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Symmetry Groups of A_n Hypergeometric Series*

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Abstract. Structures of symmetries of transformations for Holman–Biedenharn–Louck A_n hypergeometric series: A_n terminating balanced ${}_4F_3$ series and A_n elliptic ${}_{10}E_9$ series are discussed. Namely the description of the invariance groups and the classification all of possible transformations for each types of A_n hypergeometric series are given. Among them, a "periodic" affine Coxeter group which seems to be new in the literature arises as an invariance group for a class of A_n ${}_4F_3$ series.

Key words: multivariate hypergeometric series; elliptic hypergeometric series; Coxeter groups

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Dedicated to Professors Anatol N. Kirillov and Tetsuji Miwa for their 65th birthday

1 Introduction

In this paper, we discuss structures of symmetries of transformations for two classes of A_n hypergeometric series: A_n terminating balanced ${}_4F_3$ series and A_n elliptic ${}_{10}E_9$ series. Namely we give descriptions of the invariance groups and classification all of possible transformations for each type of A_n hypergeometric series group-theoretically. Among them, a "periodic" affine Coxeter group which seems to be new in the literature arises as an invariance group for a class of A_n ${}_4F_3$ series.

The hypergeometric series ${}_{r+1}F_r$ is defined by

$$
{}_{r+1}F_r\left[\begin{matrix} a_0, & a_1, & a_2, & \dots, & a_r \\ & b_1, & b_2, & \dots, & b_r \end{matrix}; z\right] := \sum_{k\in\mathbb{N}} \frac{[a_0,a_1,\dots,a_r]_k}{k![b_1,\dots,b_r]_k} z^k,
$$

where $[c]_k = c(c+1)\cdots(c+k-1)$ is Pochhammer symbol and $[d_1,\dots,d_r]_k = [d_1]_k\cdots[d_r]_k$.

Investigations of the symmetry of the hypergeometric series goes back to 19th century in the case of ${}_3F_2$ series. Thomae [33] has considered the following ${}_3F_2$ transformation formula

$$
{}_3F_2\left[\begin{matrix} a,b,c \\ d,e \end{matrix}; 1\right] = \frac{\Gamma(e)\Gamma(d+e-a-b-c)}{\Gamma(e-a)\Gamma(d+e-b-c)} {}_3F_2\left[\begin{matrix} a, d-b, d-c \\ d, d+e-b-c \end{matrix}; 1\right],
$$

where $\Gamma(x)$ is the Euler gamma function. Later, Hardy [8] formulated this case as follows, where we give a refined form (see also Whipple [36]):

Theorem (Hardy). *Let $s = s(x_1,x_2,x_3,x_4,x_5) = x_1+x_2+x_3-x_4-x_5$. The function*

$$
\frac{1}{\Gamma(s)\Gamma(2x_4)\Gamma(2x_5)} {}_3F_2\left[\begin{matrix} 2x_1-s, 2x_2-s, 2x_3-s \\ 2x_4, 2x_5 \end{matrix}; 1\right]
$$

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is a symmetric function of the 5 *variables* x_1, x_2, x_3, x_4, x_5. *Thus the* ${}_3F_2$ *series have a symmetry of the symmetric group* $\mathfrak{S}_5$ *of degree* 5.

Motivated by quantum mechanics and representation theory, the symmetry of hypergeometric series has been investigated by many authors including physicists. It is also related to highest weight representations of the unitary group SU(2). (for a expository in this direction, we refer to the paper by Krattenthaler and Srinivasa Rao [22]). The Clebsch–Gordan coefficients can be expressed in terms of ${}_3F_2$ series, in other words, Hahn polynomials and by using Hardy's result, one finds non-trivial zeros of the coefficients. That is, one can clarify the structure of the highest weight representations. The Racah coefficients can be expressed in terms of terminating balanced ${}_4F_3$ series, in other words, Racah polynomials. The corresponding results for the ${}_4F_3$ series has been given by Beyer, Louck and Stein [3] (see also Section 2.2). The results of the groups of symmetry for hypergeometric series have been generalized for each types of hypergeometric series (see [23, 34, 35]).

We also mention that recently, number-theorists have investigated in this direction: Formicella, Green and Stade [5] and Mishev [26] discussed in the case of non-terminating (but) balanced ${}_4F_3$ series with a connection with Fourier coefficients of GL_n automorphic form. In [21], Krattenthaler and Rivoal presented a different but considerably interesting approach related to their investigations regarding odd values for Riemann zeta functions.

Elliptic hypergeometric series has first introduced by Frenkel and Turaev [6] in the context of elliptic $6j$-symbol. They obtained transformation and summation formulas for elliptic hypergeometric series by using invariants of links which extends the works by A.N. Kirillov and N.Yu. Reshetikhin [20] (see also [19]).

In 1970's, Holman, Biedenharn and Louck [10] and Holman [9] has introduced a class of multivariate generalization of hypergeometric series which is nowadays called as A_n hypergeometric series (or hypergeometric series in $\mathrm{SU}(n+1)$) for explicit expressions of Clebsch–Gordan and Racah coefficients of the higher dimensional unitary group $\mathrm{SU}(n+1)$. It includes A_n ${}_4F_3$ series which we discuss in Section 2. Results of transformation and summation formulas for A_n hypergeometric series including basic and elliptic generalization and extension to other (classical) root systems has known by many authors (for summary, see an excellent exposition by S.C. Milne [24]).

Among them, we obtained a number of transformation formulas for (mainly basic) hypergeometric series of type A with different dimensions in [14] (see also [13] and [15]). In the joint work with M. Noumi [17], we showed the results can be extended in the case of balanced series and proposed the notion of *duality transformation formula*. In [14] and [17], we have obtained our results by starting from the Cauchy kernels and their action of (q-)difference operators of Macdonald type. The class of hypergeometric transformations of type A with different dimensions in our previous works can be considered to involve some of previously known A_n hypergeometric transformation formulas in 20th century (see [24]). In [16] (see also [13] and [17]), we proved a number of their results by combining some special cases (hypergeometric transformations between A_n hypergeometric series and one-dimensional (A_1) hypergeometric series). This paper can be considered to be a continuation of [16].

In this paper, we discuss the symmetry of some classes of A_n hypergeometric series including $n = 1$ case. Namely we investigate the invariance forms and the groups describing the symmetry of each type of hypergeometric series. For $n \geq 2$, the symmetry of the A_n hypergeometric series is more restricted than $n = 1$ case if we fix the symmetry corresponding to the dimension of the summation. So, the groups of symmetry are subgroups of that in the case of $n = 1$. Furthermore, we classify all the hypergeometric transformations which can be obtained by the combinations of possible permutations of the parameters and the hypergeometric transformations without trivial transformations in each cases. The classifications are given by double coset decomposition of the corresponding groups.

In Section 2, we discuss symmetries of A_n terminating balanced ${}_4F_3$ series. Among these, a *"periodic affine"* Weyl group that is periodic with respect to the translations arises in a class of A_n ${}_4F_3$ series. It seems not to have previously appeared in the literature as a Coxeter group (see [4, 11] and the paper by Iwahori and Matsumoto [12] regarding the Weyl groups with translations). In Section 3, we discuss symmetries of A_n elliptic hypergeometric series. What is remarkable in this case is a subgroup structure.

It would be interesting if the discussions and results does work for future works not only for multivariate hypergeometric transformations themselves, but also for deeper investigations to the structure of irreducible decompositions of the tensor products of certain representations of higher dimensional unitary group $\mathrm{SU}(n+1)$ and elliptic quantum groups of $\mathrm{SU}(n+1)$, the original problem to introducing A_n and elliptic hypergeometric series.

On the other hand, Kajiwara et al. [18] found that elliptic hypergeometric series ${}_{10}E_9$ arises as a class of solutions of the elliptic Painlevé equation associated to the affine Weyl group $W(E_7^{(1)})$ which is the one of the family of the Painlevé equations introduced by Sakai [31] from the geometry of rational surfaces. We also mention the work of Rains [27] on relations between elliptic hypergeometric integrals and tau functions of elliptic Painlevé equations (see also [28] and [34]). It would be an interesting problem to give geometric interpretation of the symmetries of classes of A_n hypergeometric series in terms of certain rational surfaces.

2 Symmetry groups of A_n ${}_4F_3$ series

2.1 Preliminaries on A_n hypergeometric series

Here, we note the conventions for naming series as A_n (ordinary) hypergeometric series (or hypergeometric series in $\mathrm{SU}(n+1)$). Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ be a multi-index. We denote

$$\Delta(x) := \prod_{1\le i<j\le n} (x_i - x_j) \qquad \text{and} \qquad \Delta(x+\gamma) := \prod_{1\le i<j\le n} (x_i + \gamma_i - x_j - \gamma_j),$$

as the Vandermonde determinant for the sets of variables $x = (x_1, \dots, x_n)$ and $x + \gamma = (x_1 + \gamma_1, \dots, x_n + \gamma_n)$ respectively. In this paper we refer multiple series of the form

$$\sum_{\gamma\in\mathbb{N}^n} \frac{\Delta(x+\gamma)}{\Delta(x)} H(\gamma) \tag{2.1}$$

which reduce to hypergeometric series ${}_{r+1}F_r$ for a nonnegative integer r when $n = 1$ and symmetric with respect to the subscript $1 \le i \le n$ as A_n hypergeometric series. We call such a series balanced if it reduces to a balanced series when $n = 1$. Terminating, balanced and so on are defined similarly. The subscript n in the label A_n attached to the series is the dimension of the multiple series (2.1).

Before beginning our discussion, we summarize $q \to 1$ results of A_n Sears transformation from [16] which we discuss in this paper. For the procedure of $q \to 1$ limit, one can find in the book by Gasper–Rahman [7] (see also [14]).

We introduce the notation for A_n ${}_4F_3$ series as follows

$$\begin{aligned}
&{}_4F_3^n\left(\begin{array}{c|c|c|c} \{b_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) \\
&\quad := \sum_{\gamma\in\mathbb{N}^n} \frac{\Delta(x+\gamma)}{\Delta(x)} \prod_{1\le i,j\le n} \frac{[b_j + x_i - x_j]_{\gamma_i}}{[1 + x_i - x_j]_{\gamma_i}} \prod_{1\le i\le n} \left[\frac{[c + x_i]_{\gamma_i}}{[d + x_i]_{\gamma_i}}\right] \frac{[a_1, a_2]_{|\gamma|}}{[e_1, e_2]_{|\gamma|}},
\end{aligned}$$

where $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$ is a length of the multi-index γ.

Here we give two A_n Whipple transformations which discuss in this paper.

Rectangular version (the $q \to 1$ limit of A_n Sears transformation formula, Corollary 4.5 in [16])

$$
\begin{aligned}
{}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}a_1,a_2\\ e_1,e_2\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|1\right) &= \frac{[d+e_1-a_2-c]_{|M|}}{[d+e-a_1-a_2-c]_{|M|}}\prod_{1\le i\le n}\frac{[d-a_1+x_i]_{m_i}}{[d+x_i]_{m_i}}\\
&\times {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\middle|\begin{matrix}a_1,e_1-a_2\\ e_1,d+e_1-a_2-c\end{matrix}\middle|\begin{matrix}e_1-c\\ e_1+e_2-a_2-c\end{matrix}\middle|1\right),
\end{aligned} \tag{2.2}
$$

where $|M| = m_1 + m_2 + \cdots + m_n$ and $\tilde{x}_i = -m_i + |M| - x_i$ for $1 \le i \le n$. The balancing condition in this case is

$$a_1 + a_2 + c + 1 - |M| = d + e_1 + e_2.$$

Note that ${}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}a_1,a_2\\ e_1,e_2\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|1\right)$ series terminates with respect to a multi-index. In this paper, we call such series as *rectangular* and the multiple series which terminates with respect to the length of multi-indices as *triangular*.

