Symmetry, Integrability and Geometry: Methods and Applications SIGMA 9 (2013), 032, 33 pages

On Orbifold Criteria for Symplectic Toric Quotients

Carla FARSI [†], *Hans-Christian HERBIG* [‡] *and Christopher SEATON* [§]

[†] *Department of Mathematics, University of Colorado at Boulder, Campus Box 395, Boulder, CO 80309-0395, USA*
E-mail: *farsi@euclid.colorado.edu*
URL: http://www.colorado.edu/math/people/professors/farsi.html

[‡] *Centre for Quantum Geometry of Moduli Spaces, Ny Munkegade 118 Building 1530, 8000 Aarhus C, Denmark*
E-mail: *herbig@imf.au.dk*

[§] *Department of Mathematics and Computer Science, Rhodes College, 2000 N. Parkway, Memphis, TN 38112, USA*
E-mail: *seatonc@rhodes.edu*
URL: http://faculty.rhodes.edu/seaton/

Received August 07, 2012, in final form April 02, 2013; Published online April 12, 2013
http://dx.doi.org/10.3842/SIGMA.2013.032

Abstract. We introduce the notion of regular symplectomorphism and graded regular symplectomorphism between singular phase spaces. Our main concern is to exhibit examples of unitary torus representations whose symplectic quotients cannot be graded regularly symplectomorphic to the quotient of a symplectic representation of a finite group, while the corresponding GIT quotients are smooth. Additionally, we relate the question of simplicialness of a torus representation to Gaussian elimination.

Key words: singular symplectic reduction; invariant theory; orbifold

2010 Mathematics Subject Classification: 53D20; 58A40; 13A50; 14L24; 57R18

1 Introduction

Let G be a compact Lie group acting on a symplectic manifold (M, ω) by symplectomorphisms. One says that the action is *Hamiltonian* with *moment map* $J\colon M \to \mathfrak{g}^*$, $\mathfrak{g}^*$ being the dual space of the Lie algebra $\mathfrak{g}$ of G, if

1. J is a smooth G-equivariant map,
2. For each $\xi \in \mathfrak{g}$ the vector field $\{J_\xi, \ \}$ coincides with the fundamental vector field of ξ acting on M, where $J_\xi := \langle J, \xi\rangle \in \mathcal{C}^\infty(M)$ and $\{\,,\,\}$ denotes the Poisson bracket associated to the symplectic form ω.

The *symplectic quotient* $M_0 = Z/G$ is defined to be the space of G-orbits in the zero fibre $Z := J^{-1}(0)$ of the moment map.

It is well-known [19, 21] that if $0 \in \mathfrak{g}^*$ is a regular value of J, then the quotient $M_0 = Z/G$ of the closed submanifold Z by the action of G is in a canonical way a symplectic orbifold. This is the case, for instance, when the G-action is locally free. If $0 \in \mathfrak{g}^*$ is not a regular value, a theorem of E. Lerman and R. Sjamaar [29] tells us that $M_0 = Z/G$ is a stratified symplectic space; for more details see Subsections 2.1 and 4.1. Note that $0 \in \mathfrak{g}^*$ is a singular value if, for example, (M, ω) is a symplectic vector space, the G-action is linear, and the moment map is chosen to be homogeneous quadratic. We refer to this situation as *the linear case*.

It has been observed that in the linear case, the symplectic quotient can occasionally be identified, symplectically [6, 14, 18] or merely topologically [17], with a quotient by a symplectic representation of a finite group. This is the case, for instance, with the physically interesting example of angular momentum [14]. For more examples, see Subsection 4.3.

Our paper is an attempt towards a more systematic understanding of when and how this happens. If one is searching for orbifold criteria, a natural idea is to use intuition from complex algebraic toric geometry (see e.g. [4, 12]). Namely, if one considers a representation of a complex torus $\mathbb{T}^\ell_{\mathbb{C}}$ on a complex vector space W, it is well-known that the GIT-quotient $W/\!\!/\mathbb{T}^\ell_{\mathbb{C}}$ is isomorphic as a complex algebraic variety to a complex orbifold if and only if the representation is *simplicial* (see Subsection 2.1 and Section 3). By the Kempf–Ness theorem (to be recalled in Subsection 2.1), the symplectic quotient M_0 is homeomorphic to such a GIT-quotient. Hence, the question arises whether the orbifold criterion in the complex algebraic setting carries over via the Kempf–Ness homeomorphism to the symplectic setting.

Our results can be stated as follows. If the symplectic quotient of a unitary representation of a compact torus is homeomorphic to an orbifold, then the representation has to be simplicial (see Subsection 2.2). We indicate methods of determining whether a representation satisfies this property directly from the weight matrix in Section 3. This in particular resolves the conjectures stated in [17]. When the symplectic quotient has real dimension two, the representation is always simplicial; in this case, we further demonstrate an explicit *graded regular symplectomorphism* (to be defined in Subsection 4.2) to a quotient of $\mathbb{C}$ by a finite abelian group. On the other hand, we present in Subsection 5.3 examples of simplicial unitary circle representations whose symplectic quotients are homeomorphic to $\mathbb{C}^2$, for which there cannot exist a graded regular symplectomorphism to a quotient of $\mathbb{R}^4$ by a finite subgroup of the group $\mathrm{Sp}(\mathbb{R}^4)$ of linear symplectomorphisms of $\mathbb{R}^4 = T^*\mathbb{R}^2$. So, roughly speaking, the simplicialness of the representation turns out to be merely a necessary condition for the existence of a graded regular symplectomorphism with a quotient by a finite group.

The reader might have noticed that our results should be taken with a grain of salt. Namely, for our counterexamples we cannot disprove the existence of a symplectomorphism (see Definition 4) using the methods presented here, as the invariants we compute to distinguish them from quotients by finite groups are merely invariant under graded regular symplectomorphism. More precisely, what we actually do is to focus on the case of real dimension 4 and work through the list of finite subgroups of the unitary group U_2. The Hilbert series of the ring of real polynomial invariants of these finite subgroups are in principle computable by Molien's formula, and we argue that the Hilbert series of the ring of regular functions on our symplectic circle quotients cannot occur in this list. This method is admittedly brute force, but it has the potential to guide us to a classification of unitary symplectic circle representations whose symplectic quotients are graded regularly symplectomorphic to quotients of unitary representations of finite groups. We aim to complete this classification in the near future. In higher dimensions, a more intelligent approach is necessary.

Regular and graded regular symplectomorphism of singular phase spaces are roughly speaking those that can be obtained using complete sets of differentiable invariants. In all practical applications, these are provided by the theorem of Schwarz–Mather [20, 26] (see Theorem 1). Though this construction principle for symplectomorphisms might look familiar to the specialist, we propose the terminology in Section 4 to provide a clear way of thinking about maps between singular phase spaces. We expect that this language will have applications elsewhere.

2 Basic setup

2.1 Background from representation theory

Here we recall some well-known facts about quotients of linear actions of compact groups and their relationship to certain GIT-quotients. For a more systematic presentation we refer to G.W. Schwarz' article [27].

Let $G \to \mathrm{Gl}(W)$ be a representation of a compact Lie group on a finite-dimensional real vector space W. By a theorem of Hilbert and Hurwitz, there is a complete system of real homogeneous polynomial invariants $\rho_1, \ldots, \rho_k$ in $\mathbb{R}[W]^G$; one can assume that the system is minimal. This system, which we will refer to as a *Hilbert basis*, gives rise to a map $\rho = (\rho_1, \ldots, \rho_k)\colon W \to \mathbb{R}^k$, the corresponding *Hilbert map*. It is known that ρ is proper and separates G-orbits. The induced map $\overline{\rho}\colon W/G \to \mathbb{R}^k$ will be referred to as the *Hilbert embedding*. By the Tarski–Seidenberg principle $X := \mathrm{im}(\rho) \subset \mathbb{R}^k$ is a semialgebraic set. The gradients of the ρ_i can be used to calculate the inequalities that determine X (cf. [27, § 6]). The Zariski closure $\overline{X}$ of X is determined by the polynomial relations among the ρ_i's. By definition, a function f on X is *smooth* if it is the restriction $f = F_{|X}$ to X of a smooth function $F \in \mathcal{C}^\infty(\mathbb{R}^k)$. The algebra $\mathcal{C}^\infty(X)$ of smooth functions on X is a nuclear Fréchet algebra (see, e.g., [25]).

