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Dynamics on Networks of Manifolds

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Abstract. We propose a precise definition of a continuous time dynamical system made up of interacting open subsystems. The interconnections of subsystems are coded by directed graphs. We prove that the appropriate maps of graphs called *graph fibrations* give rise to maps of dynamical systems. Consequently surjective graph fibrations give rise to invariant subsystems and injective graph fibrations give rise to projections of dynamical systems.

Key words: coupled cell networks; open dynamical systems; control systems; morphisms of dynamical systems

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1 Introduction

Given a dynamical system, one often starts by trying to find invariant subsystems; these include equilibria, periodic orbits, and higher dimensional invariant submanifolds. In addition, constructing projections onto smaller systems as well as conjugacies and semi-conjugacies with simpler systems are generally useful for understanding the qualitative properties of dynamical systems. All of these objects: invariant subsystems, projections, conjugacies and semi-conjugacies can be realized as maps of dynamical systems (q.v. Definition 2.1). Thus the search for maps between dynamical systems may be considered one of the fundamental questions of the subject.

In this paper we give a precise definition of a continuous time dynamical system made up of interacting open subsystems. We then exploit the combinatorial aspect of such systems to produce maps of dynamical systems out of appropriate maps of graphs called *graph fibrations* (q.v. Definition 3.1). We show that in particular surjective graph fibrations give rise to invariant subsystems and injective graph fibrations give rise to projections of dynamical systems.

The present work is part of an ongoing project. In [5] we reformulated the groupoid formalism of Golubitsky, Pivato, Stewart and Török [7, 8] for coupled cell networks (which are systems of ordinary differential equations) in a coordinate free manner and extended it to groupoid-invariant vector fields on manifolds. A preliminary version was posted as [4]. We later realized that groupoid invariance of vector fields is not needed for the existence of invariant subspaces. With the benefit of hindsight we see that the theory developed in [5] is an equivariant version of the theory that we develop here. We would like to point out that dropping groupoid invariance makes the theory much simpler and more flexible. In particular we expect the results of this paper to readily generalize to hybrid systems.

The absence of explicit groupoid symmetries makes our work close in spirit to the approach to dynamics on networks advocated by Field [6]. Unlike Field we find it convenient to use the language of category theory. We also find it useful to borrow the notions of open systems and their interconnection from engineering (see, for example [3, 9, 12]) and the definition of a graph fibration from computer science [2] (see [11] for a history of the notion and alternative terminologies).

We believe that both Field's approach and ours is based on the existence of a certain algebraic structure which at the present time is not completely understood. Open continuous time systems form an algebra over a certain operad [10]. This operad is implicit in the work of Field [6]. A piece of this algebra shows up in our work as the interconnection maps (see Theorem 2.32). We do not understand yet how graph fibrations interact with this operad and plan to address this issue in a future work.

The goal of this paper is to construct a category of networks of continuous time systems and a functor to the category of dynamical systems. A network in our sense consists of

- a finite directed graph G with a set of nodes G_0,
- a *phase space function* $\mathcal{P}$ that assigns to each node of the graph an appropriate phase space (which we take to be a manifold),
- a family of open systems $\{w_a\}_{a\in G_0}$ (one for each node a of the graph G) consistent in an appropriate way with the structure of the graph, and
- an interconnection map $\mathscr{I}$ that turns these open systems into a vector field on the product $\prod\limits_{a\in G_0} \mathcal{P}(a)$ of the phase spaces of the nodes.

Our main result, Theorem 3.11, shows that graph fibrations compatible with phase space functions give rise to maps of dynamical systems. This allows us to define a functor from dynamical systems on networks to general dynamical systems.

The reader may wonder what motivates us to come up with these definitions and constructions. Indeed there are many different kinds of objects in engineering, science and mathematics that are called "networks". The notion of a network in the present paper arose from the following idea, which is implicit in the literature on coupled cell networks. Imagine a physical system modeled by a vector field X on a manifold M; M is the collection of all possible states of the system. Such systems are common in classical mechanics, to give one example. Suppose further that our system consists of two interacting subsystems. We can model this by saying that the collection of states of the first subsystem forms a manifold M_1 and the second a manifold M_2. We would like the states of the big system to be completely determined by the states of its subsystems. We model this by requiring that $M = M_1 \times M_2$. A vector field X on $M_1 \times M_2$ then has to be of the form

$$X(x_1, x_2) = (X_1(x_1, x_2), X_2(x_1, x_2)),$$

where

$$X_1(x_1, x_2) \in T_{x_1}M_1 \qquad \text{for all} \quad (x_1, x_2) \in M_1 \times M_2,$$

with a similar equation holding for $X_2\colon M_1 \times M_2 \to TM_2$. Note that the functions X_1, X_2 are *not* vector fields. They are open systems in the sense of Definition 2.22. Moreover the vector field X may be considered to be a result of interconnecting X_1 and X_2 (see Proposition 2.24 and Theorem 2.32).

To continue with our example, observe that the evolution of the subsystem 1 depends on its state and the state of the subsystem 2. Similarly the evolution the second subsystem depends on its state and the state of the subsystem 1. These mutual influences can be pictured graphically as

$$\begin{tikzcd} M_1 \arrow[r, bend right] & M_2. \arrow[l, bend right] \end{tikzcd}$$

Assume now that the map X_2 *does not* really depend on the points of M_1. That is, there is a map $Y : M_2 \to TM_2$ with $Y(x_2) \in T_{x_2}M_2$ and $X_2(x_1, x_2) = Y(x_2)$ for all $(x_1, x_2) \in M_1 \times M_2$. We can picture this as

$$M_1 \longleftarrow M_2$$

and say that the second subsystem drives the first but not conversely. This way of picturing a system made up of interacting subsystems generalizes to any number of subsystems. For example, a system may be made up of three interacting subsystems like this:

$$\begin{tikzcd} M_1 \arrow[r, bend right] & M_2 \arrow[l, bend right] \arrow[r] & M_3. \end{tikzcd} \tag{1.1}$$

The total phase space of such a system would be the product $M = M_1 \times M_2 \times M_3$ and the dynamics would be governed by a vector field X of the form

$$X(x_1, x_2, x_3) = (X_1(x_1, x_2), X_2(x_2, x_1), X_3(x_3, x_2)).$$

How are we then to interpret the diagram of the form $M \circlearrowleft$? And why would we want to? Here is a two part answer. We interpret this diagram as a vector field X on the manifold M of the form

$$X(x) = w(x, x),$$

where $w : M \times M \to TM$ is an open system with $w(x_1, x_2) \in T_{x_1}M$ for all $(x_1, x_2) \in M \times M$. This seems a bit strange and pedantic, but it is useful. Consider a vector field Z on $M \times M \times M$ of the form

$$Z(x_1, x_2, x_3) = (w(x_1, x_2), w(x_2, x_1), w(x_3, x_2)),$$

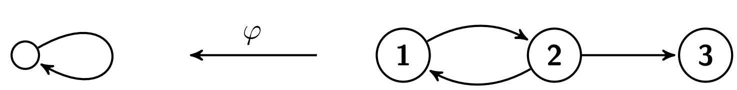
where w is the open system above. The vector field Z on M models the dynamics of a system consisting of three interacting subsystems with the first driving the second, the second driving the first and the third just as in (1.1), only now all the subsystems have isomorphic phase spaces. It is not hard to check that the diagonal

$$\Delta_M := \{(x_1, x_2, x_3) \in M \times M \times M \mid x_1 = x_2 = x_3\}$$

is an invariant submanifold for the vector field Z. According to the philosophy we brought up in the first paragraph of the paper the invariance of Δ_M should be seen as coming from a map of dynamical systems. And indeed the diagonal map

$$\delta\colon\ M \to M \times M \times M, \qquad \delta(x) = (x, x, x)$$

gives rise to a map of dynamical systems $\delta\colon (M, X) \to (M \times M \times M, Z)$. The main result of the paper, Theorem 3.11, implies that this map of dynamical systems is induced by the map of graphs



which is a graph fibration. We note that the vector field Z has groupoid symmetry in the sense of Golubitsky et al. [7, 8] and [5]. For us, however, the groupoid invariance of Z is, in some sense, incidental. It is a consequence of the fact that Z is assembled out of the triple of open systems which lies in the image of the map φ^* of Theorem 3.8 and that φ happens to be surjective.