Triangular version (the $q \to 1$ limit of A_n Sears transformation formula, Proposition 4.5 in [16])

$$
\begin{aligned}
{}_4F_3^n\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}-N,a\\ e_1,e_2\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|1\right) &= \frac{[d+e_1-a-c]_N}{[d+e_1-a-B-c]_N}\prod_{1\le i\le n}\frac{[d-b_i+x_i]_N}{[d+x_i]_N}\\
&\times {}_4F_3^n\left(\begin{matrix}\{b_i\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\middle|\begin{matrix}-N,e_1-a\\ e_1,d+e_1-a-c\end{matrix}\middle|\begin{matrix}e_1-c\\ e_1+e_2-a-c\end{matrix}\middle|1\right),
\end{aligned} \tag{2.3}
$$

where $|B| = b_1 + b_2 + \cdots + b_n$ and $\tilde{x}_i = b_i - B - x_i$ for $1 \le i \le n$. The balancing condition in this case is

$$a \; + B + c + 1 - N = d + e_1 + e_2.$$

Remark 2.1. In the case when $n = 1$ and $x_1 = 0$, (2.2) and (2.3) reduce to the Whipple transformation formula for terminating balanced ${}_4F_3$ series

$$
\begin{aligned}
{}_4F_3\left[\begin{matrix}-N,a_1,a_2,a_3\\ d_1,d_2,d_3\end{matrix};1\right] &= \frac{[d_2-a_1,d_1+d_2-a_2-a_3]_N}{[d_2,d_1+d_2-a_1-a_2-a_3]_N}\\
&\times {}_4F_3\left[\begin{matrix}-N,a_1,d_1-a_3,d_1-a_2\\ d_1,d_1+d_3-a_2-a_3,d_1+d_2-a_2-a_3\end{matrix};1\right].
\end{aligned} \tag{2.4}
$$

Note that identity above (2.4) is valid if the balancing condition

$$a_1 + a_2 + a_3 + 1 - N = d_1 + d_2 + d_3$$

holds.

2.2 Symmetries of ${}_4F_3$ transformations (A_1 case)

Here, we discuss the symmetry for terminating balanced ${}_4F_3$ series, namely the A_1 case:

$$
{}_4F_3\left[\begin{matrix}-N,a_1,a_2,a_3\\ d_1,d_2,d_3\end{matrix};1\right] \tag{2.5}
$$

with the balancing condition

$$a_1 + a_2 + a_3 + 1 - N = d_1 + d_2 + d_3. \tag{2.6}$$

Though most of all the results were originally obtained in [3] with a different formulation, we continue our discussion in order to give a correspondence to the results in Section 2.5.

The action of the parameters a_i, d_i, $i = 1, 2, 3$, for the Whipple transformation (2.4) is given as follows

$$s : \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ d_1 \\ d_2 \\ d_3 \end{bmatrix} \to \begin{bmatrix} a_1 \\ d_1 - a_3 \\ d_1 - a_2 \\ d_1 \\ d_1 + d_2 - a_2 - a_3 \\ d_1 + d_3 - a_2 - a_3 \end{bmatrix}.$$

One can consider it as a linear transformation acting on the vector $\vec{v}_1 = {}^t(a_1, a_2, a_3, d_1, d_2, d_3)$. The matrix realization S for transformation s is given as follows

$$S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \end{bmatrix}.$$

It is easy to see that the ${}_4F_3$ series is invariant under the action of the permutation in the two sets of parameters $\{a_1, a_2, a_3\}$ and $\{d_1, d_2, d_3\}$. For $i = 1, 2$, let r_i be the permutation of a_i and a_{i+1} and let t_i be the permutation of d_i and d_{i+1}. The matrix realizations R_i (resp. T_i) of r_i (resp. t_i) is given by its action on the vector $\vec{v}_1$. For example,

$$R_1 := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_1 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

We have the invariant form for the terminating balanced ${}_4F_3$ series:

Proposition 2.1 (invariance for terminating balanced ${}_4F_3$ series)**.**

$${}_4\tilde{F}_3\,[\vec{v}_1] := [d_1, d_2, d_3]_N {}_4F_3 \begin{bmatrix} -N, a_1, a_2, a_3 \\ d_1, d_2, d_3 \end{bmatrix}; 1$$

is invariant under all of the actions r_i, t_i, $i = 1, 2$, and s.

Obviously, the transformations r_1 and r_2 enjoy the braid relation $r_1 r_2 r_1 = r_2 r_1 r_2$ and $r_i^2 = \mathrm{id}$ for $i = 1, 2$. So do t_1 and t_2. The relations among the element s and others are summarized as follows:

Lemma 2.1. *We have $(st_1)^3 = (sr_1)^3 = \mathrm{id}$ and $(st_2)^2 = (sr_2)^2 = \mathrm{id}$.*

We define the mapping π_1 as

$$s_1 \to \sigma_3, \qquad r_i \to \sigma_{3-i}, \qquad t_i \to \sigma_{3+i}, \qquad i = 1, 2.$$

Then, by braid relations among r_i and t_i and the lemma above, we see that the following relation holds:

$$\begin{cases} \sigma_i \neq \mathrm{id}, \quad \sigma_i^2 = \mathrm{id}, & i = 1,2,3,4,5, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1,2,3,4, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2. \end{cases}$$

Thus we have the following.

Proposition 2.2. *The set generated by r_i, t_i for $i = 1, 2$ and s forms a Coxeter group. Furthermore, the group is isomorphic to $\mathfrak{S}_6$.*

Here we classify the possible transformations for terminating balanced ${}_4F_3$ series of the form (2.5). Recall that ${}_4F_3$ series is invariant under the action σ_k for $k = 1, 2, 4, 5$. Thus our problem reduces to give an orbit decomposition of the double coset $H\backslash G/H$, where $G := \{\sigma_i \,|\, i = 1,2,3,4,5\}$, $G_1 := \{\sigma_i \,|\, i = 1, 2\}$, $G_2 := \{\sigma_i \,|\, i = 4, 5\}$ and $H = G_1 \times G_2$. The representatives of orbits of $H\backslash G/H$ is given by

(*i*) $\omega_0 = \mathrm{id}$,

(*ii*) $\omega_1 = \sigma_3$,

(*iii*) $\omega_2 = \sigma_3\sigma_4\sigma_2\omega_1 = \sigma_3\sigma_4\sigma_2\sigma_3$,

(*iv*) $\omega_3 = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\omega_2 = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_3\sigma_4\sigma_2\sigma_3$.

Thus we are ready to present a list of the ${}_4F_3$ transformations according to the representatives above. We frequently make the simplification of the product factor by using the balancing condition (2.6).

The transformation associated with (*i*) is identical. The second one (*ii*) is the Whipple transformation (2.4) itself. The third one (*iii*) is given by

$$\begin{aligned} {}_4F_3\left[\begin{matrix}-N, a_1, a_2, a_3 \\ d_1, d_2, d_3\end{matrix}; 1\right] &= \frac{[d_1+d_2-a_1-a_3, d_1+d_2-a_2-a_3, a_3]_N}{[d_1, d_2, d_1+d_2-a_1-a_2-a_3]_N} \\ &\times {}_4F_3\left[\begin{matrix}-N, d_1-a_3, d_2-a_3, d_1+d_2-a_1-a_2-a_3 \\ d_1+d_2-a_2-a_3, d_1+d_2-a_1-a_3, d_1+d_2+d_3-a_1-a_2-2a_3\end{matrix}; 1\right]. \end{aligned} \tag{2.7}$$

The forth one (*iv*) is

$$\begin{aligned} {}_4F_3\left[\begin{matrix}-N, a_1, a_2, a_3 \\ d_1, d_2, d_3\end{matrix}; 1\right] &= \frac{[a_1, a_2, a_3]_N}{[d_1, d_2, d_1+d_2-a_1-a_2-a_3]_N} \\ &\times {}_4F_3\left[\begin{matrix}-N, d_1+d_2-a_1-a_2-a_3, d_1+d_3-a_1-a_2-a_3, \\ d_1+d_2+d_3-a_1-a_2-2a_3, d_1+d_2+d_3-a_1-2a_2-a_3, \end{matrix}\right. \\ &\qquad \left.\begin{matrix} d_2+d_3-a_1-a_2-a_3 \\ d_1+d_2+d_3-2a_1-a_2-a_3\end{matrix}; 1\right]. \end{aligned} \tag{2.8}$$

The transformation (2.8) has an alternative expression

$$\begin{aligned} {}_4F_3\left[\begin{matrix}-N, a_1, a_2, a_3 \\ d_1, d_2, d_3\end{matrix}; 1\right] &= (-1)^N \frac{[a_1, a_2, a_3]_N}{[d_1, d_2, d_3]_N} \\ &\times {}_4F_3\left[\begin{matrix}-N, 1-N-d_1, 1-N-d_2, 1-N-d_3 \\ 1-N-a_1, 1-N-a_2, 1-N-a_3\end{matrix}; 1\right]. \end{aligned} \tag{2.9}$$

Note also that (2.9) is an inversion of the order of the summation in the ${}_4F_3$ series.

2.3 Symmetry of A_n ${}_4F_3$ series of rectangular type

Here, we describe the invariance group for A_n Whipple transformation of rectangular type (2.2), namely the series of the form

$$ {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}a_1,a_2\\ e_1,e_2\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|\, 1\right) \tag{2.10}$$

with the balancing condition

$$a_1+a_2+c+1-|M| = d+e_1+e_2. \tag{2.11}$$

Suppose that $n \geq 2$ till stated otherwise. Hereafter, we also fix the symmetry of the dimension of the summation in the multiple series.

Recall that the transformation of coordinates in the right hand side of (2.2) is given as follows

$$s:\begin{bmatrix}a_1\\ a_2\\ c\\ d\\ e_1\\ e_2\end{bmatrix}\mapsto\begin{bmatrix}a_1^1\\ a_2^1\\ c^1\\ d^1\\ e_1^1\\ e_2^1\end{bmatrix}=\begin{bmatrix}a_1\\ e_1-a_2\\ e_1-c\\ e_1+e_2-a_2-c\\ e_1\\ e_1+d-a_2-c\end{bmatrix}.$$

It is easy to see that this transformation of coordinates is linear for a_1, a_2, c, d, e_1 and e_2. Thus we give a 6×6 matrix realization for transformation for s acting on the vector $\vec{v} = {}^t[a_1,a_2,c,d,e_1,e_2]$ as follows

$$\begin{bmatrix}a_1^1\\ a_2^1\\ c^1\\ d^1\\ e_1^1\\ e_2^1\end{bmatrix}=S_1\vec{v},\qquad S_1:=\begin{bmatrix}1&0&0&0&0&0\\ 0&-1&0&0&1&0\\ 0&0&-1&0&1&0\\ 0&-1&-1&0&1&1\\ 0&0&0&0&1&0\\ 0&-1&-1&1&1&0\end{bmatrix}.$$

Note that the series (2.10) is symmetric with respect to the two sets of parameters $\{a_1,a_2\}$ and $\{e_1,e_2\}$. Let s_0 be a permutation of a_1 and a_2 and let s_2 be a permutation of e_1 and e_2. The matrix realization S_0 (resp. S_2) of s_0 (resp. s_2) is given by

$$S_0:=\begin{bmatrix}0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1\end{bmatrix}\quad\text{and}\quad S_2:=\begin{bmatrix}1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&0&1\\ 0&0&0&0&1&0\end{bmatrix}.$$

For the action on the variables x_i, we set to be $s_1\cdot x_i = \tilde{x}_i = -m_i+|M|-x_i$ and otherwise to be identical.

We introduce the normalized ${}_4F_3^n$ series ${}_4\widetilde{F}_3^n\left((\vec{v},x)\right)$ as

$$ {}_4\widetilde{F}_3^n\left((\vec{v},x)\right) := [e_1,e_2]_{|M|}\prod_{1\leq i\leq n}[d+x_i]_{m_i}\,{}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\middle|\begin{matrix}a_1,a_2\\ e_1,e_2\end{matrix}\middle|\begin{matrix}c\\ d\end{matrix}\middle|\, 1\right), \tag{2.12}$$

under the balancing condition (2.11).

Proposition 2.3 (invariance form for multiple series of type (2.10))**.** *${}_4\widetilde{F}_3^n\left((\vec{v},x)\right)$ is invariant under the action of s_0, s_1 and s_2.*

In the course of the proof of this proposition, we use the following lemma.

Lemma 2.2. *If* (2.11) *holds, then we have the following*

1) $[b]_{|M|} = (-1)^{|M|}[d+e_1+e_2-a_1-a_2-c-b]_{|M|}$,
2) $[f+x_i]_{m_i} = (-1)^{m_i}[d+e_1+e_2-a_1-a_2-c+\tilde{x}_i-f]_{m_i}$.

One can prove this lemma by using an elementary manipulation of shifted factorials $[z]_m = (-1)^m[1-z-m]_m$ and the balancing condition (2.11).

