Construction of Two Parametric Deformation of KdV-Hierarchy and Solution in Terms of Meromorphic Functions on the Sigma Divisor of a Hyperelliptic Curve of Genus 3

Takanori AYANO † and *Victor M. BUCHSTABER* ‡

† *Osaka City University, Advanced Mathematical Institute, 3-3-138 Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan*
E-mail: *ayano@sci.osaka-cu.ac.jp*
URL: `https://researchmap.jp/ayano75/?lang=english`

‡ *Steklov Mathematical Institute of Russian Academy of Sciences, 8 Gubkina Street, Moscow, 119991, Russia*
E-mail: *buchstab@mi-ras.ru*

Received November 21, 2018, in final form April 11, 2019; Published online April 27, 2019
https://doi.org/10.3842/SIGMA.2019.032

Abstract. Buchstaber and Mikhailov introduced the polynomial dynamical systems in $\mathbb{C}^4$ with two polynomial integrals on the basis of commuting vector fields on the symmetric square of hyperelliptic curves. In our previous paper, we constructed the field of meromorphic functions on the sigma divisor of hyperelliptic curves of genus 3 and solutions of the systems for $g = 3$ by these functions. In this paper, as an application of our previous results, we construct two parametric deformation of the KdV-hierarchy. This new system is integrated in the meromorphic functions on the sigma divisor of hyperelliptic curves of genus 3. In Section 8 of our previous paper [*Funct. Anal. Appl.* **51** (2017), 162–176], there are miscalculations. In appendix of this paper, we correct the errors.

Key words: Abelian functions; hyperelliptic sigma functions; polynomial dynamical systems; commuting vector fields; KdV-hierarchy

2010 Mathematics Subject Classification: 14K25; 14H40; 14H42; 14H70

1 Introduction

Let V_g be a hyperelliptic curve of genus g defined by

$$V_g = \left\{(X,Y) \in \mathbb{C}^2 \,|\, Y^2 = X^{2g+1} + y_4X^{2g-1} - y_6X^{2g-2} + \cdots + y_{4g}X - y_{4g+2},\, y_i \in \mathbb{C}\right\}. \quad (1.1)$$

A meromorphic function on the Jacobian of V_g is called hyperelliptic function. The theory of hyperelliptic functions has deep relations with that of KdV-hierarchy. The KdV-hierarchy is an infinite system of differential equations defined by

$$U_{t_k} = \chi_k U, \qquad k = 1, 2, \dots,$$

for a function $U = U(t_1, t_2, \dots)$. The functions $\chi_k U$ are determined by the recursion

$$\chi_{k+1}U = \mathcal{R}\chi_k U$$

with the initial condition $\chi_1 = \partial/\partial t_1$, where $\mathcal{R}$ is the Lenard operator

$$\mathcal{R} = \frac{1}{4}\frac{\partial^2}{\partial t_1^2} - U - \frac{1}{2}U_{t_1}\left(\frac{\partial}{\partial t_1}\right)^{-1},$$

where $(\partial/\partial t_1)^{-1}$ implies an integral with respect to t_1. The KdV-equation is obtained for $k=2$

$$U_{t_2} = \frac{1}{4}U_{t_1t_1t_1} - \frac{3}{2}UU_{t_1}.$$

In the theory of hyperelliptic functions associated with the model (1.1), the hyperelliptic sigma functions play an important role. The hyperelliptic sigma functions $\sigma(w_1, w_3, \dots, w_{2g-1})$ are entire functions of g complex variables, which are originally introduced by Klein as a generalization of the Weierstrass elliptic sigma functions. Baker made a significant contribution of the theory of sigma functions: for hyperelliptic curves of genera 2 and 3, he obtained explicit expressions for higher logarithmic derivatives of sigma functions of many variables in the form of polynomials in the second and the third logarithmic derivatives of these functions [2, 3, 4]. Relatively recently it was shown that these differential polynomials give the fundamental equations of mathematical physics, including KdV-hierarchy and KP-equations (see [5, 7, 11]).

The surface determined by the equation $\sigma(w_1, \dots, w_{2g-1}) = 0$ in the Jacobian of V_g is called the sigma divisor and denoted by (σ). Let $\mathcal{F}((\sigma))$ be the field of meromorphic functions on the sigma divisor of the hyperelliptic curves of genus 3. The functions $f \in \mathcal{F}((\sigma))$ are considered as meromorphic functions on $\mathbb{C}^3$ whose restrictions to the sigma divisor (σ) are 6-periodic. In [9], the polynomial dynamical systems in $\mathbb{C}^4$ with two polynomial integrals are constructed on the basis of commuting vector fields on the symmetric square of the hyperelliptic curves V_g. In [1], for $g=3$, the solutions of the systems are constructed in terms of the functions of $\mathcal{F}((\sigma))$.

For $g=2$, the dynamical systems of [9] are related to the KdV-equation [5, 10]. In this paper we consider the case of $g=3$ and construct two parametric deformation of the KdV-hierarchy by using the dynamical systems of [9] (Theorem 5.5). We construct a solution of the new system in terms of functions of $\mathcal{F}((\sigma))$ (Theorem 7.2). If $y_{12} = y_{14} = 0$, then the new system goes to the system of the KdV-hierarchy in [5, Theorem 5.2] (Proposition 6.5). The result of this paper is one of the applications of the results in [1]. In [12], an extention of the sine-Gordon equation is given and a solution is constructed in terms of the al-function on the subvariety in the hyperelliptic Jacobian. The results of this paper can be regarded as an analog of the results of [12] for the KdV-hierarchy. In Section 8, we consider the rational case $(y_4, \dots, y_{14}) = (0, \dots, 0)$ and derive a rational solution of the KdV-hierarchy. This solution is equal to the solution obtained by the rational limit of the hyperelliptic functions of genus 2. This result would give an insight into the degeneration of the sigma functions.

In Section 8 of [1], we derived a solution of the dynamical systems introduced in [9] in the rational case $(y_4, \dots, y_{14}) = (0, \dots, 0)$ for $g=3$. Unfortunately, there are miscalculations. In Appendix A, we correct the errors.

2 The sigma function

For a positive integer g, we set

$$\Delta_g = \big\{(y_4, y_6, \dots, y_{4g+2}) \in \mathbb{C}^{2g} \,|\, Q_g(X) \text{ has a multiple root}\big\},$$

where

$$Q_g(X) = X^{2g+1} + y_4X^{2g-1} - y_6X^{2g-2} + \dots + y_{4g}X - y_{4g+2},$$

and $B_g = \mathbb{C}^{2g}\backslash\Delta_g$. Consider a nonsingular hyperelliptic curve of genus g

$$V_g = \big\{(X,Y) \in \mathbb{C}^2 \,|\, Y^2 = Q_g(X)\big\},$$

where $(y_4, y_6, \ldots, y_{4g+2}) \in B_g$. In this section we recall the definition of the sigma function for the curve V_g (see [7]) and give facts about it which will be used later on. For $(X, Y) \in V_g$, let

$$\mathrm{d}u_{2i-1} = -\frac{X^{g-i}}{2Y}\mathrm{d}X, \qquad 1 \le i \le g,$$

be a basis of the vector space of holomorphic 1-forms on V_g, and let $\mathrm{d}u = {}^t(\mathrm{d}u_1, \mathrm{d}u_3, \ldots, \mathrm{d}u_{2g-1})$. Further, let

$$\mathrm{d}r_{2i-1} = \frac{1}{2Y}\sum_{k=g-i+1}^{g+i-1}(-1)^{g+i-k}(k+i-g)y_{2g+2i-2k-2}X^k\mathrm{d}X, \qquad 1 \le i \le g, \tag{2.1}$$

be meromorphic one forms on V_g with a pole only at ∞. In (2.1) we set $y_0 = 1$ and $y_2 = 0$. For example, for $g = 2$

$$\mathrm{d}r_1 = -\frac{X^2}{2Y}\mathrm{d}X, \qquad \mathrm{d}r_3 = \frac{-y_4X - 3X^3}{2Y}\mathrm{d}X$$

and for $g = 3$

$$\mathrm{d}r_1 = -\frac{X^3}{2Y}\mathrm{d}X, \qquad \mathrm{d}r_3 = -\frac{y_4X^2 + 3X^4}{2Y}\mathrm{d}X,$$
$$\mathrm{d}r_5 = -\frac{y_8X - 2y_6X^2 + 3y_4X^3 + 5X^5}{2Y}\mathrm{d}X.$$