The paper is organized as follows. We start by defining the category DS of continuous time dynamical systems. We recall the definition of a directed multigraph, define the notion of a network of manifolds and the total space of the network. We recall the notion of an open system and discuss interconnections of open systems. We show how a network of manifolds naturally leads to a collection of spaces of open systems that can be interconnected. We then prove our main result, Theorem 3.11: fibrations of networks of manifolds give rise to maps of dynamical systems. We end the paper with a collection of examples.

2 Definitions and constructions

We start by defining what we mean by a continuous time dynamical system and by a map between two such systems.

Definition 2.1. A continuous time *dynamical system* is a vector field on a manifold. More formally it is a pair (M, X), where X is a vector field on a manifold M.

A *map* from a dynamical system (M, X) to a dynamical system (N, Y) is a smooth map $f : M \to N$ that intertwines the two vector fields:

$$Df \circ X = Y \circ f,$$

where $Df : TM \to TN$ denotes the differential of f. One also says that the vector fields X and Y are f-related.

Notation 2.2 (the category DS of dynamical systems)**.** Continuous time dynamical systems and maps of dynamical systems form a category. We denote it by DS.

2.1 Graphs and manifolds

Throughout the paper *graphs* are finite directed multigraphs, possibly with loops. More precisely, we use the following definition:

Definition 2.3. A *graph* G consists of two finite sets G_1 (of arrows, or edges), G_0 (of nodes, or vertices) and two maps $\mathfrak{s}, \mathfrak{t}\colon G_1 \to G_0$ (source, target); we write

$$G = \{G_1 \rightrightarrows G_0\}.$$

The set G_1 may be empty, i.e., we may have $G = \{\varnothing \rightrightarrows G_0\}$, making G a disjoint collection of vertices with no arrows between them.

Definition 2.4. A *map of graphs* $\varphi\colon A \to B$ from a graph A to a graph B is a pair of maps $\varphi_1\colon A_1 \to B_1$, $\varphi_0\colon A_0 \to B_0$ taking edges of A to edges of B, nodes of A to nodes of B so that for any edge γ of A we have

$$\varphi_0(\mathfrak{s}(\gamma)) = \mathfrak{s}(\varphi_1(\gamma)) \qquad \text{and} \qquad \varphi_0(\mathfrak{t}(\gamma)) = \mathfrak{t}(\varphi_1(\gamma)).$$

We often omit the indices 0 and 1 and write $\varphi(\gamma)$ for $\varphi_1(\gamma)$ and $\varphi(a)$ for $\varphi_0(a)$.

Remark 2.5. The collection of finite (directed multi-)graphs and maps of graphs form a category Graph.

In order to construct networks from graphs we need to have a consistent way of assigning manifolds to nodes of our graphs. We formalize this idea by making the collection of graphs with manifolds assigned to vertices into a category $\mathsf{Graph}/\mathsf{Man}$.

Definition 2.6 (category of networks of manifolds $\mathsf{Graph}/\mathsf{Man}$)**.** A *network of manifolds* is a pair $(G, \mathcal{P})$, where G is a (finite directed multi-)graph and $\mathcal{P}\colon G_0 \to \mathsf{Man}$ is a function that assigns to each node a of G a manifold $\mathcal{P}(a)$. We think of $\mathcal{P}$ as an assignment of phase spaces to the nodes of the graph G, and for this reason we refer to $\mathcal{P}$ as a *phase space function.*

Networks of manifolds form a category $\mathsf{Graph}/\mathsf{Man}$. Its objects are are pairs $(G, \mathcal{P})$ as above. A morphism φ from $(G, \mathcal{P})$ to$(G', \mathcal{P}')$ is a map of graphs $\varphi\colon G \to G'$ with

$$\mathcal{P}' \circ \varphi = \mathcal{P}.$$

Notation 2.7. Given a category $\mathscr{C}$ we denote the opposite category by $\mathscr{C}^{\mathsf{op}}$, i.e. the category with all of the same objects and all of the arrows reversed. We adhere to the convention that a *contravariant functor* from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a covariant functor

$$F\colon\ \mathscr{C}^{\mathsf{op}} \to \mathscr{D}.$$

Then for any morphism $c \xrightarrow{\gamma} c'$ of $\mathscr{C}$ we have $F(c) \xleftarrow{F(\gamma)} F(c')$ in $\mathscr{D}$.

Next we recall the notion of a *product* in a category $\mathscr{C}$. We will use them in two instances: when $\mathscr{C}$ is the category Man of smooth finite dimensional manifolds and smooth maps and when $\mathscr{C}$ is the category Vect of real (but not necessarily finite dimensional) vector spaces and linear maps.

Definition 2.8. A *product* of a family $\{c_s\}_{s\in S}$ of objects in a category $\mathscr{C}$ indexed by a set S is an object $\prod_{s'\in S} c_{s'}$ of $\mathscr{C}$ together with a family of morphisms $\{\pi_s\colon \prod_{s'\in S} c_{s'} \to c_s\}_{s\in S}$ with the following universal property: given an object c' of $\mathscr{C}$ and a family of morphisms $\{f_s\colon c' \to c_s\}_{s\in S}$ there is a unique morphism $f\colon c' \to \prod_{s\in S} c_s$ with

$$\pi_s \circ f = f_s \qquad \text{for all} \quad s \in S.$$

Remark 2.9. If a product exists then it is unique up to a unique isomorphism [1].

Lemma 2.10. *The category of manifolds* Man *has* (*finite*) *categorical products.*

Proof. There are several ways to *construct* categorical products in Man. The first one uses Cartesian products: given a family $\{M_s\}_{s\in S}$ of manifolds indexed by an n-element set S, order the elements of S: $S = \{s_1, \ldots, s_n\}$. Set

$$\prod_{s\in S} M_s = \prod_{i=1}^{n} M_{s_i},$$

where the right hand side is the Cartesian product. The projections $p_{s_j} : \prod\limits_{i=1}^{n} M_{s_i} \to M_{s_j}$ are just projections on the j-th factor. It is easy to check that a product constructed this way has the requisite universal property. In particular, if we choose two different orderings of elements of S, the resulting products are canonically isomorphic. This construction is convenient for writing down examples.

However, for proving the results below, such as Proposition 2.15, it is better to have a construction of the product that does not involve a choice of ordering of the indexing set in question. This may be done as follows. Given a family $\{M_s\}_{s\in S}$ of manifolds, denote by $\bigsqcup_{s\in S} M_s$ their disjoint union[1]. Now define

$$\prod_{s\in S} M_s := \left\{ x\colon S \to \bigsqcup_{s\in S} M_s \,\middle|\, x(s) \in M_s \text{ for all } s \in S \right\}.$$

[1]It may be defined by $\bigsqcup\limits_{s\in S} M_s = \bigcup\limits_{s\in S} M_s \times \{s\}$.

The projection maps $\pi_s\colon \prod_{s'\in S} M_{s'} \to M_s$ are defined by

$$\pi_s(x) = x(s).$$

We denote $x(s) \in M_s$ by x_s and think of it as the s^{th} "coordinate" of an element $x \in \prod_{s\in S} M_s$. Equivalently we may think of elements of the categorical product $\prod_{s\in S} M_s$ as *unordered* tuples $(x_s)_{s\in S}$ with $x_s \in M_s$. ■

Lemma 2.11. *The category of vector spaces* Vect *has finite categorical products.*

Sketch of proof. Just as in the proof of Lemma 2.10 the finite products in Vect can be constructed as vector spaces of ordered tuples of vectors, that is, as Cartesian products. Categorical products in Vect can also be constructed as *unordered* tuples of vectors. ■

Definition 2.12 (total phase space of a network $(G,\mathcal{P})$)**.** For a pair $(G,\mathcal{P})$ consisting of a finite graph G and an assignment $\mathcal{P}\colon G_0 \to \mathsf{Man}$, that is, for an object $(G,\mathcal{P})$ of $\mathsf{Graph}/\mathsf{Man}$ we set

$$\mathbb{P}G \equiv \mathbb{P}(G,\mathcal{P}) := \prod_{a\in G_0} \mathcal{P}(a),$$

the categorical product of manifolds attached to the nodes of the graph G by the *phase space function* $\mathcal{P}$ and call the resulting manifold $\mathbb{P}G$ the *total phase space* of the network $(G,\mathcal{P})$.

