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On a Certain Subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ Related to the Degenerate q-Onsager Algebra*

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Abstract. In [*Kyushu J. Math.* **64** (2010), 81–144], it is discussed that a certain subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ controls the second kind TD-algebra of type I (the degenerate q-Onsager algebra). The subalgebra, which we denote by $U'_q(\widehat{\mathfrak{sl}}_2)$, is generated by e_0^+, $e_1^\pm$, $k_i^{\pm 1}$ $(i = 0, 1)$ with e_0^- missing from the Chevalley generators $e_i^\pm$, $k_i^{\pm 1}$ $(i = 0, 1)$ of $U_q(\widehat{\mathfrak{sl}}_2)$. In this paper, we determine the finite-dimensional irreducible representations of $U'_q(\widehat{\mathfrak{sl}}_2)$. Intertwiners are also determined.

Key words: degenerate q-Onsager algebra; quantum affine algebra; TD-algebra; augmented TD-algebra; TD-pair

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1 Introduction

Throughout this paper, the ground field is $\mathbb{C}$ and q stands for a nonzero scalar that is not a root of unity. The symbols ε, ε^* stand for an integer chosen from $\{0, 1\}$. Let $\mathcal{A}_q = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ denote the associative algebra with 1 generated by z, z^* subject to the defining relations [4]

$$\text{(TD)} \quad \begin{cases} \big[z, [z, [z, z^*]_q]_{q^{-1}}\big] = -\varepsilon\big(q^2 - q^{-2}\big)^2 [z, z^*], \\ \big[z^*, [z^*, [z^*, z]_q]_{q^{-1}}\big] = -\varepsilon^*\big(q^2 - q^{-2}\big)^2 [z^*, z], \end{cases}$$

where $[X, Y] = XY - YX$, $[X, Y]_q = qXY - q^{-1}YX$. This paper deals with a subalgebra of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ that is closely related to $\mathcal{A}_q$ in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$. If $(\varepsilon, \varepsilon^*) = (0, 0)$, $\mathcal{A}_q$ is isomorphic to the positive part of $U_q(\widehat{\mathfrak{sl}}_2)$. If $(\varepsilon, \varepsilon^*) = (1, 1)$, $\mathcal{A}_q$ is called the q-Onsager algebra. If $(\varepsilon, \varepsilon^*) = (1, 0)$, $\mathcal{A}_q$ may well be called the *degenerate q-Onsager algebra*.

The algebra $\mathcal{A}_q$ arises in the course of the classification of TD-pairs of type I, which is a critically important step in the study of representations of Terwilliger algebras for P- and Q-polynomial association schemes [3]. For this reason, $\mathcal{A}_q$ is called the TD-algebra of type I. Precisely speaking, the TD-algebra of type I is standardized to be the algebra $\mathcal{A}_q$, where q is the main parameter for TD-pairs of type I; so $q^2 \neq \pm 1$ and q is allowed to be a root of unity. In our case where we assume q is not a root of unity, the classification of the TD-pairs of type I is equivalent to determining the finite-dimensional irreducible representations $\rho : \mathcal{A}_q \to \mathrm{End}(V)$ with the property that $\rho(z)$, $\rho(z^*)$ are both diagonalizable. Such irreducible representations

*This paper is a contribution to the Special Issue on New Directions in Lie Theory. The full collection is available at http://www.emis.de/journals/SIGMA/LieTheory2014.html of $\mathcal{A}_q$ are determined in [4] via embeddings of $\mathcal{A}_q$ into the augmented TD-algebra $\mathcal{T}_q$. (In the case of $(\varepsilon, \varepsilon^*) = (1, 1)$, the diagonalizability condition of $\rho(z)$, $\rho(z^*)$ can be dropped, because it turns out that this condition always holds for every finite-dimensional irreducible representation ρ of the q-Onsager algebra $\mathcal{A}_q$.) $\mathcal{T}_q$ is easier than $\mathcal{A}_q$ to study representations for, and each finite-dimensional irreducible representation $\rho : \mathcal{A}_q \to \mathrm{End}(V)$ with $\rho(z)$, $\rho(z^*)$ diagonalizable can be extended to a finite-dimensional irreducible representation of $\mathcal{T}_q$ via a certain embedding of $\mathcal{A}_q$ into $\mathcal{T}_q$.

The augmented TD-algebra $\mathcal{T}_q = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is the associative algebra with 1 generated by x, y, $k^{\pm 1}$ subject to the defining relations

$$(\mathrm{TD})_0 \quad \begin{cases} kk^{-1} = k^{-1}k = 1, \\ kxk^{-1} = q^2 x, \\ kyk^{-1} = q^{-2} y, \end{cases} \tag{1}$$

and

$$(\mathrm{TD})_1 \quad \begin{cases} \big[x, [x, [x, y]_q]_{q^{-1}}\big] = \delta\big(\varepsilon^* x^2 k^2 - \varepsilon k^{-2} x^2\big), \\ \big[y, [y, [y, x]_q]_{q^{-1}}\big] = \delta\big(-\varepsilon^* k^2 y^2 + \varepsilon y^2 k^{-2}\big), \end{cases} \tag{2}$$

where $\delta = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})q^4$. The finite-dimensional irreducible representations of $\mathcal{T}_q$ are determined in [4] via embeddings of $\mathcal{T}_q$ into the $U_q(\mathfrak{sl}_2)$-loop algebra $U_q(L(\mathfrak{sl}_2))$.

Let $e_i^{\pm}$, $k_i^{\pm 1}$ $(i = 0, 1)$ be the Chevalley generators of $U_q(L(\mathfrak{sl}_2))$. So the defining relations of $U_q(L(\mathfrak{sl}_2))$ are

$$\begin{aligned} &k_0 k_1 = k_1 k_0 = 1, \qquad k_i k_i^{-1} = k_i^{-1} k_i = 1, \qquad k_i e_i^{\pm} k_i^{-1} = q^{\pm 2} e_i^{\pm}, \\ &k_i e_j^{\pm} k_i^{-1} = q^{\mp 2} e_j^{\pm}, \quad i \neq j, \qquad [e_i^+, e_i^-] = \frac{k_i - k_i^{-1}}{q - q^{-1}}, \qquad [e_i^+, e_j^-] = 0, \quad i \neq j, \\ &\big[e_i^{\pm}, [e_i^{\pm}, [e_i^{\pm}, e_j^{\pm}]_q]_{q^{-1}}\big] = 0, \qquad i \neq j. \end{aligned} \tag{3}$$

Note that if $k_0 k_1 = k_1 k_0 = 1$ is replaced by $k_0 k_1 = k_1 k_0$ in (3), we have the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$: $U_q(L(\mathfrak{sl}_2))$ is the quotient algebra of $U_q(\widehat{\mathfrak{sl}}_2)$ by the two-sided ideal generated by $k_0 k_1 - 1$. For a nonzero scalar s, define the elements $x(s)$, $y(s)$, $k(s)$ of $U_q(L(\mathfrak{sl}_2))$ by

$$\begin{aligned} &x(s) = -q^{-1}\big(q - q^{-1}\big)^2\big(s e_0^+ + \varepsilon s^{-1} e_1^- k_1\big), \\ &y(s) = \varepsilon^* s e_0^- k_0 + s^{-1} e_1^+, \\ &k(s) = s k_0. \end{aligned} \tag{4}$$

Then the mapping

$$\varphi_s : \ \mathcal{T}_q \to U_q(L(\mathfrak{sl}_2)), \qquad x, y, k \mapsto x(s), y(s), k(s), \tag{5}$$

gives an injective algebra homomorphism. If $(\varepsilon, \varepsilon^*) = (0, 0)$, the image $\varphi_s(\mathcal{T}_q)$ coincides with the Borel subalgebra generated by e_i^+, $k_i^{\pm 1}$ $(i = 0, 1)$. If $(\varepsilon, \varepsilon^*) = (1, 0)$, the image $\varphi_s(\mathcal{T}_q)$ is properly contained in the subalgebra generated by e_0^+, $e_1^{\pm}$, $k_i^{\pm 1}$ $(i = 0, 1)$ with e_0^- missing from the generators; we denote this subalgebra by $U_q'(L(\mathfrak{sl}_2))$. Through the natural homomorphism $U_q(\widehat{\mathfrak{sl}}_2) \to U_q(L(\mathfrak{sl}_2))$, pull back the subalgebra $U_q'(L(\mathfrak{sl}_2))$ and denote the pre-image by $U_q'(\widehat{\mathfrak{sl}}_2)$:

$$U_q'(\widehat{\mathfrak{sl}}_2) = \big\langle e_0^+, e_1^{\pm}, k_i^{\pm 1} \,|\, i = 0, 1 \big\rangle \subset U_q(\widehat{\mathfrak{sl}}_2).$$