Proof of Proposition 2.3. Since the ${}_4\widetilde{F}_3^n\left((\vec{v},x)\right)$ is symmetric with respect to the sets of parameters $\{a_1,a_2\}$ and $\{e_1,e_2\}$, it is obvious in the case of s_0 and s_2. For the case of s_1, by using the transformation formula (2.2) and Lemma 2.2

$$
\begin{aligned}
{}_4\widetilde{F}_3^n\left(s_1(\vec{v},x)\right) &= {}_4\widetilde{F}_3^n\left((S_1\vec{v},\tilde{x})\right) = [e_1, d+e_1-a_2-c]_{|M|} \prod_{1\le i\le n} [d-a_1+x_i]_{m_i} \\
&\quad\times {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\,\middle|\,\begin{matrix}a_1, e_1-a_2\\ e_1, d+e_1-a_2-c\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,1\right)\\
&= [e_1, d+e_1-a_2-c]_{|M|} \prod_{1\le i\le n} [d-a_1+x_i]_{m_i}\\
&\quad\times \frac{[d+e_1-a_1-a_2-c]_{|M|}}{[d+e-a_2-c]_{|M|}} \prod_{1\le i\le n} \frac{[d+x_i]_{m_i}}{[d-a_1+x_i]_{m_i}}\, {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}a_1, a_2\\ e_1, e_2\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,1\right)\\
&= {}_4\widetilde{F}_3^n\left((\vec{v},x)\right).
\end{aligned}
$$

Thus we complete the proof of the proposition. ■

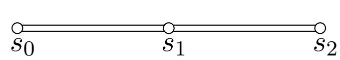
The set $\{s_0,s_1,s_2\}$ form a Coxeter group. Let G be the group generated by s_0, s_1 and s_2. The relations can be summarized as follows:

Lemma 2.3. *The generators s_0, s_1, s_2 of the group G satisfy the following relations:*

1) $s_0^2 = s_1^2 = s_2^2 = \mathrm{id}$, (2.13)
2) $(s_0s_2)^2 = \mathrm{id}, \qquad (s_0s_1)^4 = (s_1s_2)^4 = \mathrm{id}$, (2.14)
3) $(s_2s_1s_0s_1)^3 = (s_1s_2s_1s_0)^3 = \mathrm{id}\,.$ (2.15)

Proof. One can check by direct computation using the matrix realization given above. We shall leave to readers. ■

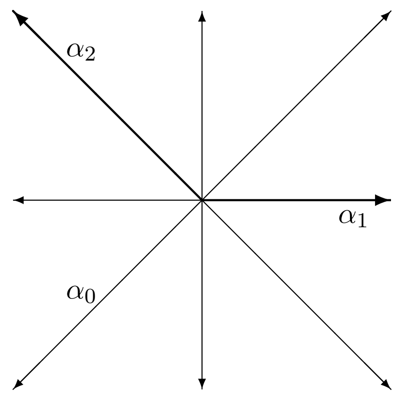
Remark 2.2. The relations (2.13) and (2.14) in Lemma 2.3 are the relations are same as that of the affine Weyl group $W(\widetilde{C}_2)$:



Dynkin diagram of $\widetilde{C}_2$

We utilize properties of affine Weyl group $W(\widetilde{C}_2)$, especially translations in $W(\widetilde{C}_2)$ to describe the structure of the group G (For properties of affine Weyl groups, see Iwahori–Matsumoto [12] and Humphreys' book [11]). Here we follow the notation of [11].

In general, it is well known that affine Weyl group is a semidirect product of a Weyl group of the corresponding finite root system and the translation group corresponding to the coroot lattice. We define the root vectors in the two dimensional Euclidean space V for the root system C_2 as the following picture:



Roots of root system C_2

The null root α_0 for $\widetilde{C}_2$ is given by $-2\alpha_1 - \alpha_2$. For a root α, we denote by the corresponding coroot $\alpha^\vee$ given by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$, where $(\cdot,\cdot)$ is the Killing form. In this case, the generators of the Weyl group of the root system C_2 be given by s_1 and s_2. We denote $L^\vee$ by the coroot lattice of the root system C_2. For $d \in V$, let $t(d)$ be the translation which sends $\lambda \in V$ to $d + \lambda$.

Lemma 2.4. *The group G is of order 72. Furthermore, G is isomorphic to a semidirect product of $W(C_2)$ and $L^\vee/3L^\vee$.*

Proof. Note that $s_2 s_1 s_0 s_1$ is the translation $t(\alpha_2^\vee)$ of minimum length in V. It is obvious to see that $s_1 t(\alpha_2^\vee) s_1 = s_1 s_2 s_1 s_0$ is $t(s_1\alpha_2^\vee) = t(\alpha_1^\vee + \alpha_2^\vee)$. Thus the group G is a subgroup of the group $W(C_2) \ltimes L^\vee/3L^\vee$. In order to see G is isomorphic to $W(C_2) \ltimes L^\vee/3L^\vee$, it suffices to check that $t(\alpha_2^\vee) \neq \mathrm{id}$ and $t(\alpha_1^\vee + \alpha_2^\vee) \neq \mathrm{id}$. Both of them can be done by direct computation using the matrix realization. ■

Remark 2.3. The Coxeter group G can be considered as a "periodic" affine Weyl group. In particular, the relation (2.15) implies "periodicity" with respect to translations for the coroot lattice. David Bessis informed us that the group G is *not* one of complex reflection groups [32]. He proved this by calculating the character of the group G.

We are going to classify possible and non-trivial transformation for the A_n ${}_4F_3$ series of rectangular type (2.10).

Let H be the subgroup of the group G generated by s_0 and s_2. Recall that the ${}_4F_3^n$ series of type (2.10) is invariant under the action of s_0 and s_2. Then our problem reduces to give an orbit decomposition of the double coset $H\backslash G/H$. The representatives of this decomposition are given by

$$\begin{gathered}(i)\quad \mathrm{id}, \qquad (ii)\quad s_1, \qquad (iii)\quad s_1s_2s_1, \qquad (iv)\quad s_1s_0s_1, \qquad (v)\quad s_1s_0s_2s_1, \\ (vi)\quad s_1s_2s_1s_0s_1, \qquad (vii)\quad s_1s_0s_1s_2s_1, \qquad (viii)\quad s_1s_0s_2s_1s_0s_2s_1. \end{gathered} \tag{2.16}$$

We are ready to exhibit a complete list of possible transformations for the series of form (2.10) according to the representative (2.16) of each orbit in G. We use Lemma 2.2 frequently without stated otherwise in simplifying the factors.

The first one (i) in (2.16) is identical. The second is (2.2). The transformation corresponding (iii) is

$$
{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) = \frac{[d+e_1-a_2-c, e_1-a_1]_{|M|}}{[d+e-a_1-a_2-c, e_1]_{|M|}} \times {}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, d-c & d-a_2 & \multirow{2}{*}{1} \\ \{x_i\}_n & d+e_2-a_2-c, d+e_1-a_2-c & d & \end{array}\right), \tag{2.17}
$$

which is equivalent to $q \to 1$ limit of the first A_n Sears transformation (4.23) of [16]. The one (iv) is

$$
{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) = \frac{[d-c]_{|M|}}{[d+e_1-a_1-a_2-c]_{|M|}} \prod_{1\le i\le n} \frac{[d+e_1-a_1-a_2+x_i]_{m_i}}{[d+x_i]_{m_i}} \times {}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & e_1-a_1, e_1-a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1+e_2-a_1-a_2, e_1 & d+e_1-a_1-a_2 & \end{array}\right). \tag{2.18}
$$

The fifth one (v) is

$$
\begin{aligned}
&{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) \\
&= \frac{[d+e_1-a_2-c, a_2]_{|M|}}{[d+e-a_1-a_2-c, e_1]_{|M|}} \prod_{1\le i\le n} \frac{[d+e_1-a_1-a_2+x_i]_{m_i}}{[d+x_i]_{m_i}} \\
&\quad \times {}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & e_1-a_2, d+e_1-a_1-a_2-c & d-a_2 & \multirow{2}{*}{1} \\ \{x_i\}_n & d+e_1+e_2-a_1-2a_2-c, d+e_1-a_2-c & d+e_1-a_1-a_2 & \end{array}\right).
\end{aligned} \tag{2.19}
$$

The one (vi) is

$$
\begin{aligned}
&{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) \\
&= \frac{[d-c, d+e_1+e_2-a_1-2a_2-c]_{|M|}}{[d+e_1-a_1-a_2-c, d+e_2-a_1-a_2-c]_{|M|}} \prod_{1\le i\le n} \frac{[d-a_1+x_i]_{m_i}}{[d+x_i]_{m_i}} \\
&\times {}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & e_1-a_2, e_2-a_2 & e_1+e_2-a_1-a_2-c & \multirow{2}{*}{1} \\ \{\tilde{x}_i\}_n & e_1+e_2-a_1-a_2, d+e_1+e_2-a_1-2a_2-c & e_1+e_2-a_2-c & \end{array}\right).
\end{aligned} \tag{2.20}
$$

The seventh one (vii) is

$$
{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) = \frac{[d+e_1-a_2-c, d+e_1-a_1-c]_{|M|}}{[d+e-a_1-a_2-c, e_1]_{|M|}} \prod_{1\le i\le n} \frac{[c+x_i]_{m_i}}{[dx_i]_{m_i}} \times {}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & d-c, d+e_1-a_1-a_2-c & e_1-c & \multirow{2}{*}{1} \\ \{\tilde{x}_i\}_n & d+e_1-a_1-c, d+e_1-a_2-c & d+e_1+e_2-a_1-a_2-2c & \end{array}\right). \tag{2.21}
$$

The one $(viii)$ is

$$
{}_4F_3^n\left(\begin{array}{c|c|c|c} \{-m_i\}_n & a_1, a_2 & c & \multirow{2}{*}{1} \\ \{x_i\}_n & e_1, e_2 & d & \end{array}\right) = \frac{[a_1, a_2]_{|M|}}{[e_1, d+e_1-a_1-a_2-c]_{|M|}} \prod_{1\le i\le n} \frac{[c+x_i]_{m_i}}{[d+x_i]_{m_i}}
$$

$$\times\, {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\,\middle|\,\begin{matrix}d+e_1-a_1-a_2-c, d+e_2-a_1-a_2-c\\ d+e_1+e_2-a_1-2a_2-c, d+e_1+e_2-2a_1-a_2-c\end{matrix}\right.$$
$$\left.\middle|\,\begin{matrix}e_1+e_2-a_1-a_2-c\\ d+e_1+e_2-a_1-a_2-2c\end{matrix}\,\middle|\,1\right). \tag{2.22}$$

(2.22) has an alternative expression:

$$ {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}a_1,a_2\\ e_1,e_2\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,1\right)=\frac{[a_1,a_2]_{|M|}}{[e_1,e_2]_{|M|}}\prod_{1\le i\le n}\frac{[c+x_i]_{m_i}}{[d+x_i]_{m_i}}$$
$$\times {}_4F_3^n\left(\begin{matrix}\{-m_i\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\,\middle|\,\begin{matrix}1-|M|-e_1,1-|M|-e_2\\ 1-|M|-a_1,1-|M|-a_2\end{matrix}\,\middle|\,\begin{matrix}1-|M|-d\\ 1-|M|-c\end{matrix}\,\middle|\,1\right). \tag{2.23}$$

Note that this expression of the formula implies the reversing the order of the summation as ${}_4F_3^n$ series of the form (2.10).

2.4 A_n ${}_4F_3$ series of triangular type

We describe the invariance group for triangular A_n Whipple transformation (2.3), namely the series of the form

$$ {}_4F_3^n\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}-N,a\\ e_1,e_2\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,1\right)$$

with the balancing condition

$$a+B+c+1-N=d+e_1+e_2.$$

Note that, on contrast to the case of (2.10), the action of the permutation s_0 in Section 2.3 is *not* valid. So what we are to consider is the action of the permutation s_2 of the parameters e_1 and e_2 and the transformation of the parameters in (2.3). The action of each parameters of the transformation (2.3) is given by

$$s_1:\begin{bmatrix}b\\a\\c\\d\\e_1\\e_2\end{bmatrix}\to\begin{bmatrix}b\\e_1-a\\e_1-c\\e_1+e_2-a-c\\e_1\\d+e_1-a-c\end{bmatrix}=S_1\begin{bmatrix}b\\a\\c\\d\\e_1\\e_2\end{bmatrix},\qquad S_1:=\begin{bmatrix}1&0&0&0&0&0\\0&-1&0&0&1&0\\0&0&-1&0&1&0\\0&-1&-1&0&1&1\\0&0&0&0&1&0\\0&-1&-1&1&1&0\end{bmatrix}.$$

Let G_t be the group generated by the transformations s_1 and s_2. The relations between s_1 and s_2 are completely same as that between s_1 and s_2 in Section 2.3. Namely,

$$s_1^2=s_2^2=\mathrm{id},\qquad (s_1s_2)^4=\mathrm{id}.$$

It follows that the group G_t is isomorphic to $W(C_2)$, the Weyl group associated to the root system C_2.