Example 2.13. Consider the graph

$$G = \begin{tikzcd} \textsf{a} \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] & \textsf{b} \end{tikzcd}$$

Define $\mathcal{P}\colon G_0 \to \mathsf{Man}$ by $\mathcal{P}(a) = S^2$ (the two sphere) and $\mathcal{P}(b) = S^3$. Then

$$\mathbb{P}(G,\mathcal{P}) = S^2 \times S^3.$$

Notation 2.14. If $G = \{\varnothing \rightrightarrows \{a\}\}$ is a graph with one node a and no arrows, we write $G = \{a\}$. Then for any phase space function $\mathcal{P}\colon G_0 = \{a\} \to \mathsf{Man}$ we abbreviate $\mathbb{P}(\{\varnothing \rightrightarrows \{a\}\}, \mathcal{P}\colon \{a\} \to \mathsf{Man})$ as $\mathbb{P}a$.

Proposition 2.15. *The assignment*

$$(G,\mathcal{P}) \mapsto \mathbb{P}G := \prod_{a\in G_0} \mathcal{P}(a)$$

of phase spaces to networks extends to a contravariant functor

$$\mathbb{P}\colon\ (\mathsf{Graph}/\mathsf{Man})^{\mathsf{op}} \to \mathsf{Man}.$$

Proof. Suppose $\varphi\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ is a morphism in $\mathsf{Graph}/\mathsf{Man}$. That is, suppose $\varphi\colon G \to G'$ is a map of graphs with $\mathcal{P}'\circ\varphi = \mathcal{P}$. We need to define a map of manifolds

$$\mathbb{P}\varphi\colon\ \mathbb{P}G' \to \mathbb{P}G.$$

Since by definition $\mathbb{P}G$ is the product $\prod_{a\in G_0} \mathcal{P}(a)$, the universal property of products implies that in order to define $\mathbb{P}\varphi$ it is enough to define a family of maps

$$\left\{(\mathbb{P}\varphi)_a\colon\ \mathbb{P}G' \to \mathcal{P}(a)\right\}_{a\in G_0}.$$

For any node a' of G' we have the canonical projection

$$\pi'_{a'}\colon\ \mathbb{P}G' \to \mathcal{P}'(a').$$

We therefore define

$$(\mathbb{P}\varphi)_a := \pi'_{\varphi_a}\colon\ \mathbb{P}G' \to \mathcal{P}'(\varphi(a)) = \mathcal{P}(a)$$

for all $a \in G_0$. By the universal property of the product $\mathbb{P}G = \prod_{a\in G_0} \mathcal{P}(a)$ this defines the desired map $\mathbb{P}\varphi\colon \mathbb{P}G' \to \mathbb{P}G$.

The universal property of products also implies that the map $\mathbb{P}$ on morphisms of $\mathsf{Graph}/\mathsf{Man}$ as defined above is actually a functor. That is,

$$\mathbb{P}(\psi \circ \varphi) = \mathbb{P}\varphi \circ \mathbb{P}\psi$$

for any pair (ψ, φ) of composable morphisms in $\mathsf{Graph}/\mathsf{Man}$. ■

Remark 2.16. Proposition 2.15 is an instance of a category-theoretic result that holds in greater generality. Namely, given a category $\mathscr{C}$ with finite products consider the category $\mathsf{FinSet}/\mathscr{C}$ whose objects are pairs (X, P), where X is a finite set and P is a function that assignes to each element of X an object of $\mathscr{C}$. The morphisms are commuting triangles. There is a contravariant functor $\mathbb{P} : (\mathsf{FinSet}/\mathscr{C})^{\mathsf{op}} \to \mathscr{C}$ which on objects is given by

$$\mathbb{P}(X, P) = \prod_{x\in X} P(x).$$

Example 2.17. Suppose G is a graph with two nodes a, b and no edges, G' is a graph with one node $\{c\}$ and no edges, $\mathcal{P}'(c)$ is a manifold M, and $\varphi\colon G \to G'$ is the only possible map of graphs (it sends both nodes to c). Suppose further that $\mathcal{P}\colon G_0 \to \mathsf{Man}$ is given by $\mathcal{P}(a) = M = \mathcal{P}(b)$ (so that $\mathcal{P}' \circ \varphi = \mathcal{P}$). Then $\mathbb{P}G' \simeq M$,

$$\mathbb{P}G = \{(x_a, x_b) \,|\, x_a \in \mathcal{P}(a), x_b \in \mathcal{P}(b)\} \simeq M \times M,$$

and $\mathbb{P}\varphi\colon M \to M \times M$ is the unique map with $(\mathbb{P}\varphi(x))_a = x$ and $(\mathbb{P}\varphi(x))_b = x$ for all $x \in \mathbb{P}G'$. Thus $\mathbb{P}\varphi\colon M \to M \times M$ is the diagonal map $x \mapsto (x, x)$.

Example 2.18. Let $(G, \mathcal{P})$, $(G', \mathcal{P}')$ be as in Example 2.17 above and $\psi\colon (G', \mathcal{P}') \to (G, \mathcal{P})$ be the map that sends the node c to a. Then $\mathbb{P}\psi\colon \mathbb{P}G \to \mathbb{P}G'$ is the map that sends (x_a, x_b) to x_a.

Remark 2.19. If $(G, \mathcal{P})$ is a graph with a phase function, that is, an object of $\mathsf{Graph}/\mathsf{Man}$, and $\varphi\colon H \to G$ a map of graphs then $\mathcal{P}\circ\varphi\colon H \to \mathsf{Man}$ is a phase function and $\varphi\colon (H, \mathcal{P}\circ\varphi) \to (G, \mathcal{P})$ is a morphism in $\mathsf{Graph}/\mathsf{Man}$. We then have a map of manifolds

$$\mathbb{P}\varphi\colon\ \mathbb{P}(H, \mathcal{P} \circ \varphi) \to \mathbb{P}(G, \mathcal{P}).$$

Similarly, a commutative diagram

$$\begin{tikzcd} K \arrow[rr, "j"] \arrow[dr, "\psi"'] & & H \arrow[dl, "\varphi"] \\ & G & \end{tikzcd}$$

of maps of graphs and a phase space function $\mathcal{P}\colon G \to \mathsf{Man}$ give rise to the commutative diagram of maps of manifolds

$$\begin{tikzcd} \mathbb{P}(K, \mathcal{P} \circ \psi) & & \mathbb{P}(H, \mathcal{P} \circ \varphi) \arrow[ll, "\mathbb{P}j"'] \\ & \mathbb{P}(G, \mathcal{P}) \arrow[ul, "\mathbb{P}\psi"] \arrow[ur, "\mathbb{P}\varphi"'] & \end{tikzcd}$$

2.2 Embeddings and submersions from maps of graphs

As we said in the introduction, the main goal of this paper is to construct maps of dynamical systems from graph fibrations. In Proposition 2.15 we showed that a map of networks $\varphi\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ defines a map of manifolds $\mathbb{P}\varphi\colon \mathbb{P}(G',\mathcal{P}') \to \mathbb{P}(G,\mathcal{P})$. In this subsection we prove that:

1) If the map of graphs $\varphi\colon G \to G'$ is injective on nodes, then $\mathbb{P}\varphi$ is a surjective submersion,
2) if the map of graphs $\varphi\colon G \to G'$ is surjective on nodes, then $\mathbb{P}\varphi$ is an embedding.

(Recall that a smooth map between two manifolds is a submersion if its differential is onto at every point. A smooth map between two manifolds is an embedding if it is 1-1, its differential is 1-1 everywhere and it is a homeomorphism onto its image.) Combined with Theorem 3.11 below, this shows that surjective fibrations of networks of manifolds give rise to invariant dynamical subsystems and injective fibrations give rise to projections of dynamical systems.