Let $\{\alpha_i, \beta_i\}_{i=1}^g$ be a canonical basis in the one-dimensional homology group of the curve V_g. We define the matrices of periods by

$$2\omega_1 = \left(\int_{\alpha_j}\mathrm{d}u_i\right), \qquad 2\omega_2 = \left(\int_{\beta_j}\mathrm{d}u_i\right), \qquad -2\eta_1 = \left(\int_{\alpha_j}\mathrm{d}r_i\right), \qquad -2\eta_2 = \left(\int_{\beta_j}\mathrm{d}r_i\right).$$

The matrix of normalized periods has the form $\tau = \omega_1^{-1}\omega_2$. Let $\delta = \tau\delta' + \delta''$, $\delta', \delta'' \in \mathbb{R}^g$, be the vectors of Riemann's constants with respect to $(\{\alpha_i, \beta_i\}, \infty)$ and $\delta := {}^t\big({}^t\delta', {}^t\delta''\big)$. Then we have $\delta' = {}^t\big(\frac{1}{2}, \ldots, \frac{1}{2}\big)$ and $\delta'' = {}^t\big(\frac{g}{2}, \frac{g-1}{2}, \ldots, \frac{1}{2}\big)$. The sigma function $\sigma(w)$, $w = {}^t(w_1, w_3, \ldots, w_{2g-1}) \in \mathbb{C}^g$, is defined by

$$\sigma(w) = C\exp\left(\frac{1}{2}{}^tw\eta_1\omega_1^{-1}w\right)\theta[\delta]\big((2\omega_1)^{-1}w, \tau\big),$$

where $\theta[\delta](w)$ is the Riemann's theta function with characteristics δ, which is defined by

$$\theta[\delta](w) = \sum_{n\in\mathbb{Z}^g}\exp\big\{\pi\sqrt{-1}\,{}^t(n+\delta')\tau(n+\delta') + 2\pi\sqrt{-1}\,{}^t(n+\delta')(w+\delta'')\big\},$$

and C is a constant. We set $\wp_{i,j}(w) = -\partial_i\partial_j\log\sigma(w)$, $\sigma_i = \partial_i\sigma$, and $\sigma_{i,j} = \partial_i\partial_j\sigma$, where $\partial_i = \partial/\partial w_i$. We define the period lattice $\Lambda_g = \{2\omega_1m_1 + 2\omega_2m_2 \,|\, m_1, m_2 \in \mathbb{Z}^g\}$ and set $W = \{w \in \mathbb{C}^g \,|\, \sigma(w) = 0\}$.

Proposition 2.1 ([7, Theorem 1.1] and [13, p. 193])**.** *For $m_1, m_2 \in \mathbb{Z}^g$, let $\Omega = 2\omega_1m_1 + 2\omega_2m_2$, and let*

$$A = (-1)^{2({}^t\delta'm_1 - {}^t\delta''m_2) + {}^tm_1m_2}\exp\big({}^t(2\eta_1m_1 + 2\eta_2m_2)(w + \omega_1m_1 + \omega_2m_2)\big).$$

Then

(*i*) $\sigma(w+\Omega) = A\sigma(w)$, *where* $w \in \mathbb{C}^g$,

(*ii*) $\sigma_i(w+\Omega) = A\sigma_i(w)$, $i = 1, 3, \dots, 2g-1$, *where* $w \in W$.

Proposition 2.1(i) implies that $w + \Omega \in W$ for any $w \in W$ and $\Omega \in \Lambda_g$. The surface

$$(\sigma) := \{ w \in \mathbb{C}^g/\Lambda_g \mid \sigma(w) = 0 \}$$

is called the sigma divisor. We set $\deg w_{2k-1} = -(2k-1)$ and $\deg y_{2i} = 2i$, where $1 \le k \le g$, $2 \le i \le 2g+1$. Let $S_{\mu_g}(w)$ be the Schur function associated with the partition $\mu_g = (g, g-1, \dots, 1)$ and set $|\mu_g| = g + (g-1) + \dots + 1$ (see [13, Section 4]).

Theorem 2.2 ([6, Theorem 6.3], [7, Theorem 7.7], [8], [13, Theorem 3])**.** *The sigma function $\sigma(w)$ is an entire function on $\mathbb{C}^g$, and it is given by the series*

$$\sigma(w) = S_{\mu_g}(w) + \sum_{i_1+3i_3+\dots+(2g-1)i_{2g-1} > |\mu_g|} \lambda_{i_1,i_3,\dots,i_{2g-1}} w_1^{i_1} w_3^{i_3} \cdots w_{2g-1}^{i_{2g-1}},$$

where the coefficients $\lambda_{i_1,i_3,\dots,i_{2g-1}} \in \mathbb{Q}[y_4, y_6, \dots, y_{4g+2}]$ are homogeneous polynomials of degree $i_1 + 3i_3 + \dots + (2g-1)i_{2g-1} - |\mu_g|$ if $\lambda_{i_1,i_3,\dots,i_{2g-1}} \neq 0$.

Example 2.3 ([6, Example 4.5], [8], [13, p. 192])**.**

$$S_{(2,1)}(w) = -w_3 + \frac{1}{3}w_1^3, \qquad S_{(3,2,1)}(w) = w_1w_5 - w_3^2 - \frac{1}{3}w_1^3w_3 + \frac{1}{45}w_1^6.$$

3 Rational functions on the symmetric square

In [1], for $g = 3$, the structure of the field of rational functions on the symmetric square of the curve V_3 is described explicitly. These results can be extended for any genus similarly. In this section we describe the structure of the field of rational functions on the symmetric square of the curve V_g.

Let $\mathcal{F}(V_g^2)$ be the field of rational functions on V_g^2 and let J_g be the ideal in $\mathbb{C}[X_1, Y_1, X_2, Y_2]$ generated by the polynomials $Y_1^2 - Q_g(X_1)$ and $Y_2^2 - Q_g(X_2)$. We denote the quotient field of an integral domain R by $\langle R \rangle$. We have

$$\mathcal{F}(V_g^2) = \langle \mathbb{C}[X_1, Y_1, X_2, Y_2]/J_g \rangle.$$

Let $\mathrm{Sym}^2(\mathbb{C}^2)$ be the symmetric square of $\mathbb{C}^2$ and let $\mathcal{F}(\mathrm{Sym}^2(\mathbb{C}^2))$ be the set of rational functions $f(X_1, Y_1, X_2, Y_2) \in \mathbb{C}(X_1, Y_1, X_2, Y_2)$ such that $f(X_1, Y_1, X_2, Y_2) = f(X_2, Y_2, X_1, Y_1)$. Let $\mathrm{Sym}^2(V_g)$ be the symmetric square of the curve V_g and let $\mathcal{F}(\mathrm{Sym}^2(V_g))$ be the set of elements $h \in \mathcal{F}(V_g^2)$ such that there exists a representative $\widetilde{h} \in \mathcal{F}(\mathrm{Sym}^2(\mathbb{C}^2))$ of h. In [1, 9], the following elements of $\mathcal{F}(\mathrm{Sym}^2(\mathbb{C}^2))$ are used

$$a = \frac{X_1 + X_2}{2}, \quad b = \frac{(X_1 - X_2)^2}{4}, \qquad c = \frac{Y_1 - Y_2}{X_1 - X_2}, \qquad d = \frac{Y_1 + Y_2}{2}.$$