To classify possible and non-trivial transformation formula, what is going to see is to give an orbit decomposition of the double coset $H\backslash G_t/H$, where H is a subgroup of G_t generated by s_2. Note that H is isomorphic to $\mathfrak{S}_2$. The representatives of each orbits associated to this decomposition are given by (i) id, (ii) s_1 and (iii) $s_1s_2s_1$.

We now present the corresponding A_n ${}_4F_3$ transformations attached to each representative given above. The transformation for (i) is identical and the second one (ii) is (2.3). The third one (iii) is

$$
{}_4F_3^n\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}-N,a\\ e_1,e_2\end{matrix}\,\middle|\,\begin{matrix}c\\ d\end{matrix}\,\middle|\,1\right)=\frac{[e_1-B,d+e_1-a\ -c]_N}{[e,d+e_1-a\ -B-c]_N}
$$
$$
\times\,{}_4F_3^n\left(\begin{matrix}\{b_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\,\begin{matrix}-N,d-c\\ d+e_2-a\ -c,d+e_1-a\ -c\end{matrix}\,\middle|\,\begin{matrix}d-a\\ d\end{matrix}\,\middle|\,1\right), \tag{2.24}
$$

which is the $q \to 1$ limit case of A_n Sears transformation ((4.2) in [13]).

2.5 Remarks on the results of Section 2

Finally, we close the present paper to give remarks on the structure of the corresponding groups of the A_n hypergeometric series of each cases.

Remark 2.4 (the case when $n=1$ in A_n ${}_4F_3$ series)**.** In the case when $n=1$, all the transformations (2.17), (2.2) and (2.18) of rectangular type and (2.24) of triangular type reduce to the Whipple transformation formula (2.4). All the transformations (2.20), (2.19), and (2.21) of rectangular type reduce to (2.7). The transformation (2.23) of rectangular type reduces to (2.9) and implies the reversing of the order of the summation in the A_n ${}_4F_3$ series of rectangular type.

Remark 2.5 (correspondence of the groups in Sections 2.2 and 2.3)**.** By direct manipulation of the matrix realization in Section 2.3, we have $s=\sigma_4\sigma_3\sigma_1\sigma_5\sigma_4$. Thus we find that the group G is isomorphic to the subgroup of $\mathfrak{S}_6$ generated by σ_2, σ_5 and s.

Remark 2.6. Except for Hardy type invariant form (2.12) for ${}_4F_3^n$ series of the form (2.10), all other results are valid in the basic case and one can obtain in the same line as the discussion in this section. For Hardy type invariant form for terminating balanced ${}_4\phi_3$ series have already appeared in Van der Jeugt and Srinivasa Rao [35].

3 Symmetry groups of A_n elliptic hypergeometric series

3.1 Preliminaries on A_n elliptic hypergeometric series

Here, we give notations for (multiple) elliptic hypergeometric series and recall the results of our previous paper with M. Noumi [17].

Let $[[x]]$ be a non-zero and homomorphic odd function in $\mathbb{C}$ which satisfies the Riemann relation:

$$
\begin{aligned}
&1)\quad [[-x]]=-[[x]],\\
&2)\quad [[x+y]]\,[[x-y]]\,[[u+v]]\,[[u-v]]\\
&\qquad\qquad =[[x+u]]\,[[x-u]]\,[[y+v]]\,[[y-v]]-[[x+v]]\,[[x-v]]\,[[y+u]]\,[[y-u]].
\end{aligned}\tag{3.1}
$$

There are following three classes of such functions:

- $\sigma(x;\omega_1,\omega_2)$: Weierstrass sigma function with the periods (ω_1,ω_2) (elliptic),
- $\sin(\pi x)$: the sine function (trigonometric),
- x: rational.

It is classically known [37] that all function $[[x]]$ satisfy the condition (3.1) are obtained from above three functions by transformation of the form $e^{ax^2+b}[[cx]]$ for complex numbers $a, b, c \in \mathbb{C}$.

Fix a generic constant $\delta \in \mathbb{C}$ so that for all integer $k \in \mathbb{Z}$, $[[k\delta]]$ does not equal to zero. In the case when $[[x]]$ is Weierstrass sigma function $\sigma(x; \omega_1, \omega_2)$ (the elliptic case for short), the condition for δ is given by $\delta \notin \mathbb{Q}\omega_1 + \mathbb{Q}\omega_2$.

Throughout the present paper, we consider the function $[[x]]$ as the elliptic case unless otherwise stated.

Next a shifted factorial $[[x]]_k$ is defined by

$$[[x]]_k := [[x]][[x+\delta]]\cdots[[x+(k-1)\delta]], \qquad k = 0, 1, 2, \dots.$$

Further, we denote

$$[[x_1, \dots, x_r]]_k := [[x_1]]_k \cdots [[x_r]]_k.$$

The elliptic hypergeometric series ${}_{r+3}E_{r+2}$ is defined as follows

$${}_{r+3}E_{r+2}(s; \{u_k\}_r) = {}_{r+3}E_{r+2}(s; u_1, \dots, u_r) := \sum_{m \in \mathbb{N}} \frac{[[s+2m\delta]]}{[[s]]} \frac{[[s]]_m}{[[\delta]]_m} \prod_{1 \le i \le r} \frac{[[u_i]]_m}{[[\delta + s - u_i]]_m}.$$

In the case when $[[x]]$ is a trigonometric function $\sin x$, this series reduces to the basic very well-poised hypergeometric series ${}_{r+3}W_{r+2}$. Note that ${}_{r+3}E_{r+2}$ series are also symmetric with respect to the parameter u_k for $1 \le k \le r$.

All the ${}_{r+3}E_{r+2}$ series discussed in this paper is balanced, namely we assume

$$u_1 + \cdots + u_r = \frac{r-1}{2}s + \frac{r-3}{2}.$$

Now, we note the conventions for naming series as A_n elliptic hypergeometric series (or referred as elliptic hypergeometric series in $\mathrm{SU}(n+1)$). Let $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{N}^n$ be a multi-index. We denote generalizations of the Vandermonde determinant

$$\Delta[x] := \prod_{1 \le i < j \le n} [[x_i - x_j]] \qquad \text{and} \qquad \Delta[x + \gamma\delta] := \prod_{1 \le i < j \le n} [[x_i + \gamma_i\delta - x_j - \gamma_j\delta]],$$

for the sets of variables $x = (x_1, \dots, x_n)$ and $x + \gamma\delta = (x_1 + \gamma_1\delta, \dots, x_n + \gamma_n\delta)$ respectively. In this paper we refer multiple series of the form

$$\sum_{\gamma \in \mathbb{N}^n} \frac{\Delta[x+\gamma\delta]}{\Delta[x]} H(\gamma) \tag{3.2}$$

which reduce to elliptic hypergeometric series ${}_{r+1}E_r$ for a nonnegative integer r when $n = 1$ and symmetric with respect to the subscript $1 \le i \le n$ as A_n elliptic hypergeometric series. Other terminology are similar to the case of A_n (ordinary) hypergeometric series. The subscript n in the label A_n attached to the series is the dimension of the multiple series (3.2).

We are going to introduce the multiple elliptic hypergeometric series $E^{n,m}$ which is defined by

$$E^{n,m}\left(\begin{matrix} \{a_i\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, s; \{u_k\}_m; \{v_k\}_m \right) := \sum_{\gamma \in \mathbb{N}^n} \frac{\Delta[x+\gamma\delta]}{\Delta[x]} \prod_{1 \le i \le n} \frac{[[(|\gamma| + \gamma_i)\delta + s + x_i]]}{[[s + x_i]]}$$
$$\times \prod_{1 \le j \le n} \frac{[[s + x_j]]_{|\gamma|}}{[[\delta + s - a_j + x_j]]_{|\gamma|}} \left(\prod_{1 \le i \le n} \frac{[[a_j + x_i - x_j]]_{\gamma_i}}{[[\delta + x_i - x_j]]_{\gamma_i}} \right)$$

$$\times \prod_{1\le k\le m} \frac{[[v_k]]_{|\gamma|}}{[[\delta+s-u_k]]_{|\gamma|}} \left(\prod_{1\le i\le n} \frac{[[u_k+x_i]]_{\gamma_i}}{[[\delta+s-v_k+x_i]]_{\gamma_i}} \right).$$

In the case when $n=1$, $E^{1,m}$ series reduces to (one dimensional) elliptic hypergeometric series ${}_{2m+4}E_{2m+3}\left(s;\{u_k\}_m,\{v_k\}_m\right)$.

Note that $E^{n,m}$ series is symmetric within two sets of parameters $\{u_k\}_m$ and $\{v_k\}_m$ respectively. This fact will be a key of the latter discussion of the symmetry for the $E^{n,m}$ series.

Here, we present the balanced duality transformation formula for multiple elliptic hypergeometric series from [17].

Under the balancing condition

$$c_1+c_2+d_1+\sum_{1\le i\le n} a_i + \sum_{1\le k\le m}(u_k+v_k) = (m+N+1)\delta + (m+2)s.$$

We have the following transformation formula between $E^{n,m+2}$ ($A_n\ {}_{2m+8}E_{2m+7}$) series and $E^{m,n+2}$ ($A_m\ {}_{2n+8}E_{2n+7}$) series ((3.17) in [17]):

$$\begin{aligned} &E^{n,m+2}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix} \,\middle|\, s;c_1,c_2,\{u_k\}_m;d_1,-N\delta,\{v_k\}_m \right) \\ &\quad = \frac{[[\delta+s-c_1-d_1,\delta+s-c_2-d_1]]_N}{[[\delta+s-c_1,\delta+s-c_2]]_N} \prod_{1\le k\le m} \frac{[[v_k,\delta+s-u_k-d_1]]_N}{[[\delta+s-u_k,v_k-d_1]]_N} \\ &\qquad \times \prod_{1\le i\le n} \frac{[[\delta+s+x_i,\delta+s+x_i-a_i-d_1]]_N}{[[\delta+s+x_i-a_i,\delta+s+x_i-d_1]]_N} \\ &\qquad \times E^{m,n+2}\left(\begin{matrix}\{b_k\}_m\\ \{y_k\}_m\end{matrix} \,\middle|\, t;-c_1,-c_2,\{z_i\}_n;d_1,-N\delta,\{w_i\}_n \right), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &t=d_1+d_2-s-\delta, \qquad b_k=\delta+s-u_k-v_k, \qquad y_k=\delta+s-v_k, \qquad k=1,\dots,m,\\ &z_i=x_i-a_i, \qquad w_i=d_1+d_2-s-x_i, \qquad i=1,\dots,n. \end{aligned}$$

Remark 3.1. In [30], Rosengren also obtained (3.3) by a different way from [17]. In the case when $m=n=1$ and $x_1=y_1=0$, (3.3) reduces to the following transformation formula for terminating balanced ${}_{10}E_9$ series:

$$\begin{aligned} &{}_{10}E_9\left(s;c_0,c_1,c_2,c_3,d_0,d_1,-N\delta\right) = \frac{[[d_0,\delta+s]]_N}{[[d_0-d_1,\delta+s-d_1]]_N} \\ &\qquad \times \prod_{0\le k\le 3} \frac{[[\delta+s-c_k-d_1]]_N}{[[\delta+s-c_k]]_N}\, {}_{10}E_9\big(\widetilde{s};\widetilde{c}_0,\widetilde{c}_1,\widetilde{c}_2,\widetilde{c}_3,\widetilde{d}_0,d_1,-N\delta\big), \\ &(c_0+c_1+c_2+c_3+d_0+d_1=(2+N)\delta+3s), \end{aligned} \tag{3.4}$$

where

$$\widetilde{s}=d_1+d_2-d_0, \qquad \widetilde{d}_0=d_1+d_2-s, \qquad \widetilde{c}_k=\delta+s-d_0-c_k, \qquad k=0,1,2,3.$$

Note that the ${}_{10}E_9$ transformation (3.4) can also be obtained by iterating twice in an appropriate manner the (rather well-known) elliptic version of the Bailey transformation [1] (see also [2]) for ${}_{10}E_9$ series due to Frenkel and Turaev [6]:

$${}_{10}E_9\left(s;c_0,c_1,c_2,d_0,d_1,d_2,-N\delta\right) = \frac{[[\delta+s]]_N}{[[\delta+s-d_0-d_1-d_2]]_N}$$

$$\times \prod_{0\le k\le 2} \frac{[[\delta+s-d_0-d_1-d_2+d_k]]_N}{[[\delta+s-d_k]]_N} {}_{10}E_9\left(\widetilde{s};\widetilde{c_0},\widetilde{c_1},\widetilde{c_2},d_0,d_1,d_2,-N\delta\right), \tag{3.5}$$

$$\widetilde{s}=\delta+2s-c_0-c_1-c_2, \qquad \widetilde{c_0}=\delta+s-c_1-c_2, \qquad \widetilde{c_1}=\delta+s-c_0-c_2,$$
$$\widetilde{c_2}=\delta+s-c_0-c_1, \qquad c_0+c_1+c_2+d_0+d_1+d_2=(2+N)\delta+3s.$$

Similarly, (3.5) can also be obtained by iterating (3.4) (see [17] and latter discussions).

