

Haantjes Structures for the Jacobi–Calogero Model and the Benenti Systems*

Giorgio TONDO [†] and Piergiulio TEMPESTA ^{‡§}

[†] *Dipartimento di Matematica e Geoscienze, Università degli Studi di Trieste, piazz.le Europa 1, I-34127 Trieste, Italy*
E-mail: tondo@units.it

[‡] *Departamento de Física Teórica II, Facultad de Físicas, Universidad Complutense, 28040 – Madrid, Spain*

[§] *Instituto de Ciencias Matemáticas (CSIC-UAM-UC3M-UCM), C/ Nicolás Cabrera, No 13–15, 28049 Madrid, Spain*
E-mail: ptempest@ucm.es
URL: <http://www.icmat.es/p.tempesta>

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Abstract. In the context of the theory of symplectic-Haantjes manifolds, we construct the Haantjes structures of generalized Stäckel systems and, as a particular case, of the quasi-bi-Hamiltonian systems. As an application, we recover the Haantjes manifolds for the rational Calogero model with three particles and for the Benenti systems.

Key words: Haantjes tensor; symplectic-Haantjes manifolds; Stäckel systems; quasi-bi-Hamiltonian systems; Benenti systems

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To Sergio Benenti, on the occasion of his 70th birthday.

1 Introduction

The purpose of this paper is to present an application of the theory recently developed in [39], aiming at a characterization of integrability and separability for classical Hamiltonian systems by means of the geometry of Haantjes operators.

In [39] the notion of Haantjes manifolds has been introduced in the realm of integrability for finite-dimensional systems (see also [15, 24] and [25] for a treatment of integrable hierarchies of PDEs from a different perspective). Haantjes tensors [18] represent a natural generalization of the well known Nijenhuis tensors [33, 34, 35]; the $(1, 1)$ tensor fields with vanishing Haantjes tensor encode many crucial features of an integrable system, especially in relation with the property of separability of the associated Hamilton–Jacobi (H–J) equation. Due to the fact that these new geometrical structures are also endowed with a standard symplectic structure, we call them symplectic-Haantjes or $\omega\mathcal{H}$ manifolds.

The theory of Haantjes manifolds is very general: it encompasses essentially all known results concerning integrability of finite classical systems.

A particularly relevant class of integrable models are the *separable* ones: one can find a coordinate system in which the H–J equation takes a separated form. In this field, the contribution of Benenti has been crucial. One of his theorems [2], particularly useful for us, states that a family

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of Hamiltonian functions $\{H_i\}_{1 \leq i \leq n}$ is separable in a set of canonical coordinates $(\mathbf{q}; \mathbf{p})$ if and only if they are in separable involution, that is, they satisfy the conditions

$$\{H_i, H_j\}_k = \frac{\partial H_i}{\partial q_k} \frac{\partial H_j}{\partial p_k} - \frac{\partial H_i}{\partial p_k} \frac{\partial H_j}{\partial q_k} = 0, \quad 1 \leq k \leq n,$$

where no summation over k is understood.

The problem of separation of variables (SoV) can be recast and treated in our approach. Indeed, under mild hypotheses, from the Haantjes structure associated with an integrable system one can derive a set of coordinates, that we shall call the *Darboux–Haantjes coordinates*, representing separation coordinates for the system.

The Haantjes structure associated with an integrable system allows a tensorial description (i.e., intrinsic) of its main properties.

In this paper, we shall prove that some of the most relevant separable systems, namely the generalized Stäckel systems, are described in terms of suitable Haantjes structures, which are responsible of their separation properties. On one hand, by identifying the Stäckel matrix in their definition with a Vandermonde type matrix, one can obtain the class of quasi-bi-Hamiltonian systems. On the other hand, the standard Stäckel systems are re-obtained by specializing properly the Stäckel functions.

Another important aspect of our analysis is that two particularly relevant integrable systems, namely the Jacobi–Calogero model and the family of Benenti systems, can be studied in the framework of Haantjes geometry.

The paper is organized as follows. In Section 2, we review the main properties of Haantjes operators and Lenard–Haantjes chains, together with the problem of separation of variables in the context of Haantjes geometry. In Section 3, we construct the Haantjes structure of the generalized Stäckel systems. As a particular case, the Haantjes structure for the quasi-bi-Hamiltonian systems and for a Goldfish model [12] is obtained in Section 4. In Section 5, the classical Stäckel systems and their Killing tensors are discussed. In Section 6, three Haantjes structures for the Jacobi–Calogero model are presented. In the final Section 7, Haantjes manifolds for the family of Benenti systems are obtained.

2 The Haantjes geometry: a brief review

We shall first summarize some basic results of the theory of Nijenhuis and Haantjes tensors, following [18, 33] (see also [16, 34, 35]).

2.1 Nijenhuis and Haantjes operators

Let M be a differentiable manifold and $\mathbf{L}: TM \rightarrow TM$ be a (1,1) tensor field (that is, a field of linear operators on the tangent space at each point of M).

Definition 2.1. The *Nijenhuis torsion* of \mathbf{L} is the skew-symmetric (1,2) tensor field defined by

$$\mathcal{T}_{\mathbf{L}}(X, Y) := \mathbf{L}^2[X, Y] + [\mathbf{L}X, \mathbf{L}Y] - \mathbf{L}([X, \mathbf{L}Y] + [\mathbf{L}X, Y]),$$

where $X, Y \in TM$ and $[\cdot, \cdot]$ denotes the commutator of two vector fields.

Given a set of local coordinates $\mathbf{x} = (x_1, \dots, x_n)$ in M , the Nijenhuis torsion takes the form

$$(\mathcal{T}_{\mathbf{L}})_{jk}^i = \sum_{\alpha=1}^n \left(\frac{\partial \mathbf{L}_k^i}{\partial x_\alpha} \mathbf{L}_j^\alpha - \frac{\partial \mathbf{L}_j^i}{\partial x_\alpha} \mathbf{L}_k^\alpha + \left(\frac{\partial \mathbf{L}_j^\alpha}{\partial x_k} - \frac{\partial \mathbf{L}_k^\alpha}{\partial x_j} \right) \mathbf{L}_\alpha^i \right),$$

with $n^2(n-1)/2$ independent components.