In [4], it is shown that in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$, all the finite-dimensional irreducible representations of $\mathcal{T}_q$ are produced by tensor products of evaluation modules for $U_q'(L(\mathfrak{sl}_2))$

via the embedding φ_s of $\mathcal{T}_q$ into $U_q'(L(\mathfrak{sl}_2))$. Using this fact and the Drinfel'd polynomials, we show in this paper that there are no other finite-dimensional irreducible representations of $U_q'(L(\mathfrak{sl}_2))$ and hence of $U_q'(\widehat{\mathfrak{sl}}_2)$ than those afforded by tensor products of evaluation modules, if we apply suitable automorphisms of $U_q'(L(\mathfrak{sl}_2))$, $U_q'(\widehat{\mathfrak{sl}}_2)$ to adjust the types of the representations to be $(1,1)$. Here we note that the evaluation parameters are allowed to be zero for $U_q'(L(\mathfrak{sl}_2))$, $U_q'(\widehat{\mathfrak{sl}}_2)$. Details will be discussed in Sections 2 and 3, where the isomorphism classes of finite-dimensional irreducible representations of $U_q'(\widehat{\mathfrak{sl}}_2)$ are also determined. In Section 4, intertwiners will be determined for finite-dimensional irreducible $U_q'(\widehat{\mathfrak{sl}}_2)$-modules.

In our approach, Drinfel'd polynomials are the key tool for the classification of finite-dimensional irreducible representations of $U_q(\widehat{\mathfrak{sl}}_2)$, $U_q'(\widehat{\mathfrak{sl}}_2)$, although they are not the main subject of this paper. They are defined in [4], and the point is that they are directly attached to $\mathcal{T}_q$-modules, not to $U_q(\widehat{\mathfrak{sl}}_2)$- or $U_q'(\widehat{\mathfrak{sl}}_2)$-modules. (In the case of $(\varepsilon,\varepsilon^*)=(0,0)$, they turn out to coincide with the original ones up to the reciprocal of the variable.) So in our approach to the case of $(\varepsilon,\varepsilon^*)=(0,0)$, finite-dimensional irreducible representations are naturally classified firstly for the Borel subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$ and then for $U_q(\widehat{\mathfrak{sl}}_2)$ itself. This will be briefly demonstrated in Section 3 as a warm-up for the case of $(\varepsilon,\varepsilon^*)=(1,0)$, thus giving another proof to the classical classification theorem of Chari–Pressley [2] and to the main theorems (Theorems 1.16 and 1.17) of [1].

We now review Drinfel'd polynomials for $\mathcal{T}_q$-modules [4, p. 119]. Let V be a finite-dimensional $\mathcal{T}_q$-module. We assume the following properties for V:

(D)$_0$: k is diagonalizable on V with $V=\bigoplus_{i=0}^{d} U_i$, $k|_{U_i}=sq^{2i-d}$, $0\le i\le d$, for some nonzero constant s;

(D)$_1$: $\dim U_0=1$.

By the relations $(\mathrm{TD})_0: kk^{-1}=k^{-1}k=1$, $kxk^{-1}=q^2x$, $kyk^{-1}=q^{-2}y$, it holds that $xU_i\subseteq U_{i+1}$, $yU_i\subseteq U_{i-1}$ $(0\le i\le d)$, where $U_{-1}=U_{d+1}=0$. So the one-dimensional subspace U_0 is invariant under y^ix^i and we have the sequence $\{\sigma_i\}_{i=0}^{\infty}$ of eigenvalues σ_i of y^ix^i on U_0: $\sigma_i=y^ix^i|_{U_0}$. Notice that $\sigma_0=1$ and $\sigma_i=0$, $d+1\le i$. The Drinfel'd polynomial $P_V(\lambda)$ of the $\mathcal{T}_q$-module V is defined by

$$P_V(\lambda)=\sum_{i=0}^{d}(-1)^i\frac{\sigma_i}{(q-q^{-1})^{2i}([i]!)^2}\prod_{j=i+1}^{d}\big(\lambda-\varepsilon s^{-2}q^{2(d-j)}-\varepsilon^*s^2q^{-2(d-j)}\big),$$

where $[i]=[i]_q=(q^i-q^{-i})/(q-q^{-1})$ and $[i]!=[1][2]\cdots[i]$ with the understanding of $[0]!=1$. Since $\sigma_0=1$, $P_V(\lambda)$ is a monic polynomial of degree d.

If V is an irreducible $\mathcal{T}_q$-module, it is known that V in fact satisfies the properties (D)$_0$, (D)$_1$ [4, Lemma 1.2, Theorem 1.8], and these properties provide a rather simple short proof for the 'injective' part of [4, Theorem 1.9], i.e., for the fact that the isomorphism class of the irreducible $\mathcal{T}_q$-module V is determined by the trio $(\{\sigma_i\}_{i=0}^{\infty},s,d)$.

If V is a tensor product of evaluation modules for $U_q(L(\mathfrak{sl}_2))$ in the case of $(\varepsilon,\varepsilon^*)=(1,1),(0,0)$ or for $U_q'(L(\mathfrak{sl}_2))$ in the case of $(\varepsilon,\varepsilon^*)=(1,0)$, we regard V as a $\mathcal{T}_q$-module via the embedding φ_s of (5). Then it is apparent that the $\mathcal{T}_q$-module V satisfies the properties (D)$_0$, (D)$_1$. Moreover it is known that a product formula holds for the Drinfel'd polynomial $P_V(\lambda)$ and it turns out that $P_V(\lambda)$ does not depend on the parameter s of the embedding φ_s [4, Theorem 5.2]. The 'surjective' part of [4, Theorem 1.9] follows from the structure of the zeros of the Drinfel'd polynomial for such a tensor product of evaluation modules regarded as a $\mathcal{T}_q$-module via the embedding φ_s.

2 Finite-dimensional irreducible representations of $U'_q(\widehat{\mathfrak{sl}}_2)$

The subalgebra $U'_q(\widehat{\mathfrak{sl}}_2)$ of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ is generated by e_0^+, $e_1^\pm$, $k_i^{\pm 1}$ $(i=0,1)$, e_0^- missing from the generators, and has by the triangular decomposition of $U_q(\widehat{\mathfrak{sl}}_2)$ the defining relations

$$\begin{aligned} &k_0k_1=k_1k_0, \qquad k_ik_i^{-1}=k_i^{-1}k_i=1, \qquad k_0e_0^+k_0^{-1}=q^2e_0^+, \qquad k_1e_1^\pm k_1^{-1}=q^{\pm 2}e_1^\pm,\\ &k_1e_0^+k_1^{-1}=q^{-2}e_0^+, \qquad k_0e_1^\pm k_0^{-1}=q^{\mp 2}e_1^\pm, \qquad [e_1^+,e_1^-]=\frac{k_1-k_1^{-1}}{q-q^{-1}},\\ &[e_0^+,e_1^-]=0, \qquad \big[e_i^+,[e_i^+,[e_i^+,e_j^+]_q]_{q^{-1}}\big]=0, \quad i\neq j. \end{aligned} \tag{6}$$

Note that if $k_0k_1=k_1k_0$ is replaced by $k_0k_1=k_1k_0=1$ in (6), we have the defining relations for $U'_q(L(\mathfrak{sl}_2))$.