Lemma 2.20. *Suppose $\varphi\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ is a map of networks of manifolds such that the map on nodes, $\varphi_0\colon G_0 \to G'_0$, is surjective. Then $\mathbb{P}\varphi\colon \mathbb{P}G' \to \mathbb{P}G$ is an embedding whose image is the "polydiagonal"*

$$\Delta_\varphi = \{x \in \mathbb{P}G \,|\, x_a = x_b \textit{ whenever } \varphi(a) = \varphi(b)\}.$$

Proof. Assume first for simplicity that G' has only one vertex $*$ and $\mathcal{P}'(*) = M$. Then for any vertex a of G we have

$$\mathcal{P}(a) = \mathcal{P}'(\varphi(a)) = \mathcal{P}'(*) = M,$$

$\mathbb{P}G' = M$ and $\mathbb{P}G = M \times \cdots \times M$ ($|G_0|$ copies), where as before G_0 is the set of vertices of the graph G. In this case the proof of Proposition 2.15 shows that the map $\mathbb{P}\varphi\colon M \to M^{G_0}$ is of the form

$$\mathbb{P}\varphi(x) = (x,\ldots,x)$$

for all $x \in M$. This is clearly an embedding. In general,

$$\mathbb{P}\varphi\colon\ \mathbb{P}G' = \prod_{a'\in G'_0} \mathcal{P}'(a') \to \prod_{a'\in G'_0}\left(\prod_{a\in\varphi^{-1}(a')} \mathcal{P}(a)\right) = \mathbb{P}G$$

is the product of maps of the form

$$\mathcal{P}'(a') \to \prod_{a\in\varphi^{-1}(a')} \mathcal{P}(a), \qquad x \mapsto (x,\ldots,x).$$

■

Lemma 2.21. *Suppose $\varphi\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ is a map of networks of manifolds such that the map $\varphi_0\colon G_0 \to G'_0$ on nodes is injective. Then $\mathbb{P}\varphi\colon \mathbb{P}G' \to \mathbb{P}G$ is a surjective submersion.*

Proof. Since $\varphi\colon G \to G'$ is injective, the set of nodes G'_0 of G' can be partitioned as the disjoint union of the image $\varphi(G_0)$, which is a copy of G_0, and the complement. Hence

$$\mathbb{P}G' \simeq \prod_{a\in G_0} \mathcal{P}(\varphi(a)) \times \prod_{a'\notin\varphi(G_0)} \mathcal{P}'(a') \simeq \mathbb{P}G \times \prod_{a'\notin\varphi(G_0)} \mathcal{P}'(a').$$

With respect to this identification of $\mathbb{P}G'$ with $\mathbb{P}G \times \prod_{a'\notin\varphi(G_0)} \mathcal{P}'(a')$ the map $\mathbb{P}\varphi\colon \mathbb{P}G' \to \mathbb{P}G$ is the projection

$$\mathbb{P}G \times \prod_{a'\notin\varphi(G_0)} \mathcal{P}'(a') \to \mathbb{P}G.$$

which is a surjective submersion. ■

2.3 Open systems and their interconnections

Having set up a consistent way of assigning phase spaces to graphs, we now take up continuous time dynamical systems. We start by recalling a definition of an open (control) systems, which is essentially due to Brockett [3]. It is not the most general definition; it is more than enough for this paper.

Definition 2.22. A *continuous time control system* (or an *open system*) on a manifold M is a surjective submersion $p\colon Q \to M$ from some manifold Q together with a smooth map $F\colon Q \to TM$ so that

$$F(q) \in T_{p(q)}M$$

for all $q \in Q$. That is, the diagram $\begin{tikzcd} Q \arrow[r, "F"] \arrow[dr, "p"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$ commutes. Here $\pi\colon TM \to M$ is the canonical projection.

Definition 2.23 (Control$(M \times U \to M)$)**.** Given a manifold M of "state variables" and a manifold U of "control variables" we may consider control systems of the form

$$F\colon\ M \times U \to TM, \qquad F(x,u) \in T_xM \qquad \text{for all} \quad (x,u) \in M \times U.$$

The collection of all such control systems forms a vector space $\mathsf{Control}(M \times U \to M)$. Explicitly

$$\mathsf{Control}(M \times U \to M) := \{F\colon M \times U \to TM \mid F(x,u) \in T_xM \text{ for all } (x,u) \in M \times U\}.$$

Now suppose we are given a finite family $\{F_i\colon M_i \times U_i \to TM_i\}_{i=1}^N$ of control systems and we want to somehow interconnect them to obtain a closed system $\mathscr{I}(F_1, \dots, F_N)$, that is, a vector field on the product $\prod_i M_i$. What additional data do we need to define the interconnection map

$$\mathscr{I}\colon\ \prod_i \mathsf{Control}(M_i \times U_i \to M_i) \to \Gamma\left(T\left(\prod_i M_i\right)\right)?$$

An answer is given by the following proposition:

Proposition 2.24. *Given a family $\{p_j\colon M_j \times U_j \to M_j\}_{j=1}^N$ of projections on the first factor and a family of smooth maps $\{s_j\colon \prod M_i \to M_j \times U_j\}$ so that the diagrams*

$$\begin{tikzcd} M_j \times U_j \arrow[dr, "p_j"] & \\ \prod M_i \arrow[u, "s_j"] \arrow[r, "pr_j"'] & M_j \end{tikzcd}$$

commute for each index j, there is an interconnection map $\mathscr{I}$ making the diagrams

$$\begin{tikzcd} \prod_i \mathsf{Control}(M_i \times U_i \to M_i) \arrow[r, "\mathscr{I}"] \arrow[d] & \Gamma(T(\prod_i M_i)) \arrow[d, "\varpi_j = D(pr_j)\circ -"] \\ \mathsf{Control}(M_j \times U_j \to M_j) \arrow[r, "\mathscr{I}_j"'] & \mathsf{Control}(\prod_i M_i \xrightarrow{pr_j} M_j) \end{tikzcd}$$

commute for each j. The components $\mathscr{I}_j$ of the interconnection map $\mathscr{I}$ are defined by $\mathscr{I}_j(F_j) := F_j \circ s_j$ for all j, where $D(pr_j)\colon T\prod M_i \to TM_j$ denotes the differential of the canonical projection $pr_j\colon \prod M_i \to M_j$.

Proof. The space of vector fields $\Gamma(T(\prod_i M_i))$ on the product $\prod_i M_i$ is the product of vector spaces $\mathsf{Control}(\prod_i M_i \to M_j)$:

$$\Gamma\left(T\left(\prod_i M_i\right)\right) = \prod_j \mathsf{Control}\left(\prod_i M_i \xrightarrow{pr_j} M_j\right).$$

In other words a vector field X on the product $\prod_i M_i$ is a tuple $X = (X_1, \ldots, X_N)$, where

$$X_j := D(pr_j) \circ X.$$

Each component $X_j \colon \prod_i M_i \to TM_i$ is a control system.

To define a map from a vector space into a product of vector spaces it is enough to define a map into each of the factors. We have canonical projections

$$\pi_j \colon \prod_i \mathsf{Control}(M_i \times U_i \to M_i) \to \mathsf{Control}(M_j \times U_j \to M_j), \qquad j = 1, \ldots, N.$$

Consequently to define the interconnection map $\mathscr{I}$ it is enough to define the maps

$$\mathscr{I}_j \colon \mathsf{Control}(M_j \times U_j \to M_j) \to \mathsf{Control}\left(\prod_i M_i \xrightarrow{pr_j} M_j\right).$$

for each index j. We therefore define the maps $\mathscr{I}_j \colon \mathsf{Control}(M_j \times U_j \to M_j) \to \mathsf{Control}(\prod_i M_i \xrightarrow{pr_j} M_j)$, $1 \le j \le N$, by

$$\mathscr{I}_j(F_j) := F_j \circ s_j. \qquad \blacksquare$$

Remark 2.25. It will be useful for us to remember that the canonical projections

$$\varpi_j \colon \Gamma\left(T \prod M_i\right) \to \mathsf{Control}\left(\prod M_i \to M_j\right)$$

are given by

$$\varpi_j(X) = D(pr_j) \circ X,$$

where as before $D(pr_j) \colon T \prod M_i \to TM_j$ are the differentials of the canonical projections $pr_j \colon \prod M_i \to M_j$.

2.4 Interconnections and graphs

We next explain how finite directed graphs whose nodes are decorated with phase spaces, that is, networks of manifolds in the sense of Definition 2.6 give rise to interconnection maps. To do this precisely it is useful to have a notion of *input trees* of a directed graph. This notion is a generalization of the notion of an *input set* of Golubitsky et al. (*op. cit.*) [7, 8]. Given a graph, an input tree $I(a)$ of a vertex a is – roughly – the vertex itself and all of the arrows leading into it. We want to think of this as a graph in its own right, as follows.