Definition 2.2. The *Haantjes tensor* associated with \mathbf{L} is the (1,2) tensor field defined by

$$\mathcal{H}_{\mathbf{L}}(X, Y) := \mathbf{L}^2 \mathcal{T}_{\mathbf{L}}(X, Y) + \mathcal{T}_{\mathbf{L}}(\mathbf{L}X, \mathbf{L}Y) - \mathbf{L}(\mathcal{T}_{\mathbf{L}}(X, \mathbf{L}Y) + \mathcal{T}_{\mathbf{L}}(\mathbf{L}X, Y)).$$

The Haantjes tensor is skew-symmetric due to the skew-symmetry of the Nijenhuis torsion. Locally, we have

$$(\mathcal{H}_{\mathbf{L}})_{jk}^i = \sum_{\alpha, \beta=1}^n (\mathbf{L}_{\alpha}^i \mathbf{L}_{\beta}^{\alpha} (\mathcal{T}_{\mathbf{L}})_{jk}^{\beta} + (\mathcal{T}_{\mathbf{L}})_{\alpha\beta}^i \mathbf{L}_j^{\alpha} \mathbf{L}_k^{\beta} - \mathbf{L}_{\alpha}^i ((\mathcal{T}_{\mathbf{L}})_{\beta k}^{\alpha} \mathbf{L}_j^{\beta} + (\mathcal{T}_{\mathbf{L}})_{j\beta}^{\alpha} \mathbf{L}_k^{\beta})).$$

A discussion of some basic examples can be found in [39].

The powers of a single Haantjes operator generate a module over the ring of smooth functions on M , as can be inferred from the results in [8, 9]. Our main definitions are the following.

Definition 2.3. A Haantjes (Nijenhuis) field of operators is a field of operators whose associated Haantjes (Nijenhuis) tensor vanishes identically.

Definition 2.4. A field of operators \mathbf{L} is said to be *semisimple* if it is diagonalizable at each point \mathbf{x} of M .

The following important result, proved by Haantjes, establishes the conditions assuring that the generalized eigen-distribution $\mathcal{D}_i := \text{Ker}(\mathbf{L} - l_i \mathbf{I})^{\rho_i}$ of \mathbf{L} (ρ_i denotes the Riesz index of the eigenvalue l_i) is integrable.

Theorem 2.5 ([18]). *Let \mathbf{L} be a generic field of operators, and assume that the rank of each generalized eigen-distribution \mathcal{D}_i is independent of $\mathbf{x} \in M$. The vanishing of the Haantjes tensor*

$$\mathcal{H}_{\mathbf{L}}(X, X') = 0, \quad \forall X, X' \in TM \tag{2.1}$$

is a sufficient condition to ensure the integrability of each distribution \mathcal{D}_i and of any direct sum $\mathcal{D}_i \oplus \mathcal{D}_j \oplus \dots \oplus \mathcal{D}_k$ (where all indices i, j, \dots, k are different). In addition, if \mathbf{L} is semisimple, the converse is also true.

We remind that a reference frame is a set of n vector fields $\{Y_1, \dots, Y_n\}$ which form a basis of the tangent space $T_{\mathbf{x}}U$ at each point \mathbf{x} belonging to an open set $U \subseteq M$. Two frames $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_n\}$ are said to be equivalent if n nowhere vanishing smooth functions f_i do exist such that

$$X_i = f_i(\mathbf{x}) Y_i, \quad i = 1, \dots, n.$$

A natural or coordinate frame $\{\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\}$ is the frame associated to a local chart $\{(x_1, \dots, x_n)\}$.

Definition 2.6. An *integrable* frame is a reference frame equivalent to a natural frame.

In other words, to say that a frame $\{Y_1, \dots, Y_n\}$ is integrable there must exist a local chart (x_1, \dots, x_n) and n nowhere vanishing functions f_i such that

$$Y_i = f_i(\mathbf{x}) \frac{\partial}{\partial x_i}, \quad i = 1, \dots, n.$$

Proposition 2.7 ([6]). *A reference frame in a differentiable manifold M is an integrable frame if and only if it satisfies any of the two equivalent conditions:*

- *each two-dimensional distribution generated by any two vector fields Y_i, Y_j is Frobenius integrable;*
- *each $(n - 1)$ -dimensional distribution E_i generated by all the vector fields except Y_i is Frobenius integrable.*

Then, under the hypotheses of Theorem 2.5, we can interpret equation (2.1) as the sufficient condition that ensures the existence of a suitable integrable generalized eigen-frame of \mathbf{L} . Furthermore, if \mathbf{L} is semisimple, condition (2.1) is also necessary.

2.2 Symplectic-Haantjes manifolds: an outline

The symplectic-Haantjes manifolds, or $\omega\mathcal{H}$ manifolds, have been introduced in [39]. As we shall see in the subsequent sections, these structures allow to formulate the theory of Hamiltonian integrable systems in a natural geometric language.

Definition 2.8. A symplectic-Haantjes or $\omega\mathcal{H}$ manifold $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$ is a symplectic manifold of dimension $2n$, endowed with n endomorphisms of the tangent bundle of M

$$\mathbf{K}_\alpha: TM \mapsto TM, \quad \alpha = 0, \dots, n-1,$$

which satisfy the following conditions:

- The operator \mathbf{K}_0 is the identity operator in TM

$$\mathbf{K}_0 = I.$$

- Their Haantjes tensor vanishes identically, that is

$$\mathcal{H}_{\mathbf{K}_\alpha}(X, Y) = 0, \quad \forall X, Y \in TM, \quad \alpha = 0, \dots, n-1.$$

- The endomorphisms are compatible with ω (or equivalently, with the corresponding symplectic operator $\mathbf{\Omega} := \omega^\flat$), namely

$$\mathbf{K}_\alpha^T \mathbf{\Omega} = \mathbf{\Omega} \mathbf{K}_\alpha, \quad \alpha = 0, \dots, n-1, \quad (2.2)$$

where $\mathbf{K}_\alpha^T: T^*M \mapsto T^*M$ is the transposed operator of \mathbf{K}_α .

- The endomorphisms are compatible with each others, in the sense that they form a commutative ring

$$\mathbf{K}_\alpha \mathbf{K}_\beta = \mathbf{K}_\beta \mathbf{K}_\alpha, \quad \alpha, \beta = 0, \dots, n-1, \quad (2.3)$$

and generate a module \mathcal{K} over the ring of smooth functions on M , that is, they satisfy

$$\mathcal{H}_{\left(\sum_{\alpha=0}^{n-1} a_\alpha(\mathbf{x}) \mathbf{K}_\alpha\right)}(X, Y) = 0, \quad \forall X, Y \in TM, \quad (2.4)$$

where $a_\alpha(\mathbf{x})$ are arbitrary smooth functions on M .

