemma 2.6, we can apply Lemma 2.7 for the case $A=\mathcal{A}_{\boldsymbol{z},\mu}$, $\mathfrak{A}=\mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$, $V=\mathcal{M}_{\boldsymbol{\lambda},\mu}$, and $(\cdot|\cdot)=\mathcal{S}_{\boldsymbol{\lambda}}(\cdot,\cdot)$. Therefore we conclude that the algebra $\mathfrak{A}_{\boldsymbol{\lambda},\boldsymbol{z},\mu}$ is a Frobenius algebra. ■

Theorem 2.8 gives the following important facts. By Theorem 2.8, Lemma 1.3, and [21, Corollary 3.3], we see that the joint eigenvectors (up to proportionality) of the Gaudin algebra in $V_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ are in one-to-one correspondence with monodromy-free ${}^L\mathfrak{g}$-opers on the projective line with regular singularities at the points $z_1, \dots, z_\ell, \infty$ and the prescribed residues at the singular points. Here $z_1, \dots, z_\ell$ are *arbitrary* pairwise distinct complex numbers, cf. [21, Corollary 3.4]. Similarly, when $\mathfrak{g}$ is of type B or C (resp. G_2), one deduces from [11, Theorem 4.5] (resp. [9, Theorem 5.8]) that there exists a bijection between joint eigenvectors (up to proportionality) of the Gaudin algebra in $V_{\boldsymbol{\lambda}}^{\mathrm{sing}}$ and self-dual (resp. self-self-dual) spaces of polynomials in a suitable intersection of Schubert cells in Grassmannian.

2.8 Conjecture for general $\boldsymbol{\mu} \in \mathfrak{g}^*$

For an arbitrary $\mu \in \mathfrak{g}^* \cong \mathfrak{g}$, there exists an element $g \in G$ such that $g\mu g^{-1}$ is in the negative Borel part $\mathfrak{b}_- = \mathfrak{n}_- \oplus \mathfrak{h}$. Thus, without loss of generality, we can assume that $\mu \in \mathfrak{b}_-$.

Let $\mathfrak{z}_\mu(\mathfrak{g})$ be the centralizer of μ in $\mathfrak{g}$. It is known that $\mathcal{A}_{\boldsymbol{z},\mu}$ commutes with the diagonal action of $\mathfrak{z}_\mu(\mathfrak{g})$, see [19, Proposition 4].

Let $V_{\boldsymbol{\lambda}}$ be as before. Define $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ as a subspace of $V_{\boldsymbol{\lambda}}$ by

$$\mathcal{M}_{\boldsymbol{\lambda},\mu} := \{v \in V_{\boldsymbol{\lambda}} \,|\, xv = 0, \ \text{for all } x \in \mathfrak{z}_\mu(\mathfrak{g}) \cap \mathfrak{n}_+\}.$$

Then $\mathcal{A}_{\boldsymbol{z},\mu}$ acts on $\mathcal{M}_{\boldsymbol{\lambda},\mu}$.

Conjecture 2.9. *The $\mathcal{A}_{\boldsymbol{z},\mu}$-module $\mathcal{M}_{\boldsymbol{\lambda},\mu}$ is perfectly integrable.*

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