Definition 2.26 (input tree)**.** Given a vertex a of a graph G we define the *input tree* $I(a)$ to be a graph with the set of vertices $I(a)_0$ given by

$$I(a)_0 := \{a\} \sqcup \mathfrak{t}^{-1}(a),$$

where, as before, the set $\mathfrak{t}^{-1}(a)$ is the set of arrows in G with target a. The set of edges $I(a)_1$ of the input tree is the set of pairs

$$I(a)_1 := \{(a, \gamma) \mid \gamma \in G_1, \mathfrak{t}(\gamma) = a\},$$

and the source and target maps $I(a)_1 \rightrightarrows I(a)_0$ are defined by

$$\mathfrak{s}(a,\gamma) = \gamma \qquad \text{and} \qquad \mathfrak{t}(a,\gamma) = a.$$

In pictures,

$$\begin{tikzcd} \gamma \arrow[r, bend left, "(a,\gamma)"] & a \end{tikzcd}$$

Example 2.27. Consider the graph

$$G = \begin{tikzcd} a \arrow[r, bend left, "\beta"] \arrow[r, bend right, "\alpha"'] & b \end{tikzcd}$$

as in Example 2.13. Then the input tree $I(a)$ is the graph with one node a and no edges: $I(a) = \{a\}$ (see Notation 2.14). The input tree $I(b)$ has three nodes and two edges:

$$I(b) = \begin{tikzcd} \alpha \arrow[dr, bend left, "(b,\alpha)"] & \\ & b \\ \beta \arrow[ur, bend right, "(b,\beta)"'] & \end{tikzcd}$$

Notice that our definition of input tree "pulls apart" multiple edges coming from a common vertex.

Remark 2.28. For each node a of a graph G we have a natural map of graphs

$$\xi = \xi_a \colon\ I(a) \to G.$$

It is defined by sending the edge of the form $\gamma \xrightarrow{(a,\gamma)} a$ to the edge $\mathfrak{s}(\gamma) \xrightarrow{\gamma} a$. Note that the map ξ need not be injective on vertices.

Proposition 2.29. *Given a graph G with a phase space function $\mathcal{P}\colon G_0 \to \mathsf{Man}$, that is, a network $(G,\mathcal{P})$ of manifolds, we have commutative diagrams of maps of manifolds*

$$\begin{tikzcd} \mathbb{P}I(a) = \mathcal{P}(a) \times \prod_{\gamma \in \mathfrak{t}^{-1}(a)} \mathcal{P}(\mathfrak{s}(\gamma)) \arrow[r, "\mathbb{P}j_a"] & \mathcal{P}(a) = \mathbb{P}a \\ \mathbb{P}G = \prod_{b \in G_0} \mathcal{P}(b) \arrow[u, "\mathbb{P}\xi"] \arrow[ur, "\mathbb{P}\iota_a"'] & \end{tikzcd}$$

for each node a of the graph G.

Proof. Let a be a node of the graph G. We then have a graph $\{a\}$ with one node and no arrows. Denote the inclusion of $\{a\}$ in G by ι_a and the inclusion into its input tree $I(a)$ by j_a. Then the diagram of maps of graphs

$$\begin{tikzcd} \{a\} \arrow[r, "j_a"] \arrow[dr, "\iota_a"'] & I(a) \arrow[d, "\xi"] \\ & G \end{tikzcd}$$

commutes. By Remark 2.19 we have a commuting diagram of maps of manifolds

$$\begin{tikzcd} \mathbb{P}\{a\} & & \mathbb{P}I(a) \arrow[ll, "\mathbb{P}j_a"'] \\ & \mathbb{P}G \arrow[ul, "\mathbb{P}\iota_a"] \arrow[ur, "\mathbb{P}\xi"'] & \end{tikzcd}$$

Let us now examine more closely the map $\mathbb{P}j_a \colon \mathbb{P}I(a) \to \mathbb{P}a$.

Since the set of nodes $I(a)_0$ of the input tree $I(a)$ is the disjoint union

$$I(a)_0 = \{a\} \sqcup \mathfrak{t}^{-1}(a),$$

and since $\xi_a(\gamma) = \mathfrak{s}(\gamma)$ for any $\gamma \in \mathfrak{t}^{-1}(a) \subset I(a)_0$, we have

$$\mathbb{P}I(a) = \mathcal{P}(a) \times \prod_{\gamma \in \mathfrak{t}^{-1}(a)} \mathcal{P}(\mathfrak{s}(\gamma)).$$

Since $j_a \colon \{a\} \to I(a)_0 = \{a\} \sqcup \mathfrak{t}^{-1}(a)$ is the inclusion,

$$\mathbb{P}j_a \colon\ \mathbb{P}I(a) \to \mathbb{P}a$$

is the projection

$$\mathcal{P}(a) \times \prod_{\gamma \in \mathfrak{t}^{-1}(a)} \mathcal{P}(\mathfrak{s}(\gamma)) \to \mathbb{P}a.$$

Similarly

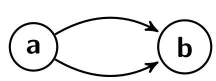
$$\mathbb{P}\iota_a \colon\ \mathbb{P}G \to \mathbb{P}a$$

is the projection

$$\prod_{b \in G_0} \mathcal{P}(b) \to \mathcal{P}(a).$$

The proposition follows from these two observations. ■

Example 2.30. Suppose

$G =$ 

is a graph as in Example 2.13 and suppose $\mathcal{P} \colon G_0 \to \mathsf{Man}$ is a phase space function. Then

$$\mathbb{P}I(b) \simeq \mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b),$$

$\mathbb{P}j_b$ is the projection $\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \to \mathcal{P}(b)$, and

$$\mathsf{Control}(\mathbb{P}I(b) \to \mathbb{P}b) = \mathsf{Control}(\mathcal{P}(a) \times \mathcal{P}(a) \times \mathcal{P}(b) \to \mathcal{P}(b)).$$

On the other hand $\mathbb{P}I(a) = \mathcal{P}(a)$, $\mathbb{P}j_a \colon \mathcal{P}(a) \to \mathcal{P}(a)$ is the identity map and

$$\mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a) = \Gamma(T\mathcal{P}(a)),$$

the space of vector fields on the manifold $\mathcal{P}(a)$.

Notation 2.31. Given a network $(G, \mathcal{P})$ of manifolds we have a product of vector spaces

$$\mathcal{Ctrl}(G, \mathcal{P}) := \prod_{a \in G_0} \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a).$$

The elements of $\mathcal{Ctrl}(G, \mathcal{P})$ are unordered tuples of $(w_a)_{a \in G_0}$ of control systems (q.v. Lemma 2.11). We may think of them as sections of the vector bundle $\bigsqcup_{a \in G_0} \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a) \to G_0$ over the vertices of G.

It is easy to see that Propositions 2.24 and 2.29 give us

Theorem 2.32. *Given a network $(G, \mathcal{P})$ of manifolds, there exists a natural interconnection map*

$$\mathscr{I} \colon \prod_{a \in G_0} \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a) \to \Gamma(T\mathbb{P}G)$$

with

$$\varpi_a \circ \mathscr{I}((w_b)_{b \in G_0}) = w_a \circ \mathbb{P}j_a$$

for all nodes $a \in G_0$. Here $\varpi_a \colon \Gamma(T\mathbb{P}G) \to \mathsf{Control}(\mathbb{P}G_0 \xrightarrow{\mathbb{P}\iota_a} \mathbb{P}a)$ are the projection maps; $\varpi_a = D(\mathbb{P}\iota_a)$ (q.v. Remark 2.25).

Example 2.33. Consider the graph G as in Examples 2.13 and 2.30 with a phase space function $\mathcal{P} \colon G_0 \to \mathsf{Man}$. Then the vector field

$$X = \mathscr{I}(w_a, w_b) \colon \; \mathcal{P}(a) \times \mathcal{P}(b) \to T\mathcal{P}(a) \times T\mathcal{P}(b)$$

is of the form

$$X(x, y) = (w_a(x), w_b(x, x, y)) \qquad \text{for all} \quad (x, y) \in \mathcal{P}(a) \times \mathcal{P}(b).$$

Example 2.34. Consider the graph

$$G = \begin{tikzcd} \mathsf{a} \arrow[r, bend left] \arrow[r, bend right] & \mathsf{b} \arrow[r] & \mathsf{c} \end{tikzcd}$$

and let $\mathcal{P} \colon G_0 \to \mathsf{Man}$ be a phase space function. Then

$$(\mathscr{I}(w_a, w_b, w_c))\,(x, y, z) = (w_a(x), w_b(x, x, y), w_c(y, z))$$

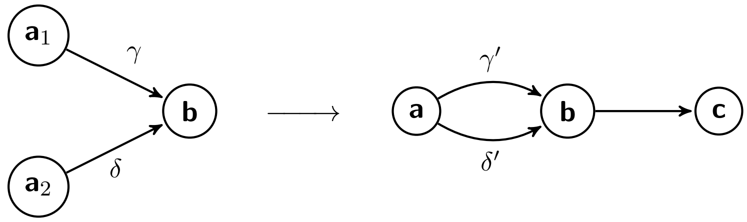
for all $(w_a, w_b, w_c) \in \mathcal{Ctrl}(G, \mathcal{P})$ and all $(x, y, z) \in \mathcal{P}(a) \times \mathcal{P}(b) \times \mathcal{P}(c)$.