The $(n+1)$ -tuple $(\omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$ is called the $\omega\mathcal{H}$ structure associated with the $\omega\mathcal{H}$ manifold, and the module (ring) \mathcal{K} is called the Haantjes module (ring).

In other words, we require that the endomorphisms \mathbf{K}_α and any operator belonging to the module (ring) \mathcal{K} be a Haantjes operator compatible with ω and with the original Haantjes operators $\{\mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1}\}$.

2.3 Lenard–Haantjes chains

Despite the relevance of Lenard chains in soliton hierarchies, especially in the construction of integrals of motion in involution [21, 22, 27], their importance in the theory of separation of variables for finite-dimensional Hamiltonian systems has been acknowledged only recently (see [13, 14, 23, 29, 31, 32, 38, 41, 42]). The natural extension of the original notion of Lenard chain to the context of the Haantjes geometry is proposed below.

Definition 2.9. Let $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$ be a $2n$ -dimensional $\omega\mathcal{H}$ manifold and let $\{H_j\}_{1 \leq j \leq n}$ be n independent functions which satisfy the following relations

$$dH_j = \mathbf{K}_\alpha^T dH, \quad j = \alpha + 1, \quad \alpha = 0, \dots, n-1, \quad H := H_1.$$

Under these conditions, we shall say that the functions $\{H_j\}_{1 \leq j \leq n}$ form a Lenard–Haantjes chain generated by the function H .

The relevance of Lenard–Haantjes chains is clarified by the following

Proposition 2.10. *Let M be a $2n$ -dimensional $\omega\mathcal{H}$ manifold and $\{H_j\}_{1 \leq j \leq n}$ be n smooth independent functions forming a Lenard–Haantjes chain. Then, the foliation generated by these functions is Lagrangian. Consequently, each Hamiltonian system with Hamiltonian functions H_j , $1 \leq j \leq n$ is integrable by quadratures.*

Proof. By virtue of the classical Arnold–Liouville theorem, it is sufficient to prove that the functions H_j belonging to a Lenard–Haantjes chain are in involution w.r.t. the Poisson bracket defined by the symplectic form ω . In fact, if we denote by $\mathbf{P} := \Omega^{-1}$ the Poisson operator induced by the symplectic form, we obtain

$$\{H_j, H_k\} = \langle dH_j, \mathbf{P}dH_k \rangle = \langle \mathbf{K}_\alpha^T dH, \mathbf{P}\mathbf{K}_\beta^T dH \rangle = \langle dH, \mathbf{K}_\alpha \mathbf{P}\mathbf{K}_\beta^T dH \rangle = 0,$$

as the operator $\mathbf{K}_\alpha \mathbf{P}\mathbf{K}_\beta^T$ is skew-symmetric by virtue of the compatibility condition (2.2). ■

Remark 2.11. Given a $\omega\mathcal{H}$ manifold, the purpose of its Haantjes operators is to provide a Lenard–Haantjes chain of n integrals of motion in involution. To this end, n independent Haantjes operators are required.

2.4 Darboux–Haantjes coordinates and separation of variables in $\omega\mathcal{H}$ manifolds

By analogy with the classical Darboux coordinates, in Haantjes geometry the Darboux–Haantjes coordinates (DH) are a set of distinguished local symplectic coordinates, which simultaneously diagonalize every Haantjes operator.

Definition 2.12. Let $(M, \omega, \mathbf{K}_0, \mathbf{K}_1, \dots, \mathbf{K}_{n-1})$ be a $\omega\mathcal{H}$ manifold. A set of local coordinates $(q_1, \dots, q_n; p_1, \dots, p_n)$ will be said to be a set of Darboux–Haantjes (DH) coordinates if the symplectic form in these coordinates assumes the *Darboux form*

$$\omega = \sum_{i=1}^n dp_i \wedge dq_i$$

and each Haantjes operator diagonalizes:

$$\mathbf{K}_\alpha = \sum_{i=1}^n l_i^{(\alpha)}(\mathbf{q}, \mathbf{p}) \left(\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right), \quad \alpha = 0, \dots, n-1,$$

with $l_i^{(0)} = 1$, $i = 1, \dots, n$.

In [39] we have shown that a semisimple Haantjes structure which admits a maximal generator, that is a cyclic Haantjes operator with n distinct eigenvalues, provides DH coordinates. They turn out to be *separation variables* for each Hamiltonian function belonging to an associated Lenard–Haantjes chain. In particular, see Theorems 57 and 59 of [39] for the statement and the

proof of the main results concerning the existence of separation variables for $\omega\mathcal{H}$ manifolds. By virtue of these general theorems, such structures take the form

$$\mathbf{K}_\alpha = \sum_{i=1}^n \frac{\frac{\partial H_{\alpha+1}}{\partial p_i}}{\frac{\partial H_1}{\partial p_i}} \left(\frac{\partial}{\partial q_i} \otimes dq_i + \frac{\partial}{\partial p_i} \otimes dp_i \right), \quad \alpha = 0, \dots, n-1, \quad (2.5)$$

in any set of separable Darboux coordinates (\mathbf{q}, \mathbf{p}) for the Hamiltonian functions H_α , $\alpha = 0, \dots, n-1$.

3 Generalized Stäckel systems

The purpose of this section is to determine the Haantjes structure for a huge class of Hamiltonian systems of Stäckel type, that we shall call the generalized Stäckel systems. Specializing conveniently their Stäckel matrix and functions, we can also derive, in a direct way, both families of quasi-bi-Hamiltonian and classical Stäckel systems.