A key result for the analytic study of such an orbit space W/G is the theorem of Schwarz and Mather [20, 26] on differentiable invariants.

Theorem 1 (G.W. Schwarz, J. Mather)**.** *With the notation above the pullback $\rho^*\colon \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(W)^G$, $f \mapsto f \circ \rho$ is split surjective onto the Fréchet algebra $\mathcal{C}^\infty(W)^G$ of smooth invariants on W.*

In [20, 26], the authors use Theorem 1 to prove the existence of a complete set of differentiable invariants for a G-manifold using Mostov's embedding theorem, i.e. a generating set for the algebra of smooth G-invariant functions. In the case of a G-representation, a complete set of differentiable invariants is given by a Hilbert basis. Using the language of Section 4, this theorem implies that the Hilbert embedding $\overline{\rho}$ is actually a diffeomorphism from the differential space $(W/G, \mathcal{C}^\infty(W)^G)$ onto the differential space $(X, \mathcal{C}^\infty(X))$.

Now suppose $G \to \mathrm{U}(V)$ is a unitary representation of the compact Lie group G on a finite-dimensional complex vector space V with hermitian scalar product $\langle\ ,\ \rangle$. By convention, $\langle\ ,\ \rangle$ is complex antilinear in the first argument. Note that we can make any symplectic representation of G unitary by using an invariant compatible complex structure. In order to express equation (2.2) transparently, it will be convenient to express real polynomials using complex coordinates. Let $\overline{V}$ be the complex conjugate vector space of V, and then the identity map on V induces a complex antilinear map $^-\colon V \to \overline{V}$, $v \mapsto \overline{v}$. The complex conjugation $^-$ extends to a real structure on the algebra $\mathbb{C}[V \times \overline{V}]$, and the ring of *real regular functions on* V is defined to be the subring of invariants with respect to $^-$, i.e. $\mathbb{R}[V] := \mathbb{C}[V \times \overline{V}]^-$. It is of course isomorphic to the ring of regular functions on the real vector space $V_\mathbb{R}$ underlying V.

The group G acts on $\overline{V}$ by $\overline{v} \mapsto (g^{-1})^t\overline{v}$. Letting G act on $V \times \overline{V}$ diagonally, and observing that this action commutes with $^-$, we obtain an action of G on $\mathbb{R}[V]$ by $\mathbb{R}$-algebra automorphisms. This action can be seen as coming from the obvious $\mathbb{R}$-linear G-action on $V_\mathbb{R}$. Hence $\mathbb{R}[V]^G$ is a $\mathbb{Z}$-graded Noetherian $\mathbb{R}$-algebra, we can find a Hilbert basis $\rho_1, \ldots, \rho_k \in \mathbb{R}[V]^G$ and Theorem 1 applies. Note that $v \mapsto \langle v, v\rangle$ is always a quadratic invariant.

It is well-known that the unitary action of G on V extends uniquely to a $\mathbb{C}$-linear action of the complexification $G_\mathbb{C}$ of G on V. Note also that the complexification of the G-action on $V \times \overline{V}$ turns out to be the cotangent lifted $G_\mathbb{C}$-action on $V \times V^*$. Moreover, we have the following isomorphism of invariant rings

$$\mathbb{R}[V]^G \otimes_\mathbb{R} \mathbb{C} \cong \mathbb{C}[V \times V^*]^{G_\mathbb{C}} \tag{2.1}$$

as $\mathbb{Z}$-graded $\mathbb{C}$-algebras.

The (infinitesimal) information of the unitary representations $G \to \mathrm{U}(V)$ can be encoded into the moment map J. This is the regular quadratic map

$$J\colon\ V \to \mathfrak{g}^*, \qquad J_\xi(v) = \langle J(v), \xi\rangle := \frac{\sqrt{-1}}{2}\langle v, \xi v\rangle \tag{2.2}$$

for $\xi \in \mathfrak{g}$. Alternatively, we can think of J as a linear map $\mathfrak{g} \to \mathbb{R}[V]$. Often it is convenient to write $J_i := J_{e_i}$ for some fixes basis $e_1, \dots, e_\ell$ of $\mathfrak{g}$. The moment map is of particular importance when it comes to discussing the symplectic geometry of our unitary representation. Let us, for convenience, identify V with $\mathbb{C}^n$ by choosing an orthonormal basis and denote the corresponding coordinates by $(\boldsymbol{z}, \overline{\boldsymbol{z}}) = (z_1, \dots, z_n, \overline{z}_1, \dots, \overline{z}_n)$. It follows that $\mathbb{R}[V]$ is identified with $\mathbb{R}[\mathbb{C}^n] = \mathbb{C}[\boldsymbol{z}, \overline{\boldsymbol{z}}]^-$. The Poisson bracket corresponding to the symplectic form $\omega \in \Omega^2(V)$, $\omega(v, w) = \operatorname{Im}\langle v, w\rangle$, is given by the relation

$$\{z_i, \overline{z}_j\} = \frac{2}{\sqrt{-1}}\delta_{i,j},$$

all other brackets between coordinates being zero. This makes $\mathcal{C}^\infty(\mathbb{C}^n)$ into a Poisson algebra with Poisson subalgebra $\mathbb{R}[\mathbb{C}^n]$. It turns out that $\{J_\xi, J_\eta\} = J_{[\xi,\eta]}$, which is equivalent to the equivariance of the map $J\colon V \to \mathfrak{g}^*$.

In the situation of a unitary representation the zero fibre $Z = J^{-1}(0)$ of the moment map always has a conical singularity at 0. The symplectic quotient $M_0 = Z/G$ is a stratified symplectic space (this will be further explained in Section 4). In general, it is not a real variety but a semialgebraic set. In contrast, the GIT quotient $V/\!\!/G_{\mathbb{C}}$ is defined to be the complex variety underlying the $\mathbb{C}$-algebra $\mathbb{C}[V]^{G_{\mathbb{C}}}$. It might happen that $V/\!\!/G_{\mathbb{C}}$ is actually smooth (cf. Section 5). Due to the following theorem of Kempf and Ness (see [27, Corollary 4.7]), $Z = J^{-1}(0)$ is sometimes called the *Kempf–Ness set*.

Theorem 2 (G. Kempf and L. Ness)**.** *The map $Z \hookrightarrow V \mapsto V/\!\!/G_{\mathbb{C}}$ is proper and induces a homeomorphism $Z/G \to V/\!\!/G_{\mathbb{C}}$.*

In view of equation (2.1), the Kempf–Ness theorem actually comes as a surprise, as the invariant theory of a cotangent lifted representation is more involved than that of the original representation. The theorem is a useful tool to count dimensions of symplectic quotients. The aim of the paper is to give examples where $V/\!\!/G_{\mathbb{C}}$ is smooth while Z/G is not an orbifold in an appropriate sense.

2.2 Background from toric geometry

Next we would like to specialize the discussion to the case when our compact group G is actually an ℓ-dimensional torus. By this we mean an ℓ-fold copy $\mathbb{T}^\ell := (\mathbb{S}^1)^\ell$ of the unit sphere $\mathbb{S}^1 \subset \mathbb{C}$. We are interested in unitary representations

$$G = \mathbb{T}^\ell \to \mathrm{U}_n := \mathrm{U}(\mathbb{C}^n),$$

where $\mathbb{C}^n$ is understood with its standard hermitian scalar product as in the previous section. We identify the Lie algebra $\mathfrak{g}$ of $G = \mathbb{T}^\ell$ with $\mathbb{R}^\ell$ by writing an arbitrary element $(t_1, \dots, t_\ell) \in G = \mathbb{T}^\ell$ in the form $t_i = \exp(2\pi\sqrt{-1}\xi_i)$, for the vector $(\xi_1, \dots, \xi_\ell) \in \mathfrak{g} = \mathbb{R}^n$. Since the factors $\mathbb{S}^1$ of our torus action can be simultaneously diagonalized, the unitary representation can actually be encoded into a *weight matrix* $A = (a_{ij}) \in \mathbb{Z}^{\ell\times n}$. More specifically, setting $(\eta_1, \dots, \eta_n) := (\xi_1, \dots, \xi_\ell) \cdot A \in \mathbb{R}^n$, the $G = \mathbb{T}^\ell$-action corresponding to the weight matrix A is given by the formula

$$(t_1, \dots, t_\ell).(z_1, \dots, z_n) = \big(\exp\big(2\pi\sqrt{-1}\eta_1\big)z_1, \dots, \exp\big(2\pi\sqrt{-1}\eta_n\big)z_n\big).$$

Elementary row operations with integer scalars for A, i.e. row operations that correspond to left multiplication by elements of $\mathrm{GL}_\ell(\mathbb{Z})$, correspond to the changing of a basis of $\mathfrak{g}$, while permutations of the columns of A correspond to changing coordinates for $\mathbb{C}^n$.