Note that the elements a, b, c, and d are algebraically independent and generate the field $\mathcal{F}(\mathrm{Sym}^2(\mathbb{C}^2))$ over $\mathbb{C}$, i.e., $\mathcal{F}(\mathrm{Sym}^2(\mathbb{C}^2)) = \mathbb{C}(a, b, c, d)$. We set

$$M_g = \frac{Y_1^2 - Q_g(X_1) - Y_2^2 + Q_g(X_2)}{X_1 - X_2}, \qquad N_g = Y_1^2 - Q_g(X_1) + Y_2^2 - Q_g(X_2),$$

and $\widetilde{N}_g = -N_g/2 + aM_g$. For example, for $g = 2$, we obtain

$$M_2(a,b,c,d) = -5a^4 - 10a^2b - b^2 + 2cd - y_4(3a^2 + b) + 2y_6a - y_8,$$
$$\widetilde{N}_2(a,b,c,d) = -4a^5 + 4ab^2 - c^2b + 2acd - d^2 + 2y_4(-a^3 + ab) + y_6(a^2 - b) - y_{10},$$

and for $g = 3$, we obtain[1]

$$\begin{aligned} M_3(a,b,c,d) &= 2cd - 7a^6 - 35a^4b - 21a^2b^2 - b^3 - y_4\big(5a^4 + 10a^2b + b^2\big) \\ &\quad + 4y_6\big(a^3 + ab\big) - y_8\big(3a^2 + b\big) + 2y_{10}a - y_{12}, \\ \widetilde{N}_3(a,b,c,d) &= -d^2 - bc^2 + 2acd - 6a^7 - 14a^5b + 14a^3b^2 + 6ab^3 \\ &\quad - 4y_4\big(a^5 - ab^2\big) + y_6\big(3a^4 - 2a^2b - b^2\big) - 2y_8\big(a^3 - ab\big) + y_{10}\big(a^2 - b\big) - y_{14}. \end{aligned}$$

Let A_g be the ideal generated by the polynomials M_g and N_g in the ring $\mathbb{C}[a,b,c,d]$ and let u_i, $i = 2,4,2g-1,2g+1$, denote the elements of $\mathcal{F}\big(V_g^2\big)$ such that $u_2 = a$, $u_4 = b$, $u_{2g-1} = c$, and $u_{2g+1} = d$ in the field $\mathbb{C}(X_1,Y_1,X_2,Y_2)$, i.e., u_2, u_4, u_{2g-1}, and u_{2g+1} are the equivalence classes of a, b, c, and d in $\mathcal{F}\big(V_g^2\big)$, respectively. Note that u_2, u_4, u_{2g-1}, and u_{2g+1} are contained in $\mathcal{F}\big(\mathrm{Sym}^2(V_g)\big)$. Consider the homomorphism

$$\Gamma_g\colon\ \mathbb{C}[a,b,c,d] \to \mathcal{F}\big(\mathrm{Sym}^2(V_g)\big), \qquad a \mapsto u_2, \qquad b \mapsto u_4, \qquad c \mapsto u_{2g-1}, \qquad d \mapsto u_{2g+1}.$$

Then we have $\mathrm{Ker}(\Gamma_g) = A_g$ and the isomorphism [1, Lemma 3.3 and Theorem 3.4]

$$\widetilde{\Gamma}_g\colon\ \langle \mathbb{C}[a,b,c,d]/A_g\rangle \to \mathcal{F}\big(\mathrm{Sym}^2(V_g)\big).$$

The following two *commuting* derivations acting on the field $\mathcal{F}\big(V_g^2\big)$ were used in [1, 9]:

$$\mathcal{L}_{2g-3}^{(g)} = \frac{1}{X_1 - X_2}(\mathcal{D}_2 - \mathcal{D}_1), \qquad \mathcal{L}_{2g-1}^{(g)} = \frac{1}{X_1 - X_2}(X_2\mathcal{D}_1 - X_1\mathcal{D}_2),$$

where

$$\mathcal{D}_k = 2Y_k\partial_{X_k} + Q_g'(X_k)\partial_{Y_k}, \qquad k = 1,2.$$

In [9, Lemmas 16 and 17], $\mathcal{L}_i^{(g)}u_j$ is expressed as a polynomial of u_2, u_4, u_{2g-1}, and u_{2g+1} whose coefficients are in $\mathbb{Q}[y_4,y_6,\dots,y_{4g+2}]$ for any $i = 2g-3, 2g-1$ and $j = 2,4,2g-1,2g+1$. These can be regarded as polynomial dynamical systems in $\mathbb{C}^4$ with coordinates u_2, u_4, u_{2g-1}, and u_{2g+1}. We assume $g = 3$.

Theorem 3.1 ([9, Lemmas 16 and 17])**.** *In the space $\mathbb{C}^4$ with coordinates u_2, u_4, u_5, and u_7, we have the following families of dynamical systems with constant parameters y_4, y_6, y_8, and y_{10}:*

$$\begin{aligned} (I)\qquad & \mathcal{L}_3^{(3)}u_2 = -u_5, \qquad \mathcal{L}_3^{(3)}u_4 = -2u_7, \\ & \mathcal{L}_3^{(3)}u_5 = -35u_2^4 - 42u_2^2u_4 - 3u_4^2 - 2y_4(5u_2^2 + u_4) + 4y_6u_2 - y_8, \\ & \mathcal{L}_3^{(3)}u_7 = -7\big(3u_2^5 + 10u_2^3u_4 + 3u_2u_4^2\big) - 10y_4\big(u_2^3 + u_2u_4\big) \\ & \qquad\qquad + 2y_6\big(3u_2^2 + u_4\big) - 3y_8u_2 + y_{10}, \\ (II)\qquad & \mathcal{L}_5^{(3)}u_2 = u_2u_5 - u_7, \qquad \mathcal{L}_5^{(3)}u_4 = 2(u_2u_7 - u_4u_5), \\ & \mathcal{L}_5^{(3)}u_5 = u_5^2 + 14u_2^5 - 28u_2^3u_4 - 18u_2u_4^2 - 8y_4u_2u_4 + 2y_6\big(u_2^2 + u_4\big) - 2y_8u_2 + y_{10}, \\ & \mathcal{L}_5^{(3)}u_7 = -u_5u_7 + 21u_2^6 + 35u_2^4u_4 - 21u_2^2u_4^2 - 3u_4^3 + 2y_4\big(5u_2^4 - u_4^2\big) \\ & \qquad\qquad - 2y_6\big(3u_2^3 - u_2u_4\big) + y_8\big(3u_2^2 - u_4\big) - y_{10}u_2. \end{aligned}$$

The systems (I) and (II) have common first integrals $H_{12} := M_3(u_2,u_4,u_5,u_7) + y_{12}$ and $H_{14} := \widetilde{N}_3(u_2,u_4,u_5,u_7) + y_{14}$ [9, 10], [1, Theorem 7.1]. Moreover, the system (I) is a Hamiltonian system with the Hamiltonian H_{12} and the Poisson structure determined by $\{u_2,u_7\} = -1/2$, $\{u_4,u_5\} = -1$, $\{u_2,u_4\} = \{u_2,u_5\} = \{u_4,u_7\} = \{u_5,u_7\} = 0$ [10]. The system (II) is a Hamiltonian system with the Hamiltonian H_{14} and the Poisson structure determined by $\{u_2,u_7\} = 1/2$, $\{u_4,u_5\} = 1$, $\{u_2,u_4\} = \{u_2,u_5\} = \{u_4,u_7\} = \{u_5,u_7\} = 0$ [10]. These Hamiltonians are in involution with respect to the Poisson structures and the systems are Liouville integrable [10].

[1] In [1, p. 165], the expression of H_{14} is that of $-H_{14}/2 + u_2H_{12}$ in the notation of [1].

4 Meromorphic functions on the sigma divisor

In [1], for $g = 3$, the field of meromorphic functions on the sigma divisor of the curve V_3 is described. In this section we recall these results.

We assume $g = 3$. Fix any constant vector $(y_4, y_6, y_8, y_{10}, y_{12}, y_{14}) \in B_3$. Let $\mathcal{F}$ be the field of all meromorphic functions on $\mathbb{C}^3$ and let $\mathcal{F}[(\sigma)]$ be the set of meromorphic functions $f \in \mathcal{F}$ satisfying the following two conditions:

- for any point $w \in W$, there exist an open neighborhood $U_1 \subset \mathbb{C}^3$ of this point and two holomorphic functions g and h on U_1 such that the function h does not identically vanish on $U_1 \cap W$ and $f = g/h$ on U_1;
- $f(w + \Omega) = f(w)$ for any $w \in W$ and $\Omega \in \Lambda_3$.