Proposition 3.1 (generalized Stäckel systems). *Let us consider the Hamiltonian functions [1]*

$$H_j = \sum_{k=1}^n \frac{\tilde{S}_{jk}}{\det(\mathbf{S})} f_k(q_k, p_k), \quad j = 1, \dots, n, \quad (3.1)$$

where S_{ij} are the elements of a Stäckel matrix $\mathbf{S}(\mathbf{q})$ (i.e., an invertible matrix whose i -th row depends on the coordinate q_i only), and \tilde{S}_{jk} denotes the cofactor of the element S_{kj} . They belong to the Lenard–Haantjes chain

$$\mathbf{K}_{j-1}^T dH_1 = dH_j, \quad j = 1, \dots, n, \quad (3.2)$$

where \mathbf{K}_{j-1} are the Haantjes operators defined by

$$\mathbf{K}_{j-1} := \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \left(\frac{\partial}{\partial q_r} \otimes dq_r + \frac{\partial}{\partial p_r} \otimes dp_r \right), \quad j = 1, \dots, n. \quad (3.3)$$

Proof. The operators (3.3) are a special case of the operators (2.5) for the Stäckel Hamiltonian functions (3.1). ■

Remark 3.2. It is known that the Stäckel matrix $\mathbf{S}(\mathbf{q})$ is not unique. In fact, multiplying the i -th row of a given Stäckel matrix for an arbitrary function $F_i(q_i)$, one obtains a different Stäckel matrix for the same coordinate web. However, it should be noted that, although the Hamiltonian functions (3.1) transform into

$$H_j \mapsto \sum_{k=1}^n \frac{\tilde{S}_{jk}}{F_k(q_k) \det(\mathbf{S})} f_k(q_k, p_k), \quad j = 1, \dots, n,$$

the Haantjes operators (3.3) stay *invariant*, as their eigenvalues turn out to be

$$\frac{\tilde{S}_{jr}}{F_r(q_r)} \frac{F_r(q_r)}{\tilde{S}_{1r}} \equiv \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}}.$$

For this reason, we could say that the Haantjes operators (3.3) are the tensorial representation of a given Stäckel web in $T^*\mathcal{Q}$.

Remark 3.3. The Haantjes operators (3.3) are independent of the functions $f_k(q_k, p_k)$, that appear in the Hamiltonians (3.1) only. They are called the *Stäckel* functions and are characteristic functions of the Haantjes web, according to [39].

By choosing as Stäckel functions $f_k = \psi_k(p_k)$, $f_k = W_k(q_k)$ and $f_k = \psi_k(p_k) + W_k(q_k)$, where $\psi_k(p_k)$ and $W_k(q_k)$ are arbitrary smooth functions of their argument, from equation (3.2) and Proposition 2.10, we obtain the following result.

Corollary 3.4. *The functions*

$$T_j := \sum_{k=1}^n \frac{\tilde{S}_{jk}}{\det(\mathbf{S})} \psi_k(p_k), \quad V_j := \sum_{k=1}^n \frac{\tilde{S}_{jk}}{\det(\mathbf{S})} W_k(q_k) \quad j = 1, \dots, n, \quad (3.4)$$

are elements of the Haantjes chains

$$\mathbf{K}_{j-1}^T dT_j = dT_j, \quad \mathbf{K}_{j-1}^T dV_j = dV_j, \quad j = 1, \dots, n. \quad (3.5)$$

Therefore, they fulfill the involution relations

$$\{T_i, T_j\} = 0, \quad \{V_i, V_j\} = 0, \quad \{T_i, V_j\} + \{V_i, T_j\} = 0, \quad i, j = 1, \dots, n. \quad (3.6)$$

Using [39, Theorem 57], stating the existence of generators of a Haantjes structure, we can derive the following result.

Proposition 3.5. *The Haantjes structure of a (generalized) Stäckel system admits as generator the Haantjes operator*

$$\mathbf{K} := \sum_{r=1}^n \lambda_r(\mathbf{q}) \left(\frac{\partial}{\partial q_r} \otimes dq_r + \frac{\partial}{\partial p_r} \otimes dp_r \right), \quad (3.7)$$

where $\lambda_r(\mathbf{q})$ are arbitrary smooth functions of the coordinates (q_1, \dots, q_n) , with $\lambda_r \neq \lambda_j$ at any point of \mathcal{Q} , except possibly for a closed set.

In fact,

$$\mathbf{K}_{j-1} = \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \pi_r(\mathbf{K}), \quad j = 1, \dots, n,$$

where the operators

$$\pi_r(\mathbf{K}) := \frac{\prod_{\substack{i=1 \\ i \neq r}}^n (\mathbf{K} - \lambda_i \mathbf{I})}{\prod_{\substack{i=1 \\ i \neq r}}^n (\lambda_r - \lambda_i)} = \frac{\partial}{\partial q_r} \otimes dq_r + \frac{\partial}{\partial p_r} \otimes dp_r, \quad r = 1, \dots, n,$$

are the elements of the so-called Lagrange interpolation basis $\mathcal{B}_{\text{int}} = \{\pi_1(\mathbf{K}), \dots, \pi_n(\mathbf{K})\}$ associated to the operator \mathbf{K} , and represent a basis of the Haantjes module \mathcal{K} .

Remark 3.6. The representation of the Haantjes operators (3.3) on the cyclic basis $\mathcal{B}_{\text{cycl}} = \{\mathbf{I}, \mathbf{K}, \mathbf{K}^2, \dots, \mathbf{K}^{n-1}\}$ associated to the Haantjes operator (3.7) can be obtained by observing that the transition matrix between the cyclic basis and the interpolation basis is given by the Vandermonde matrix of the eigenvalues of (3.7)

$$[\mathbf{I}]_{\mathcal{B}_{\text{cycl}}}^{\mathcal{B}_{\text{int}}} = \mathbf{V}(\mathbf{q}) = \begin{bmatrix} 1 & \lambda_1^2 & \dots & \lambda_1^{n-1} \\ 1 & \lambda_2^2 & \dots & \lambda_2^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \lambda_n^2 & \dots & \lambda_n^{n-1} \end{bmatrix}.$$

Proposition 3.7. *The Haantjes operators (3.3) can be projected along the fibers $T^*\mathcal{Q}$ onto the operators*

$$\tilde{\mathbf{K}}_{j-1} := \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \frac{\partial}{\partial q_r} \otimes dq_r, \quad j = 1, \dots, n-1, \quad (3.12)$$

that are still Haantjes operators in the configuration space \mathcal{Q} and are compatible with each other, that is, fulfill the relations (2.3), (2.4). Moreover, the Haantjes generator (3.7) as well can be projected onto the operator

$$\tilde{\mathbf{K}} = \sum_{r=1}^n \lambda_r(\mathbf{q}) \frac{\partial}{\partial q_r} \otimes dq_r, \quad j = 1, \dots, n-1. \quad (3.13)$$