Let V be a finite-dimensional irreducible $U'_q(\widehat{\mathfrak{sl}}_2)$-module. Let us first observe that the $U'_q(\widehat{\mathfrak{sl}}_2)$-module V is obtained from a $U'_q(L(\mathfrak{sl}_2))$-module by applying an automorphism of $U'_q(\widehat{\mathfrak{sl}}_2)$ as follows. Since the element k_0k_1 belongs to the centre of $U'_q(\widehat{\mathfrak{sl}}_2)$, k_0k_1 acts on V as a scalar s by Schur's lemma. Since k_0k_1 is invertible, the scalar s is nonzero: $k_0k_1|_V=s\in\mathbb{C}^\times$. Observe that $U'_q(\widehat{\mathfrak{sl}}_2)$ admits an automorphism that sends k_0 to $s^{-1}k_0$ and preserves k_1. Hence we may assume $k_0k_1|_V=1$. Then we can regard V as a $U'_q(L(\mathfrak{sl}_2))$-module.

Let V be a finite-dimensional irreducible $U'_q(L(\mathfrak{sl}_2))$-module. For a scalar θ, set $V(\theta)=\{v\in V\,|\,k_0v=\theta v\}$. So if $V(\theta)\neq 0$, θ is an eigenvalue of k_0 and $V(\theta)$ is the corresponding eigenspace of k_0. For an eigenvalue θ and an eigenvector $v\in V(\theta)$, it holds that $e_0^+v\in V(q^2\theta)$ by the relation $k_0e_0^+=q^2e_0^+k_0$ and $e_1^\pm v\in V(q^{\mp 2}\theta)$ by $k_0e_1^\pm=q^{\mp 2}e_1^\pm k_0$. We have

$$e_0^+V(\theta)\subseteq V\big(q^2\theta\big), \qquad e_1^\pm V(\theta)\subseteq V\big(q^{\mp 2}\theta\big). \tag{7}$$

If $\dim V=1$, then $e_0^+V=0$, $e_1^\pm V=0$ by (7) and $k_0|_V=\pm 1$ by $[e_1^+,e_1^-]=(k_1-k_1^{-1})/(q-q^{-1})=(k_0^{-1}-k_0)/(q-q^{-1})$. Such a $U'_q(L(\mathfrak{sl}_2))$-module V is said to be *trivial*. Assume $\dim V\geq 2$. Choose an eigenvalue θ of k_0 on V. Then $\sum_{i\in\mathbb{Z}} V(q^{\pm 2i}\theta)$ is invariant under the actions of the generators by (7), and so we have $V=\sum_{i\in\mathbb{Z}} V(q^{\pm 2i}\theta)$ by the irreducibility of the $U'_q(L(\mathfrak{sl}_2))$-module V. Since V is finite-dimensional, there exists a positive integer d and a nonzero scalar s_0 such that the eigenspace decomposition of k_0 is

$$V=\bigoplus_{i=0}^{d} V\big(s_0q^{2i-d}\big). \tag{8}$$

We want to show that $s_0=\pm 1$ holds in (8).

Consider the subalgebra of $U'_q(L(\mathfrak{sl}_2))$ generated by $e_1^\pm$, $k_1^{\pm 1}$ and denote it by $\mathcal{U}:\mathcal{U}=\langle e_1^\pm, k_1^{\pm 1}\rangle$. Regard V as a $\mathcal{U}$-module. Since $\mathcal{U}$ is isomorphic to the quantum algebra $U_q(\mathfrak{sl}_2)$, V is a direct sum of irreducible $\mathcal{U}$-modules, and for each irreducible $\mathcal{U}$-submodule W of V, the eigenvalues of $k_1=k_0^{-1}$ on W are either $\{q^{2i-\ell}\,|\,0\leq i\leq \ell\}$ or $\{-q^{2i-\ell}\,|\,0\leq i\leq\ell\}$ for some nonnegative integer ℓ. This implies that (i) $s_0=\pm q^m$ for some $m\in\mathbb{Z}$ and (ii) if θ is an eigenvalue of k_0, so is θ^{-1}. It follows from (i) that $V=\bigoplus_{i=0}^{d} V(\pm q^{2i-d+m})$, and so by (ii), we obtain $m=0$, i.e., $s_0=\pm 1$.

Observe that $U'_q(L(\mathfrak{sl}_2))$ admits an automorphism that sends k_i to $-k_i$ $(i=0,1)$ and e_1^+ to $-e_1^+$. Hence we may assume $s_0=1$ in (8). Note that in this case, k_1 has the eigenvalues $\{s_1q^{2i-d}\,|\,0\leq i\leq d\}$ with $s_1=1$. Such an irreducible module or the irreducible representation afforded by such is said to be of *type* $(1,1)$, indicating $(s_0,s_1)=(1,1)$. We conclude that the determination of finite-dimensional irreducible representations for $U'_q(\widehat{\mathfrak{sl}}_2)$ is, via automorphisms, reduced to that of type $(1,1)$ for $U'_q(L(\mathfrak{sl}_2))$.

In the rest of this section, we shall introduce evaluation modules for $U'_q(L(\mathfrak{sl}_2))$ and state our main theorem that every finite-dimensional irreducible representation of type $(1,1)$ of $U'_q(L(\mathfrak{sl}_2))$ is afforded by a tensor product of them. For $a\in\mathbb{C}$ and $\ell\in\mathbb{Z}_{\geq 0}$, let $V(\ell,a)$ denote the $(\ell+1)$-dimensional vector space with a basis $v_0,v_1,\dots,v_\ell$. Using (6), it can be routinely verified that $U'_q(L(\mathfrak{sl}_2))$ acts on $V(\ell,a)$ by

$$\begin{aligned}&k_0v_i=q^{2i-\ell}v_i,\qquad k_1v_i=q^{\ell-2i}v_i,\qquad e_0^+v_i=aq[i+1]v_{i+1},\\&e_1^+v_i=[\ell-i+1]v_{i-1},\qquad e_1^-v_i=[i+1]v_{i+1},\end{aligned}\tag{9}$$

where $v_{-1}=v_{\ell+1}=0$ and $[t]=[t]_q=(q^t-q^{-t})/(q-q^{-1})$. This $U'_q(L(\mathfrak{sl}_2))$-module $V(\ell,a)$ is irreducible and called an *evaluation module*. The basis $v_0,v_1,\dots,v_\ell$ of the $U'_q(L(\mathfrak{sl}_2))$-module $V(\ell,a)$ is called a *standard basis*. The vector v_0 is called the *highest weight vector*. Note that the evaluation parameter a is allowed to be zero. Also note that if $\ell=0$, $V(\ell,a)$ is the trivial module. We denote the evaluation module $V(\ell,0)$ by $V(\ell)$, allowing $\ell=0$, and use the notation $V(\ell,a)$ only for an evaluation module with $a\neq 0$ and $\ell\geq 1$.

The $U_q(\mathfrak{sl}_2)$-loop algebra $U_q(L(\mathfrak{sl}_2))$ has the coproduct $\Delta: U_q(L(\mathfrak{sl}_2))\to U_q(L(\mathfrak{sl}_2))\otimes U_q(L(\mathfrak{sl}_2))$ defined by

$$\begin{aligned}&\Delta(k_i^{\pm1})=k_i^{\pm1}\otimes k_i^{\pm1},\qquad \Delta(e_i^+)=k_i\otimes e_i^++e_i^+\otimes 1,\\&\Delta(e_i^-k_i)=k_i\otimes e_i^-k_i+e_i^-k_i\otimes 1.\end{aligned}\tag{10}$$

The subalgebra $U'_q(L(\mathfrak{sl}_2))$ is closed under Δ. Thus given a set of evaluation modules $V(\ell)$, $V(\ell_i,a_i)$ $(1\leq i\leq n)$ for $U'_q(L(\mathfrak{sl}_2))$, the tensor product

$$V(\ell)\otimes V(\ell_1,a_1)\otimes\cdots\otimes V(\ell_n,a_n)\tag{11}$$

becomes a $U'_q(L(\mathfrak{sl}_2))$-module via Δ. Note that by the coassociativity of Δ, the way of putting parentheses in the tensor product of (11) does not affect the isomorphism class as a $U'_q(L(\mathfrak{sl}_2))$-module. Also note that if $\ell=0$ in (11), then $V(0)$ is the trivial module and the tensor product of (11) is isomorphic to $V(\ell_1,a_1)\otimes\cdots\otimes V(\ell_n,a_n)$ as $U'_q(L(\mathfrak{sl}_2))$-modules. Finally we allow $n=0$, in which case we understand that the tensor product of (11) means $V(\ell)$.