3 Maps of dynamical systems from fibrations

Following Boldi and Vigna [2] (see also [11]) we single out a class of maps of graphs called graph fibrations.

Definition 3.1. A map $\varphi \colon G \to G'$ of directed graphs is a *graph fibration* if for any vertex a of G and any edge e' of G' ending at $\varphi(a)$ there is a unique edge e of G ending at a with $\varphi(e) = e'$.

Example 3.2. The map of graphs



sending the edge γ to γ' and the edge δ to δ' is a graph fibration.

Remark 3.3. Given any map $\varphi\colon G \to G'$ of graphs and a node a of G there is an induced map of input trees

$$\varphi_a\colon\ I(a) \to I(\varphi(a)).$$

On edges of $I(a)$ the map is defined by

$$\varphi(a, \gamma) := (\varphi(a), \varphi(\gamma))$$

(cf. Definition 2.26). Moreover the diagram of graphs

$$\begin{tikzcd} I(a) \arrow[r, "\varphi_a"] \arrow[d, "\xi_a"'] & I(\varphi(a)) \arrow[d, "\xi_{\varphi(a)}"] \\ G \arrow[r, "\varphi"] & G' \end{tikzcd}$$

commutes (the map $\xi_a\colon I(a) \to G$ from an input tree to the original graph is defined in Remark 2.28).

Lemma 3.4. *If $\varphi\colon G \to G'$ is a graph fibration then the induced maps*

$$\varphi_a\colon\ I(a) \to I(\varphi(a))$$

of input trees defined above are isomorphisms for all nodes a of G.

Proof. Given an edge $(\varphi(a), \gamma')$ of $I(\varphi(a))$ there is a unique edge γ of G with $\varphi(\gamma) = \gamma'$ and $\mathsf{t}(\gamma) = a$ and consequently $\varphi_a(a, \gamma) = (\varphi(a), \gamma')$. It follows that φ_a is bijective on vertices and edges. ∎

Remark 3.5. The converse is true as well: if the induced maps $\varphi_a\colon I(a) \to I(\varphi(a))$ are isomorphisms for all nodes a of G then $\varphi\colon G \to G'$ is a graph fibration.

Recall that a map from a network $(G, \mathcal{P})$ to a network $(G', \mathcal{P}')$ is a map of graphs $\varphi\colon G \to G'$ with the property that

$$\mathcal{P}' \circ \varphi = \mathcal{P}.$$

Definition 3.6 (fibration of networks of manifolds)**.** A map of networks $\varphi\colon (G, \mathcal{P}) \to (G', \mathcal{P}')$ of manifolds is a *fibration* if $\varphi\colon G \to G'$ is a graph fibration.

Remark 3.7 (the category $(\mathsf{Man}/\mathsf{Graph})_{\mathsf{fib}}$ of networks of manifolds and fibrations)**.** We note that the composit of two fibrations is again a fibration. Consequently networks of manifolds and fibrations form a category which we denote by $(\mathsf{Man}/\mathsf{Graph})_{\mathsf{fib}}$.

Theorem 3.8 below is our reason for singling out fibrations of networks.

Theorem 3.8. *A fibration $\varphi\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ of networks induces a linear map*

$$\varphi^*\colon\ \mathcal{C}trl(G',\mathcal{P}') \to \mathcal{C}trl(G,\mathcal{P}).$$

Proof. Since

$$\mathcal{C}trl(G,\mathcal{P}) = \prod_{a\in G_0} \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a)$$

is a product of vector spaces, the map φ^* is uniquely determined by maps from $\mathcal{C}trl(G',\mathcal{P}')$ to the factors $\mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a)$, $a \in G_0$. On the other hand we have canonical projections

$$\pi_b\colon\ \mathcal{C}trl(G',\mathcal{P}') = \prod_{c\in G_0'} \mathsf{Control}(\mathbb{P}I(c) \to \mathbb{P}c) \to \mathsf{Control}(\mathbb{P}I(b) \to \mathbb{P}b)$$

for all $b \in G_0'$. Hence in order to define the map φ^* it is enough to define maps of vector spaces

$$\varphi_a^*\colon\ \mathsf{Control}(\mathbb{P}I(\varphi(a)) \to \mathbb{P}\varphi(a)) \to \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a)$$

for all nodes a of the graph G. By Remark 3.3 the diagram

$$\begin{tikzcd}
I(a) \arrow[r, "\varphi_a"] \arrow[d, "\xi_a"'] & I(\varphi(a)) \arrow[d, "\xi_{\varphi(a)}"] \\
G \arrow[r, "\varphi"] \arrow[dr, "\mathcal{P}"'] & G' \arrow[d, "\mathcal{P}'"] \\
& \mathsf{Man}
\end{tikzcd}$$

commutes for each $a \in G_0$. Let

$$\varphi|_{\{a\}}\colon\ \{a\} \to \{\varphi(a)\}$$

denote the restriction of $\varphi\colon G \to G'$ to the subgraph $\{a\} \hookrightarrow G$. It is easy to see that the diagrams

$$\begin{tikzcd}
I(a) \arrow[r, "\varphi_a"] & I(\varphi(a)) \\
\{a\} \arrow[u, "j_a"'] \arrow[r, "\varphi|_{\{a\}}"] & \{\varphi(a)\} \arrow[u, "j_{\varphi(a)}"']
\end{tikzcd}$$

commutes as well. By Lemma 3.4 the map φ_a is an isomorphism of graphs. Hence

$$\mathbb{P}\varphi_a\colon\ \mathbb{P}I(a) \to \mathbb{P}I(\varphi(a))$$

is an isomorphism of manifolds. Define

$$\varphi_a^*\colon\ \mathsf{Control}(\mathbb{P}I(\varphi(a)) \to \mathbb{P}\varphi(a)) \to \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a)$$

by

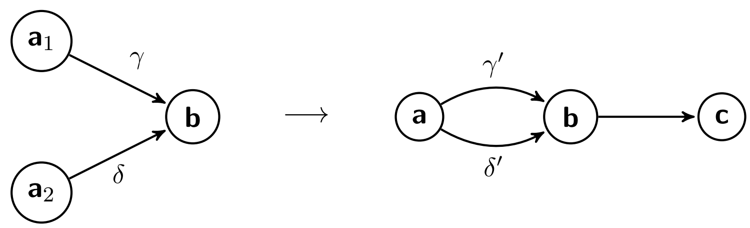
$$\varphi_a^*(F) = D\mathbb{P}(\varphi|_{\{a\}}) \circ F \circ (\mathbb{P}\varphi_a)^{-1}$$

for all $F \in \mathsf{Control}(\mathbb{P}I(\varphi(a)))$. By the universal property of products this gives us the desired map φ^*. Moreover the diagrams

$$\begin{tikzcd}
\mathit{Ctrl}(G', \mathcal{P}') \arrow[r, "\varphi^*"] \arrow[d, "\pi_{\varphi(a)}"'] & \mathit{Ctrl}(G, \mathcal{P}) \arrow[d, "\pi_a"] \\
\mathsf{Control}(\mathbb{P}I(\varphi(a)) \to \mathbb{P}\varphi(a)) \arrow[r, "\varphi_a^*"] & \mathsf{Control}(\mathbb{P}I(a) \to \mathbb{P}a)
\end{tikzcd}$$

commute for all $a \in G_0$. ■

Example 3.9. We write down an example of the map φ^* constructed in Theorem 3.8. Consider the graph fibration $\varphi\colon G \to G'$:



as in Example 3.2. Let $\mathcal{P}'\colon G_0' \to \mathsf{Man}$ be a phase space function. Then

$$\begin{aligned}
\mathit{Ctrl}(G', \mathcal{P}') = \{&(w_a\colon \mathcal{P}(a) \to T\mathcal{P}(a), w_b\colon \mathcal{P}'(a) \times \mathcal{P}'(a) \times \mathcal{P}'(b) \to T\mathcal{P}(b),\\
&w_c\colon \mathcal{P}(b) \times \mathcal{P}(c) \to T\mathcal{P}(c))\},\\
\mathit{Ctrl}(G, \mathcal{P}' \circ \varphi) = \{&(w'_{a_1}\colon \mathcal{P}'(a) \to \mathcal{P}'(a), w_{a_2}\colon \mathcal{P}'(a) \to \mathcal{P}'(a),\\
&w_b\colon \mathcal{P}'(a) \times \mathcal{P}'(a) \times \mathcal{P}'(b) \to T\mathcal{P}'(b))\}
\end{aligned}$$

and

$$\varphi^*(w'_a, w'_b, w'_c) = (w'_a, w'_a, w'_b).$$

Remark 3.10 (the category DSN of dynamical systems on networks of manifolds)**.** It is easy to see that if $\varphi : (G, \mathcal{P}) \to (G', \mathcal{P}')$ and $\psi : (G', \mathcal{P}') \to (G'', \mathcal{P}'')$ are two fibrations then