Such an operator is a Haantjes operator in \mathcal{Q} and generates the Haantjes operators (3.12) according to the relations

$$\tilde{\mathbf{K}}_{j-1} = \sum_{r=1}^n \frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} \pi_r(\tilde{\mathbf{K}}) = \sum_{i=0}^{n-1} a_i^{(j-1)}(\mathbf{q}) \tilde{\mathbf{K}}^i, \quad j = 1, \dots, n.$$

Proof. The components $\tilde{S}_{jr}/\tilde{S}_{1r}$ of the Haantjes operators (3.3) in $T^*\mathcal{Q}$, as well as the eigenvalues of \mathbf{K} , in the separation coordinates $(\mathbf{q}; \mathbf{p})$ depend on the coordinates \mathbf{q} only. Therefore, the operators (3.3) and (3.7) can be projected along the fibers of $T^*\mathcal{Q}$. Moreover, the projected operators inherit the properties (2.3), (2.4) from (3.3). ■

4 Quasi-bi-Hamiltonian systems

We derive here the Haantjes structure of a large class of separable systems with n degrees of freedom introduced in [29], that includes a Goldfish system by F. Calogero and the L -systems of Benenti (whose discussion is postponed to Section 7). Geometrically, such systems can be interpreted as reductions of Gelfand–Zakarevich systems of maximal rank to a symplectic submanifold of a suitable bi-Hamiltonian manifold [14]. To this aim, we have to choose each eigenvalue of the Haantjes generator to be dependent only on the homologous coordinate, so that the generator (3.7) becomes the following *Nijenhuis* operator

$$\mathbf{N} := \sum_{r=1}^n \lambda_r(q_r) \left(\frac{\partial}{\partial q_r} \otimes dq_r + \frac{\partial}{\partial p_r} \otimes dp_r \right). \quad (4.1)$$

As before, we assume that its eigenvalues $\lambda_r(q_r)$ are arbitrary smooth functions of their argument, with the restriction that $\lambda_r \neq \lambda_j$ at any point of \mathcal{Q} , except possibly for a closed set. Accordingly, we can choose as Stäckel matrix in equation (3.1) the (reverse) Vandermonde matrix of the eigenvalues of (4.1)

$$\mathbf{S}(\mathbf{q}) = \mathbf{V}_R = \begin{bmatrix} \lambda_1^{n-1}(q_1) & \lambda_1^{n-2}(q_1) & \dots & 1 \\ \lambda_2^{n-1}(q_2) & \lambda_2^{n-2}(q_2) & \dots & 1 \\ \dots & \dots & \dots & \dots \\ \lambda_n^{n-1}(q_n) & \lambda_n^{n-2}(q_n) & \dots & 1 \end{bmatrix}.$$

Computing its inverse, we find that

$$(\mathbf{V}_R^{-1})_{jk} = \frac{(\tilde{V}_R)_{jk}}{\det(\tilde{V}_R)} = \frac{\partial c_k}{\partial \lambda_j} \frac{1}{\prod_{\substack{r=1 \\ r \neq j}}^n (\lambda_j - \lambda_r)}.$$

Here the functions c_k are the (opposite of the) coefficients of the minimal polynomial of \mathbf{N} .

Thus, we have obtained the class of separable Hamiltonian functions

$$H_k = \sum_{i=1}^n \frac{\partial c_k}{\partial \lambda_i} \frac{f_i(q_i, p_i)}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)}, \quad k = 1, \dots, n, \quad (4.2)$$

that has been discussed in [7, 43] in the framework of quasi-bi-Hamiltonian (QBH) systems.

Using equation (3.3), and the relations

$$\frac{\tilde{S}_{jr}}{\tilde{S}_{1r}} = \frac{(\tilde{V}_R)_{jr}}{(\tilde{V}_R)_{1r}} = \frac{\partial c_j}{\partial \lambda_r},$$

we find that such systems admit the simple Haantjes structure $(T^*\mathcal{Q}, \omega, \mathbf{K}_0 = \mathbf{I}, \mathbf{K}_1, \dots, \mathbf{K}_n)$, given by

$$\mathbf{K}_{j-1} := \sum_{r=1}^n \frac{\partial c_j}{\partial \lambda_r} \left(\frac{\partial}{\partial q_r} \otimes dq_r + \frac{\partial}{\partial p_r} \otimes dp_r \right), \quad j = 1, \dots, n-1. \quad (4.3)$$

They can be projected onto the Haantjes operators on \mathcal{Q} :

$$\tilde{\mathbf{K}}_{j-1} := \sum_{r=1}^n \frac{\partial c_j}{\partial \lambda_r} \frac{\partial}{\partial q_r} \otimes dq_r, \quad j = 1, \dots, n-1. \quad (4.4)$$

Proposition 4.1. *The Haantjes operators (4.3) of a QBH system are the elements of the control basis (3.9) associated to the Nijenhuis operator (4.1).*

Proof. It is sufficient to compute explicitly the transition matrix (3.11) and to observe that the i -th column of such matrix coincides with the eigenvalues of the Haantjes operators (4.3). ■

Due to the fact that the Haantjes operators (4.3) are generated by the Nijenhuis operator (4.1) through the relations (3.8), the Lenard–Haantjes chain formed by the Hamiltonian functions (4.2) is an example of generalized Lenard chain (see, e.g., [38, 42]).

4.1 A Goldfish system

In 1996, Calogero studied a solvable system (already introduced by him in 1978) whose Hamiltonian function in canonical coordinates (q_i, p_i) reads

$$H = \sum_{i=1}^n \frac{e^{ap_i}}{\prod_{\substack{j=1 \\ j \neq i}}^n (q_i - q_j)}. \quad (4.5)$$

The corresponding Newton equations are

$$\ddot{q}_k = 2 \sum_{\substack{i=1 \\ i \neq k}}^n \frac{\dot{q}_k \dot{q}_i}{(q_k - q_i)}.$$

This model is the simplest representative of a large class of solvable models called *Goldfish* systems (see [12] and reference therein).