In the case when $m=1$, $y_1=0$, (3.3) reduces to the transformation formula between n-dimensional $E^{n,3}$ series and 1-dimensional ${}_{2n+8}E_{2n+7}$ series

$$\begin{aligned}
&E^{n,3}\left(\begin{array}{c|c}\{a_i\}_n & \\ \{x_i\}_n & s;c_0,c_1,c_2;d_0,d_1,-N\delta\end{array}\right)\\
&\quad=\frac{[[d_0,\delta+s-c_0-d_1,\delta+s-c_1-d_1,\delta+s-c_2-d_1]]_N}{[[d_0-d_1,\delta+s-c_0,\delta+s-c_1,\delta+s-c_2]]_N}\\
&\qquad\times\prod_{1\le i\le n}\frac{[[\delta+s+x_i,\delta+s+x_i-a_i-d_1]]_N}{[[\delta+s+x_i-a_i,\delta+s+x_i-d_1]]_N}\\
&\qquad\times {}_{2n+8}E_{2n+7}\left(t;e_0,e_1,e_2,\{u_i\}_n,\{v_i\}_n,d_1,-N\delta\right),
\end{aligned} \tag{3.6}$$

where

$$t=d_1-N\delta-d_0, \qquad e_k=\delta+s-d_0-c_k, \qquad k=0,1,2,$$
$$u_i=\delta+s-d_0+x_i-a_i, \qquad v_i=d_1-N\delta-s-x_i, \qquad i=1,\dots,m,$$

under the balancing condition for $E^{n,3}$ series

$$\sum_{1\le i\le m} a_i+(c_0+c_1+c_2)+(d_0+d_1)=(2+N)\delta+3s.$$

3.2 Symmetry of ${}_{10}E_9$ series (A_1 case)

Here, we describe the symmetry of 1-dimensional elliptic Bailey transformation for ${}_{10}E_9$ series (3.5), namely for the ${}_{10}E_9$ series of the form

$${}_{10}E_9\left(s;c_0,c_1,c_2,c_3,c_4,c_5,-N\delta\right),$$

with the balancing condition

$$c_0+c_1+c_2+c_3+c_4+c_5=(2+N)\delta+3s. \tag{3.7}$$

The result here has appeared in the paper by S. Lievens and J. Van der Jeugt [23] in the case of very well-poised basic hypergeometric series ${}_{10}W_9$. But our description given here is modified for the sake of the connection of the results in this section.

For $k=1,\dots,5$, let s_k be the permutation for the parameters c_{k-1} and c_k. Let b be the transformation of parameters for the Bailey transformation (3.5). Note that these are affine transformations in 7-dimensional vector space. Here we shall give a 8×8 matrix realization acting on the vector $\vec{v}_1={}^t[s,c_0,c_1,c_2,c_3,c_4,c_5,\delta]$ for these transformations. The transformation of parameters b for Bailey transformation (3.5) and its matrix realization are given by

$$b:\ \vec{v}_1=\begin{bmatrix}s\\c_0\\c_1\\c_2\\c_3\\c_4\\c_5\\\delta\end{bmatrix}\mapsto\begin{bmatrix}2s+\delta-c_0-c_1-c_2\\s+\delta-c_1-c_2\\s+\delta-c_0-c_2\\s+\delta-c_0-c_1\\c_3\\c_4\\c_5\\\delta\end{bmatrix}=B\cdot\vec{v}_1,$$

and

$$B = \begin{bmatrix} 2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively. The matrix realization for s_1 is given by

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and so on.

Proposition 3.1 (Hardy type invariant form for ${}_{10}E_9$ series with the condition (3.7))**.** *If the balancing condition* (3.7) *holds,*

$${}_{10}\widetilde{E}_9\left((\vec{v}_1)\right) := \frac{\prod\limits_{0\le k\le 5} [[\delta + s - c_k]]_N}{[[s]]_N} {}_{10}E_9\left(s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta\right)$$

is invariant under the action of b and s_k for all $1 \le k \le 5$.

Note that $B^2 = \mathrm{id}_8$, namely $b^2 = \mathrm{id}$. By definition, $\{s_i \,|\, i = 1, 2, 3, 4, 5\}$ satisfy the relation

$$\begin{cases} s_i \neq \mathrm{id}, \qquad s_i^2 = \mathrm{id}, & i = 1, 2, 3, 4, 5, \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, & i = 1, 2, 3, 4, \\ s_i s_j = s_j s_i, & |i - j| \ge 2. \end{cases} \tag{3.8}$$

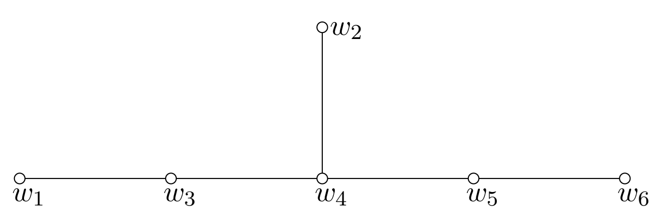
The relations between b and s_i, $i = 1, 2, 3, 4, 5$ summarized as the following lemma:

Lemma 3.1. *We have $(s_3 b)^3 = \mathrm{id}$. And b commutes with any other s_i, $i = 1, 2, 4, 5$.*

The proof of this lemma can be given by using the matrix realization given above.

By combining this lemma and (3.8), we conclude:

Theorem 3.1. *The group G_1 generated by the permutations of the parameters $\{c_k \,|\, k = 0, \dots, 5\}$ and the Bailey transformation* (3.5) *for ${}_{10}E_9$ series is isomorphic to the Weyl group $W(E_6)$ associated to the root system E_6:*



Dynkin diagram of E_6

Here we classify the possible non-trivial transformation formulas for ${}_{10}E_9$ series. Notice again that ${}_{10}E_9$ series is symmetric for the permutation of parameters $c_0, \dots, c_5$. That is, it is invariant under the action s_i for all $i = 1, \dots, 5$. Thus our problem turn out to give an orbit decomposition of the double coset $\mathfrak{S}_6 \backslash W(E_6)/\mathfrak{S}_6$.

We define the mapping π_1 according to Bourbaki [4] as follows

$$s_1 \mapsto w_1, \qquad b \mapsto w_2, \qquad s_i \mapsto w_{i+1}, \qquad i = 2, 3, 4, 5, 6.$$

The representatives of orbits in the double coset $\mathfrak{S}_6 \backslash W(E_6)/\mathfrak{S}_6$ are given as follows:

1) $\tau_1 = \mathrm{id}$,
2) $\tau_2 = w_2$,
3) $\tau_3 = w_2w_4w_3w_5w_4w_2$,
4) $\tau_4 = w_2w_4w_3w_1w_5w_4w_3w_6w_5w_4w_2$,
5) $\tau_5 = w_2w_4w_3w_1w_5w_4w_2w_3w_4w_5w_6w_5w_4w_2w_3w_1w_4w_3w_5w_4w_2$.

Thus we are ready and we shall exhibit a list of the possible ${}_{10}E_9$ transformations. We assume that all the ${}_{10}E_9$ series in the formulas listed here satisfy the balancing condition (3.7).

The transformation corresponding to $\tau_1 = \mathrm{id}$ is identical as ${}_{10}E_9$ transformation. The transformation corresponding to τ_2 is equivalent to the Bailey transformation due to Frenkel–Turaev (3.5). The third one corresponding to τ_3 is equivalent to (3.4). The forth one (τ_4) is

$$\begin{aligned}
{}_{10}E_9\left(s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta\right) &= \frac{[[\delta + s]]_N}{[[3\delta + 4s - c_0 - c_1 - c_2 - 2c_3 - 2c_4 - 2c_5]]_N} \\
&\times \prod_{0 \le k \le 2} \frac{[[c_{k+3}, \delta + s - c_0 - c_1 - c_2 + c_k]]_N}{[[\delta + s - c_k, \delta + s - c_{k+3}]]_N} \, {}_{10}E_9\left(\widehat{s}; \widehat{c}_0, \widehat{c}_1, \widehat{c}_2, \widehat{d}_0, \widehat{d}_1, \widehat{d}_2, -N\delta\right), \qquad (3.9)
\end{aligned}$$

$$\begin{aligned}
&\widehat{s} = (1 - N)\delta + s - c_3 - c_4 - c_5 = 3\delta + 4s - c_0 - c_1 - c_2 - 2c_3 - 2c_4 - 2c_5, \\
&\widehat{c}_k = \delta + s - c_3 - c_4 - c_5 + c_{k+3}, \\
&\widehat{c}_{k+3} = -N\delta + c_k - s = 2\delta + 2s - c_0 - c_1 - c_2 - c_3 - c_4 - c_5 + c_k, \qquad k = 0, 1, 2.
\end{aligned}$$

Finally, the fifth one corresponding to τ_5 is

$$\begin{aligned}
{}_{10}E_9\left(s; c_0, c_1, c_2, c_3, c_4, c_5, -N\delta\right) &= \frac{[[\delta + s]]_N}{[[4\delta + 5s - 2c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 - 2c_5]]_N} \\
&\times \prod_{0 \le k \le 5} \frac{[[c_k]]_N}{[[\delta + s - c_k]]_N} \, {}_{10}E_9\left(\check{s}; \check{c}_0, \check{c}_1, \check{c}_2, \check{c}_3, \check{c}_4, \check{c}_5, -N\delta\right), \qquad (3.10)
\end{aligned}$$

$$\begin{aligned}
&\check{s} = -2N\delta - s = 4\delta + 5s - 2c_0 - 2c_1 - 2c_2 - 2c_3 - 2c_4 - 2c_5, \\
&\check{c}_k = -N\delta + c_k - s = 2\delta + 2s - c_0 - c_1 - c_2 - c_3 - c_4 - c_5 + c_k, \qquad k = 0, 1, 2, 3, 4, 5.
\end{aligned}$$

Note that (3.10) implies reversing order of the summation in ${}_{10}E_9$ series.

3.3 Symmetry of A_n Bailey transformations of rectangular type

Here we discuss the symmetry for two A_n elliptic Bailey transformation formulas (3.13) and (3.14). The corresponding series is A_n elliptic hypergeometric series of rectangular type, which the multiple series terminates with respect to a multi-index. Namely the $E^{n,3}$ series of the form

$$E^{n,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, s; c_0, c_1, c_2; d_0, d_1, d_2\right) \qquad (3.11)$$

with the balancing condition

$$(c_0 + c_1 + c_2) + (d_0 + d_1 + d_2) = (2 + |M|)\delta + 3s. \tag{3.12}$$

In [17], we obtained several A_n generalizations of the elliptic Bailey transformation formula (3.5) for $E^{n,3}$ series by iterating (3.6) twice. Among these, here we give two transformations which $E^{n,3}$ series of rectangular type which we discuss here. These can be obtained in a similar way as in Section 3.2.