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

This can be interpreted as saying that the assignment

$$(G, \mathcal{P}) \mapsto \mathit{Ctrl}(G, \mathcal{P})$$

extends to a contravariant functor Ctrl from the category $(\mathsf{Man}/\mathsf{Graph})_{\mathsf{fib}}$ of networks of manifolds and fibrations to the category Vect of real vector spaces and linear maps. That is, on arrows,

$$\mathit{Ctrl}\,\varphi := \varphi^*.$$

Grothendieck's construction (see for example [1]) applied to this functor produces a category DSN which we would like to call the category of (continuous time) dynamical systems on networks of manifolds. More explicitly the objects of the category DSN are triples

$$(G, \mathcal{P}\colon G_0 \to \mathsf{Man}, w \in \mathit{Ctrl}(G, \mathcal{P})),$$

where as before G is a finite directed graph, $\mathcal{P}$ is a phase space function and $w = (w_a)_{a \in G_0}$ is a tuple of control systems associated with the input trees of the graph G and the function $\mathcal{P}$.

A morphism from $(G', \mathcal{P}', w')$ to $(G, \mathcal{P}, w)$ is a graph fibration $\varphi\colon G \to G'$ with $\mathcal{P}' \circ \varphi = \mathcal{P}$ and $\varphi^* w' = w$. Alternatively we may think of a map from $(G', \mathcal{P}', w')$ to $(G, \mathcal{P}, w)$ as a fibration of networks of manifolds $\varphi\colon (G, \mathcal{P}) \to (G', \mathcal{P}')$ with $\varphi^* w' = w$.

Note that the Grothendieck construction also gives us a forgetful functor

$$\mathsf{DSN} \to (\mathsf{Man}/\mathsf{Graph})_{\mathsf{fib}}{}^{\mathsf{op}}$$

that simply forgets the open systems. On objects it is given by sending the triple $(G, \mathcal{P}, w)$ to the pair $(G, \mathcal{P})$.

Of course just because we can define a category and call it a category of dynamical systems on networks does not mean that this is a right thing to do. This said, Theorem 2.32 tells us that to every dynamical system on a network $(G, \mathcal{P}, w)$ we can assign a dynamical system $(\mathbb{P}G, \mathscr{I} w)$. We will next argue that this assignment actually extends to a functor

$$\mathbb{P}\colon\ \mathsf{DSN} \to \mathsf{DS}$$

from dynamical systems on networks to the category DS of dynamical systems (q.v. Definition 2.1 and Remark 2.2). The first step is to define the functor on arrows. We do it in Theorem 3.11 below which may be considered the main result of the paper.

Theorem 3.11. *Let $\varphi\colon (G, \mathcal{P}) \to (G', \mathcal{P}')$ be a fibration of networks of manifolds. Then the pullback map*

$$\varphi^*\colon\ \mathcal{C}trl(G', \mathcal{P}') \to \mathcal{C}trl(G, \mathcal{P})$$

constructed in Theorem 3.8 is compatible with the interconnection maps

$$\mathscr{I}'\colon\ \mathcal{C}trl(G', \mathcal{P}') \to \Gamma(T\mathbb{P}G') \qquad \text{and} \qquad \mathscr{I}\colon\ \mathcal{C}trl(G, \mathcal{P}) \to \Gamma(T\mathbb{P}G).$$

Namely for any collection $w' \in \mathcal{C}trl(G', \mathcal{P}')$ of open systems on the network $(G', \mathcal{P}')$ the diagram

$$\begin{tikzcd} T\mathbb{P}G' \arrow[r, "D\mathbb{P}\varphi"] & T\mathbb{P}G \\ \mathbb{P}G' \arrow[u, "\mathscr{I}'(w')"] \arrow[r, "\mathbb{P}\varphi"'] & \mathbb{P}G \arrow[u, "\mathscr{I}(\varphi^* w')"'] \end{tikzcd} \tag{3.1}$$

commutes. Consequently

$$\mathbb{P}\varphi\colon\ (\mathbb{P}(G', \mathcal{P}'), \mathscr{I}'(w')) \to (\mathbb{P}(G, \mathcal{P}), \mathscr{I}(\varphi^* w'))$$

is a map of dynamical systems.

Proof. Recall that the manifold $\mathbb{P}G$ is the product $\prod_{a \in G_0} \mathbb{P}a$. Hence the tangent bundle bundle $T\mathbb{P}G$ is the product $\prod_{a \in G_0} T\mathbb{P}a$. The canonical projections

$$T\mathbb{P}G \to T\mathbb{P}a$$

are the differentials of the maps $\mathbb{P}\iota_a\colon \mathbb{P}G \to \mathbb{P}a$, where, as before, $\iota_a\colon \{a\} \hookrightarrow G$ is the canonical inclusion of graphs. Hence by the universal property of products, two maps into $T\mathbb{P}G$ are equal if and only if all their components are equal. Therefore, in order to prove that (3.1) commutes it is enough to show that

$$D\mathbb{P}\iota_a \circ \mathscr{I}(\varphi^* w') \circ \mathbb{P}\varphi = D\mathbb{P}\iota_a \circ D\mathbb{P}\varphi \circ \mathscr{I}'(w')$$

for all nodes $a \in G_0$. By definition of the restriction $\varphi|_{\{a\}}$ of $\varphi\colon G \to G'$ to $\{a\} \hookrightarrow G$, the diagram

$$\begin{tikzcd}
\{a\} \arrow[r, "\varphi|_{\{a\}}"] \arrow[d, "\iota_a"'] & \{\varphi(a)\} \arrow[d, "\iota_{\varphi(a)}"'] \\
G \arrow[r, "\varphi"'] & G'
\end{tikzcd} \tag{3.2}$$

commutes. By the definition of the pullback map φ^* and the interconnection maps $\mathscr{I}$, $\mathscr{I}'$ the diagram

$$\begin{tikzcd}
T\mathbb{P}a & T\mathbb{P}\varphi(a) \arrow[l, "D\mathbb{P}\varphi|_{\{a\}}"] \\
\mathbb{P}I(a) \arrow[u, "(\varphi^* w')_a"] & \mathbb{P}I(\varphi(a)) \arrow[l, "\mathbb{P}\varphi_a"'] \arrow[u, "w'_{\varphi(a)}"'] \\
\mathbb{P}G \arrow[u, "\mathbb{P}\xi_a"] \arrow[uu, bend left=70, "\mathscr{I}(\varphi^* w')_a"] & \mathbb{P}G' \arrow[l, "\mathbb{P}\varphi"] \arrow[u, "\mathbb{P}\xi_{\varphi(a)}"'] \arrow[uu, bend right=70, "\mathscr{I}'(w')_{\varphi(a)}"']
\end{tikzcd} \tag{3.3}$$

commutes as well. We now compute:

$$\begin{aligned}
D\mathbb{P}\iota_a \circ \mathscr{I}(\varphi^* w') \circ \mathbb{P}\varphi &= (\mathscr{I}(\varphi^* w'))_a \circ \mathbb{P}\varphi \\
&= D\mathbb{P}(\varphi|_{\{a\}}) \circ \mathscr{I}'(w')_{\varphi(a)} && \text{by (3.3)} \\
&= D\mathbb{P}(\varphi|_{\{a\}}) \circ D\mathbb{P}\iota_{\varphi(a)} \circ \mathscr{I}'(w') && \text{by definition of } \mathscr{I}'(w')_{\varphi(a)} \\
&= D\mathbb{P}\left(\iota_{\varphi(a)} \circ \varphi|_{\{a\}}\right) \circ \mathscr{I}'(w') && \text{since } \mathbb{P} \text{ is a contravariant functor} \\
&= D\mathbb{P}\left(\varphi \circ \iota_a\right) \circ \mathscr{I}'(w') && \text{by (3.2)} \\
&= D\mathbb{P}(\iota_a) \circ D\mathbb{P}\varphi \circ \mathscr{I}'(w').
\end{aligned}$$

And we are done. ∎

Corollary 3.12. *The map*

$$\mathsf{DSN} \to \mathsf{DS}$$

$$\left((G', \mathcal{P}', w') \xrightarrow{\varphi} (G, \mathcal{P}, w)\right) \;\mapsto\; \left((\mathbb{P}G', \mathscr{I}(w')) \xrightarrow{\mathbb{P}\varphi} (\mathbb{P}G, \mathscr{I}(w))\right)$$

is a functor.