Note that $\mathcal{F}[(\sigma)]$ is a subring in $\mathcal{F}$, but it is not generally a field. Let us consider the Abel–Jacobi map

$$I_3\colon\ \mathrm{Sym}^2(V_3) \to \mathrm{Jac}(V_3) = \mathbb{C}^3/\Lambda_3, \qquad (P_1, P_2) \mapsto \int_\infty^{P_1} \mathrm{d}u + \int_\infty^{P_2} \mathrm{d}u.$$

The Abel–Jacobi map I_3 induces a ring homomorphism

$$I_3^*\colon\ \mathcal{F}[(\sigma)] \to \mathcal{F}\big(\mathrm{Sym}^2(V_3)\big), \qquad f \mapsto f \circ I_3.$$

Let J^* be the set of meromorphic functions $f \in \mathcal{F}[(\sigma)]$ identically vanishing on W. Thus, we have $\mathrm{Ker}\, I_3^* = J^*$. We set $\mathcal{F}((\sigma)) = \mathcal{F}[(\sigma)]/J^*$. Then $\mathcal{F}((\sigma))$ is a field and, by construction, there is an isomorphism of fields (see [1, Section 4])

$$\overline{I_3^*}\colon\ \mathcal{F}((\sigma)) \to \mathcal{F}\big(\mathrm{Sym}^2(V_3)\big).$$

The following meromorphic functions on $\mathbb{C}^3$ are introduced in [1]:

$$f_1 = \frac{\sigma_{1,1}}{\sigma_1}, \qquad f_2 = \frac{\sigma_3}{\sigma_1}, \qquad f_3 = \frac{\sigma_{1,3}}{\sigma_1}, \qquad f_4 = \frac{\sigma_5}{\sigma_1},$$
$$f_5 = \frac{\sigma_{3,3}}{\sigma_1}, \qquad g_5 = \frac{\sigma_{1,5}}{\sigma_1}, \qquad f_7 = \frac{\sigma_{3,5}}{\sigma_1},$$
$$F_2 = -\frac{1}{2} f_2, \qquad F_4 = \frac{1}{4} f_2^2 - f_4, \qquad F_5 = \frac{1}{2}\big(f_1 f_2^2 + f_5 - 2 f_2 f_3\big),$$
$$F_7 = \frac{1}{4}\big(2 f_2^2 f_3 - 2 f_3 f_4 - f_1 f_2^3 + 2 f_1 f_2 f_4 - f_2 f_5 + 2 f_7 - 2 f_2 g_5\big).$$

We have $F_i \in \mathcal{F}[(\sigma)]$ and $I_3^*(F_i) = u_i$ for $i = 2, 4, 5, 7$ (see [1, Proposition 4.1]). In [1], the following derivations acting on the field $\mathcal{F}((\sigma))$ are introduced

$$L_3^{(3)} = \partial_3 - \frac{\sigma_3}{\sigma_1}\partial_1, \qquad L_5^{(3)} = \partial_5 - \frac{\sigma_5}{\sigma_1}\partial_1.$$

Lemma 4.1 ([1, Lemma 6.4])**.** *The relations $\mathcal{L}_3^{(3)} \circ \overline{I_3^*} = \overline{I_3^*} \circ L_3^{(3)}$ and $\mathcal{L}_5^{(3)} \circ \overline{I_3^*} = \overline{I_3^*} \circ L_5^{(3)}$ hold.*

5 Two parametric deformation of KdV-hierarchy

We assume $g = 3$ and consider the following derivations:

$$T_1 := \partial_1 - \frac{\sigma_1}{\sigma_5}\partial_5 = -f_4^{-1} L_5^{(3)}, \qquad T_3 := \partial_3 - \frac{\sigma_3}{\sigma_5}\partial_5 = L_3^{(3)} - f_2 f_4^{-1} L_5^{(3)}.$$

From [1, Lemma 6.2], the commutation relation $[T_1, T_3] = 0$ holds in the Lie algebra of derivations of $\mathcal{F}$. Since the operators $L_3^{(3)}$ and $L_5^{(3)}$ are the derivations of the field $\mathcal{F}((\sigma))$ and $f_2, f_4^{-1} \in \mathcal{F}[(\sigma)]$, the operators T_1 and T_3 are also the derivations of the field $\mathcal{F}((\sigma))$. We consider the following derivations acting on the field $\mathcal{F}\big(V_g^2\big)$

$$\mathcal{T}_1 = -\frac{1}{X_1X_2}\mathcal{L}_5^{(3)} = -\frac{1}{X_1X_2(X_1-X_2)}(X_2\mathcal{D}_1 - X_1\mathcal{D}_2), \tag{5.1}$$

$$\mathcal{T}_3 = \mathcal{L}_3^{(3)} + \frac{X_1+X_2}{X_1X_2}\mathcal{L}_5^{(3)} = \frac{1}{X_1-X_2}(\mathcal{D}_2 - \mathcal{D}_1) + \frac{X_1+X_2}{X_1X_2(X_1-X_2)}(X_2\mathcal{D}_1 - X_1\mathcal{D}_2). \tag{5.2}$$

Proposition 5.1. *The commutation relation* $[\mathcal{T}_1, \mathcal{T}_3] = 0$ *holds.*

Proof. Since $\big[\mathcal{L}_3^{(3)}, \mathcal{L}_5^{(3)}\big] = 0$, the direct calculation shows the proposition. ■

Proposition 5.2. *We have*

$$\mathcal{T}_1X_1 = \frac{-2Y_1}{X_1(X_1-X_2)}, \qquad \mathcal{T}_1Y_1 = \frac{-Q_3'(X_1)}{X_1(X_1-X_2)},$$
$$\mathcal{T}_1X_2 = \frac{2Y_2}{X_2(X_1-X_2)}, \qquad \mathcal{T}_1Y_2 = \frac{Q_3'(X_2)}{X_2(X_1-X_2)},$$
$$\mathcal{T}_3X_1 = \frac{2X_2Y_1}{X_1(X_1-X_2)}, \qquad \mathcal{T}_3Y_1 = \frac{X_2Q_3'(X_1)}{X_1(X_1-X_2)},$$
$$\mathcal{T}_3X_2 = \frac{-2X_1Y_2}{X_2(X_1-X_2)}, \qquad \mathcal{T}_3Y_2 = \frac{-X_1Q_3'(X_2)}{X_2(X_1-X_2)}.$$

Proof. The direct calculation shows the proposition. ■

Lemma 5.3. *The relations* $\mathcal{T}_1 \circ \overline{I_3^*} = \overline{I_3^*} \circ T_1$ *and* $\mathcal{T}_3 \circ \overline{I_3^*} = \overline{I_3^*} \circ T_3$ *hold.*

Proof. From Lemma 4.1, $I_3^*(f_2) = -(X_1+X_2)$, and $I_3^*(f_4) = X_1X_2$, we obtain the lemma. ■

Proposition 5.4. *In the space* $\mathbb{C}^4$ *with coordinates* u_2*,* u_4*,* u_5*, and* u_7*, we have the following families of rational dynamical systems with constant parameters* y_4*,* y_6*,* y_8*, and* y_{10}*:*