The components $J_i := J_{e_i} = \langle J, e_i\rangle$ of the moment map $J\colon \mathbb{C}^n \to \mathbb{R}^\ell \cong \mathfrak{g}^*$ can also be expressed in terms of the weight matrix

$$J_i(\boldsymbol{z}, \overline{\boldsymbol{z}}) = \frac{1}{2}\sum_{j=1}^{n} a_{ij} z_j \overline{z}_j, \qquad i = 1, \dots, \ell.$$

Note 1. Sometimes it will be convenient to emphasize the dependency on A in the notation. In these cases we will write J_A for the moment map, $Z_A = J_A^{-1}(0)$ for the zero fiber, and $M_A = Z_A/\mathbb{T}^\ell$ for the reduced space. We will also let $X_A = Z_A \cap \mathbb{S}^{2n-1}$ denote the intersection of the zero fiber with the unit sphere in $\mathbb{C}^n$ and $Y_A = X_A/\mathbb{T}^\ell$ the link. Note that X_A is clearly $\mathbb{T}^\ell$-invariant.

The case of toric moment maps has certain peculiarities; for example, the components of toric moment maps are actually invariants. We will have to say more about this in Subsection 4.3.

Let us introduce some further notation. We denote by $\mathrm{sq}\colon \mathbb{C}^n \to \mathbb{R}^n$ the map $(z_1, \dots, z_n) \mapsto (z_1\overline{z}_1, \dots, z_n\overline{z}_n)$. It is clear that sq is actually $G = \mathbb{T}^\ell$-invariant and hence induces a map $\widetilde{\mathrm{sq}}\colon \mathbb{C}^n/\mathbb{T}^\ell \to \mathbb{R}^n$.

We will primarily be interested in the case where the action of $\mathbb{T}^\ell$ on $\mathbb{C}^n$ is effective, i.e. if for some $t \in \mathbb{T}^\ell$ we have $tz = z$ for all $z \in \mathbb{C}^n$, then $t = 1$. We will see below (Lemma 2) that this introduces no loss of generality. In order to do so, we first interpret this condition in terms of the weight matrix A.

It is easy to see that there is a subgroup $K \leq \mathbb{T}^\ell$ of positive dimension that acts trivially on $\mathbb{C}^n$ if and only if $\operatorname{rank}(A) < \ell$. In particular, choosing a basis for $\mathfrak{g}$ that contains an element of the Lie algebra $\mathfrak{k}$ of K, it is easy to see that the corresponding row of A is the zero row. Hence, A has full rank if and only if the subgroup of $\mathbb{T}^\ell$ that acts trivially on $\mathbb{C}^n$ is finite. In this case, we have the following lemma; we include the proof since we do not know of an appropriate reference.

Lemma 1. *Suppose $A \in \mathbb{Z}^{\ell\times n}$ has full rank $\ell \leq n$. Then the action of $\mathbb{T}^\ell$ on $\mathbb{C}^n$ is effective if and only if the nonzero $\ell \times \ell$-minors of A are relatively prime. Moreover, if p is a prime that divides each of the $\ell \times \ell$-minors of A, by elimination with integer scalars and permuting columns, A can be expressed in a form where each entry of its first row is divisible by p.*

Proof. Suppose $t = \big(\exp(2\pi\sqrt{-1}\xi_1), \dots, \exp(2\pi\sqrt{-1}\xi_\ell)\big) \in \mathbb{T}^\ell$ is nontrivial and acts trivially on $\mathbb{C}^n$. As A has full rank, t must be of finite order. Thus there is a $j \in \{1, \dots, \ell\}$ such that $\xi_j = k/q$ for some coprime integers k, q with $q \geq 2$. By assumption, $(\xi_1, \dots, \xi_\ell)A \in \mathbb{Z}^n$. Let B be a nonsingular $\ell \times \ell$-submatrix of A. Since $(\xi_1, \dots, \xi_\ell)B \in \mathbb{Z}^\ell$, we can use Cramer's rule to conclude that $q \mid \det(B)$.

Conversely, let $\gcd_\ell(A)$ denote the gcd of the $\ell \times \ell$-minors of A. We prove by induction on $n-\ell$ that if p is a prime that divides $\gcd_\ell(A)$, then A can be row-reduced with integer scalars and the coordinates of $\mathbb{C}^n$ can be permuted so that $(\exp(2\pi\sqrt{-1}/p), 0, \dots, 0)$ acts trivially on $\mathbb{C}^n$, i.e. $(1/p, 0, \dots, 0)A \in \mathbb{Z}^n$.

Let $A \in \mathbb{Z}^{\ell\times n}$ and let p be a prime such that $p | \gcd_\ell(A)$. Assume the result holds for all $\ell \times (\ell + k)$-weight matrices with $k < n - \ell$. By row operations and permutations of coordinates, we can assume that $A = [D \mid C]$ where $D = \operatorname{diag}(d_1, \dots, d_\ell) \in \mathbb{Z}^{\ell\times\ell}$ and $C \in \mathbb{Z}^{\ell\times(n-\ell)}$. Then $p|\det(D)$ so that by further permuting coordinates, we can assume that $p|d_1$. If $n - \ell = 0$ it follows that $(1/p, 0, \dots, 0)A \in \mathbb{Z}^n$.

Otherwise, let A' denote the matrix formed by removing the first column of A. Consider the case when A' does not have full rank. This means in particular that each $\ell \times \ell$-submatrix of A' corresponding to the columns $2, 3, \dots, \ell, \ell + j$ of A for $1 \leq j \leq n - \ell$ is singular. It follows that the first row of A' is the zero row, which implies $(1/p, 0, \dots, 0)A \in \mathbb{Z}^n$.

On the other hand, suppose A' has full rank. Then by the inductive hypothesis, we can row-reduce A' with integer scalars and permute the columns of A' to yield a matrix R' such that that $(1/p, 0, \ldots, 0)R' \in \mathbb{Z}^{n-1}$. If we apply the same row-reduction to A, however, and permute columns $2, 3, \ldots, n$ in the same way, it is easy to see that the resulting matrix R is of the form $[\boldsymbol{r} \mid R']$ where $R' \in \mathbb{Z}^{\ell\times(n-1)}$ and $\boldsymbol{r}$ is a column with each entry divisible by p. It follows that $(1/p, 0, \ldots, 0)R \in \mathbb{Z}^n$, completing the proof. ■

Now, suppose the action of $\mathbb{T}^\ell$ on $\mathbb{C}^n$ is not effective, and let $K \leq \mathbb{T}^\ell$ denote the subgroup that acts trivially. Then $\mathbb{T}^\ell$ fibers over $\mathbb{T}^\ell/K$, which is itself a torus, and we may consider the Hamiltonian action of $\mathbb{T}^\ell/K$ on $\mathbb{C}^n$. If K is infinite and connected, then $\mathbb{T}^\ell/K$ is a torus of dimension smaller than ℓ. The row-reduced weight matrix A has zero rows, and the moment maps of the $\mathbb{T}^\ell$- and $\mathbb{T}^\ell/K$-actions differ only by extending by zero. If K is finite, then the moment maps of the two actions coincide up to an isomorphism between the Lie algebra of $\mathbb{T}^\ell/K$ with that of $\mathbb{T}^\ell$. Combining these two arguments for an arbitrary K yields the following.

Lemma 2. *Let A' denote the weight matrix of the $\mathbb{T}^\ell/K$-action on $\mathbb{C}^n$. Then $J_A^{-1}(0) = J_{A'}^{-1}(0)$.*

As a consequence, if the action of $\mathbb{T}^\ell$ is not effective, then we may replace $\mathbb{T}^\ell$ with $\mathbb{T}^\ell/K$ without changing the reduced space. Hence, in the sequel, we assume without loss of generality that $\mathbb{T}^\ell$ acts effectively on $\mathbb{C}^n$, and in particular that $\ell \leq n$.