With the evaluation module $V(\ell,a)$, we associate the set $S(\ell,a)$ of scalars $aq^{-\ell+1}$, $aq^{-\ell+3},\dots$, $aq^{\ell-1}$:

$$S(\ell,a)=\{aq^{2i-\ell+1}\,|\,0\leq i\leq\ell-1\}.$$

The set $S(\ell,a)$ is called a *q-string* of length ℓ. Two q-strings $S(\ell,a)$, $S(\ell',a')$ are said to be *in general position* if either

(i) the union $S(\ell,a)\cup S(\ell',a')$ is not a q-string, or

(ii) one of $S(\ell,a)$, $S(\ell',a')$ includes the other.

Below is the main theorem of this paper. It classifies the isomorphism classes of the finite-dimensional irreducible $U'_q(L(\mathfrak{sl}_2))$-modules of type $(1,1)$.

Theorem 1. *The following (i), (ii), (iii), (iv) hold.*

(i) A tensor product $V=V(\ell)\otimes V(\ell_1,a_1)\otimes\cdots\otimes V(\ell_n,a_n)$ of evaluation modules is irreducible as a $U'_q(L(\mathfrak{sl}_2))$-module if and only if $S(\ell_i,a_i)$, $S(\ell_j,a_j)$ are in general position for all $i,j\in\{1,2,\dots,n\}$. In this case, V is of type $(1,1)$.

(*ii*) *Consider two tensor products* $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell') \otimes V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_m, a'_m)$ *of evaluation modules and assume that they are both irreducible as a* $U'_q(L(\mathfrak{sl}_2))$*-module. Then* V, V' *are isomorphic as* $U'_q(L(\mathfrak{sl}_2))$*-modules if and only if* $\ell = \ell'$, $n = m$ *and* $(\ell_i, a_i) = (\ell'_i, a'_i)$ *for all* i, $1 \leq i \leq n$, *with a suitable reordering of the evaluation modules* $V(\ell_1, a_1), \ldots, V(\ell_n, a_n)$.

(*iii*) *Every non-trivial finite-dimensional irreducible* $U'_q(L(\mathfrak{sl}_2))$*-module of type* $(1, 1)$ *is isomorphic to some tensor product* $V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ *of evaluation modules.*

(*iv*) *If a tensor product* $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ *of evaluation modules is irreducible as a* $U'_q(L(\mathfrak{sl}_2))$*-module, then any change of the orderings of the evaluation modules* $V(\ell)$, $V(\ell_1, a_1), \ldots, V(\ell_n, a_n)$ *for the tensor product does not change the isomorphism class of the* $U'_q(L(\mathfrak{sl}_2))$*-module* V.

3 Proof of Theorem 1(i), (ii), (iii)

Discard the evaluation module $V(\ell)$ from the statement of Theorem 1 and replace $U'_q(L(\mathfrak{sl}_2))$ by $U_q(L(\mathfrak{sl}_2))$ or by $\mathcal{B}$, where $\mathcal{B}$ is the Borel subalgebra of $U_q(L(\mathfrak{sl}_2))$ generated by e_i^+, $k_i^{\pm 1}$ $(i = 0, 1)$. Then it precisely describes the classification of the isomorphism classes of finite-dimensional irreducible modules of type $(1, 1)$ for $U_q(L(\mathfrak{sl}_2))$ [2] or for $\mathcal{B}$ [1]. There is a one-to-one correspondence of finite-dimensional irreducible modules of type $(1, 1)$ between $U_q(L(\mathfrak{sl}_2))$ and $\mathcal{B}$: every finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type $(1, 1)$ is irreducible as a $\mathcal{B}$-module and every finite-dimensional irreducible $\mathcal{B}$-module of type $(1, 1)$ is uniquely extended to a $U_q(L(\mathfrak{sl}_2))$-module. This sort of correspondence of finite-dimensional irreducible modules exists between $U'_q(L(\mathfrak{sl}_2))$ and $\mathcal{T}_q$ via the embedding φ_s of (5), where $\mathcal{T}_q$ is the augmented TD-algebra with $(\varepsilon, \varepsilon^*) = (1, 0)$. Namely, we shall show that (i) every finite-dimensional irreducible $U'_q(L(\mathfrak{sl}_2))$-module of type $(1, 1)$ is irreducible as a $\mathcal{T}_q$-module via certain embedding φ_s of (5), and (ii) every finite-dimensional irreducible $\mathcal{T}_q$-module is uniquely extended to a $U'_q(L(\mathfrak{sl}_2))$-module of type $(1, 1)$ via the embedding φ_s of (5) with s uniquely determined. Since finite-dimensional irreducible $\mathcal{T}_q$-modules are classified in [4], this gives a proof of Theorem 1.

Apart from the Drinfel'd polynomials, the key to our understanding of the correspondence is the following two lemmas about $U_q(\mathfrak{sl}_2)$-modules. Let $\mathcal{U}$ denote the quantum algebra $U_q(\mathfrak{sl}_2)$: $\mathcal{U}$ is the associative algebra with 1 generated by $X^{\pm}$, $K^{\pm 1}$ subject to the defining relations

$$KK^{-1} = K^{-1}K = 1, \qquad KX^{\pm}K^{-1} = q^{\pm 2}X^{\pm}, \qquad [X^+, X^-] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{12}$$

Lemma 1 ([4, Lemma 7.5])**.** *Let* V *be a finite-dimensional* $\mathcal{U}$*-module that has the following weight-space* (K*-eigenspace*) *decomposition:*

$$V = \bigoplus_{i=0}^{d} U_i, \qquad K|_{U_i} = q^{2i-d}, \qquad 0 \leq i \leq d.$$

Let W *be a subspace of* V *as a vector space. Assume that* W *is invariant under the actions of* X^+ *and* K*:*

$$X^+W \subseteq W, \qquad KW \subseteq W.$$

If it holds that

$$\dim(W \cap U_i) = \dim(W \cap U_{d-i}), \qquad 0 \leq i \leq d,$$

then $X^-W \subseteq W$, *i.e.,* W *is a* $\mathcal{U}$*-submodule.*

Lemma 2. *If V is a finite-dimensional $\mathcal{U}$-module, the action of X^- on V is uniquely determined by those of X^+, $K^{\pm 1}$ on V.*

Proof. The claim holds if V is irreducible as a $\mathcal{U}$-module. By the semi-simplicity of $\mathcal{U}$, it holds generally. ■

As a warm-up for the proof of Theorem 1, we shall demonstrate how to use these lemmas in the case of the corresponding theorem [2] for $U_q(L(\mathfrak{sl}_2))$. We want, and it is enough, to show part (iii) of the theorem for $U_q(L(\mathfrak{sl}_2))$ by using the classification of finite-dimensional irreducible $\mathcal{B}$-modules. This is because the parts (i), (ii), (iv) are well-known in advance of [2], while the finite-dimensional irreducible $\mathcal{B}$-modules are classified in [4] rather straightforward by the product formula of Drinfel'd polynomials without using the part (iii) in question.