$$\mathcal{T}_1u_2 = \frac{u_2u_5 - u_7}{u_4 - u_2^2}, \qquad \mathcal{T}_1u_4 = \frac{2(u_2u_7 - u_4u_5)}{u_4 - u_2^2},$$
$$\mathcal{T}_1u_5 = \big(u_4 - u_2^2\big)^{-1}\big\{u_5^2 + 14u_2^5 - 28u_2^3u_4 - 18u_2u_4^2 - 8y_4u_2u_4 + 2y_6\big(u_2^2 + u_4\big) - 2y_8u_2 + y_{10}\big\},$$
$$\mathcal{T}_1u_7 = \big(u_4 - u_2^2\big)^{-1}\big\{-u_5u_7 + 21u_2^6 + 35u_2^4u_4 - 21u_2^2u_4^2 - 3u_4^3 + 2y_4\big(5u_2^4 - u_4^2\big) - 2y_6\big(3u_2^3 - u_2u_4\big) + y_8\big(3u_2^2 - u_4\big) - y_{10}u_2\big\},$$
$$\mathcal{T}_3u_2 = \frac{2u_2u_7 - u_4u_5 - u_2^2u_5}{u_4 - u_2^2}, \qquad \mathcal{T}_3u_4 = \frac{2\big(2u_2u_4u_5 - u_4u_7 - u_2^2u_7\big)}{u_4 - u_2^2},$$
$$\mathcal{T}_3u_5 = \big(u_4 - u_2^2\big)^{-1}\big\{-2u_2u_5^2 + 7u_2^6 + 63u_2^4u_4 - 3u_2^2u_4^2 - 3u_4^3 + 2y_4\big(5u_2^4 + 4u_2^2u_4 - u_4^2\big) - 8y_6u_2^3 + y_8\big(5u_2^2 - u_4\big) - 2y_{10}u_2\big\},$$
$$\mathcal{T}_3u_7 = \big(u_4 - u_2^2\big)^{-1}\big\{2u_2u_5u_7 - 21u_2^7 - 21u_2^5u_4 - 7u_2^3u_4^2 - 15u_2u_4^3 - 2y_4\big(5u_2^5 + 3u_2u_4^2\big) + 2y_6\big(3u_2^4 + u_4^2\big) - y_8\big(3u_2^3 + u_2u_4\big) + y_{10}\big(u_2^2 + u_4\big)\big\}.$$

Proof. From Theorem 3.1, (5.1), and (5.2), we obtain the proposition. ■

Let $u = 4u_2$ and $v = 2\big(u_4 - u_2^2\big)$. For any $w \in \mathcal{F}\big(\mathrm{Sym}^2(V_3)\big)$, we use the notation $w' = \mathcal{T}_1w$ and $\dot{w} = \mathcal{T}_3w$. Then we obtain the main result of this paper.

Theorem 5.5. *We obtain the following new system that can be called two parametric deformed KdV-hierarchy:*

$$v^4(u''' - 4\dot{u} - 6uu') - 32y_{12}v\dot{u} + 32y_{14}(vu' - 3u\dot{u}) = 0, \tag{5.3}$$

$$v^4(\dot{u}'' - 4u\dot{u} - 2u'v) - 32y_{12}v\dot{v} + 32y_{14}(vv' - 3u\dot{v}) = 0, \tag{5.4}$$

$$\dot{u} = v', \tag{5.5}$$

$$2\dot{v} = vu' - uv'. \tag{5.6}$$

Proof. From Proposition 5.4, the direct calculation shows

$$u_2'' = 2u_4 + 10u_2^2 + y_4 - \frac{y_{12}}{\left(u_4 - u_2^2\right)^2} - \frac{4y_{14}u_2}{\left(u_4 - u_2^2\right)^3}, \tag{5.7}$$

where in the above calculation we deleted the terms u_5u_7 and u_7^2 by using the relations

$$M_3(u_2, u_4, u_5, u_7) = \widetilde{N}_3(u_2, u_4, u_5, u_7) = 0$$

in $\mathcal{F}\left(\mathrm{Sym}^2(V_3)\right)$ (see Section 3). By differentiating the both sides of (5.7) with respect to $\mathcal{T}_1$, we obtain

$$\begin{aligned} u_2''' &= -\frac{4\left(4u_2u_7 + u_4u_5 - 5u_2^2u_5\right)}{u_4 - u_2^2} + 4y_{12}\frac{2u_2u_7 - u_4u_5 - u_2^2u_5}{\left(u_4 - u_2^2\right)^4} \\ &\quad + 4y_{14}\frac{u_4u_7 + 11u_2^2u_7 - 7u_2u_4u_5 - 5u_2^3u_5}{\left(u_4 - u_2^2\right)^5} \\ &= 4\dot{u}_2 + 24u_2u_2' + \frac{4y_{12}\dot{u}_2}{\left(u_4 - u_2^2\right)^3} + \frac{4y_{14}\left\{\left(u_2^2 - u_4\right)u_2' + 6u_2\dot{u}_2\right\}}{\left(u_4 - u_2^2\right)^4}. \end{aligned}$$

Therefore we obtain the equation (5.3). By differentiating the both sides of (5.7) with respect to $\mathcal{T}_3$, we obtain

$$\begin{aligned} \dot{u}_2'' &= -4\frac{u_7\left(u_4 - 9u_2^2\right) + u_2u_5\left(3u_4 + 5u_2^2\right)}{u_4 - u_2^2} + 4y_{12}\frac{u_2u_5\left(3u_4 + u_2^2\right) - u_7\left(u_4 + 3u_2^2\right)}{\left(u_4 - u_2^2\right)^4} \\ &\quad + 4y_{14}\frac{u_5\left(u_4^2 + 18u_2^2u_4 + 5u_2^4\right) - 8u_2u_7\left(u_4 + 2u_2^2\right)}{\left(u_4 - u_2^2\right)^5}, \\ &= 16u_2\dot{u}_2 + 4\left(u_4 - u_2^2\right)u_2' + 2y_{12}\frac{\left(u_4 - u_2^2\right)^{\cdot}}{\left(u_4 - u_2^2\right)^3} - 2y_{14}\frac{6u_2\left(u_2^2 - u_4\right)^{\cdot} + \left(u_2^2 - u_4\right)\left(u_2^2 - u_4\right)'}{\left(u_4 - u_2^2\right)^4}. \end{aligned}$$

Therefore we obtain the equation (5.4). From Proposition 5.4, we obtain the equations (5.5) and (5.6). ■

6 Relation with a curve of genus 2

Let us consider the homomorphism of the field of rational functions $\mathbb{C}(X_1, Y_1, X_2, Y_2)$

$$\psi\colon\ \mathbb{C}(X_1, Y_1, X_2, Y_2) \to \mathbb{C}(X_1, Y_1, X_2, Y_2), \qquad X_i \mapsto X_i, \qquad Y_i \mapsto \frac{Y_i}{X_i}, \qquad i = 1, 2.$$

The map ψ induces the homomorphism

$$\mathrm{Sym}(\psi)\colon\ \mathcal{F}\left(\mathrm{Sym}^2\left(\mathbb{C}^2\right)\right) \to \mathcal{F}\left(\mathrm{Sym}^2\left(\mathbb{C}^2\right)\right).$$

In Section 3, we noted $\mathcal{F}\big(\mathrm{Sym}^2(\mathbb{C}^2)\big) = \mathbb{C}(a,b,c,d)$. The map $\mathrm{Sym}(\psi)$ transforms the generators a, b, c, and d as follows

$$a \mapsto a, \qquad b \mapsto b, \qquad c \mapsto \frac{ac-d}{a^2-b}, \qquad d \mapsto \frac{ad-bc}{a^2-b}. \tag{6.1}$$

Fix any constant vector $(y_4, y_6, y_8, y_{10}) \in \mathbb{C}^4$. We consider a curve V_2

$$V_2 = \big\{(X,Y) \in \mathbb{C}^2 \,|\, Y^2 = X^5 + y_4 X^3 - y_6 X^2 + y_8 X - y_{10}\big\}$$

and a curve $V_{3,2}$

$$V_{3,2} = \big\{(X,Y) \in \mathbb{C}^2 \,|\, Y^2 = X^7 + y_4 X^5 - y_6 X^4 + y_8 X^3 - y_{10} X^2\big\}.$$

The map ψ induces the homomorphism

$$\psi_1\colon\ \mathcal{F}\big(V_2^2\big) \to \mathcal{F}\big(V_{3,2}^2\big), \qquad X_i \mapsto X_i, \qquad Y_i \mapsto \frac{Y_i}{X_i}, \qquad i = 1,2.$$

Proposition 6.1. *The map ψ_1 is an isomorphism between $\mathcal{F}\big(V_2^2\big)$ and $\mathcal{F}\big(V_{3,2}^2\big)$.*

Proof. We can consider the map

$$\psi_2\colon\ \mathcal{F}\big(V_{3,2}^2\big) \to \mathcal{F}\big(V_2^2\big), \qquad X_i \to X_i, \qquad Y_i \to X_i Y_i, \qquad i = 1,2.$$