Now, let $\mathbb{T}^\ell_\mathbb{C}$ denote the complexification of $\mathbb{T}^\ell$. Then $\mathbb{T}^\ell_\mathbb{C}$ acts on $\mathbb{C}^n$ via

$$(w_1, \ldots, w_\ell)(z_1, \ldots, z_n) = \big(w_1^{a_{11}} w_2^{a_{21}} \cdots w_\ell^{a_{\ell 1}} z_1, \ldots, w_1^{a_{1n}} w_2^{a_{2n}} \cdots w_\ell^{a_{\ell n}} z_n\big),$$

and this action induces an injective homomorphism $\mathbb{T}^\ell_\mathbb{C} \to \mathbb{T}^n_\mathbb{C}$. Then the GIT quotient $\mathbb{C}^n/\!\!/\mathbb{T}^\ell_\mathbb{C}$ is equipped with an effective action of $\mathbb{T}^n_\mathbb{C}/\mathbb{T}^\ell_\mathbb{C} \cong \mathbb{T}^{n-\ell}_\mathbb{C}$ with a single, dense orbit and hence has the structure of an $(n-\ell)$-dimensional toric variety $\mathcal{X}$, see e.g. [4] or [12]. In particular, $\mathbb{C}^n/\!\!/\mathbb{T}^\ell_\mathbb{C}$ is the affine toric variety given by the spectrum of the semigroup $\ker(A) \cap \mathbb{Z}^n_{\geq 0}$ and hence is associated to the cone given by the kernel of A intersected with the positive n-ant in $\mathbb{R}^n$.

Definition 1. The cone σ_A associated to the weight matrix A is the intersection of the kernel of A with the positive n-ant in $\mathbb{R}^n$.

Recall that the cone σ_A is *simplicial* if it is generated by a collection of linearly independent vectors. It is well-known, see e.g. [12, Section 2.2], that if σ_A is simplicial, then the affine toric variety associated to σ_A is a complex orbifold. In particular, applying the Cox construction, see [4, Chapter 5], we have that that $\mathcal{X} = \mathbb{C}^{n-\ell}/\Gamma$ for a finite group Γ as follows.

We have the short exact sequence [4, Theorem 4.1.3]

$$0 \longrightarrow M \longrightarrow \mathrm{Div}_{\mathbb{T}^{n-\ell}_\mathbb{C}}(\mathcal{X}) \longrightarrow \mathrm{Cl}(\mathcal{X}) \longrightarrow 0,$$

where M denotes the character lattice of the algebraic torus $\mathbb{T}^{n-\ell}_\mathbb{C}$, $\mathrm{Div}_{\mathbb{T}^{n-\ell}_\mathbb{C}}(\mathcal{X})$ denotes the group of $\mathbb{T}^{n-\ell}_\mathbb{C}$-invariant Weil divisors of $\mathcal{X}$, and $\mathrm{Cl}(\mathcal{X})$ denotes the class group of $\mathcal{X}$. Choosing bases, this sequence can be expressed as

$$0 \longrightarrow \mathbb{Z}^{n-\ell} \xrightarrow{(*)} \mathbb{Z}^{n-\ell} \longrightarrow \mathrm{Cl}(\mathcal{X}) \longrightarrow 0,$$

where the map $(*)$ is given by the matrix whose rows are the coordinates of the $n-\ell$ linearly independent minimal generators of the cone σ_A and hence has maximal rank. In particular, $\mathrm{Cl}(\mathcal{X})$ is finite. Applying $\mathrm{Hom}_\mathbb{Z}(\cdot, \mathbb{T}^1_\mathbb{C})$ and setting $\Gamma = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(\mathcal{X}), \mathbb{T}^1_\mathbb{C})$ yields the exact sequence

$$1 \longrightarrow \Gamma \longrightarrow \mathbb{T}^{n-\ell}_\mathbb{C} \xrightarrow{(*)^T} \mathbb{T}^{n-\ell}_\mathbb{C} \longrightarrow 1,$$

defining an action of Γ on $\mathbb{C}^{n-\ell}$. Hence, as σ_A consists of a single cone so that the exceptional set is empty, the toric variety $\mathcal{X}$ is given by the complex orbifold $\mathbb{C}^{n-\ell}/\Gamma$.

In particular, if $n-\ell=1$, it is easy to see that the cone σ_A is simply $\mathbb{R}_{\geq 0}$ with minimal generator 1. Therefore, the map $\mathbb{Z} \xrightarrow{(*)} \mathbb{Z}$ above is simply the identity, and $\mathrm{Cl}(\mathcal{X})$ and Γ are both trivial. It follows that $\mathcal{X}=\mathbb{C}$.

For any complex orbifold Q, each local group action preserves the complex structure. It follows that Q is a *locally orientable* orbifold, i.e. each local group action preserves a local orientation. By [15, 4.2.4], the underlying topological space of a locally orientable orbifold of (real) dimension m is an *m-dimensional rational homology manifold*. That is, if $\mathbb{X}_Q$ denotes the underlying space of Q, then the local homology groups with rational coefficients $H_k(\mathbb{X}_Q, \mathbb{X}_Q - x; \mathbb{Q})$ at each point $x \in \mathbb{X}_Q$ satisfy

$$H_k(\mathbb{X}_Q, \mathbb{X}_Q - x; \mathbb{Q}) = \begin{cases} \mathbb{Q}, & k = m, \\ 0, & k \neq m. \end{cases}$$

If, on the other hand, σ_A is not simplicial, then the recursion formula given in [1, p. 2] for the local intersection cohomology Betti numbers in terms of the cone generators indicates that the second local intersection cohomology is nontrivial. Because the local intersection cohomology of a rational homology manifold is trivial, it follows that the toric variety $\mathcal{X}$ associated to σ_A is not a rational homology manifold. With this, applying the Kempf–Ness homeomorphism between $M_A = Z_A/\mathbb{T}^\ell$ and $\mathcal{X}$, we have the following.

Theorem 3. *Using the notation of Note 1 and Definition 1, the reduced space $M_A = Z_A/\mathbb{T}^\ell$ associated to $A \in \mathbb{Z}^{\ell\times n}$ is a rational homology manifold if and only if the cone σ_A is simplicial.*

In particular, note that symplectic orbifolds are locally orientable and hence rational homology manifolds. Therefore, if the cone σ_A is not simplicial, then the topological space M_A does not admit a homeomorphism to a symplectic orbifold.

In the sequel, it will be convenient to use the following terminology.

Definition 2. We say that the weight matrix $A \in \mathbb{Z}^{\ell\times n}$ is *simplicial* if the corresponding cone σ_A is simplicial. In this case we also say that the corresponding unitary $\mathbb{T}^\ell$-action and its complexified $\mathbb{T}^\ell_{\mathbb{C}}$-action are *simplicial*.

2.3 Other topological indications

In many examples of non-simplicial weight matrices A, it is possible to demonstrate that the reduced space is not homeomorphic to a symplectic orbifold directly without appealing to the Kempf–Ness homeomorphism. In this subsection, we briefly indicate results in this direction.

In [6, Example 2.4], the reduced space corresponding to the weight matrix $[-1,-1,1,1]$ was described as the cone on $\mathbb{S}^3 \times_{\mathbb{S}^1} \mathbb{S}^3$, implying that the local homology in degree 3 at the cone point is nontrivial. It follows that the reduced space is not a rational homology manifold and hence not an orbifold.

By [16, Proposition 3.1], the quotient of an n-dimensional sphere by a finite group acting linearly and preserving orientation is a rational homology n-sphere, i.e. has the homology with rational coefficients of the n-dimensional sphere $\mathbb{S}^n$. It follows that the link $Y_A = X_A/\mathbb{T}^\ell$, see Note 1, of a locally orientable n-dimensional orbifold singularity is a rational homology n-sphere. In [17], this observation was used to show that the reduced space $M_A = Z_A/\mathbb{T}^\ell$ cannot be an orbifold if the link Y_A is not a rational homology sphere. In particular, in the case $\ell = 1$, [17, Proposition 3.1] demonstrates that Y_A is not a rational homology sphere if the weight matrix A has at least two positive and two negative entries; this condition is clearly equivalent to the negation of Theorem 4(2) below in this case.

Similarly, in cases where X_A consist of points of a single orbit type, the quotient map $X_A \to Y_A$ is a torus fibration with fiber given by the quotient of $\mathbb{T}^\ell$ by the isotropy group of X_A. In this case, formulas for the homology of X_A have been developed in [2], and in some cases, the exact sequence [30, Theorem 2, p. 482] can be used to demonstrate that X_A does not admit such a torus fibration over a rational homology sphere of the appropriate dimension.