Let V be a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type $(1,1)$. Then V has the weight-space decomposition

$$V = \bigoplus_{i=0}^{d} U_i, \qquad k_0|_{U_i} = q^{2i-d}, \qquad 0 \le i \le d.$$

Regard V as a $\mathcal{B}$-module. Let W be a minimal $\mathcal{B}$-submodule of V. Note that W is irreducible as a $\mathcal{B}$-module. We want to show $W = V$, i.e., V is irreducible as a $\mathcal{B}$-module. Since the mapping $(e_0^+)^{d-2i}\colon U_i \to U_{d-i}$ is a bijection and $W \cap U_i$ is mapped into $W \cap U_{d-i}$ by $(e_0^+)^{d-2i}$, we have $\dim(W \cap U_i) \le \dim(W \cap U_{d-i})$, $0 \le i \le [d/2]$. Similarly from the bijection $(e_1^+)^{d-2i}\colon U_{d-i} \to U_i$, we get $\dim(W \cap U_{d-i}) \le \dim(W \cap U_i)$. Thus it holds that

$$\dim(W \cap U_i) = \dim(W \cap U_{d-i}), \qquad 0 \le i \le d.$$

Consider the algebra homomorphism from $\mathcal{U}$ to $U_q(L(\mathfrak{sl}_2))$ that sends X^+, X^-, $K^{\pm 1}$ to e_0^+, e_0^-, $k_0^{\pm 1}$, respectively. Regard V as a $\mathcal{U}$-module via this algebra homomorphism. Then $X^+W \subseteq W$, $KW \subseteq W$. Since $\dim(W \cap U_i) = \dim(W \cap U_{d-i})$, $0 \le i \le d$, we have by Lemma 1 that $X^-W \subseteq W$, i.e., $e_0^- W \subseteq W$. Similarly, Lemma 1 can be applied to the $\mathcal{U}$-module V via the algebra homomorphism from $\mathcal{U}$ to $U_q(L(\mathfrak{sl}_2))$ that sends X^+, X^-, $K^{\pm 1}$ to e_1^+, e_1^-, $k_1^{\pm 1}$, respectively, in which case the weight-space decomposition of the $\mathcal{U}$-module V is $V = \bigoplus_{i=0}^{d} U_{d-i}$, $K|_{U_{d-i}} = q^{2i-d}$, $0 \le i \le d$. Consequently, we get $X^-W \subseteq W$, i.e., $e_1^- W \subseteq W$. Thus W is $U_q(L(\mathfrak{sl}_2))$-invariant and we have $W = V$ by the irreducibility of the $U_q(L(\mathfrak{sl}_2))$-module V. We conclude that every finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type $(1,1)$ is irreducible as a $\mathcal{B}$-module.

Now consider the class of finite-dimensional irreducible $\mathcal{B}$-modules V, where V runs through all tensor products of evaluation modules that are irreducible as a $U_q(L(\mathfrak{sl}_2))$-module:

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n).$$

Then it turns out that the Drinfel'd polynomials $P_V(\lambda)$ of the irreducible $\mathcal{B}$-modules V exhaust all that are possible for finite-dimensional irreducible $\mathcal{B}$-modules of type $(1,1)$, as shown in [4, Theorem 5.2] by the product formula

$$P_V(\lambda) = \prod_{i=1}^{n} P_{V(\ell_i,a_i)}(\lambda), \qquad P_{V(\ell_i,a_i)}(\lambda) = \prod_{\zeta \in S(\ell_i,a_i)} (\lambda + \zeta).$$

Since the Drinfel'd polynomial $P_V(\lambda)$ determines the isomorphism class of the $\mathcal{B}$-module V of type $(1,1)$ [4, the injectivity part of Theorem 1.9′], there are no other finite-dimensional irreducible $\mathcal{B}$-modules of type $(1,1)$. This means that every finite-dimensional irreducible $\mathcal{B}$-module of type $(1,1)$ comes from some tensor product of evaluation modules for $U_q(L(\mathfrak{sl}_2))$.

Let V be a finite-dimensional irreducible $U_q(L(\mathfrak{sl}_2))$-module of type $(1,1)$. Then V is irreducible as a $\mathcal{B}$-module and so there exists an irreducible $U_q(L(\mathfrak{sl}_2))$-module $V' = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ such that V, V' are isomorphic as $\mathcal{B}$-modules. By Lemma 2, V, V' are isomorphic as $U_q(L(\mathfrak{sl}_2))$-modules. This completes the proof of part (iii) of the theorem for $U_q(L(\mathfrak{sl}_2))$.

The proof of Theorem 1 can be given by an argument very similar to the one we have seen above for the case of $U_q(L(\mathfrak{sl}_2))$. We prepare two more lemmas for the case of $U_q'(L(\mathfrak{sl}_2))$ to make the point clearer. Set $(\varepsilon, \varepsilon^*) = (1, 0)$ and let $\mathcal{T}_q$ be the augmented TD-algebra defined by $(\mathrm{TD})_0$, $(\mathrm{TD})_1$ in (1), (2). For $s \in \mathbb{C}^\times$, let φ_s be the embedding of $\mathcal{T}_q$ into $U_q'(L(\mathfrak{sl}_2))$ given by (4), (5).

Lemma 3. *Let V_1, V_2 be finite-dimensional irreducible $U_q'(L(\mathfrak{sl}_2))$-modules. If V_1, V_2 are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules for some $s \in \mathbb{C}^\times$, then V_1, V_2 are isomorphic as $U_q'(L(\mathfrak{sl}_2))$-modules.*

Proof. By (4), $\varphi_s(\mathcal{T}_q)$ is generated by $se_0^+ + s^{-1}e_1^-k_1$, e_1^+ and $k_i^{\pm 1}$ $(i = 0, 1)$. Since $\langle e_1^\pm, k_1^{\pm 1}\rangle$ is isomorphic to the quantum algebra $U_q(\mathfrak{sl}_2)$, the action of e_1^- on V_i, $i = 1, 2$, is uniquely determined by those of e_1^+, $k_1^{\pm 1} \in \varphi_s(\mathcal{T}_q)$ by Lemma 2. Apparently the action of e_0^+ on V_i, $i = 1, 2$, is uniquely determined by those of $se_0^+ + s^{-1}e_1^-k_1$, e_1^-, k_1, and hence by that of $\varphi_s(\mathcal{T}_q)$. So the action of $U_q'(L(\mathfrak{sl}_2))$ on V_i, $i = 1, 2$, is uniquely determined by that of $\varphi_s(\mathcal{T}_q)$. ■

Lemma 4. *Let V be a finite-dimensional irreducible $U_q'(L(\mathfrak{sl}_2))$-module of type $(1,1)$. Then there exists a finite set Λ of nonzero scalars such that V is irreducible as a $\varphi_s(\mathcal{T}_q)$-module for each $s \in \mathbb{C}^\times \setminus \Lambda$.*

Proof. For $s \in \mathbb{C}^\times$, regard V as a $\varphi_s(\mathcal{T}_q)$-module. Let W be a minimal $\varphi_s(\mathcal{T}_q)$-submodule of V. It is enough to show that $W = V$ holds if s avoids finitely many scalars. By (8) with $s_0 = 1$, the eigenspace decomposition of $k_1 = k_0^{-1}$ on V is $V = \bigoplus_{i=0}^{d} U_{d-i}$, $k_1|_{U_{d-i}} = q^{2i-d}$, $0 \le i \le d$. The subalgebra $\langle e_1^\pm, k_1^{\pm 1}\rangle$ of $U_q'(L(\mathfrak{sl}_2))$ is isomorphic to the quantum algebra $\mathcal{U} = U_q(\mathfrak{sl}_2)$ in (12) via the correspondence of $e_1^\pm$, $k_1^{\pm 1}$ to $X^\pm$, $K^{\pm 1}$. The element $(e_1^+)^{d-2i}$ maps U_{d-i} onto U_i bijectively, $0 \le i \le [d/2]$. Also $(e_1^-k_1)^{d-2i}$ maps U_i onto U_{d-i} bijectively, $0 \le i \le [d/2]$.

The element $(e_1^+)^{d-2i}$ belongs to $\varphi_s(\mathcal{T}_q)$. So $(e_1^+)^{d-2i}W \subseteq W$. Since the mapping $(e_1^+)^{d-2i}$: $U_{d-i} \to U_i$ is a bijection, we have $\dim(W \cap U_{d-i}) \le \dim(W \cap U_i)$, $0 \le i \le [d/2]$.