A_n Bailey transformation for $E^{n,3}$ of rectangular type (3.11) (Theorem 4.2 in [17]). Suppose that $a_i = -m_i\delta$, $m_i \in \mathbb{N}$ for all $i = 1, \ldots, n$. For c_k, d_k, $k = 0, 1, 2$, suppose that the balancing condition (3.12). Then we have two types of A_n Bailey transformation formula.

A_n Bailey I (Milne–Newcomb type)

$$\begin{aligned} E^{n,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \middle| s; c_0, c_1, c_2; d_0, d_1, d_2 \right) &= \frac{[[\delta + s - c_1 - d_0, \delta + s - c_2 - d_0]]_{|M|}}{[[\delta + s - c_1, \delta + s - c_2]]_{|M|}} \\ &\times \prod_{1 \le i \le n} \frac{[[\delta + s + x_i, 2\delta + 2s - c_0 - d_0 - d_1 - d_2 + x_i]]_{m_i}}{[[\delta + s - d_0 + x_i, 2\delta + 2s - c_0 - d_1 - d_2 + x_i]_{m_i}} \\ &\times E^{n,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \middle| \widetilde{s}; \widetilde{c}_0, c_1, c_2; d_0, \widetilde{d}_1, \widetilde{d}_2 \right), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} &\widetilde{s} = \delta + 2s - c_0 - d_1 - d_2, \qquad \widetilde{c}_0 = \delta + s - d_1 - d_2, \\ &\widetilde{d}_1 = \delta + s - c_0 - d_2, \qquad \widetilde{d}_2 = \delta + s - c_0 - d_1. \end{aligned}$$

A_n Bailey II (Kajihara–Noumi type)

$$\begin{aligned} E^{n,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \middle| s; c_0, c_1, c_2; d_0, d_1, d_2 \right) &= \prod_{1 \le i \le n} \left[\frac{[[\delta + s + x_i, \delta + s - d_0 - d_1 + x_i]]_{m_i}}{[[\delta + s - d_0 + x_i, \delta + s - d_1 + x_i]]_{m_i}} \right. \\ &\times \left. \frac{[[\delta + s - d_0 - d_2 + x_i, \delta + s - d_1 - d_2 + x_i]]_{m_i}}{[[\delta + s - d_2 + x_i, \delta + s - d_0 - d_1 - d_2 + x_i]]_{m_i}} \right] \\ &\times E^{n,3}\left(\begin{matrix} \{-m_i\delta\} \\ \{\widetilde{x}_i\} \end{matrix} \middle| \widetilde{s}; \widetilde{c}_0, \widetilde{c}_1, \widetilde{c}_2; d_0, d_1, d_2 \right), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} &\widetilde{s} = \delta + 2s - c_0 - c_1 - c_2, \qquad \widetilde{c}_0 = \delta + s - c_1 - c_2, \qquad \widetilde{c}_1 = \delta + s - c_0 - c_2, \\ &\widetilde{c}_2 = \delta + s - c_0 - c_1, \qquad \widetilde{x}_i = -m_i\delta - x_i + |M|\delta, \qquad i = 1, \ldots, n. \end{aligned}$$

Remark 3.2. In the case when $n = 1$, $x_1 = 0$, both (3.13) and (3.14) reduce to the elliptic Bailey transformation for ${}_{10}E_9$ series (3.5). Bailey I (3.13) is originally due to Rosengren [29] which is a elliptic version of A_n Bailey transformation formula by Milne–Newcomb [25]. Bailey II (3.14) has originally appeared in our previous work [17] together with the basic case.

First, we shall show the invariance property for the transformations for $E^{n,3}$ series of rectangular type (3.11). Suppose that $n \ge 2$ till we will state otherwise. Recall that the transformations of coordinates in the right hand side of the Bailey I (3.13) and Bailey II (3.14) are described as follows

$$b_1 : \begin{bmatrix} s \\ c_0 \\ c_1 \\ c_2 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} s^{1,0} \\ c_0^{1,0} \\ c_1^{1,0} \\ c_2^{1,0} \\ d_0^{1,0} \\ d_1^{1,0} \\ d_2^{1,0} \end{bmatrix} = \begin{bmatrix} 2s+\delta-c_0-d_1-d_2 \\ s+\delta-d_1-d_2 \\ c_1 \\ c_2 \\ d_0 \\ s+\delta-c_0-d_2 \\ s+\delta-c_0-d_1 \end{bmatrix},$$

$$b_2 : \begin{bmatrix} s \\ c_0 \\ c_1 \\ c_2 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix} \mapsto \begin{bmatrix} s^{0,1} \\ c_0^{0,1} \\ c_1^{0,1} \\ c_2^{0,1} \\ d_0^{0,1} \\ d_1^{0,1} \\ d_2^{0,1} \end{bmatrix} = \begin{bmatrix} 2s+\delta-c_0-c_1-c_2 \\ s+\delta-c_1-c_2 \\ s+\delta-c_0-c_2 \\ s+\delta-c_0-c_1 \\ d_0 \\ d_1 \\ d_2 \end{bmatrix}.$$

Note that these are compositions of linear transformations for parameters s, c_0, c_1, c_2, d_0, d_1, d_2 and shift by δ, namely affine transformations of 7-dimensional vector space. Thus we give a realization for these transformations in terms of 8×8 matrices acting on the vector $\vec{v} = {}^t[s, c_0, c_1, c_2, d_0, d_1, d_2, \delta]$ as follows

$$\begin{bmatrix} s^{1,0} \\ c_0^{1,0} \\ c_1^{1,0} \\ c_2^{1,0} \\ d_0^{1,0} \\ d_1^{1,0} \\ d_2^{1,0} \\ \delta \end{bmatrix} = B_1 \cdot \vec{v}, \qquad \begin{bmatrix} s^{0,1} \\ c_0^{0,1} \\ c_1^{0,1} \\ c_2^{0,1} \\ d_0^{0,1} \\ d_1^{0,1} \\ d_2^{0,1} \\ \delta \end{bmatrix} = B_2 \cdot \vec{v},$$

where the matrix B_1 is given by

$$B_1 = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & -1 & -1 & 1 \\ 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 1 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and B_2 is given by

$$B_2 = \begin{bmatrix} 2 & -1 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & 0 & -1 & 0 & 0 & 0 & 1 \\ 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Note that the matrix B_2 is the same as B in Section 3.2. Recall that $E^{n,3}$ series is invariant under the permutations within each sets of parameters $\{c_0, c_1, c_2\}$ and $\{d_0, d_1, d_2\}$. For $i = 0, 1$, set s_i (resp. t_i) to be the permutation of c_i and c_{i+1} (resp. d_i and d_{i+1}). The matrix realizations S_i (resp. T_i) for s_i (resp. t_i) is given by its action on the vector $\vec{v}$. For example,

$$S_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad T_0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the action on the variable x_i, we set to be $b_2 \cdot x_i = \tilde{x}_i = -m_i\delta - x_i + |M|\delta$ and identical otherwise. We denote I_n as the unit $n \times n$ matrix.

We introduce the normalized elliptic hypergeometric series $\widetilde{E}^{n,3}((\vec{v}, x))$ as follows

$$\widetilde{E}^{n,3}\left((\vec{v}, x)\right) := \frac{\prod\limits_{0\le k\le 2} [[\delta + s - c_k]]_{|M|} \left(\prod\limits_{1\le i\le n} [[\delta + s - d_k + x_i]]_{m_i} \right)}{\prod\limits_{1\le i\le n} [[\delta + s + x_i]]_{m_i}}$$
$$\times E^{n,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \middle| s; c_0, c_1, c_2; d_0, d_1, d_2 \right). \tag{3.15}$$

Now, we show an invariance property for $E^{n,3}$ series of type (3.11).

Proposition 3.2 (Hardy type invariant form for $E^{n,3}$ series of type (3.11))**.** *Under the balancing condition* (3.12)*, $\widetilde{E}^{n,3}((\vec{v}, x))$ is invariant under the action of b_1, b_2, s_0, s_1, t_0 and t_1.*

In the course of the proof of this proposition, we use the following lemma.

Lemma 3.2. *If* (3.12) *holds, we have the following*

1) $$[[a]]_{|M|} = (-1)^{|M|}[[3s + 3\delta - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a]]_{|M|}, \tag{3.16}$$
2) $$[[a + x_i]]_{m_i} = (-1)^{m_i}[[3s + 3\delta - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a + \tilde{x}_i]]_{m_i}. \tag{3.17}$$

Proof. Since $[[x]]$ is a odd function of x,

$$[[a]]_{|M|} = [[a]][[a+\delta]] \cdots [[a + (|M|-1)\delta]] = (-1)^{|M|}[[-a + (1 - |M|)\delta]] \cdots [[-a]]$$
$$= (-1)^{|M|}[[-a + (1 - |M|)\delta]]_{|M|}.$$

By the balancing condition (3.12), we have

$$-a + (1 - |M|)\delta = 3\delta + 3s - c_0 - c_1 - c_2 - d_0 - d_1 - d_2 - a.$$

Thus we have (3.16). Further, one can check (3.17) in a similar fashion. ■

Proof of Proposition 3.2. It is not hard to see in the case of s_0, s_1, t_0, t_1 since $\widetilde{E}^{n,3}$ (3.15) is symmetric with respect to the subscript k. For the case of b_2,

$$\widetilde{E}^{n,3}\left(b_2 \cdot (\vec{v}, x)\right) = \widetilde{E}^{n,3}\left((B_2\vec{v}, \tilde{x})\right)$$

$$= \frac{\prod\limits_{0\le k\le 2}[[\delta+s-c_k]]_{|M|}\left(\prod\limits_{1\le i\le n}[[\delta+s+d_k-d_0-d_1-d_2+x_i]]_{m_i}\right)}{\prod\limits_{1\le i\le n}[[\delta+s-d_0-d_1-d_2+x_i]]_{m_i}}$$
$$\times E^{n,3}\left(\begin{matrix}\{-m_i\delta\}_n\\ \{\tilde{x}_i\}_n\end{matrix}\,\middle|\, s^{0,1}; c_0^{0,1}, c_1^{0,1}, c_2^{0,1}; d_0^{0,1}, d_1^{0,1}, d_2^{0,1}\right)$$
$$= \frac{\prod\limits_{0\le k\le 2}[[\delta+s-c_k]]_{|M|}\left(\prod\limits_{1\le i\le n}[[\delta+s+d_k-d_0-d_1-d_2+x_i]]_{m_i}\right)}{\prod\limits_{1\le i\le n}[[\delta+s-d_0-d_1-d_2+x_i]]_{m_i}}$$
$$\times \prod_{1\le i\le n}\frac{[[\delta+s-d_0-d_1-d_2+x_i,\delta+s-d_0+x_i]]_{m_i}}{[[\delta+s-d_0-d_1+x_i,\delta+s-d_0-d_2+x_i]]_{m_i}}$$
$$\times \prod_{1\le i\le n}\frac{[[\delta+s-d_1+x_i,\delta+s-d_2+x_i]]_{m_i}}{[[\delta+s-d_1-d_2+x_i,\delta+s+x_i]]_{m_i}}$$
$$\times E^{n,3}\left(\begin{matrix}\{-m_i\delta\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s; c_0, c_1, c_2; d_0, d_1, d_2\right) = \widetilde{E}^{n,3}\left((\vec{v},x)\right).$$

Here we used the transformation (3.14) and Lemma 3.2. For b_1, one can check similarly. ■

Here we shall investigate the compositions of the transformations b_1, b_2, s_0, s_1, t_0 and t_1.

Lemma 3.3. *The relations $b_1^2 = b_2^2 = s_0^2 = s_1^2 = t_0^2 = t_1^2 = \mathrm{id}$ holds (where id stands for identical as a transformation). Thus the set of the transformations $\{b_1, b_2, s_0, s_1, t_0, t_1\}$ constitutes the generators of a Coxeter group by compositions.*

Proof. For s_0, s_1, t_0 and t_1, it is obvious since these are the permutation for the coordinates. For b_1 and b_2, we can check by direct computations of matrices B_1 and B_2 that $B_1^2 = B_2^2 = I_8$. ■

Remark 3.3. Recall that the variables x_i in Bailey II (3.14) change to $\tilde{x}_i = m_i - |M| - x_i$. It is easy to see that by iterating twice,

$$\tilde{\tilde{x}}_i = m_i - |M| - \tilde{x}_i = x_i.$$

Thus we see that it turn out to be identity as transformation for $E^{n,3}$ series by iterating Bailey II (3.14) twice.