Then we can check

$$\psi_2 \circ \psi_1 = \mathrm{id}_{\mathcal{F}\left(V_2^2\right)}, \qquad \psi_1 \circ \psi_2 = \mathrm{id}_{\mathcal{F}\left(V_{3,2}^2\right)}.$$ ■

Proposition 6.2. *The map ψ_1 is an isomorphism between $\mathcal{F}\big(\mathrm{Sym}^2(V_2)\big)$ and $\mathcal{F}(\mathrm{Sym}^2(V_{3,2}))$, and we have*

$$\psi_1(u_2) = u_2, \qquad \psi_1(u_4) = u_4, \qquad \psi_1(u_3) = \frac{u_2u_5 - u_7}{u_2^2 - u_4}, \qquad \psi_1(u_5) = \frac{u_2u_7 - u_4u_5}{u_2^2 - u_4}. \tag{6.2}$$

Proof. Since $\psi_1\big(\mathcal{F}\big(\mathrm{Sym}^2(V_2)\big)\big) \subset \mathcal{F}\big(\mathrm{Sym}^2(V_{3,2})\big)$ and $\psi_2\big(\mathcal{F}\big(\mathrm{Sym}^2(V_{3,2})\big)\big) \subset \mathcal{F}\big(\mathrm{Sym}^2(V_2)\big)$, the map ψ_1 is an isomorphism between $\mathcal{F}\big(\mathrm{Sym}^2(V_2)\big)$ and $\mathcal{F}\big(\mathrm{Sym}^2(V_{3,2})\big)$. From (6.1) we obtain the relations (6.2). ■

Proposition 6.3. *We have*

$$\mathcal{T}_1 \circ \psi_1 = \psi_1 \circ \mathcal{L}_1^{(2)}, \qquad \mathcal{T}_3 \circ \psi_1 = \psi_1 \circ \mathcal{L}_3^{(2)}.$$

Proof. By the direct calculation we can check $\mathcal{T}_1 \circ \psi_1(X_i) = \psi_1 \circ \mathcal{L}_1^{(2)}(X_i)$ and $\mathcal{T}_1 \circ \psi_1(Y_i) = \psi_1 \circ \mathcal{L}_1^{(2)}(Y_i)$ for $i = 1,2$. Therefore we obtain $\mathcal{T}_1 \circ \psi_1 = \psi_1 \circ \mathcal{L}_1^{(2)}$. Similary, we obtain $\mathcal{T}_3 \circ \psi_1 = \psi_1 \circ \mathcal{L}_3^{(2)}$. ■

We assume $(y_4, y_6, y_8, y_{10}) \in B_2$. Let us consider the Abel–Jacobi map of the curve V_2

$$I_2\colon\ \mathrm{Sym}^2(V_2) \to \mathrm{Jac}(V_2) = \mathbb{C}^2/\Lambda_2, \qquad (P_1, P_2) \mapsto \int_\infty^{P_1} \mathrm{d}u + \int_\infty^{P_2} \mathrm{d}u.$$

Let $\mathcal{F}(\mathrm{Jac}(V_2))$ be the field of meromorphic functions on the Jacobian $\mathrm{Jac}(V_2)$. The Abel–Jacobi map I_2 induces the isomorphism of the fields:

$$I_2^*\colon\ \mathcal{F}(\mathrm{Jac}(V_2)) \to \mathcal{F}\big(\mathrm{Sym}^2(V_2)\big), \qquad f \mapsto f \circ I_2.$$

As derivations of $\mathcal{F}(V_2^2)$, the derivations $\mathcal{L}_1^{(2)}$ and $\mathcal{L}_3^{(2)}$ can be expressed as [1, Section 6][2]

$$\mathcal{L}_1^{(2)} = \frac{1}{X_1 - X_2}\{-2Y_1(\mathrm{d}_{P_1}/\mathrm{d}X_1) + 2Y_2(\mathrm{d}_{P_2}/\mathrm{d}X_2)\},$$
$$\mathcal{L}_3^{(2)} = \frac{1}{X_1 - X_2}\{2X_2Y_1(\mathrm{d}_{P_1}/\mathrm{d}X_1) - 2X_1Y_2(\mathrm{d}_{P_2}/\mathrm{d}X_2)\},$$

where $P_i = (X_i, Y_i) \in V_2$, $i = 1, 2$, we regard X_i and Y_i as meromorphic functions on V_2^2, and $\mathrm{d}X_i$ and $\mathrm{d}Y_i$ are the total differentials of X_i and Y_i for $i = 1, 2$. Let us describe the action of these operators in more detail. For $g(P_1, P_2) \in \mathcal{F}(V_2^2)$, $\mathrm{d}_{P_i}(g)$ is the total differential of g as a meromorphic function of P_i. Then $\mathrm{d}_{P_i}(g)/\mathrm{d}X_i$ is the meromorphic function on V_2^2 determined uniquely by $\mathrm{d}_{P_i}(g) = (\mathrm{d}_{P_i}(g)/\mathrm{d}X_i) \cdot \mathrm{d}X_i$. We consider the following derivations of $\mathcal{F}(\mathrm{Jac}(V_2))$

$$L_1^{(2)} = \frac{\partial}{\partial w_1}, \qquad L_3^{(2)} = \frac{\partial}{\partial w_3}.$$

Lemma 6.4. *We have* $\mathcal{L}_1^{(2)} \circ I_2^* = I_2^* \circ L_1^{(2)}$ *and* $\mathcal{L}_3^{(2)} \circ I_2^* = I_2^* \circ L_3^{(2)}$.

Proof. Set $h \in \mathcal{F}(\mathrm{Jac}(V_2))$ and $w = I_2((P_1, P_2))$. We have

$$\begin{aligned}\mathcal{L}_1^{(2)} \circ I_2^*(h) &= \mathcal{L}_1^{(2)}(h(w)) = \frac{1}{X_1 - X_2}\{-2Y_1(\mathrm{d}_{P_1}(h(w))/\mathrm{d}X_1) + 2Y_2(\mathrm{d}_{P_2}(h(w))/\mathrm{d}X_2)\}\\ &= \frac{1}{X_1 - X_2}\left\{-2Y_1\left(-\frac{X_1}{2Y_1}h_1(w) - \frac{1}{2Y_1}h_3(w)\right) + 2Y_2\left(-\frac{X_2}{2Y_2}h_1(w) - \frac{1}{2Y_2}h_3(w)\right)\right\}\\ &= h_1(w) = I_2^* \circ L_1^{(2)}(h),\end{aligned}$$

where $h_i = \partial_{w_i} h$. The lemma's assertions for the operator $L_3^{(2)}$ are proved similarly. ■

By the isomorphism

$$I_2^*\colon\ \mathcal{F}(\mathrm{Jac}(V_2)) \simeq \mathcal{F}(\mathrm{Sym}^2(V_2)),$$

the operators $L_1^{(2)}$ and $L_3^{(2)}$ transform into $\mathcal{L}_1^{(2)}$ and $\mathcal{L}_3^{(2)}$, respectively. By the isomorphism

$$\psi_1\colon\ \mathcal{F}(\mathrm{Sym}^2(V_2)) \simeq \mathcal{F}(\mathrm{Sym}^2(V_{3,2})),$$

the operators $\mathcal{L}_1^{(2)}$ and $\mathcal{L}_3^{(2)}$ transform into the operators $\mathcal{T}_1$ and $\mathcal{T}_3$, respectively.

Proposition 6.5. *If* $y_{12} = y_{14} = 0$ *and* $v \neq 0$, *the system* (5.3), (5.4), *and* (5.5) *in Theorem* 5.5 *goes to the system of the KdV-hierarchy in* [5, Theorem 5.2].

Proof. If $y_{12} = y_{14} = 0$ and $v \neq 0$, the equation (5.3) goes to

$$u''' - 4\dot{u} - 6uu' = 0.$$

Therefore we obtain

$$u''' = 3(u^2)' + 4\dot{u}.$$

On the other hand, the equation (5.4) goes to

$$\dot{u}'' - 4u\dot{u} - 2u'v = 0.$$

[2]In [1], these expressions are given for $g = 3$ and we can prove them for any g similarly.