The element $(e_1^-k_1)^{d-2i}$ does not belong to $\varphi_s(\mathcal{T}_q)$, but $(e_0^+ + s^{-2}e_1^-k_1)^{d-2i}$ does. By (7), $(e_0^+ + s^{-2}e_1^-k_1)^{d-2i}$ maps U_i to U_{d-i}, $0 \le i \le [d/2]$. We want to show it is a bijection if s avoids finitely many scalars. Identify U_{d-i} with U_i as vector spaces by the bijection $(e_1^+)^{d-2i}$ between them. Then it makes sense to consider the determinant of a linear map from U_i to U_{d-i}. Set $t = s^{-2}$ and expand $(e_0^+ + te_1^-k_1)^{d-2i}$ as

$$t^{d-2i}(e_1^-k_1)^{d-2i} + \text{lower terms in } t.$$

Each term of the expansion gives a linear map from U_i to U_{d-i}. So the determinant of $(e_0^+ + te_1^-k_1)^{d-2i}|_{U_i}$ equals

$$t^{(d-2i)\dim U_i} \det(e_1^-k_1)^{d-2i}\big|_{U_i} + \text{lower terms in } t, \tag{13}$$

and this is a polynomial in t of degree $(d-2i)\dim U_i$, since $\det(e_1^-k_1)^{d-2i}\big|_{U_i} \neq 0$. Let Λ_i be the set of nonzero s such that $t = s^{-2}$ is a root of the polynomial in (13). Then if $s \in \mathbb{C}^\times \setminus \Lambda_i$, $(e_0^+ + s^{-2}e_1^-k_1)^{d-2i}$ maps U_i to U_{d-i} bijectively.

Set $\Lambda = \cup_{i=0}^{[d/2]}\Lambda_i$. Choose $s \in \mathbb{C}^\times \setminus \Lambda$. Then the mapping $(e_0^+ + s^{-2}e_1^-k_1)^{d-2i} : U_i \to U_{d-i}$ is a bijection for $0 \le i \le [d/2]$. Since $e_0^+ + s^{-2}e_1^-k_1$ belongs to $\varphi_s(\mathcal{T}_q)$, we have $(e_0^+ +$

$s^{-2}e_1^- k_1)^{d-2i}W \subseteq W$ and so $\dim(W \cap U_i) \leq \dim(W \cap U_{d-i})$. Since we have already shown $\dim(W \cap U_{d-i}) \leq \dim(W \cap U_i)$, we obtain $\dim(W \cap U_i) = \dim(W \cap U_{d-i})$, $0 \leq i \leq [d/2]$. Therefore by Lemma 1, we have $e_1^- W \subseteq W$. Since $(e_0^+ + s^{-2}e_1^- k_1)W \subseteq W$, the inclusion $e_0^+ W \subseteq W$ follows from $e_1^- W \subseteq W$ and so W is $U_q'(L(\mathfrak{sl}_2))$-invariant. Thus $W = V$ holds by the irreducibility of V as a $U_q'(L(\mathfrak{sl}_2))$-module. ■

Proof of Theorem 1. We use the classification of finite-dimensional irreducible $\mathcal{T}_q$-modules in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$ [4, Theorem 1.18]:

(i) A tensor product $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ of evaluation modules is irreducible as a $\varphi_s(\mathcal{T}_q)$-module if and only if $-s^{-2} \notin S(\ell_i, a_i)$ for all $i \in \{1, \ldots, n\}$ and $S(\ell_i, a_i)$, $S(\ell_j, a_j)$ are in general position for all $i, j \in \{1, \ldots, n\}$.

(ii) Consider two tensor products $V = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell') \otimes V(\ell_1', a_1') \otimes \cdots \otimes V(\ell_m', a_m')$ of evaluation modules and assume that they are both irreducible as a $\varphi_s(\mathcal{T}_q)$-module. Then V, V' are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules if and only if $\ell = \ell'$, $n = m$ and $(\ell_i, a_i) = (\ell_i', a_i')$ for all $i \in \{1, \ldots, n\}$ with a suitable reordering of the evaluation modules $V(\ell_1, a_1), \ldots, V(\ell_n, a_n)$.

(iii) Every finite-dimensional irreducible $\mathcal{T}_q$-module V, $\dim V \geq 2$, is isomorphic to a $\mathcal{T}_q$-module $V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ on which $\mathcal{T}_q$ acts via some embedding $\varphi_s : \mathcal{T}_q \to U_q'(L(\mathfrak{sl}_2))$.

Part (i) of Theorem 1 follows immediately from the part (i) above, due to Lemma 4. Part (ii) of Theorem 1 follows immediately from the part (ii) above, the 'if' part due to Lemma 3 (and Lemma 4) and the 'only if' part due to Lemma 4.

We want to show part (iii) of Theorem 1. Let V be a finite-dimensional irreducible $U_q'(L(\mathfrak{sl}_2))$-module of type $(1, 1)$. By Lemma 4, there exists a nonzero scalar s such that V is irreducible as a $\varphi_s(\mathcal{T}_q)$-module. By the part (iii) above, for the proof of which Drinfel'd polynomials play the key role, the $\mathcal{T}_q$-module V via φ_s is isomorphic to some $\mathcal{T}_q$-module $V' = V(\ell) \otimes V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ via some embedding $\varphi_{s'}$ of $\mathcal{T}_q$ into $U_q'(L(\mathfrak{sl}_2))$. Since k_0 has the same eigenvalues on V, V', we have $s = s'$ and so V, V' are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules. By Lemma 3, V, V' are isomorphic as $U_q'(L(\mathfrak{sl}_2))$-modules. Part (iv) will be shown in the next section. ■

4 Intertwiners: Proof of Theorem 1(iv)

In this section, we show that for ℓ, $m \in \mathbb{Z}_{>0}$, $a \in \mathbb{C}^\times$, there exists an intertwiner between the $U_q'(L(\mathfrak{sl}_2))$-modules $V(\ell, a) \otimes V(m)$, $V(m) \otimes V(\ell, a)$, i.e., a nonzero linear map R from $V(\ell, a) \otimes V(m)$ to $V(m) \otimes V(\ell, a)$ such that

$$R\Delta(\xi) = \Delta(\xi)R, \qquad \forall\, \xi \in U_q'(L(\mathfrak{sl}_2)), \tag{14}$$

where Δ is the coproduct from (10). If such an intertwiner R exists, then it is routinely concluded that $V(\ell, a) \otimes V(m)$ is isomorphic to $V(m) \otimes V(\ell, a)$ as $U_q'(L(\mathfrak{sl}_2))$-modules and any other intertwiner is a scalar multiple of R, since $V(m) \otimes V(\ell, a)$ is irreducible as a $U_q'(L(\mathfrak{sl}_2))$-module by Theorem 1.

Using the theory of Drinfel'd polynomials [4] for the augmented TD-algebra $\mathcal{T}_q = \mathcal{T}_q^{(\varepsilon, \varepsilon^*)}$ with $(\varepsilon, \varepsilon^*) = (1, 0)$, we shall firstly show that $V(\ell, a) \otimes V(m)$ is isomorphic to $V(m) \otimes V(\ell, a)$ as $U_q'(L(\mathfrak{sl}_2))$-modules. This proves Theorem 1(iv), since it is well-known [2, Theorem 4.11] that $V(\ell_i, a_i) \otimes V(\ell_j, a_j)$ and $V(\ell_j, a_j) \otimes V(\ell_i, a_i)$ are isomorphic as $U_q(\widehat{\mathfrak{sl}}_2)$-modules, if $S(\ell_i, a_i)$ and $S(\ell_j, a_j)$ are in general position. Finally we shall construct an intertwiner explicitly.

Let us denote the $U_q'(L(\mathfrak{sl}_2))$-modules $V(\ell, a) \otimes V(m)$, $V(m) \otimes V(\ell, a)$ by V, V':

$$V = V(\ell, a) \otimes V(m), \qquad V' = V(m) \otimes V(\ell, a).$$

Recall we assume that the integers ℓ, m and the scalar a are nonzero. Let us denote a standard basis of the $U_q'(L(\mathfrak{sl}_2))$-module $V(\ell,a)$ (resp. $V(m)$) by $v_0, v_1, \dots, v_\ell$ (resp. $v_0', v_1', \dots, v_m'$) in the sense of (9). Recall $V(m)$ is an abbreviation of $V(m,0)$ and the action of $U_q'(L(\mathfrak{sl}_2))$ on V, V' are via the coproduct Δ of (10).