More generally, note that this argument can be applied to the closed orbit-type strata of the link Y_A to show that the reduced space does not admit a stratum-preserving homeomorphism to an orbifold. To see this, suppose G is a finite group acting on a sphere $\mathbb{S}^n$. For each $H \leq G$, we let $\mathbb{S}^n_H$ denote the set of points with isotropy group H and $\mathbb{S}^n_{(H)}$ the set of points with orbit type (H). Then H acts trivially on $\mathbb{S}^n_H$, $N_G(H)/H$ acts freely on $\mathbb{S}^n_H$, and $\mathbb{S}^n_{(H)}/G$ is diffeomorphic to $\mathbb{S}^n_H/N_G(H)$; see [25, Theorem 4.3.10 and Corollary 4.3.11]. If $\mathbb{S}^n_{(H)}$ has minimal dimension among the strata, then $\mathbb{S}^n_H = (\mathbb{S}^n)^H$, and hence $\mathbb{S}^n_{(H)}/G$ is diffeomorphic to the quotient of a sphere by the free action of a finite group. Similarly, if $\mathbb{S}^n_H$ is closed, then it is locally a stratum of minimal dimension, and we can draw the same conclusion. It follows that the closed orbit-type strata of the link of an orbifold singularity are as well rational homology spheres, so that this must also be true for a reduced space that admits a stratum-preserving homeomorphism to an orbifold.

We illustrate these observations with the following.

Example 1. Consider the case of $\mathbb{T}^2$ acting on $\mathbb{C}^6$ with weight matrix

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}.$$

Then Z_A is described by

$$|z_1|^2 + |z_3|^2 = |z_2|^2 + |z_4|^2, \qquad |z_5|^2 = |z_6|^2.$$

The isotropy types away from the origin are given by $(z_1, z_2, z_3, z_4, 0, 0)$ with isotropy $1 \times \mathbb{T}^1$, $(0, \dots, 0, z_5, z_6)$ with isotropy $\mathbb{T}^1 \times 1$, and $(z_1, \dots, z_6)$ with trivial isotropy. If $z_5 = z_6 = 0$, then the intersection with the unit sphere is $|z_1|^2 + |z_2|^2 = |z_3|^2 + |z_4|^2 = 1/2$, and the corresponding orbit-type stratum is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^3/\mathbb{T}^1$. Using [30, Theorem 2, p. 482], it is an easy exercise to show that $\mathbb{S}^3 \times \mathbb{S}^3$ does not admit a $\mathbb{T}^1$-fibration over a rational homology 5-sphere, and hence that a closed stratum of $Y_A = X_A/\mathbb{T}^2$ is not a rational homology 5-sphere. It follows that $M_A = Z_A/\mathbb{T}^2$ does not admit a stratum-preserving homeomorphism with an orbifold.

3 Gaussian elimination and the simplicial condition

In this section, we will use Theorem 3 to determine necessary and sufficient conditions for the reduced space $M_A = Z_A/\mathbb{T}^\ell$ to be a rational homology manifold directly in terms of the matrix A. Given a subset X of $\mathbb{R}^n$ we write $\operatorname{aff}(X)$ for its affine hull and $\operatorname{cch}(X)$ for its closed convex hull. By X° we mean its relative interior, i.e., the interior of X seen as a subspace of $\operatorname{aff}(X)$. We also use the shorthand X^c for the complement $\mathbb{R}^n \backslash X$.

Let $A \in \mathbb{Z}^{\ell \times n}$ with $\ell \leq n$. Let Δ^{n-1} denote the standard simplex in $\mathbb{R}^n$, and let $\mathcal{P}_A := \ker(A) \cap \Delta^{n-1} \subset \mathbb{R}^n$ denote the intersection of the kernel of A in $\mathbb{R}^n$ with the standard simplex Δ^{n-1}. Then the cone σ_A defined in Definition 1 is spanned by $\mathcal{P}_A$. Note that if $\mathcal{P}_A \neq \varnothing$, then $\mathcal{P}_A$ is a polytope by [3, Corollary 9.4]. Each element of $\mathcal{P}_A$ is a convex combination of its vertices by definition, so that the vertices of $\mathcal{P}_A$ clearly span the linear space spanned by $\mathcal{P}_A$. It follows that if $\mathcal{P}_A$ has dimension m, then the vertices are linearly independent if and only if there are exactly $m + 1$ vertices. This is the case if and only if $\mathcal{P}_A$ is combinatorially equivalent to a standard simplex, see [3, Chapter 2, § 10], so that the matrix A is simplicial if and only if $\mathcal{P}_A$ is combinatorially equivalent to a simplex.

In examples, the most direct method of determining whether A is simplicial is to compute the vertices of $\mathcal{P}_A$ using the results of Lemma 3 below. However, in the sequel, we will need to use a standard row-reduced form of a simplicial weight matrix A, and hence we will reformulate the simplicial condition (cf. Definition 2) in these terms in Theorem 4. In addition, we give a geometric formulation to aid in the reader's intuition.

We use $e_1, \dots, e_n$ to denote the standard basis vectors of $\mathbb{R}^n$ so that $\Delta^{n-1} = \operatorname{cch}(e_1, \dots, e_n)$ is the closed convex hull of the set of standard basis vectors. If $I \subset \{1, \dots, n\}$ is a subset of indices, we let

$$V_I = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j = 0 \;\forall\, j \in \{1, \dots, n\} \setminus I\big\}$$

denote the coordinate subspace associated to I. Recall that $X_A = Z_A \cap \mathbb{S}^{2n-1}$ denotes the intersection of the zero fiber Z_A with the unit sphere in $\mathbb{C}^n$ and $Y_A = X_A/\mathbb{T}^\ell$ denotes the link, see Note 1. Then we have that $\operatorname{sq}(X_A) = \widetilde{\operatorname{sq}}(Y_A) = \mathcal{P}_A$, where sq and $\widetilde{\operatorname{sq}}$ are the maps defined in Subsection 2.2. As well, note that the combinatorial type of $\mathcal{P}_A$ is clearly invariant under row reduction and permuting the columns of A.

In general, it may happen that $\mathcal{P}_A$ is contained in a coordinate subspace of $\mathbb{R}^n$ and hence a proper face of Δ^{n-1}. To address this possibility, let

$$I_A = \big\{j \in \{1, \dots, n\} \,|\, \exists\, (x_1, \dots, x_n) \in \mathcal{P}_A : x_j \neq 0\big\}$$

denote the set of coordinates x_j that are not identically 0 on $\mathcal{P}_A$. Equivalently, I_A is the set of indices j such that there is an element of $\ker(A)$ with non-negative entries and positive jth entry. Let A' denote the $\ell \times |I_A|$ submatrix of A given by the columns corresponding to elements of I_A. Let V_{I_A} denote the coordinate subspace of $\mathbb{R}^n$ associated to I_A, i.e., the subspace $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_j = 0 \;\forall\, j \notin I_A\}$. In examples, I_A can be determined by computing the vertices of $\mathcal{P}_A$. Let $r \leq \ell$ denote the rank of A. We will establish the following criteria for the a simplicial weight matrix. Note that $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

Theorem 4. *Let A be an $n \times \ell$ weight matrix. The following are equivalent.*

1. *The polytope $\mathcal{P}_A$ is combinatorially equivalent to a simplex.*
2. *By permuting the indices in I_A and performing elementary row operations with integer scalar multiples, the matrix A' can be expressed in the form*

$$\left[\begin{array}{c|cc} D & C & 0 \\ 0 & 0 & 0 \end{array}\right],$$

   where D is an $r \times r$ diagonal matrix with strictly negative entries on the diagonal and C is an $r \times q$ matrix such that each entry is nonnegative and $q \leq |I_A| - r$.
3. *There are vectors $\mu_1, \dots, \mu_r \in V_{I_A}$ and indices $j_1, \dots, j_r \in I_A$ such that* $\ker(A') = V_{I_A} \cap \big(\bigcap_{i=1}^r \mu_i^\perp\big)$*, where $\mu_i^\perp$ denotes the orthogonal complement in $\mathbb{R}^n$, and for each $i = 1, \dots, r$, $\langle e_{j_i}, \mu_i \rangle < 0$, $\langle e_{j_k}, \mu_i \rangle = 0$ for $k \neq i$, and $\langle e_j, \mu_i \rangle \geq 0$ for $j \in I_A$, $j \neq j_i$.*

These conditions are trivially satisfied if $n \leq r + 2$.