Remark 3.13. Given a dynamical system on a network $(G, \mathcal{P}, w)$ we can forget the dynamics. This defines a functor

$$\mathsf{DSN} \to (\mathsf{Man}/\mathsf{Graph})_{\mathsf{fib}}$$

from the category of dynamical systems on networks to a subcategory of the category of networks of manifolds whose maps are fibrations of networks (hence the subscript $_{\mathsf{fib}}$). Composing the functor above with the functor $\mathsf{Man}/\mathsf{Graph} \to \mathsf{Graph}^{\mathsf{op}}$ forgets all the information except for the graph. This gives rise to a functor

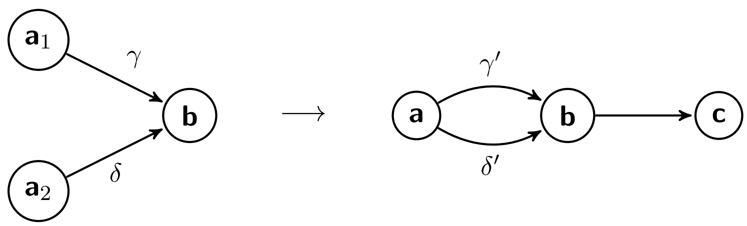
$$\mathsf{DSN} \to {\mathsf{Graph}_{\mathsf{fib}}}^{\mathsf{op}}.$$

Here the superscript $^{\mathsf{op}}$ indicates that the functor reverses the direction of arrows and the subscript $_{\mathsf{fib}}$ reminds us that the morphisms in the target category are the (opposite of the) graph fibrations.

These two functors from DSN to DS and to ${\mathsf{Graph}_{\mathsf{fib}}}^{\mathsf{op}}$, respectively, allow us to interpret continuous time dynamical systems on networks both as dynamical systems and as graphs.

We end the paper with examples.

Example 3.14. Consider the graph fibration



as in Examples 3.2 and 3.9. Let $\mathcal{P}' \colon G_0' \to \mathsf{Man}$ be a phase space function and let $\mathcal{P} = \mathcal{P}' \circ \varphi$. Then

$$
\begin{aligned}
&\mathbb{P}G' = \mathcal{P}'(a) \times \mathcal{P}'(b) \times \mathcal{P}'(c),\\
&\mathbb{P}G = \mathcal{P}'(a) \times \mathcal{P}'(a) \times \mathcal{P}'(b),\\
&\mathbb{P}\varphi(x,y,z) = (x,x,y),
\end{aligned}
$$

and

$$D\mathbb{P}\varphi(p,q,r) = (p,p,q).$$

For any $w' = (w_a', w_b', w_c') \in \mathcal{C}\mathit{trl}(G', \mathcal{P}')$,

$$
\begin{aligned}
&\big(\mathscr{I}'(w')\big)(x,y,z) = (w_a'(x), w_b'(x,x,y), w_c'(y,z)),\\
&\varphi^* w' = (w_a', w_a', w_b'),\\
&\big(\mathscr{I}(\varphi^* w')\big)(x_1,x_2,y) = (w_a'(x_1), w_a'(x_2), w_b'(x_1,x_2,y))
\end{aligned}
$$

and

$$\big(\mathscr{I}(\varphi^* w') \circ \mathbb{P}\varphi\big)(x,y,z) = (w_a'(x), w_a'(x), w_b'(x,x,y))$$

while

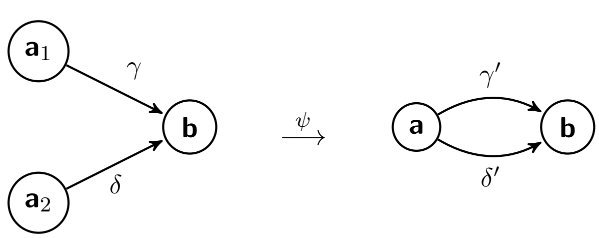
$$\big(D\mathbb{P}\varphi \circ \mathscr{I}'(w')\big)(x,y,z) = D\mathbb{P}\varphi(w_a'(x), w_b'(x,x,y), w_c'(y,z)) = (w_a'(x), w_a'(x), w_b'(x,x,y)).$$

Hence

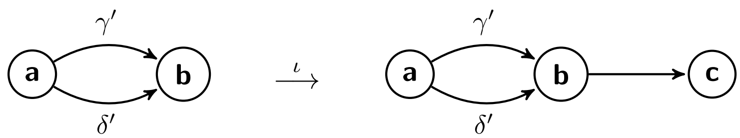
$$\big(\mathscr{I}(\varphi^* w') \circ \mathbb{P}\varphi\big) = \big(D\mathbb{P}\varphi \circ \mathscr{I}'(w')\big)$$

as expected.

Example 3.15. In Example 3.14 above the map $\varphi \colon G \to G'$ is neither injective nor surjective. It can, of course, be factored as a surjection $\psi \colon G \to G''$:



followed by an injection $\iota\colon G'' \to G$:



The map $\mathbb{P}\psi\colon \mathbb{P}G'' \to \mathbb{P}G$ is easily seen to be given by

$$\mathbb{P}\psi(x,y) = (x,x,y).$$

It is an embedding, as it should be (q.v. Lemma 2.20). The map $\mathbb{P}\imath\colon \mathbb{P}G' \to \mathbb{P}G''$ is given by

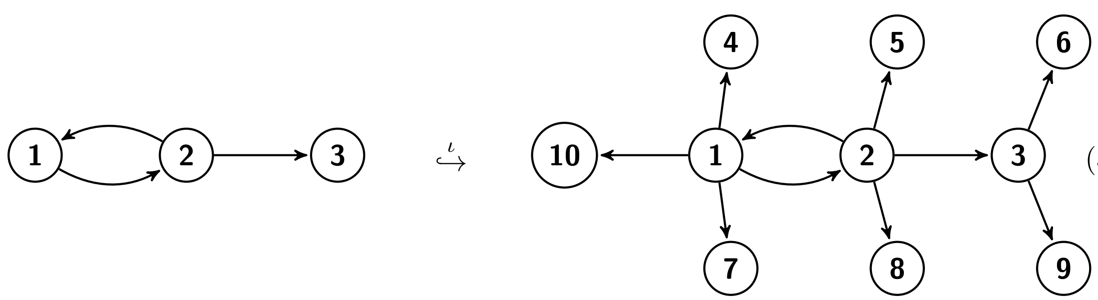
$$\mathbb{P}\imath(x,y,z) = (x,y).$$

It is a submersion (q.v. Lemma 2.21). Since $\mathbb{P}$ is a contravariant functor,

$$\mathbb{P}\varphi = \mathbb{P}(\imath \circ \psi) = \mathbb{P}\psi \circ \mathbb{P}\imath.$$

Theorem 3.11 tells us that for any $w' = (w'_a, w'_b, w'_c) \in \mathcal{C}trl(G', \mathcal{P}')$, the map $\mathbb{P}\imath$ projects the integral curves of the vector field $\mathscr{I}(w')$ to the integral curves of the vector field $\mathscr{I}(\imath^* w')$ on $\mathbb{P}G''$. Furthermore, $\mathbb{P}\psi$ embeds the dynamical system $(\mathbb{P}G'', \mathscr{I}(\imath^* w'))$ into the dynamical system $(\mathbb{P}G, \mathscr{I}(\varphi^* w'))$. An interested reader can check these two assertions directly.

Example 3.16. Consider the injective graph fibration $\iota\colon G \to G'$:



(3.4)

Choose phase space functions $\mathcal{P}$, $\mathcal{P}'$ so that $i\colon (G,\mathcal{P}) \to (G',\mathcal{P}')$ is a map of networks. By Theorem 3.11, for any collection $w' \in \mathcal{C}trl(G',\mathcal{P}')$ of open systems on the network $(G',\mathcal{P}')$ the dynamics in the subsystem $(\mathbb{P}G, \mathscr{I}(i^* w'))$ drives the entire system $(\mathbb{P}G', \mathscr{I}(w'))$. This is intuitively clear from the graph (3.4) since there are no “feedbacks” from vertices $4, \dots, 10$ back into 1, 2, 3.

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