Let G_r be the group generated by b_1, b_2, s_0, s_1, t_0 and t_1. Now we shall give the relations between the generators of the group G_r. By definition of s_0, s_1, t_0 and t_1, the following two braid relations hold:

$$(s_0s_1)^3 = (t_0t_1)^3 = \mathrm{id}. \tag{3.18}$$

Note also that, for $i, j \in \{0, 1\}$, s_i and t_j mutually commute. Other relations, among b_1, b_2 and others, can be summarized as follows:

Lemma 3.4. *The relations*

$$(b_1s_0)^3 = (b_1t_0)^3 = \mathrm{id}$$

hold. Other pairs of generators of G_r commute. Especially, b_2 commutes with any other generators.

Proof. One can check by direct computation for the matrix realization given above. So we shall leave to readers. ■

We define the mapping π as

$$b_1 \mapsto \sigma_3, \qquad b_2 \mapsto \tau, \qquad s_i \mapsto \sigma_{2-i}, \qquad t_i \mapsto \sigma_{4+i}, \qquad i = 0, 1.$$

Then, by braid relations (3.18) and two lemmas above, we see that the following relation holds:

$$\begin{cases} \sigma_i \neq \mathrm{id}, \qquad \sigma_i^2 = \mathrm{id}, & i = 1,2,3,4,5, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & i = 1,2,3,4, \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2, \end{cases}$$

and $\tau^2 = \mathrm{id}$. In other words, $\{\sigma_j\}$ and $\{\tau\}$ is a realization of $\mathfrak{S}_6$ and $\mathfrak{S}_2$. Thus we have

Proposition 3.3. *The group G_r is isomorphic to the direct product of $\mathfrak{S}_6$ and $\mathfrak{S}_2$.*

To summarize the results here, we state the following:

Theorem 3.2. *Under the balancing condition (3.12), $\tilde{E}^{n,3}\left((\vec{v}, x)\right)$ is invariant under the action of the direct product of $\mathfrak{S}_6$ and $\mathfrak{S}_2$ realized by the mapping π^{-1}.*

We are going to classifying non-trivial transformations for $E^{n,3}$ of rectangular type (3.11) by using the realization σ_i and τ.

Proposition 3.3 tells us that the group G_r of the symmetry of $E^{n,3}$ of type (3.11) is isomorphic to a direct product of the $\mathfrak{S}_6$ and $\mathfrak{S}_2$ and is of order $6! \times 2! = 1440$. Recall again that $E^{n,3}$ series of rectangular type is symmetric with respect to the c_k, $k = 0,1,2$ and d_k, $k = 0,1,2$. Then it is not hard to see that the the right action of σ_1, σ_2, σ_4 and σ_5 corresponds to the permutations of the subscript in the sets of parameters $\{c_0, c_1, c_2\}$ and $\{d_0, d_1, d_2\}$ and the left action corresponds to the permutation of the location of coordinates. Thus our problem turns out to give an orbit decomposition of the double coset $H_r \backslash G_r / H_r$, where H_r is a subgroup generated by σ_1, σ_2, σ_4, σ_5, which is isomorphic to a direct product of two $\mathfrak{S}_3$. The representatives of orbits in $H_r \backslash G_r / H_r$ are given by the following:

0) $\omega_0 = \mathrm{id}$,
1) $\omega_1 = \sigma_3$,
2) $\omega_2 = \sigma_3\sigma_4\sigma_2\omega_1 = \sigma_3\sigma_4\sigma_2\sigma_3$,
3) $\omega_3 = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\omega_2 = \sigma_3\sigma_4\sigma_5\sigma_2\sigma_1\sigma_3\sigma_4\sigma_2\sigma_3$.

Since the element τ commutes with σ_i for all $i = 1,2,3,4,5$, one can find that τ is also commutative with all the representatives ω_i for $i = 0,1,2,3$.

Before going to present a list of non-trivial possible transformations for $E^{n,3}$ series of type (3.11), we define a transformation $\pi^{r,t}$ as $\pi^{r,t} := \tau^t\omega_r$ for $r = 0,1,2,3$ and $t = 0,1$. We call the transformation formula corresponding to $\pi^{r,t}$ as $T(r,t)$ and express it as follows

$$E^{m,3}\left(\begin{matrix} \{-m_i\delta\}_n \\ \{x_i\}_n \end{matrix} \,\middle|\, s; c_0, c_1, c_2; d_0, d_1, d_2 \right) = \\ P^{r,t}(x; s; C; D) E^{m,3}\left(\begin{matrix} \{-m_i\delta\} \\ \{x_i^t\} \end{matrix} \,\middle|\, s^{r,t}; c_0^{r,t}, c_1^{r,t}, c_2^{r,t}; d_0^{r,t}, d_1^{r,t}, d_2^{r,t} \right),$$

where $s^{r,t}$, $c_k^{r,t}$, $d_k^{r,t}$ ($k = 0,1,2$) is parameters associated to the transformation $\pi^{r,t}$ and $P^{r,t}(x; s; C; D) = P^{r,t}(\{x_i\}_n; s; c_0, c_1, c_2; d_0, d_1, d_2)$ is the corresponding product factor. For the variables x_i^t, we set $x_i^0 = x_i$ and $x_i^1 = \widetilde{x}_i = (|M| - m_i)\delta - x_i$. Note that Bailey I (3.13) and Bailey II (3.14) correspond to $T(1,0)$ and $T(0,1)$ respectively. Note also that $T(0,0)$ is identical.

Here we exhibit a list of the product factors and transformations for parameters in $T(r,t)$. In order to simplify each product factor, we frequently use Lemma 3.2. Note that the expressions of each product factors have ambiguity because of the balancing condition (3.12).

$\boldsymbol{T(2,0)}$

- Product factor

$$\begin{aligned}P^{2,0}(x;s;C;D) &= (-1)^{|M|}\frac{[[d_2,\delta+s-c_2-d_0,\delta+s-c_2-d_1]]_{|M|}}{[[\delta+s-c_0,\delta+s-c_1,\delta+s-c_2]]_{|M|}}\\ &\times\prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i,2\delta+2s-c_0-d_0-d_1-d_2+x_i]]_{m_i}}{[[\delta+s-d_0+x_i,\delta+s-d_1+x_i]]_{m_i}}\right.\\ &\left.\times\frac{[[2\delta+2s-c_1-d_0-d_1-d_2+x_i]]_{m_i}}{[[3\delta+3s-c_0-c_1-d_0-d_1-2d_2+x_i]]_{m_i}}\right].\end{aligned}$$

- Parameters

$$\begin{bmatrix}s^{2,0}\\c_0^{2,0}\\c_1^{2,0}\\c_2^{2,0}\\d_0^{2,0}\\d_1^{2,0}\\d_2^{2,0}\end{bmatrix}=\begin{bmatrix}3s+2\delta-c_0-c_1-d_0-d_1-2d_2\\s+\delta-d_0-d_2\\s+\delta-d_1-d_2\\c_2\\s+\delta-c_0-d_2\\s+\delta-c_1-d_2\\2s+2\delta-c_0-c_1-d_0-d_1-d_2\end{bmatrix}.$$

$\boldsymbol{T(3,0)}$

- Product factor

$$\begin{aligned}P^{3,0}(x;s;C;D) &= (-1)^{|M|}\frac{[[d_0,d_1,d_2]]_{|M|}}{[[\delta+s-c_0,\delta+s-c_1,\delta+s-c_2]]_{|M|}}\\ &\times\prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i,2\delta+2s-c_0-d_0-d_1-d_2+x_i]]_{m_i}}{[[\delta+s-d_0+x_i,\delta+s-d_1+x_i]]_{m_i}}\right.\\ &\left.\times\frac{[[2\delta+2s-c_1-d_0-d_1-d_2+x_i,2\delta+2s-c_2-d_0-d_1-d_2+x_i]]_{m_i}}{[[\delta+s-d_2+x_i,4\delta+4s-c_0-c_1-c_2-2d_0-2d_1-2d_2+x_i]]_{m_i}}\right].\end{aligned}$$

- Parameters

$$\begin{bmatrix}s^{3,0}\\c_0^{3,0}\\c_1^{3,0}\\c_2^{3,0}\\d_0^{3,0}\\d_1^{3,0}\\d_2^{3,0}\end{bmatrix}=\begin{bmatrix}4s+3\delta-c_0-c_1-c_2-2d_0-2d_1-2d_2\\s+\delta-d_0-d_1\\s+\delta-d_0-d_2\\s+\delta-d_1-d_2\\2s+2\delta-c_0-c_1-d_0-d_1-d_2\\2s+2\delta-c_0-c_2-d_0-d_1-d_2\\2s+2\delta-c_1-c_2-d_0-d_1-d_2\end{bmatrix}.$$

$\boldsymbol{T(1,1)}$

- Product factor

$$\begin{aligned}P^{1,1}(x;s;C;D) &= \frac{[[\delta+s-c_1-d_0,\delta+s-c_2-d_0]]_{|M|}}{[[\delta+s-c_1,\delta+s-c_2]]_{|M|}}\\ &\times\prod_{1\le i\le n}\frac{[[\delta+s+x_i,c_0+x_i]]_{m_i}}{[[\delta+s-d_0+x_i,\delta+s-d_1+x_i]]_{m_i}}\\ &\times\prod_{1\le i\le n}\frac{[[\delta+s-d_0-d_2+x_i,\delta+s-d_0-d_1+x_i]]_{m_i}}{[[\delta+s-d_2+x_i,c_0-d_0+x_i]]_{m_i}}.\end{aligned}$$

• Parameters

$$\begin{bmatrix} s^{1,1} \\ c_0^{1,1} \\ c_1^{1,1} \\ c_2^{1,1} \\ d_0^{1,1} \\ d_1^{1,1} \\ d_2^{1,1} \end{bmatrix} = \begin{bmatrix} 3s+2\delta-2c_0-c_1-c_2-d_1-d_2 \\ 2s+2\delta-c_0-c_1-c_2-d_1-d_2 \\ s+\delta-c_0-c_2 \\ s+\delta-c_0-c_1 \\ d_0 \\ s+\delta-c_0-d_2 \\ s+\delta-c_0-d_1 \end{bmatrix}.$$

***T*(2, 1)**

• Product factor

$$\begin{aligned} P^{2,1}(x;s;C;D) &= (-1)^{|M|}\frac{[[\delta+s-c_2-d_0, d_2, \delta+s-c_2-d_1]]_{|M|}}{[[\delta+s-c_0, \delta+s-c_1, \delta+s-c_2]]_{|M|}} \\ &\quad\times \prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i, c_1+x_i]]_{m_i}}{[[\delta+s-d_0+x_i, \delta+s-d_1+x_i]]_{m_i}}\right. \\ &\quad\left.\times\frac{[[c_0+x_i, \delta+s-d_0-d_1+x_i]]_{m_i}}{[[\delta+s-d_2+x_i, -\delta-s+c_0+c_1+d_2+x_i]]_{m_i}}\right]. \end{aligned}$$

• Parameters

$$\begin{bmatrix} s^{2,1} \\ c_0^{2,1} \\ c_1^{2,1} \\ c_2^{2,1} \\ d_0^{2,1} \\ d_1^{2,1} \\ d_2^{2,1} \end{bmatrix} = \begin{bmatrix} 4s+3\delta-2c_0-2c_1-c_2-d_0-d_1-2d_2 \\ 2s+2\delta-c_0-c_1-c_2-d_0-d_2 \\ 2s+2\delta-c_0-c_1-c_2-d_1-d_2 \\ s+\delta-c_0-c_1 \\ s+\delta-c_0-d_2 \\ s+\delta-c_1-d_2 \\ 2s+2\delta-c_0-c_1-d_0-d_1-d_2 \end{bmatrix}.$$

***T*(3, 1)**

• Product factor

$$\begin{aligned} P^{3,1}(x;s;C;D) &= (-1)^{|M|}\frac{[[d_0, d_1, d_2]]_{|M|}}{[[\delta+s-c_0, \delta+s-c_1, \delta+s-c_2]]_{|M|}} \\ &\quad\times \prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i]]_{m_i}}{[[-2\delta-2s+c_0+c_1+c_2+d_0+d_1+d_2+x_i]]_{m_i}}\right. \\ &\quad\left.\times\frac{[[c_0+x_i, c_1+x_i, c_2+x_i]]_{m_i}}{[[\delta+s-d_0+x_i, \delta+s-d_1+x_i, \delta+s-d_2+x_i]]_{m_i}}\right]. \end{aligned}$$

• Parameters

$$\begin{bmatrix} s^{3,1} \\ c_0^{3,1} \\ c_1^{3,1} \\ c_2^{3,1} \\ d_0^{3,1} \\ d_1^{3,1} \\ d_2^{3,1} \end{bmatrix} = \begin{bmatrix} 5s+4\delta-2c_0-2c_1-2c_2-2d_0-2d_1-2d_2 \\ 2s+2\delta-c_0-c_1-c_2-d_0-d_1 \\ 2s+2\delta-c_0-c_1-c_2-d_0-d_2 \\ 2s+2\delta-c_0-c_1-c_2-d_1-d_2 \\ 2s+2\delta-c_0-c_1-d_0-d_1-d_2 \\ 2s+2\delta-c_0-c_2-d_0-d_1-d_2 \\ 2s+2\delta-c_1-c_2-d_0-d_1-d_2 \end{bmatrix}.$$

Note that by reversing the order of the summation for $E^{n,3}$ series, namely by replacing $\gamma_i \mapsto m_i - \gamma_i$ and simplifying the factors, we also obtain $T(3,1)$. Note also that (3.14) can be obtained by combining $T(3,0)$ and $T(3,1)$.