Let $\mathcal{U}$ denote the subalgebra of $U_q'(L(\mathfrak{sl}_2))$ generated by $e_1^\pm$, $k_1^\pm$. The subalgebra $\mathcal{U}$ is isomorphic to the quantum algebra $U_q(\mathfrak{sl}_2)$. Let $V(n)$ denote the irreducible $\mathcal{U}$-module of dimension $n+1$: $V(n)$ has a standard basis $x_0, x_1, \dots, x_n$ on which $\mathcal{U}$ acts as

$$k_1 x_i = q^{n-2i} x_i, \qquad e_1^+ x_i = [n-i+1] x_{i-1}, \qquad e_1^- x_i = [i+1] x_{i+1},$$

where $[t] = [t]_q = (q^t - q^{-t})/(q - q^{-1})$, $x_{-1} = x_{n+1} = 0$. We call x_n (resp. x_0) the *lowest (highest) weight vector*: $k_1 x_n = q^{-n} x_n$, $e_1^- x_n = 0$ ($k_1 x_0 = q^n x_0$, $e_1^+ x_0 = 0$). Note that $V(\ell,a)$ is isomorphic to $V(\ell)$ as $\mathcal{U}$-modules.

By the Clebsch–Gordan formula, $V = V(\ell,a) \otimes V(m)$ is decomposed into the direct sum of $\mathcal{U}$-submodules $\widetilde{V}(n)$, $|\ell - m| \le n \le \ell + m$, $n \equiv \ell + m \bmod 2$, where $\widetilde{V}(n)$ is the unique irreducible $\mathcal{U}$-submodule of V isomorphic to $V(n)$. With $n = \ell + m - 2\nu$, we have

$$V = V(\ell,a) \otimes V(m) = \bigoplus_{\nu=0}^{\min\{\ell,m\}} \widetilde{V}(\ell + m - 2\nu). \tag{15}$$

Let $\widetilde{x}_n$ denote a lowest weight vector of the $\mathcal{U}$-module $\widetilde{V}(n)$. So

$$\Delta(k_1)\widetilde{x}_n = q^{-n}\widetilde{x}_n, \qquad \Delta(e_1^-)\widetilde{x}_n = 0.$$

Since V has a basis $\{v_{\ell-i} \otimes v_{m-j}' \mid 0 \le i \le \ell, 0 \le j \le m\}$ and k_1 acts on it by $\Delta(k_1)(v_{\ell-i} \otimes v_{m-j}') = q^{-(\ell+m)+2(i+j)} v_{\ell-i} \otimes v_{m-j}'$, the lowest weight vector $\widetilde{x}_n$ of $\widetilde{V}(n)$ can be expressed as

$$\widetilde{x}_n = \sum_{i+j=\nu} c_j v_{\ell-i} \otimes v_{m-j}', \qquad n = \ell + m - 2\nu.$$

Solving $\Delta(e_1^-)\widetilde{x}_n = 0$ for the coefficients c_j, we obtain

$$\frac{c_j}{c_{j-1}} = -q^{m-2j+2} \frac{[\ell - \nu + j]}{[m - j + 1]}$$

and so with a suitable choice of c_0

$$\widetilde{x}_n = \sum_{j=0}^{\nu} (-1)^j q^{j(m-j+1)} [\ell - \nu + j]! [m-j]! v_{\ell-\nu+j} \otimes v_{m-j}', \tag{16}$$

where $n = \ell + m - 2\nu$ and $[t]! = [t][t-1]\cdots[1]$.

Lemma 5. $\Delta(e_0^+)\widetilde{x}_n = aq\widetilde{x}_{n+2}$.

Proof. By (10), we have $\Delta(e_0^+) = e_0^+ \otimes 1 + k_0 \otimes e_0^+$. By (9), the element e_0^+ vanishes on $V(m)$ and acts on $V(\ell,a)$ as aqe_1^-. Since $e_1^- v_{\ell-\nu+j} = [\ell - (\nu-1) + j] v_{\ell-(\nu-1)+j}$, the result follows from (16), using $v_{\ell+1} = 0$. ■

Corollary 1. *Any nonzero $U_q'(L(\mathfrak{sl}_2))$-submodule of $V(\ell,a) \otimes V(m)$ contains $\widetilde{x}_{\ell+m}$, the lowest weight vector of the $\mathcal{U}$-module $V(\ell,a) \otimes V(m)$.*

We are ready to prove the following

Theorem 2. *The $U'_q(L(\mathfrak{sl}_2))$-modules $V(\ell,a)\otimes V(m)$, $V(m)\otimes V(\ell,a)$ are isomorphic for every $\ell,m\in\mathbb{Z}_{>0}$, $a\in\mathbb{C}^\times$.*

Proof. Let $\mathcal{T}_q=\mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra with $(\varepsilon,\varepsilon^*)=(1,0)$. Let $\varphi_s:\mathcal{T}_q\to U'_q(L(\mathfrak{sl}_2))$ denote the embedding of $\mathcal{T}_q$ into $U'_q(L(\mathfrak{sl}_2))$ given in (5). By [4, Theorem 5.2], the Drinfel'd polynomial $P_V(\lambda)$ of the $\varphi_s(\mathcal{T}_q)$-module $V=V(\ell,a)\otimes V(m)$ turns out to be

$$P_V(\lambda)=\lambda^m\prod_{i=0}^{\ell-1}\big(\lambda+aq^{2i-\ell+1}\big).$$

(Note that the parameter s of the embedding φ_s does not appear in $P_V(\lambda)$. So the polynomial $P_V(\lambda)$ can be called the Drinfel'd polynomial attached to the $U'_q(L(\mathfrak{sl}_2))$-module V.)

Let W be a minimal $U'_q(L(\mathfrak{sl}_2))$-submodule of $V=V(\ell,a)\otimes V(m)$; notice that we have shown the irreducibility of the $U'_q(L(\mathfrak{sl}_2))$-module $V'=V(m)\otimes V(\ell,a)$ in Theorem 1 but not yet of $V=V(\ell,a)\otimes V(m)$. By Corollary 1, W contains the lowest and hence highest weight vectors of V. In particular, the irreducible $U'_q(L(\mathfrak{sl}_2))$-module W is of type $(1,1)$. By Lemma 4, there exists a finite set Λ of nonzero scalars such that W is irreducible as a $\varphi_s(\mathcal{T}_q)$-module for any $s\in\mathbb{C}^\times\setminus\Lambda$. By the definition, the Drinfel'd polynomial $P_W(\lambda)$ of the irreducible $\varphi_s(\mathcal{T}_q)$-module W coincides with $P_V(\lambda)$:

$$P_W(\lambda)=P_V(\lambda).$$

By Theorem 1, $V'=V(m)\otimes V(\ell,a)$ is irreducible as a $U'_q(L(\mathfrak{sl}_2))$-module. So by Lemma 4, there exists a finite set Λ' of nonzero scalars such that V' is irreducible as a $\varphi_s(\mathcal{T}_q)$-module for any $s\in\mathbb{C}^\times\setminus\Lambda'$. By [4, Theorem 5.2], the Drinfel'd polynomial $P_{V'}(\lambda)$ of the irreducible $\varphi_s(\mathcal{T}_q)$-module V' coincides with $P_V(\lambda)$:

$$P_{V'}(\lambda)=P_V(\lambda).$$

Both of the irreducible $\varphi_s(\mathcal{T}_q)$-modules W, V' have type s, diameter $d=\ell+m$ and the Drinfel'd polynomial $P_V(\lambda)$. By [4, Theorem 1.9′], W and V' are isomorphic as $\varphi_s(\mathcal{T}_q)$-modules. By Lemma 3, W and V' are isomorphic as $U'_q(L(\mathfrak{sl}_2))$-modules. In particular, $\dim W=\dim V'$. Since $\dim V'=\dim V$, we have $W=V$, i.e., V and V' are isomorphic as $U'_q(L(\mathfrak{sl}_2))$-modules. ■