From (5.5), the above equation becomes

$$v''' - 4uv' - 2u'v = 0.$$

From (5.6), we obtain

$$0 = v''' - 4uv' - 2u'v = v''' - 3uv' + 2\dot{v} - vu' - 2u'v = v''' - 3uv' - 3u'v + 2\dot{v}.$$

Thus we have

$$v''' = 3(uv)' - 2\dot{v}.$$ ■

7 Solution of the two parametric deformed KdV-hierarchy

We assume $g = 3$. The two parametric deformed KdV-hierarchy introduced in Theorem 5.5 is integrated in functions of $\mathcal{F}((\sigma))$. Consider a constant vector $(y_4, y_6, y_8, y_{10}, y_{12}, y_{14}) \in B_3$. Take a point $w^{(0)} = \big(w_1^{(0)}, w_3^{(0)}, w_5^{(0)}\big) \in W$ such that $\sigma_i\big(w^{(0)}\big) \neq 0$ for $i = 1, 5$. In a sufficiently small open neighborhood $U_2 \subset \mathbb{C}^2$ of $\big(w_1^{(0)}, w_3^{(0)}\big)$, there exists a uniquely determined holomorphic function $\xi(w_1, w_3)$ on U_2 such that $\xi\big(w_1^{(0)}, w_3^{(0)}\big) = w_5^{(0)}$, $(w_1, w_3, \xi(w_1, w_3)) \in W$ for any point $(w_1, w_3) \in U_2$, and $\sigma_i(w_1, w_3, \xi(w_1, w_3)) \neq 0$ for any point $(w_1, w_3) \in U_2$ and $i = 1, 5$.

Lemma 7.1. *For any $F \in \mathcal{F}$, we have*

$$\frac{\partial}{\partial w_1} F(w_1, w_3, \xi(w_1, w_3)) = T_1(F), \qquad \frac{\partial}{\partial w_3} F(w_1, w_3, \xi(w_1, w_3)) = T_3(F).$$

Proof. According to the definition of the function ξ, we have

$$\frac{\partial \xi}{\partial w_1} = -\frac{\sigma_1}{\sigma_5}(w_1, w_3, \xi(w_1, w_3)), \qquad \frac{\partial \xi}{\partial w_3} = -\frac{\sigma_3}{\sigma_5}(w_1, w_3, \xi(w_1, w_3)).$$

Therefore

$$\frac{\partial}{\partial w_1} F(w_1, w_3, \xi(w_1, w_3)) = \partial_1 F - \frac{\sigma_1}{\sigma_5}(\partial_5 F) = T_1(F),$$
$$\frac{\partial}{\partial w_3} F(w_1, w_3, \xi(w_1, w_3)) = \partial_3 F - \frac{\sigma_3}{\sigma_5}(\partial_5 F) = T_3(F).$$ ■

We set $U(x,t) = 4F_2(x, t, \xi(x,t))$ and $V(x,t) = 2\big\{F_4(x, t, \xi(x,t)) - F_2(x, t, \xi(x,t))^2\big\}$. For a function $K(x,t)$, we use the notation $K' = \partial_x K$, $\dot{K} = \partial_t K$.

Theorem 7.2. *The functions U and V satisfy the two parametric deformed KdV-hierarchy*

$$V^4(U''' - 4\dot{U} - 6UU') - 32y_{12}V\dot{U} + 32y_{14}(VU' - 3U\dot{U}) = 0,$$
$$V^4(\dot{U}'' - 4U\dot{U} - 2U'V) - 32y_{12}V\dot{V} + 32y_{14}(VV' - 3U\dot{V}) = 0,$$
$$\dot{U} = V',$$
$$2\dot{V} = VU' - UV'.$$

Proof. From Lemmas 5.3, 7.1, and Theorem 5.5, we obtain the theorem. ■

8 The rational limit

Let the constant vector $(y_4,\dots,y_{14})\in\mathbb{C}^6$ tend to zero. Then, according to Theorem 2.2, the sigma function $\sigma(w_1,w_3,w_5)$ transforms into the Schur–Weierstrass polynomial (see [6])

$$\sigma = w_1w_5 - w_3^2 - \frac{1}{3}w_1^3w_3 + \frac{1}{45}w_1^6.$$

As a result, we obtain

$$\sigma_1 = w_5 - w_1^2w_3 + \frac{2}{15}w_1^5, \qquad \sigma_3 = -2w_3 - \frac{1}{3}w_1^3, \qquad \sigma_5 = w_1,$$
$$\sigma_{11} = -2w_1w_3 + \frac{2}{3}w_1^4, \qquad \sigma_{13} = -w_1^2, \qquad \sigma_{15} = 1, \qquad \sigma_{33} = -2, \qquad \sigma_{35} = 0.$$

Take a point $w^{(0)} = \big(w_1^{(0)}, w_3^{(0)}, w_5^{(0)}\big)\in W$ such that $\sigma_i\big(w^{(0)}\big)\neq 0$ for $i=1,5$. In a sufficiently small open neighborhood $U_2\subset\mathbb{C}^2$ of $\big(w_1^{(0)}, w_3^{(0)}\big)$, there exists a uniquely determined holomorphic function $\xi(w_1,w_3)$ on U_2 such that $\xi\big(w_1^{(0)}, w_3^{(0)}\big) = w_5^{(0)}$, $\big(w_1,w_3,\xi(w_1,w_3)\big)\in W$ for any point $(w_1,w_3)\in U_2$, and $\sigma_i(w_1,w_3,\xi(w_1,w_3))\neq 0$ for any point $(w_1,w_3)\in U_2$ and $i=1,5$. On U_2 the function $\xi(w_1,w_3)$ is expressed as

$$\xi(w_1,w_3) = \frac{w_3^2}{w_1} + \frac{1}{3}w_1^2w_3 - \frac{1}{45}w_1^5.$$

We have

$$U(x,t) = 4F_2(x,t,\xi(x,t)) = -2\frac{\sigma_3(x,t,\xi(x,t))}{\sigma_1(x,t,\xi(x,t))} = \frac{6x\big(x^3+6t\big)}{\big(x^3-3t\big)^2},$$
$$V(x,t) = 2\big\{F_4(x,t,\xi(x,t)) - F_2(x,t,\xi(x,t))^2\big\} = -2\frac{\sigma_5(x,t,\xi(x,t))}{\sigma_1(x,t,\xi(x,t))} = -\frac{18x^2}{\big(x^3-3t\big)^2}.$$

Theorem 8.1. *The function $U(x,t)$ is a solution of the KdV-hierarchy*

$$U''' - 4\dot{U} - 6UU' = 0, \tag{8.1}$$
$$\dot{U}'' - 4U\dot{U} - 2U'V = 0, \qquad \dot{U} = V'.$$

Proof. This theorem follows from Theorem 7.2. ■

Let us consider the curve V_2 of genus 2. It is well known that the function $D(x,t) = 2\wp_{1,1}(x,t)$ is a solution of the KdV-hierarchy (see [5, Theorems 5.1 and 5.2], [7, Theorem 3.6], [11, Theorem 6])

$$D''' - 4\dot{D} - 6DD' = 0, \qquad \dot{D}'' - 4D\dot{D} - 2D'E = 0, \qquad \dot{D} = E'. \tag{8.2}$$

where $E(x,t) = 2\wp_{1,3}(x,t)$. Let the constant vector $(y_4,y_6,y_8,y_{10})\in\mathbb{C}^4$ tends to zero. Then we have

$$\sigma = -w_3 + \frac{1}{3}w_1^3, \qquad \sigma_1 = w_1^2, \qquad \sigma_3 = -1, \qquad \sigma_{11} = 2w_1, \qquad \sigma_{13} = 0,$$
$$D(x,t) = 2\frac{\sigma_1^2 - \sigma_{11}\sigma}{\sigma^2} = \frac{6x\big(x^3+6t\big)}{\big(x^3-3t\big)^2}, \qquad E(x,t) = 2\frac{\sigma_1\sigma_3 - \sigma_{13}\sigma}{\sigma^2} = -\frac{18x^2}{\big(x^3-3t\big)^2},$$

which is a solution of the KdV-hierarchy (8.2). Note that $U(x,t) = D(x,t)$ and $V(x,t) = E(x,t)$ in the rational limit.