In the papers [30, 43], it was proved that the Goldfish system (4.5) and the generalized one with Hamiltonian function

$$H = \sum_{i=1}^n \left(\frac{e^{ap_i}}{\prod_{\substack{j=1 \\ j \neq i}}^n (q_i - q_j)} + bq_i \right), \quad b \in \mathbb{R}, \quad (4.6)$$

and with Newton equations

$$\ddot{q}_k = 2 \sum_{\substack{i=1 \\ i \neq k}}^n \frac{\dot{q}_k \dot{q}_i}{(q_k - q_i)} - ab\dot{q}_k,$$

admit a quasi-bi-Hamiltonian structure which ensures the separability of the associated H-J equation directly in the symplectic coordinates $(\mathbf{q}; \mathbf{p})$. We wish to point out that the generalized system (4.6) admits also the $\omega\mathcal{H}$ structure (4.3), shared by all quasi-bi-Hamiltonian systems. Therefore, in turn, it belongs to the generalized Stäckel class (3.1).

The Hamiltonian function of the Goldfish system (4.6) arises from equation (4.2) with $\lambda_i \equiv q_i$, $i = 1, \dots, n$, with the following choice of the Stäckel functions

$$f_i := e^{ap_i} + bq_i^n,$$

taking also into account the Jacobi identity [1]

$$\sum_{i=1}^n \frac{q_i^n}{\prod_{\substack{j=1 \\ j \neq i}}^n (q_i - q_j)} = \sum_{i=1}^n q_i.$$

5 Classical systems of Stäckel

The classical separable Stäckel systems arise from equation (3.1) by choosing as Stäckel functions the homogeneous quadratic functions in the momenta

$$f_k := \frac{1}{2} p_k^2 + W_k(q_k). \quad (5.1)$$

The functions $W_k(q_k)$ are components of the so-called Stäckel multiplier [36]. With this choice, the Hamiltonian function (3.1) takes the form

$$H = \frac{1}{2} \sum_{j=1}^n g^j(\mathbf{q}) p_j^2 + V(\mathbf{q}),$$

where the functions

$$g^j(\mathbf{q}) = \frac{\tilde{S}_{1j}}{\det(\mathbf{S})}$$

can be interpreted as the diagonal components $g^j := g^{jj}$ of the inverse of a metric tensor g over the configuration space \mathcal{Q} :

$$\mathbf{G} := \sum_{j=1}^n g^j \frac{\partial}{\partial q_j} \otimes \frac{\partial}{\partial q_j}. \quad (5.2)$$

Also,

$$V(\mathbf{q}) = \sum_{j=1}^n g^j W_j$$

is the potential energy. The presence of the metric (5.2) allows one to construct the contravariant form of the Haantjes operators (3.12), which are still diagonal, with components

$$(K_{j-1})^{ii} = (K_{j-1})_i^i g^i = \frac{\tilde{S}_{ji}}{\det(\mathbf{S})}, \quad i, j = 1, \dots, n.$$

Proposition 5.1. *The tensor fields (3.12) are Killing tensors for the metric (5.2) and are in involution with respect to the Schouten bracket of two symmetric contravariant tensors.*

Proof. The result follows immediately from the first involution relation (3.6). ■

6 The Jacobi–Calogero model

In this section, we analyze in the framework of Haantjes geometry the celebrated rational Calogero model describing particles on a line, interacting with an inverse square potential [11]. We shall limit ourselves to the case of three particles, already introduced by Jacobi in 1866 [20], but totally forgotten till his contribution was re-discovered in [37]. The Jacobi–Calogero Hamiltonian function reads

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V_{\text{Cal}}, \quad (6.1)$$

where the potential energy V_{Cal} is

$$V_{\text{Cal}} = \frac{a}{(x-y)^2} + \frac{a}{(y-z)^2} + \frac{a}{(z-x)^2}, \quad a \in \mathbb{R},$$

and (x, y, z) are the coordinates of the three particles on the line. The configuration space of this system is $\mathcal{Q} = \mathcal{E}^3 \setminus \Delta$, where \mathcal{E}^3 denotes the 3-dimensional Euclidean affine space and $\Delta := \{x = y, x = z, y = z\}$ the set of the collision planes. Its cotangent bundle $T^*\mathcal{Q} \simeq \mathcal{Q} \times \mathcal{E}^3$ is the phase space of the model.

Two bi-Hamiltonian structures has been worked out for this system. However, in the first one [26], the computation of a second Poisson operator \mathbf{P}_1 and therefore of a Nijenhuis operator $\mathbf{N} := \mathbf{P}_1 \Omega$ seems to be prohibitively complicated and has not been carried out explicitly. The other bi-Hamiltonian structure [44] is an “irregular” one and does not provide integrals of motion different from the Hamiltonian (6.1).

Here, we will compute five Haantjes operators that turn out to be very simple, that is, at most quadratic in the coordinates and momenta, and provide integrals of motion by means of three independent Lenard–Haantjes chains.

As is well known, the model is *maximally superintegrable*, namely, it admits five independent integrals of motion in involution [28, 40]. Furthermore, it is also *multi-separable*. In fact, in the interesting paper [5], it has been proved that, besides the known circular cylindrical coordinates [10], there are four other webs in which the associate HJ equation is separable: spherical, parabolic, oblate spheroidal and prolate spheroidal. All such webs have a common axis of rotational symmetry. Using the first three of them, we will be able to construct the Haantjes structures of the model. The other two webs, oblate and prolate spheroidal, do not provide further independent Haantjes structures. Specifically, we will write down the Calogero Hamiltonian function (6.1) (the source) and two integrals of motion (the \mathbf{K} -images) in each

of the separable webs and we will apply Theorem 59 of [39], that assures the construction of a Haantjes structure for which the separable coordinates are DH coordinates.