Finally we want to construct an intertwiner R between the irreducible $U'_q(L(\mathfrak{sl}_2))$-modules V, V'. Regard $V'=V(m)\otimes V(\ell,a)$ as a $\mathcal{U}$-module. By the Clebsch–Gordan formula, we have the direct sum decomposition

$$V'=V(m)\otimes V(\ell,a)=\bigoplus_{\nu=0}^{\min\{\ell,m\}}\widetilde{V}'(\ell+m-2\nu),\tag{17}$$

where $\widetilde{V}'(n)$ is the unique irreducible $\mathcal{U}$-submodule of V' isomorphic to $V(n)$, $n=\ell+m-2\nu$. Let $\widetilde{x}'_n$ be a lowest weight vector of the $\mathcal{U}$-module $\widetilde{V}'(n)$. By (16), we have

$$\widetilde{x}'_n=\sum_{j=0}^{\nu}(-1)^jq^{j(\ell-j+1)}[m-\nu+j]![\ell-j]!v'_{m-\nu+j}\otimes v_{\ell-j}\tag{18}$$

up to a scalar multiple, where $n=\ell+m-2\nu$. It can be easily checked as in Lemma 5 that the lowest weight vectors $\widetilde{x}'_n$, $n=\ell+m-2\nu$, $0\le\nu\le\min\{\ell,m\}$, are related by

$$(e_1^-\otimes 1)\widetilde{x}'_n=\widetilde{x}'_{n+2},\tag{19}$$

where $V'=V(m)\otimes V(\ell,a)$ is regarded as a $(\mathcal{U}\otimes\mathcal{U})$-module in the natural way.

Lemma 6. $\Delta(e_0^+)\widetilde{x}'_n = -aq\cdot q^{n+2}\widetilde{x}'_{n+2}$.

Proof. We have $\Delta(e_0^+)\widetilde{x}'_n = aq(k_1^{-1}\otimes e_1^-)\widetilde{x}'_n$, since $\Delta(e_0^+) = e_0^+\otimes 1 + k_0\otimes e_0^+$, and e_0^+ vanishes on $V(m)$ and acts on $V(\ell,a)$ as aqe_1^-. Express $k_1^{-1}\otimes e_1^-$ as $k_1^{-1}\otimes e_1^- = (k_1^{-1}\otimes 1)(1\otimes e_1^-) = (k_1^{-1}\otimes 1)(\Delta(e_1^-) - e_1^-\otimes k_1^{-1}) = (k_1^{-1}\otimes 1)\Delta(e_1^-) - k_1^{-1}e_1^-\otimes k_1^{-1} = (k_1^{-1}\otimes 1)\Delta(e_1^-) - q^2(e_1^-\otimes 1)\Delta(k_1^{-1})$. Since $\Delta(e_1^-)\widetilde{x}'_n = 0$, $\Delta(k_1^{-1})\widetilde{x}'_n = q^n\widetilde{x}'_n$, the result follows from (19). ■

There exists a unique linear map

$$R_n:\ V = V(\ell,a)\otimes V(m)\to \widetilde{V}'(n)$$

that commutes with the action of $\mathcal{U}$ and sends $\widetilde{x}_n$ to $\widetilde{x}'_n$. The linear map R_n vanishes on $\widetilde{V}(t)$ for $t\neq n$ and affords an isomorphism between $\widetilde{V}(n)$ and $\widetilde{V}'(n)$ as $\mathcal{U}$-modules. If R is an intertwiner in the sense of (14), then R can be expressed as

$$R = \sum_{\nu=0}^{\min\{\ell,m\}} \alpha_\nu R_{\ell+m-2\nu}, \tag{20}$$

regarding R as an intertwiner for the $\mathcal{U}$-modules V, V'. By (14), we have

$$R\Delta(e_0^+) = \Delta(e_0^+)R. \tag{21}$$

Apply (21) to the lowest weight vector $\widetilde{x}_n$ in (16). By Lemma 5, $\Delta(e_0^+)\widetilde{x}_n = aq\widetilde{x}_{n+2}$ and so with $n = \ell+m-2\nu$, we have

$$R\Delta(e_0^+)\widetilde{x}_n = aq\alpha_{\nu-1}\widetilde{x}'_{n+2}. \tag{22}$$

On the other hand, $R\widetilde{x}_n = \alpha_\nu\widetilde{x}'_n$, $n = \ell+m-2\nu$, and so by Lemma 6, we have

$$\Delta(e_0^+)R\widetilde{x}_n = -aq\alpha_\nu q^{n+2}\widetilde{x}'_{n+2}. \tag{23}$$

By (22), (23), we have $\alpha_\nu/\alpha_{\nu-1} = -q^{-n-2} = -q^{-\ell-m+2(\nu-1)}$ and so

$$\alpha_\nu = (-1)^\nu q^{-\nu(\ell+m-\nu+1)} \tag{24}$$

by choosing $\alpha_0 = 1$. An intertwiner exists by Theorem 2. If it exists, it has to be in the form of (20), (24). Thus we obtain the following.

Theorem 3. *The linear map*

$$R = \sum_{\nu=0}^{\min\{\ell,m\}} (-1)^\nu q^{-\nu(\ell+m-\nu+1)} R_{\ell+m-2\nu}$$

is an intertwiner between the $U_q'(L(\mathfrak{sl}_2))$-modules $V(\ell,a)\otimes V(m)$, $V(m)\otimes V(\ell,a)$. Any other intertwiner is a scalar multiple of R.

Remark 1. Let $R(a,b)$ be an intertwiner between the irreducible $U_q'(L(\mathfrak{sl}_2))$-modules $V = V(\ell,a)\otimes V(m,b)$, $V' = V(m,b)\otimes V(\ell,a)$, where $a\neq 0$, $b\neq 0$:

$$R(a,b):\ V = V(\ell,a)\otimes V(m,b)\to V' = V(m,b)\otimes V(\ell,a).$$

As in (20), we write

$$R(a,b) = \sum_{\nu=0}^{\min\{\ell,m\}} \alpha_\nu R_{\ell+m-2\nu}.$$

Recall R_n is the linear map from V to V' that commutes with the action of $\mathcal{U} = \langle e_1^{\pm}, k_1^{\pm}\rangle$ and sends $\widetilde{x}_n$ to $\widetilde{x}'_n$, where $\widetilde{x}_n$, $\widetilde{x}'_n$ are the lowest weight vectors of $\widetilde{V}(n)$, $\widetilde{V}'(n)$ from (15), (17) that are explicitly given by (16), (18) and satisfy $(e_1^- \otimes 1)\widetilde{x}_n = \widetilde{x}_{n+2}$, $(e_1^- \otimes 1)\widetilde{x}'_n = \widetilde{x}'_{n+2}$ as in (19). Since the $U'_q(L(\mathfrak{sl}_2))$-modules V, V' can be extended to $U_q(\widehat{\mathfrak{sl}}_2)$-modules, we have by [2, Theorem 5.4]

$$\alpha_\nu = \prod_{j=0}^{\nu-1} \frac{a - bq^{\ell+m-2j}}{b - aq^{\ell+m-2j}}, \tag{25}$$

where we choose $\alpha_0 = 1$. Note that the denominator and the numerator of (25) are non-zero, since V, V' are assumed to be irreducible and so $S(\ell, a)$, $S(m, b)$ are in general position.

The intertwiner $R(a, b)$ with $a \neq 0$, $b \neq 0$ is derived from the universal R-matrix for the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ [5]. If we put $b = 0$ in (25), then the spectral parameter u disappears, where $q^{2u} = a/b$, and we get (24). In this sense, the intertwiner R of Theorem 3 is related to the universal R-matrix for $U_q(\widehat{\mathfrak{sl}}_2)$, but we cannot expect that R comes from it directly, because the $U'_q(L(\mathfrak{sl}_2))$-modules $V = V(\ell, a) \otimes V(m)$, $V' = V(m) \otimes V(\ell, a)$ cannot be extended to $U_q(\widehat{\mathfrak{sl}}_2)$-modules. In order to derive both of the intertwiners $R(a, b)$, R from a universal R-matrix directly, we need to construct it for the subalgebra $U'_q(\widehat{\mathfrak{sl}}_2)$ of $U_q(\widehat{\mathfrak{sl}}_2)$.

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