Remark 8.2. The Lax form of the KdV-equation (8.1) is

$$\frac{\mathrm{d}L}{\mathrm{d}t} = [A, L],$$

where

$$L = -\partial_x^2 + U, \qquad A = \partial_x^3 - \frac{3}{2}U\partial_x - \frac{3}{4}U_x.$$

The Schrödinger equation with the potential $U(x,t) = 6x\big(x^3+6t\big)/\big(x^3-3t\big)^2$ is

$$-\frac{\mathrm{d}\eta}{\mathrm{d}x^2} + U(x,t)\eta(x,t) = \lambda\eta(x,t), \tag{8.3}$$

where $\eta(x,t)$ is an unknown function and $\lambda \in \mathbb{C}$. At each fixed time t, the function $U(x,t) = 6x\big(x^3+6t\big)/\big(x^3-3t\big)^2$ decreases as $O\big(1/x^2\big)$ for $x \to \infty$. Thus, we obtain a meaningful solution of the corresponding Schrödinger equation (8.3) in the point of view of physics.

A Correction of Section 8 in [1]

In [1, Section 8], we derived the solution of the dynamical systems introduced in [9] in the case of $(y_4, \dots, y_{14}) = (0, \dots, 0)$ for $g = 3$. Unfortunately, there are miscalculations in the expressions of F_5, F_7 and Examples 1 and 3. In this appendix we correct the errors.

We assume $g = 3$ and consider the curve V_3. Let the constant vector $(y_4, \dots, y_{14}) \in \mathbb{C}^6$ tend to zero. Then the sigma function $\sigma(w_1, w_3, w_5)$ transforms into the Schur–Weierstrass polynomial as Section 8. Take a point $w^{(0)} = \big(w_1^{(0)}, w_3^{(0)}, w_5^{(0)}\big) \in W$ such that $\sigma_1(w^{(0)}) \neq 0$. In a sufficiently small open neighborhood $U_3 \subset \mathbb{C}^2$ of $\big(w_3^{(0)}, w_5^{(0)}\big)$, there exists a uniquely determined holomorphic function $\varphi(w_3, w_5)$ on U_3 such that $\varphi\big(w_3^{(0)}, w_5^{(0)}\big) = w_1^{(0)}$, $(\varphi(w_3, w_5), w_3, w_5) \in W$ for any point $(w_3, w_5) \in U_3$, and $\sigma_1(\varphi(w_3, w_5), w_3, w_5) \neq 0$ for any point $(w_3, w_5) \in U_3$. We have the relation

$$\varphi(w_3, w_5)^6 = 15\varphi(w_3, w_5)^3 w_3 + 45w_3^2 - 45\varphi(w_3, w_5)w_5. \tag{A.1}$$

The definitions of F_2, F_4, F_5, F_7 (see Section 4) and the relation (A.1) imply[3]

$$\begin{aligned}
&F_2(\varphi(w_3, w_5), w_3, w_5) = 5\big(\varphi^3 + 6w_3\big)/K_2,\\
&F_4(\varphi(w_3, w_5), w_3, w_5) = 15\big(-\varphi^3 w_3 + 15\varphi w_5 - 15w_3^2\big)/K_4,\\
&F_5(\varphi(w_3, w_5), w_3, w_5) = \big(-15{w_5}^2 - 195\varphi^2 w_3 w_5 + 8\varphi^5 w_5 + 135\varphi {w_3}^3 + 63\varphi^4 {w_3}^2\big)/K_5,\\
&F_7(\varphi(w_3, w_5), w_3, w_5) = -15\varphi\big(25\varphi^2 {w_5}^2 - 45\varphi {w_3}^2 w_5 - 15\varphi^4 w_3 w_5 + 27{w_3}^4 + 18\varphi^3 {w_3}^3\big)/K_7,
\end{aligned}$$

where

$$\begin{aligned}
&K_2 = 2\big(2\varphi^5 - 15\varphi^2 w_3 + 15w_5\big), \qquad K_4 = 4\big(-8\varphi^5 w_5 + 27\varphi^4 w_3^2 - 30\varphi^2 w_3 w_5 + 15w_5^2\big),\\
&K_5 = 3\big(5{w_5}^3 + 165\varphi^2 w_3 {w_5}^2 + 14\varphi^5 {w_5}^2 - 585\varphi {w_3}^3 w_5 - 111\varphi^4 {w_3}^2 w_5\\
&\qquad + 405{w_3}^5 + 189\varphi^3 {w_3}^4\big),\\
&K_7 = 2\big(15{w_5}^4 - 4380\varphi^2 w_3 {w_5}^3 - 208\varphi^5 {w_5}^3 + 28620\varphi {w_3}^3 {w_5}^2 + 3042\varphi^4 {w_3}^2 {w_5}^2\\
&\qquad - 24300{w_3}^5 w_5 - 11583\varphi^3 {w_3}^4 w_5 + 2187\varphi^2 {w_3}^6 + 729\varphi^5 {w_3}^5\big).
\end{aligned}$$

The dynamical systems introduced in [9] for $g = 3$ and $y_4 = y_6 = y_8 = y_{10} = 0$ are as follows

$$(I) \qquad \partial_t G_2 = -G_5, \qquad \partial_t G_4 = -2G_7, \qquad \partial_t G_5 = -35G_2^4 - 42G_2^2 G_4 - 3G_4^2,$$

[3] We used the computer algebra system Maxima for calculation.

$$\partial_t G_7 = -7\big(3G_2^5 + 10G_2^3 G_4 + 3G_2 G_4^2\big),$$

$$(II) \quad \partial_\tau G_2 = G_2 G_5 - G_7, \qquad \partial_\tau G_4 = 2(G_2 G_7 - G_4 G_5),$$

$$\partial_\tau G_5 = G_5^2 + 14G_2^5 - 28G_2^3 G_4 - 18G_2 G_4^2,$$

$$\partial_\tau G_7 = -G_5 G_7 + 21G_2^6 + 35G_2^4 G_4 - 21G_2^2 G_4^2 - 3G_4^3.$$

Theorem A.1 ([1, Theorem 8.1])**.** *The set of functions* (G_2, G_4, G_5, G_7)*, where*

$$G_i(t,\tau) = F_i(\varphi(t,\tau), t, \tau), \qquad i = 2,4,5,7,$$

is a solution of the dynamical systems (I) *and* (II)*.*

Example 1. Let $w^{(0)} = (0,0,1)$. Then $w^{(0)} \in W$ and $\sigma_1\big(w^{(0)}\big) \neq 0$. The function $\varphi(t,1)$ of t has the following expansion in a neighborhood of the point $t = 0$:

$$\varphi(t,1) = t^2 + \frac{1}{3}t^7 + \frac{14}{45}t^{12} + \cdots.$$

According to Theorem A.1, the set of functions (G_2, G_4, G_5, G_7), where

$$G_i(t) = F_i(\varphi(t,1), t, 1), \qquad i = 2,4,5,7,$$

is a solution of the dynamical system (I).

Example 3. Let $w^{(0)} = (q,1,0)$ such that $q^6 = 15q^3 + 45$. Then $w^{(0)} \in W$, $\sigma_1\big(w^{(0)}\big) = q^2\big(2q^3 - 15\big)/15 \neq 0$, and $\varphi(t,0) = qt^{1/3}$ around $t = 1$. We have

$$F_2\big(qt^{1/3}, t, 0\big) = \frac{q}{6}t^{-2/3}, \qquad F_4\big(qt^{1/3}, t, 0\big) = -\frac{5\big(q^3+15\big)}{36q^4}t^{-4/3},$$

$$F_5\big(qt^{1/3}, t, 0\big) = \frac{q}{9}t^{-5/3}, \qquad F_7\big(qt^{1/3}, t, 0\big) = -\frac{5(2q^3+3)}{54q\big(q^3+3\big)}t^{-7/3}.$$

According to Theorem A.1, the set of functions (G_2, G_4, G_5, G_7), where

$$G_i(t) = F_i\big(qt^{1/3}, t, 0\big), \qquad i = 2,4,5,7,$$

is a solution of the dynamical system (I).

Acknowledgements

The authors are grateful to Shigeki Matsutani for telling the relations between his paper [12] and our results in this paper, and for valuable comments.

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