Note that in condition (2) of Theorem 4, by construction of the index set I_A, the matrix C cannot have rows that are identically zero.

Condition (3) of Theorem 4 can be understood as follows. For each $i = 1, \dots, r$, let $H_i = \mu_i^\perp \cap V_{I_A}$ denote the orthogonal complement $\mu_i^\perp$ in V_{I_A}. Then condition (3) states that the hyperplane H_i separates one vertex of the standard simplex in V_{I_A} from the others, and moreover that each hyperplane H_i contains all of the separated basis vectors e_{j_k} for $k \neq i$. This condition lends some intuition for the geometric meaning of simplicial condition (2).

In order to establish Theorem 4, we will first restrict to the case of weight matrices satisfying the following hypotheses to simplify the arguments.

(i) The polytope $\mathcal{P}_A$ has nonempty intersection with the relative interior of the standard simplex Δ^{n-1}.

(ii) The matrix A has full rank ℓ.

(iii) The matrix A has no columns that are identically zero.

Note that as each point in $(\Delta^{n-1})^\circ$ has nonzero x_i-coordinate for each i, hypothesis (i) is equivalent to $I_A = \{1, \ldots, n\}$. Similarly, (ii) implies that $\ker(A)$ has dimension $n-\ell$; equivalently, no positive-dimensional subgroups of $\mathbb{T}^\ell$ act trivially on $\mathbb{C}^n$. Hypothesis (iii) implies that there are no coordinate lines in $\mathbb{C}^n$ on which $\mathbb{T}^\ell$ acts trivially. Assuming (i), (ii), and (iii), it is easy to see that the relative interior of $\mathcal{P}_A$ is an open subset of the affine space given by the intersection of $\ker(A)$ and the affine hull of Δ^{n-1}, and hence $\mathcal{P}_A$ is a polytope of dimension $n - \ell - 1$.

Under these hypotheses, we first establish Lemma 3, demonstrating that the faces of $\mathcal{P}_A$ consist of the intersection of $\ker(A)$ with coordinate subspaces of $\mathbb{R}^n$. If $I \subset \{1, \ldots, n\}$ is a collection of indices, we again use the notation that $V_I = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_j = 0 \ \forall\, j \notin I\}$ is the associated coordinate subspace. We then show Proposition 1, which states Theorem 4 for matrices that satisfy (i), (ii), and (iii), and then proceed to the proof of Theorem 4.

Lemma 3. *Let $A \in \mathbb{Z}^{n\times\ell}$ satisfy hypotheses (i), (ii), and (iii).*

(a) *If $I \subset \{1, \ldots, n\}$ such that $\mathcal{P}_A \cap V_I = \{\nu\}$, then ν is a vertex of $\mathcal{P}_A$.*

(b) *Each face F of $\mathcal{P}_A$ is given by $F = \mathcal{P}_A \cap V_I$ for some I with $|I| = \ell + \dim(F) + 1$.*

As a special case of (b), note that each vertex of $\mathcal{P}_A$ is given by the intersection $\mathcal{P}_A \cap V_I$ where $I \subset \{1, \ldots, n\}$ is a subset of cardinality $\ell + 1$. Note that given a k-face F, the set I given by condition (b) need not be unique. If $\ker(A)$ intersects the simplex Δ^{n-1} *generically*, i.e., each of its vertices is contained in the relative interior of an ℓ-dimensional face of Δ^{n-1}, then the I corresponding to F is unique. In general, however, a face can be contained in the intersection of several $(\ell + k)$-dimensional faces. Given hypotheses (i), (ii), and (iii), however, it is easy to see that a vertex of $\mathcal{P}_A$ cannot correspond to a vertex of Δ^{n-1}; this would indicate that a standard basis vector e_j is contained in the kernel, and hence that the jth column of A is a zero column. Similarly, if I is a set of indices of cardinality $|I| = \ell + 1$, it need not be the case that $\mathcal{P}_A \cap V_I$ is a singleton.

Proof. (a) Assume $\mathcal{P}_A \cap V_I = \{\nu\}$ for $I \subset \{1, \ldots, n\}$ with $\nu = (v_1, \ldots, v_n)$. Suppose $\nu = tp + (1-t)q$ for $t \in]0, 1[$ and $p = (p_1, \ldots, p_n), q = (q_1, \ldots, q_n) \in \mathcal{P}_A$. For each $j \notin I$, we have that $tp_j + (1 - t)q_j = v_j = 0$. As $p_j, q_j \geq 0$, it follows that $p_j = q_j = 0$. Applying this argument to each $j \notin I$, it follows that $p, q \in V_I$. Hence $p, q \in \mathcal{P}_A \cap V_I$, which was assumed to be a singleton, so that $p = q = \nu$ and ν is a vertex of $\mathcal{P}_A$.

(b) We prove the statement by induction on the codimension c of the face F. The case of $c = 0$ is trivial. Let F be a face of codimension $c + 1$, so $k := \dim(F) = n - \ell - c - 2$. Note that F is contained in a face F' of codimension c. By our inductive hypothesis, we can write $F' = \mathcal{P}_A \cap V_{I'}$ for some $I' \subset \{1, \ldots, n\}$ of cardinality $|I'| = \ell + (k + 1) + 1 = n - c$. This means that

$$F' = \mathcal{P}_A \cap V_{I'} = \ker(A) \cap \Delta^{n+1} \cap V_{I'} = \ker(A) \cap \Delta^{n-c-1},$$

where Δ^{n-c-1} is the standard simplex in $V_{I'}$.

We claim that F is contained in a face of Δ^{n-c-1}. Letting $W := \operatorname{aff}(\Delta^{n-c-1}) \cap \ker(A)$ and $H_i^+ := \{(x_1, \dots, x_n) \in \mathbb{R}^n | x_i \geq 0\}$, we write

$$F' = \ker(A) \cap \operatorname{aff}\left(\Delta^{n-c-1}\right) \cap \left(\cap_{i \in I'} H_i^+\right) = W \cap \left(\cap_{i\in I'} H_i^+\right).$$

Setting $K_i^+ := W \cap H_i^+$, we have $F' = \cap_{i\in J} K_i^+$ for $J \subset I'$ chosen such that $K_i^+ \neq W$ if and only if $i \in J$. It is a well-known fact (see, e.g., [3, Theorem 8.2]) that each facet of F' is of the form $K_i \cap W$ for some $i \in J$, where $K_i := W \cap V_{\{i\}^c}$ is the supporting hyperplane of K_i^+. Since F is a facet of F', we conclude that

$$F = F' \cap K_i = F' \cap V_{\{i\}^c} \cap W. \tag{3.1}$$

Since $F \subset F' \subset W$, it follows that that $F \subset F' \cap V_{\{i\}^c}$ which proves the claim.

Moreover, equation (3.1) shows that $F = \mathcal{P}_A \cap V_I$ with $I := I' \setminus \{i\}$. ■

With this, we have the following.

Proposition 1. *Let A be an $n \times \ell$ weight matrix satisfying hypotheses (i), (ii), and (iii) so that $\mathcal{P}_A$ is an $(n-\ell-1)$-dimensional polytope. Then conditions (1), (2), and (3) of Theorem 4 are equivalent and are always satisfied if $n \leq \ell + 2$.*

Note that given the hypotheses, condition (1) is equivalent to $\mathcal{P}_A$ having $n - \ell$ vertices. Similarly, $A = A'$ has full rank and no zero columns, simplifying (2).

Proof. (1) ⇒ (2): Suppose $\mathcal{P}_A$ has $n-\ell$ vertices $\nu_1, \dots, \nu_{n-\ell}$. To establish (2), we will show that each vertex ν_j lies in an $(\ell+1)$-dimensional coordinate plane, and the intersection of these coordinate planes is an ℓ-dimensional coordinate plane. This will indicate the order of the vertices under which A takes the required form.