Due to the absence of external force fields, the linear momentum

$$p_x + p_y + p_z$$

is a (linear) integral of motion. In the configuration space \mathcal{Q} , it is equivalent to the conserved scalar quantity $(\vec{p} \cdot \vec{u})$, where \vec{p} and \vec{u} are the vectors

$$\vec{p} := p_x \vec{e}_x + p_y \vec{e}_y + p_z \vec{e}_z, \quad \vec{u} := \vec{e}_x + \vec{e}_y + \vec{e}_z,$$

and $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ is a basis of three orthonormal vectors in \mathbb{E}_3 . This fact amounts to the rotational symmetry of the model around the axis (O, \vec{u}) , that is, the straight line passing through the origin of the coordinates O and parallel to the vector \vec{u} . Thus, such an axis is a symmetry axis for each of the separable webs above-mentioned. Following [5], we consider the integral of motion

$$\begin{aligned} H_2 &= \frac{1}{6} (\vec{L}_0 \cdot \vec{u})^2 + \left| \vec{r} \times \frac{\vec{u}}{|\vec{u}|} \right|^2 V_{\text{Cal}} = \frac{1}{6} ((yp_z - zp_y) + (zp_x - xp_z) + (xp_y - yp_x))^2 \\ &\quad + \frac{1}{3} ((x-y)^2 + (x-z)^2 + (y-z)^2) V_{\text{Cal}}, \end{aligned}$$

related to the axial angular momentum. This integral has a privileged role for it is separable in each of the separated webs above-mentioned. Thus, writing down the integral and the Hamiltonian function (6.1) in one of the separated webs, and using equation (2.5) we obtain a diagonal operator (consequently a Haantjes one) which in cartesian coordinates reads

$$\mathbf{K}_1 = \left[\begin{array}{c|c} \mathbf{A} & \mathbf{0}_3 \\ \hline \mathbf{B} & \mathbf{A} \end{array} \right],$$

where

$$\begin{aligned} \mathbf{A} &= \frac{1}{3} \begin{bmatrix} (y-z)^2 & (y-z)(z-x) & (y-z)(x-y) \\ (y-z)(z-x) & (x-z)^2 & (z-x)(x-y) \\ (y-z)(x-y) & (z-x)(x-y) & (x-y)^2 \end{bmatrix}, \\ \mathbf{B} &= \frac{1}{3} ((x-y)p_z + (y-z)p_x + (z-x)p_y) \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}. \end{aligned}$$

Such an operator provides

$$dH_2 = \mathbf{K}_1^T dH,$$

which is the first element common to the three Lenard–Haantjes chains presented in equations (6.2).

Now, we shall focus on the separable webs.

6.1 Cylindrical Haantjes operator

Let us consider the integral of the (square of the) linear momentum

$$H_{\text{cil}} = \frac{1}{2} (\vec{p} \cdot \vec{u})^2 = \frac{1}{2} (p_x + p_y + p_z)^2.$$

Once we write it in cylindrical circular coordinates with axes (O, \vec{u}) together with the Hamiltonian function (6.1), according to equation (2.5) we can define a second uniform Haantjes operator, given in cartesian coordinates by

$$\mathbf{K}_{\text{cil}} = \left[\begin{array}{c|c} \mathbf{A}_{\text{cil}} & \mathbf{0}_3 \\ \hline \mathbf{0}_3 & \mathbf{A}_{\text{cil}} \end{array} \right], \quad \text{where} \quad \mathbf{A}_{\text{cil}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

6.2 Spherical Haantjes operator

Analogously, we consider the following integral of motion

$$\begin{aligned} H_{\text{sph}} &= \frac{1}{2} |\vec{L}_O|^2 + |\vec{r}|^2 V_{\text{Cal}} \\ &= \frac{1}{2} ((yp_z - zp_y)^2 + (zp_x - xp_z)^2 + (xp_y - yp_x)^2) + (x^2 + y^2 + z^2) V_{\text{Cal}} \end{aligned}$$

related to the (square) module of the angular momentum. From the expression of this integral in spherical coordinates and from the Hamiltonian function (6.1), we construct a Haantjes operator that in cartesian coordinates reads

$$\mathbf{K}_{\text{sph}} = \left[\begin{array}{c|c} \mathbf{A}_{\text{sph}} & 0_3 \\ \hline \mathbf{B}_{\text{sph}} & \mathbf{A}_{\text{sph}} \end{array} \right],$$

where

$$\begin{aligned} \mathbf{A}_{\text{sph}} &= \begin{bmatrix} y^2 + z^2 & -xy & -zx \\ -xy & x^2 + z^2 & -yz \\ -zx & -yz & x^2 + y^2 \end{bmatrix}, \\ \mathbf{B}_{\text{sph}} &= \begin{bmatrix} 0 & yp_x - xp_y & zp_x - xp_z \\ -(yp_x - xp_y) & 0 & zp_y - yp_z \\ -(zp_x - xp_z) & -(zp_y - yp_z) & 0 \end{bmatrix}. \end{aligned}$$

6.3 Parabolic Haantjes operator

We consider the following integral of motion

$$\begin{aligned} H_{\text{par}} &= \frac{1}{2} ((\vec{p} \cdot \vec{u})(\vec{p} \cdot \vec{r}) - (\vec{r} \cdot \vec{u})(\vec{p} \cdot \vec{p})) - (\vec{r} \cdot \vec{u}) V_{\text{Cal}} \\ &= \frac{1}{2} ((p_x + p_y + p_z)(xp_x + yp_y + zp_z) - (x + y + z)(p_x^2 + p_y^2 + p_z^2)) - (x + y + z) V_{\text{Cal}} \end{aligned}$$

related to the product of the axial with the radial linear momentum. The associated Haantjes operator is

$$\mathbf{K}_{\text{par}} = \left[\begin{array}{c|c} \mathbf{A}_{\text{par}} & 0_3 \\ \hline \mathbf{B}_{\text{par}} & \mathbf{A}_{\text{par}} \end{array} \right],$$

where

$$\begin{aligned} \mathbf{A}_{\text{par}} &= \frac{1}{2} \begin{bmatrix} -2(y+z) & (x+y) & (x+z) \\ (x+y) & -2(x+z) & (y+z) \\ (x+z) & (y+z) & -2(x+y) \end{bmatrix}, \\ \mathbf{B}_{\text{par}} &= \frac{1}{2} \begin{bmatrix} 0 & p_y - p_x & p_z - p_x \\ -(p_y - p_x) & 0 & p_z - p_y \\ -(p_z - p_x) & -(p_z - p_y) & 0 \end{bmatrix}. \end{aligned}$$

The following result holds.