3.4 The case of triangular $E^{n,3}$ series

Here, we shall discuss triangular case. That is the case of $E^{n,3}$ series of the form

$$E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s; c_0, c_1, c_2; -N\delta, d_1, d_2\right), \tag{3.19}$$

which terminates with respect to the length of multi-indices and is provided the balancing condition

$$\sum_{1\le i\le n} a_i + c_0 + c_1 + c_2 + d_1 + d_2 = (2+N)\delta + 3s. \tag{3.20}$$

In this case, we have also obtained the following A_n elliptic Bailey transformation formulas for $E^{n,3}$ series for triangular type (3.19) in [17].

***A_n* Bailey transformations for *$E^{n,3}$* series of triangular type** (Theorem 4.1 in [17]). Under the balancing condition (3.20), we have two types of A_n Bailey transformation formulas.

***A_n* Bailey I**

$$\begin{aligned}
&E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s; c_0, c_1, c_2; -N\delta, d_1, d_2\right)\\
&\quad= \frac{[[\delta+\widetilde{s}-c_1, \delta+\widetilde{s}-c_2]]_N}{[[\delta+s-c_1, \delta+s-c_2]]_N} \prod_{1\le i\le n} \frac{[[\delta+s+x_i, \delta+\widetilde{s}+x_i-a_i]]_N}{[[\delta+s+x_i-a_i, \delta+\widetilde{s}+x_i]]_N}\\
&\qquad\times E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, \widetilde{s}; \widetilde{c}_0, c_1, c_2; -N\delta, \widetilde{d}_1, \widetilde{d}_2\right),
\end{aligned} \tag{3.21}$$

where

$$\begin{aligned}
&\widetilde{s} = \delta + 2s - c_2 - d_0 - d_1, \qquad \widetilde{c}_0 = \delta + s - d_1 - d_2,\\
&\widetilde{d}_1 = \delta + s - c_0 - d_2, \qquad \widetilde{d}_2 = \delta + s - c_0 - d_1.
\end{aligned}$$

***A_n* Bailey II**

$$\begin{aligned}
&E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s; c_0, c_1, c_2; -N\delta, d_1, d_2\right) = \prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i, \delta+s+x_i-d_1-d_2]]_N}{[[\delta+s+x_i-d_1, \delta+s+x_i-d_2]]_N}\right.\\
&\qquad\left.\times\frac{[[\delta+s+x_i-a_i-d_1, \delta+s+x_i-a_i-d_2]]_N}{[[\delta+s+x_i-a_i, \delta+s+x_i-a_i-d_1-d_2]]_N}\right]\\
&\qquad\times E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{z_i\}_n\end{matrix}\,\middle|\, \widetilde{s}; \widetilde{c}_0, \widetilde{c}_1, \widetilde{c}_2; -N\delta, d_1, d_2\right),
\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}
&\widetilde{s} = \delta + 2s - c_0 - c_1 - c_2, \qquad \widetilde{c}_0 = \delta + s - c_1 - c_2, \qquad \widetilde{c}_1 = \delta + s - c_0 - c_2,\\
&\widetilde{c}_2 = \delta + s - c_0 - c_1, \qquad z_i = a_i - x_i - |a|, \qquad i = 1, \dots, m.
\end{aligned}$$

Note that, in the case when $n = 1, x_1 = 0$, (3.21) and (3.22) reduce to the elliptic Bailey transformation formula (3.5).

Recall that, though the right hand side in Bailey I in rectangular case (3.13) contains two types of d_j's: $\widetilde{d}_0 = d_0$ fixed and $\widetilde{d}_j = \delta + s - c_0 - d_1 - d_2 + d_j$, $j = 1, 2$, it consists of only $\widetilde{d}_j$ $(j = 1, 2)$ in triangular case (3.21). Thus we find that, on the contrast to rectangular case, the element $t_0 = \sigma_4$ lacks in this case. Thus we have:

Proposition 3.4. *The group describing the symmetry for the transformations* (3.21) *and* (3.22) *is isomorphic to* $\mathfrak{S}_4 \times (\mathfrak{S}_2)^2$.

It is not hard to see that the composition of (3.21) and (3.22) is the only further non-trivial transformation which can be obtained

$$\begin{aligned}
&E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{x_i\}_n\end{matrix}\,\middle|\, s; c_0, c_1, c_2; -N\delta, d_1, d_2\right)\\
&\quad = \frac{[[2\delta+s-c_0-c_1-d_1-d_2, 2\delta+s-c_0-c_2-d_1-d_2]]_N}{[[\delta+s-c_1, \delta+s-c_2]]_N}\\
&\qquad \times \prod_{1\le i\le n}\left[\frac{[[\delta+s+x_i, x_i+c_0]]_N}{[[\delta+s+x_i-a_i, x_i+c_0-a_i]]_N}\right.\\
&\qquad \times \left.\frac{[[\delta+s+x_i-d_1-a_i, \delta+s+x_i-d_2-a_i]]_N}{[[\delta+s+x_i-d_1, \delta+s+x_i-d_2]]_N}\right]\\
&\qquad \times E^{n,3}\left(\begin{matrix}\{a_i\}_n\\ \{z_i\}_n\end{matrix}\,\middle|\, \widehat{s}; \widehat{c}_0, \widehat{c}_1, \widehat{c}_2; -N\delta, \widehat{d}_1, \widehat{d}_2\right),
\end{aligned} \tag{3.23}$$

where

$$\begin{aligned}
&\widehat{s} = 2\delta+3s-2c_0-c_1-c_2-d_1-d_2, \qquad \widehat{c}_0 = 2\delta+2s-c_0-c_1-c_2-d_1-d_2,\\
&\widehat{c}_1 = \delta+s-c_0-c_2, \qquad \widehat{c}_2 = \delta+s-c_0-c_1, \qquad \widehat{d}_1 = \delta+s-c_0-d_2,\\
&\widehat{d}_2 = \delta+s-c_0-d_1, \qquad z_i = a_i - x_i - |a|, \qquad i = 1,\dots,n.
\end{aligned}$$

To simplify the product factor, we used the following lemma which can be proved just in the same line as in the rectangular case.

Lemma 3.5. *If the balancing condition* (3.20) *holds, then we have*

$$[[b]]_N = (-1)^N[[3\delta+3s-b-(c_0+c_1+c_2)-(d_1+d_2)-|a|]]_N.$$

3.5 Remarks on results of Section 3

We close this paper to give some remarks.

Remark 3.4. The transformation $T(1,1)$ in Section 3.3 has appeared as Corollary 4.3 in Rosengren [30] with a different expression and the transformation (3.23) has appeared as Corollary 4.2 in [30].

Remark 3.5 (in the case when $n = 1$, $x_1 = 0$)**.** In this case, $T(2,0)$ and $T(1,1)$ in Section 3.3. and (3.23) in Section 3.4. reduce to (3.4) in Section 3.2. $T(3,0)$ and $T(2,1)$ reduce to (3.9). Finally, $T(3,1)$ reduces to (3.10). Notice that (3.10) can also be obtained by reversing order of the summation in the ${}_{10}E_9$ series.

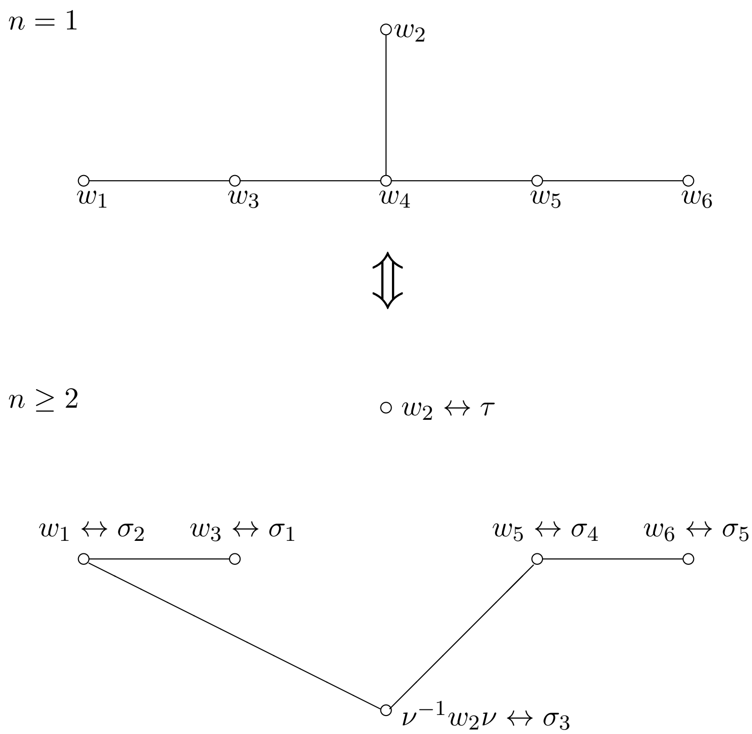
Remark 3.6 (correspondence of the group G_r in Section 3.3 and G_1 in Section 3.2)**.** By direct computation using the matrix realization in this paper, one finds that $b_1 = \pi^{-1}(\sigma_3)$ can be expressed as

$$\nu^{-1}w_2\nu, \qquad \nu = w_4w_5w_6w_3w_4w_5. \tag{3.24}$$

The correspondence between the generators of the group G_r in Section 3.3. and the elements of the group G_1 is summarized as follows:

$$\begin{array}{ccc}
G_1 \simeq W(E_6) & & G_r \simeq \mathfrak{S}_6 \times \mathfrak{S}_2 \\
w_1 & \longleftrightarrow & \sigma_2, \\
w_3 & \longleftrightarrow & \sigma_1, \\
w_5 & \longleftrightarrow & \sigma_4, \\
w_6 & \longleftrightarrow & \sigma_5, \\
w_2 & \longleftrightarrow & \tau, \\
\nu^{-1} w_2 \nu & \longleftrightarrow & \sigma_3.
\end{array}$$

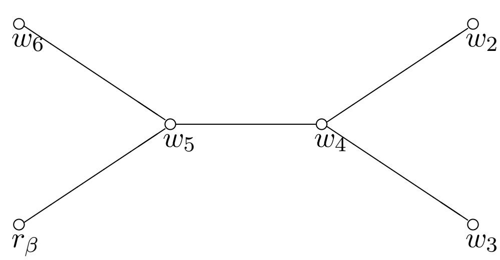
The correspondence is described diagrammatically as follows:



Correspondence of the elements

Thus we find the group generated by w_1, w_3, w_4, w_5 and $\nu^{-1}\sigma_2\nu$ is isomorphic to the symmetric group $\mathfrak{S}_6$ and all the generators commute with w_2.

Furthermore, $r_\beta = \nu^{-1} w_2 \nu \in W(E_6)$ (3.24) is the reflection of the root $\beta = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$. Note that β is the highest root of the root system D_5 whose roots are α_2, α_3, α_4, α_5 and α_6:



Extended Dynkin diagram of D_5

Note also that the expression in (3.24) of r_β is reduced.

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