For each vertex ν_j, let F_j denote the $(n-\ell-2)$-dimensional facet of $\mathcal{P}_A$ that does not contain ν_j, so that $F_j = \operatorname{cch}\{\nu_k \mid k \neq j\}$. Then by Lemma 3, each F_j is given by the intersection of $\mathcal{P}_A$ with a coordinate $n-1$-plane, and hence corresponds to setting a single coordinate equal to zero. By reordering the variables $x_1, \dots, x_n$, we may assume that $F_j = \mathcal{P}_A \cap V_{\{\ell+j\}^c}$ for $j = 1, \dots, n-\ell$. Let $v_{j,i}$ indicate the coordinates of ν_j, i.e., $\nu_j = (v_{j,1}, v_{j,2}, \dots, v_{j,n})$. Note that for each j, as $V_{\{\ell+j\}^c}$ does not contain the vertex ν_j, it follows that $v_{j,\ell+j} \neq 0$.

For each j, we claim that $\cap_{k\neq j} F_k = \{\nu_j\}$. To see this, first note that $\nu_j \in F_k$ for each $k \neq j$ so that $\{\nu_j\} \subset \cap_{k\neq j} F_k$. For the reverse inclusion, suppose $p = (p_1, \dots, p_n) \in \cap_{k\neq j} F_k$. Then as $p \in \mathcal{P}_A$, we have that p is a convex combination of the $\nu_1, \dots, \nu_{n-\ell}$, say $p = \sum_{m=1}^{n-\ell} t_m \nu_m$ with $0 \leq t_m \leq 1$ and $\sum_{m=1}^{n-\ell} t_m = 1$. For each $r \neq j$, we have that $\cap_{k\neq j} F_k \subset F_r$ so that $p \in F_r$ and $p_{\ell+r} = 0$. As $v_{r,\ell+r} \neq 0$, it then follows that $t_r = 0$. Therefore, the only nonzero t_r is $t_j = 1$, and $p = \nu_j$. Letting $I_j = \{1, 2, \dots, \ell, \ell+j\}$, it follows that

$$\{\nu_j\} = \bigcap_{k\neq j} F_k = \bigcap_{k\neq j} \left(\mathcal{P}_A \cap V_{\{\ell+k\}^c}\right) = \mathcal{P}_A \cap \bigcap_{k\neq j} V_{\{\ell+k\}^c} = \mathcal{P}_A \cap V_{I_j}.$$

Let $[D \mid C]$ denote the weight matrix A row-reduced using integer scalar multiples, where D is $\ell \times \ell$ and C is $\ell \times (n-\ell)$. We let $c_{k,j}$ denote the entries of c as usual, with $1 \leq k \leq \ell$ and $1 \leq j \leq n-\ell$. As A has full rank, it must be that $[D \mid C]$ has full rank as well. We claim that D is diagonal and nonsingular.

Suppose not, and then one of the pivot columns must be contained in C so that the last row of D is the zero row. For each j, as $v_{j,\ell+k} = 0$ for $k \neq j$, it follows that the nth entry of $[D \mid C]\nu_j$ is given by $c_{\ell,j} v_{j,\ell+j}$. Recall that $v_{j,\ell+j} \neq 0$ and $\nu_j \in \ker(A) = \ker([D \mid C])$, and then

$c_{\ell,j} = 0$. However, as this is true for each $j \leq n - \ell$, it follows that the last row of C is the zero row, contradicting the fact that $[D \mid C]$ has full rank. We conclude that D is diagonal and nonsingular. Clearly, by multiplying rows by -1, we can assume that the diagonal entries of D are all negative. Let $d_k < 0$ denote the diagonal entries of D for $1 \leq k \leq \ell$.

Finally, we claim that each $c_{k,j} \geq 0$. For each j, as ν_j has nonzero coordinates only in the $1, 2, \ldots, \ell$, and $\ell + j$ positions, we have that the kth coordinate of $[D \mid C]\nu_j$ is given by $d_k v_{j,k} + c_{k,j} v_{j,\ell+j}$. As $[D \mid C]\nu_j = 0$, we have that $d_k v_{j,k} + c_{k,j} v_{j,\ell+j} = 0$. As $d_k < 0$, as $v_{j,k} \geq 0$, and as $v_{j,\ell+j} > 0$. It follows that $c_{k,j} \geq 0$, completing the proof that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3): Assuming A is in the form $[D \mid C]$ as in (2), let μ_i denote the ith row of A. Then it is easy to see that $\langle \mu_i, e_i \rangle < 0$ and $\langle \mu_i, e_j \rangle \geq 0$ for $j \neq i$. Moreover, $\ker(A) = \bigcap_{i=1}^{\ell} \mu_i^{\perp}$ by definition.

(3) $\Rightarrow$ (1): Permute the coordinates $x_1, \ldots, x_n$ so that $j_i = i$ for $i = 1, \ldots, \ell$. Let M be the $\ell \times n$ matrix with ith row μ_i and then $\ker(M) = \ker(A)$ by hypothesis so that $\mathcal{P}_A = \mathcal{P}_M$. Clearly, M must then satisfy hypotheses (i), (ii), and (iii), and moreover M is of the form $[D \mid C]$ as described in condition (2). Let $d_k < 0$ denote the entries of D and $c_{k,j} \geq 0$ denote the entries of C.

It is easy to see that each coordinate plane corresponding to $\{1, \ldots, \ell, \ell + k\}$ intersects $\mathcal{P}_M$ at a single vertex. In particular, define

$$\nu_j = \frac{1}{1 + \sum\limits_{k=1}^{\ell} -c_{k,j}/d_k} \left(\frac{-c_{1,j}}{d_1}, \frac{-c_{2,j}}{d_2}, \ldots, \frac{-c_{\ell,j}}{d_\ell}, 0, \ldots, 0, 1, 0, \ldots, 0 \right)$$

for $j = 1, \ldots, n - \ell$, where the 1 occurs in the $(\ell + j)$th position. Simple computations show that each $\nu_j \in \ker([D \mid C]) \cap \Delta^{n-1}$ and $\ker([D \mid C]) \cap V_{\{1,\ldots,\ell,\ell+j\}}$ is a 1-dimensional subspace of $\mathbb{R}^n$. Therefore, $\{\nu_j\} = \ker([D \mid C]) \cap \Delta^{n-1} \cap V_{\{1,\ldots,\ell,\ell+j\}}$, so that by Lemma 3, each ν_j is a vertex of $\mathcal{P}_M$. It remains only to show that there are no other vertices.

However, for each $p = (p_1, \ldots, p_n) \in \mathcal{P}_M = \mathcal{P}_{[D|C]}$ the fact that $[D \mid C]p = 0$ implies that the $p_1, \ldots, p_\ell$ are uniquely determined by the $p_{\ell+1}, \ldots, p_n$. Moreover, letting $\pi\colon \mathbb{R}^n \to \mathbb{R}^{n-\ell}$ denote the projection $\pi\colon (x_1, \ldots, x_n) \mapsto (x_{\ell+1}, \ldots, x_n)$, it is obvious that $\{\pi(\nu_1), \ldots, \pi(\nu_{n-\ell})\}$ is linearly independent in $\mathbb{R}^{n-\ell}$ and hence affinely independent. Hence, given coordinates $p_{\ell+1}, \ldots, p_n$, there is a unique affine combination of the $\pi(\nu_1), \ldots, \pi(\nu_{n-\ell})$ that yields $(p_{\ell+1}, \ldots, p_n)$. Then there are unique values $p_1, \ldots, p_\ell$ such that $(p_1, \ldots, p_n) \in \ker(M) = \ker([D \mid C])$, and this affine combination of the $\pi(\nu_j)$ is a convex combination if and only $(p_1, \ldots, p_n) \in \Delta^{n-1}$. It follows that each $p \in \mathcal{P}_M = \mathcal{P}_{[D|C]}$ is a convex combination of the ν_j, and hence that there are no other vertices. We conclude that the polytope $\mathcal{P}_A = \mathcal{P}_M = \mathcal{P}_{[D|C]}$ has $n - \ell$ vertices and hence, as it is $(n - \ell - 1)$-dimensional, that it is combinatorially equivalent to the standard $(n - \ell - 1)$-simplex.

To complete the proof, we need only note that if $n \leq \ell + 2$, then $\mathcal{P}_A$ is a 0- or 1-dimensional polytope, which is necessarily a simplex. ∎

With this, we are prepared to prove Theorem 4, completing this subsection.