Proposition 6.1. *The three Haantjes structures $(T^*\mathcal{Q}, \omega, \mathbf{K}_0 = \mathbf{I}_6, \mathbf{K}_1, \mathbf{K}_{\text{cyl}})$, $(T^*\mathcal{Q}, \omega, \mathbf{K}_0 = \mathbf{I}_6, \mathbf{K}_1, \mathbf{K}_{\text{sph}})$, $(T^*\mathcal{Q}, \omega, \mathbf{K}_0 = \mathbf{I}_6, \mathbf{K}_1, \mathbf{K}_{\text{par}})$ together with the Hamiltonian function (6.1) gene-*

rate three Lenard–Haantjes chains with two common elements

$$\begin{array}{c}
 \mathbf{K}_0^T dH = dH_1, \quad \mathbf{K}_1^T dH = dH_2, \\
 \begin{array}{l}
 \nearrow \mathbf{K}_{\text{cyl}}^T dH = dH_{\text{cyl}}, \\
 \rightarrow \mathbf{K}_{\text{sph}}^T dH = dH_{\text{sph}}, \\
 \searrow \mathbf{K}_{\text{par}}^T dH = dH_{\text{par}}.
 \end{array}
 \end{array} \tag{6.2}$$

Proof. In any of the separable webs above-mentioned the operators \mathbf{K}_0 and \mathbf{K}_1 take a diagonal form. Furthermore, \mathbf{K}_{cyl} , \mathbf{K}_{sph} , \mathbf{K}_{par} , by construction, are diagonal in the cylindrical, spherical, parabolic webs, respectively. Then, they fulfill all the conditions of Definition 2.8. ■

Remark 6.2. The existence of more than one independent Lenard–Haantjes chain is due to the superintegrability of the Calogero model. However, only two of the three chains are independent as

$$dH_1 \wedge dH_2 \wedge dH_{\text{cyl}} \wedge dH_{\text{sph}} \wedge dH_{\text{par}} = 0,$$

and

$$\begin{aligned}
 dH_1 \wedge dH_2 \wedge dH_{\text{cyl}} \wedge dH_{\text{sph}} &\neq 0, \\
 dH_1 \wedge dH_2 \wedge dH_{\text{cyl}} \wedge dH_{\text{par}} &\neq 0, \\
 dH_1 \wedge dH_2 \wedge dH_{\text{sph}} \wedge dH_{\text{par}} &\neq 0.
 \end{aligned}$$

Therefore, an additional independent integral is required in order to prove the maximal superintegrability of the model. The additional integral is the cubic one in the momenta

$$H_3 := \frac{1}{3}(p_x^3 + p_y^3 + p_z^3) + a \left(\frac{p_x + p_y}{(x-y)^2} + \frac{p_x + p_z}{(x-z)^2} + \frac{p_y + p_z}{(y-z)^2} + \frac{p_x + p_z}{(x-z)^2} \right),$$

which is in involution both with H_1 and H_2 . The problem of finding a Haantjes structure that involves such an integral is under investigation.

Remark 6.3. According to Proposition 3.7, all the previous Haantjes operators can be projected onto the configuration space. Each projection is simply given by the first block of the representative matrix, that is, by \mathbf{I}_3 , \mathbf{A} , \mathbf{A}_{cil} , \mathbf{A}_{sph} , \mathbf{A}_{par} . These two-tensors in the configuration space coincide with the mixed form (1, 1) of the Killing tensors found in [5].

7 A Haantjes route to Benenti systems

In this section, we will prove that the \mathbf{L} -systems introduced by S. Benenti [3, 4] and discussed in [19] within a bi-Hamiltonian framework, can be recovered by projecting onto the configuration space the Haantjes operators of the QBH systems (4.3). This can be done by choosing the classical quadratic functions in the momenta (5.1) as Stäckel functions in the Hamiltonian (4.2). Indeed, the components of the metric (5.2) turn out to be

$$g^i = \frac{(\tilde{V}_R)_{1i}}{\det(\tilde{V}_R)} = \frac{\partial c_1}{\partial \lambda_i} \frac{1}{\prod_{\substack{j=1 \\ j \neq k}}^n (\lambda_i - \lambda_j)} = \frac{1}{\prod_{\substack{j=1 \\ j \neq i}}^n (\lambda_i - \lambda_j)}. \tag{7.1}$$

Now we are able to prove the following

Proposition 7.1. *The projected Haantjes operators (4.4) are Killing tensors w.r.t. the metric (7.1) and commute with each other. As for the QBH systems $\lambda_r(\mathbf{q}) = \lambda_r(q_r)$, the projected operator (3.13), which we shall denote by \mathbf{L} , is a Nijenhuis operator and generates the Killing tensors (4.4) by means of the relations*

$$\begin{aligned}\tilde{\mathbf{K}}_0 &= \mathbf{L}^0 = \mathbf{I}, \\ \tilde{\mathbf{K}}_\alpha &= -\sum_{j=0}^{\alpha-1} c_{\alpha-j} \mathbf{L}^j + \mathbf{L}^\alpha, \quad \alpha = 1, \dots, n-1.\end{aligned}\tag{7.2}$$

Moreover, it is a L -tensor or a conformal Killing tensor of trace-type, i.e., it fulfills the relation

$$[\mathbf{L}, \mathbf{G}] = -2\mathbf{X} \odot \mathbf{G}, \quad \mathbf{X} = \mathbf{G}d(\text{tr}(\mathbf{L})).\tag{7.3}$$

Here, the symbols $[\cdot, \cdot]$ and \odot denote the Schouten bracket and the symmetric product of two contravariant tensor fields, respectively. Furthermore, the potential functions (3.4) form the Haantjes chain in \mathcal{Q}

$$\tilde{\mathbf{K}}_{j-1}^T dV_1 = dV_j, \quad j = 1, \dots, n.\tag{7.4}$$

Proof. The first assertion is a direct consequence of Proposition 5.1 and of the compatibility condition (2.3). The generating formula (7.2) follows from Proposition 4.1. Property (7.3) is a consequence of equation (7.2), which for $\alpha = 1$ and in contravariant form implies

$$\mathbf{L} = c_1 \mathbf{G} + \tilde{\mathbf{K}}_1,$$

and of the properties of the Schouten bracket. Indeed,

$$[\mathbf{L}, \mathbf{G}] = [c_1 \mathbf{G} + \tilde{\mathbf{K}}_1, \mathbf{G}] = -2[c_1, \mathbf{G}] \odot \mathbf{G}.$$

Finally, the potential functions $V_j(\mathbf{q})$ in equation (3.4) can be projected naturally along the fibers of $T^*\mathcal{Q}$. Therefore, the second Haantjes chain of (3.5) can also be projected onto \mathcal{Q} , giving equation (7.4). \blacksquare

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