Proof of Theorem 4. First, we note that the zero-fiber $J_A^{-1}(0)$ is contained in the preimage under sq of the coordinate plane V_{I_A} so that we may identify $\mathrm{sq}(J_A^{-1}(0))$ with $\mathrm{sq}(J_{A'}^{-1}(0))$ via the embedding $\mathbb{R}^{|I_A|} \to \mathbb{R}^n$ induced by $I_A \subset \{1, \ldots, n\}$. Permute the coordinates x_i for $i \in I_A$ so that any zero columns of A' are listed last. Row reducing A' using integer scalar multiples yields a matrix with any zero rows listed last of the form

$$R = \begin{bmatrix} A'' & 0 \\ 0 & 0 \end{bmatrix}.$$

Here, A'' has dimensions $k \times (k + m)$ such that $k \leq \ell$ and $m \leq |I_A| - k$. To see this, note that A'' has a pivot in each row by construction, and moreover that A'' has at least one positive and one negative element in each row to ensure that each x_i is nonzero for some element of $\ker(A'')$.

Clearly, the reduced space of the action of $\mathbb{T}^\ell$ on $\mathbb{R}^{|I_A|}$ with weight matrix R coincides with the reduced space of the action with weight matrix $R' = [A''\,0]$. Note that by construction, A'' has full rank and no zero columns. Moreover, for each $i \in I_A$, there is a point in $\ker(A)$ with nonnegative coordinates such that $x_i \geq 0$. It follows by convexity that $\ker(A) \cap (\Delta^{k+m-1})^\circ \neq \varnothing$. Therefore, A'' satisfies hypotheses (i), (ii), and (iii).

Now,

$$\begin{aligned}\mathcal{P}_{R'} &= \ker(R') \cap \Delta^{|I_A|-1} = \big(\ker(A'') \times \mathbb{R}^{|I_A|-m}\big) \cap \Delta^{|I_A|-1} \\ &= \operatorname{cch}\big(\big(\ker\big(A''\big) \cap \Delta^{k+m-1}\big) \cup \{e_{m+1}, \dots, e_{|I_A|}\}\big),\end{aligned}$$

where $e_1, \dots, e_{|I_A|}$ denotes the standard basis of $\mathbb{R}^{|I_A|}$, Δ^{k+m-1} is the standard simplex in $\mathbb{R}^{k+m} = \operatorname{Span}\{e_1, \dots, e_{k+m}\}$, and the elements of $\ker(A'') \cap \Delta^{k+m-1}$ are identified with elements of $\mathbb{R}^{|I_A|}$ via the obvious embedding $\mathbb{R}^{k+m} \to \mathbb{R}^{|I_A|}$.

With this, it is clear that the vertices of $\mathcal{P}_{R'}$ are given by the $e_{m+1}, \dots, e_{|I_A|}$ along with the images of the vertices of $\ker(A'') \cap \Delta^{k+m-1}$ in $\mathbb{R}^{|I_A|}$ as above. Hence, $\mathcal{P}_{R'}$ is a polytope given by the closed convex hull of $\mathcal{P}_{A''}$ along with $|I_A| - m$ points that are linearly independent to the vertices of $\mathcal{P}_{A''}$. It follows that A' and hence A is simplicial if and only if A'' is simplicial. Recalling that A'' satisfies hypotheses (i), (ii), and (iii), an application of Proposition 1 to A'' completes the proof. ∎

Example 2. For the weight matrix given in Example 1, the vertices of $\mathcal{P}_A = \ker(A) \cap \Delta^5$ are given by $(1/2, 1/2, 0, 0, 0, 0)$; $(1/2, 0, 0, 1/2, 0, 0)$; $(0, 1/2, 1/2, 0, 0, 0)$; $(0, 0, 1/2, 1/2, 0, 0)$; and $(0, 0, 0, 0, 1/2, 1/2)$; so that $\mathcal{P}_A$ is a 3-dimensional polytope with 5 vertices. Hence Theorem 4(1) fails, and A is not simplicial.

4 Smooth structures on singular phase spaces

The aim of this section is to study singular phase spaces and smooth maps between them. In this paper, we use the term *singular phase space* loosely, i.e., we mean by it a space (preferably with singularities) on which one can do some sort of Hamiltonian mechanics. In order to give precise definitions, there are some choices to be made. It will be convenient for our purposes to focus on the notion of a *differential space* in the sense of Sikorski [28].

4.1 Poisson differential spaces

To begin, let us recall the definition of a stratified symplectic space and the theorem of Sjamaar and Lerman, which says that every symplectic quotient is such a space.

Definition 3. A *stratified symplectic space* is a Whitney stratified space $X = \sqcup_{i \in I} X_i$ with an algebra $\mathcal{C}^\infty(X)$ of continuous functions such that

1) each stratum X_i is a symplectic manifold,
2) $\mathcal{C}^\infty(X)$ is a Poisson algebra, and
3) the pullback $\mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X_i)$ with respect to the inclusions $X_i \hookrightarrow X$ is compatible with the Poisson bracket.

The Poisson algebra $\mathcal{C}^\infty(X)$, is called the *algebra of smooth functions* on X.

If we regard X merely as a topological space, we say that $\mathcal{C}^\infty(X)$ is a *smooth structure* on X. In many cases (for example in the case of the theorem below), it is known (see e.g. [29]) that one can reconstruct the stratification from the Poisson algebra $\mathcal{C}^\infty(X)$. The question of when one can do so without using the Poisson bracket is, to our knowledge, open. So morally, $\mathcal{C}^\infty(X)$ contains all the information about X.

Theorem 5 ([29])**.** *Let G be a compact Lie group acting on a symplectic manifold (M,ω) in a Hamiltonian way, and let $J\colon M \to \mathfrak{g}^*$ be a moment map for this action. Then the symplectic quotient $M_0 = Z/G$, with $Z = J^{-1}(0)$, is a stratified symplectic space, where the strata*

$$(M_0)_{(H)} := (M_{(H)} \cap Z)/G$$

are indexed by conjugacy classes (H) of subgroups $H \subset G$ that arise as isotropy groups of elements of Z. Here $M_{(H)}$ is the set of points in M whose isotropy group is an element of the class (H). The Poisson algebra of smooth functions $\mathcal{C}^\infty(M_0)$ is given by

$$\mathcal{C}^\infty(M_0) := \mathcal{C}^\infty(M)^G / \big(\mathcal{C}^\infty(M)^G \cap I_Z\big),$$

where $I_Z \subset \mathcal{C}^\infty(M)$ denotes the ideal of smooth functions vanishing on Z, and $\mathcal{C}^\infty(M)^G \subset C^\infty(M)$ is the Poisson subalgebra of of G-invariant smooth functions.

Note that elements of $\mathcal{C}^\infty(M_0)$ can be in fact regarded as functions on M_0. Note further that it is not difficult to check that $\mathcal{C}^\infty(M)^G \cap I_Z \subset \mathcal{C}^\infty(M)$ is actually a Poisson ideal, so that the Poisson bracket on $\mathcal{C}^\infty(M_0)$ is canonically defined.

Using the smooth structure as the key idea, one can easily talk about symplectomorphisms between stratified symplectic spaces.

Definition 4 ([18])**.** A *symplectomorphism* between the symplectic stratified spaces $(X = \sqcup_{i\in I} X_i, \mathcal{C}^\infty(X))$ and $(Y = \sqcup_{j\in J} Y_j, \mathcal{C}^\infty(Y))$ is defined to be a homeomorphism $\varphi\colon X \to Y$ whose pullback $\mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$ with $f \mapsto f\circ\varphi$ is an isomorphism of Poisson algebras.

Before refining the concept of symplectomorphism, we recall the notion of a Poisson differential space [23]. This idea will help us strip off the unnecessary details from the notion of a stratified symplectic space and widen the setup to including, e.g., orbit spaces of Poisson G-actions.

Definition 5. A *differential space* (in the sense of Sikorski) is defined as a pair $(X, \mathcal{C}^\infty(X))$, where X is a topological space and $\mathcal{C}^\infty(X)$ is an algebra of continuous functions on X such that the following axioms are fulfilled:

1. The topology of X is generated by $\mathcal{C}^\infty(X)$.
2. If $F \in \mathcal{C}^\infty(\mathbb{R}^n)$, $f_1, \dots, f_n \in \mathcal{C}^\infty(X)$, then $F(f_1, \dots, f_n) \in \mathcal{C}^\infty(X)$.
3. If $f\colon X \to \mathbb{R}$ is a function such that for every $x \in X$, there exists an open neighborhood U of x and an $f_U \in \mathcal{C}^\infty(X)$ such that $f_{|U} = f_U$, then $f \in \mathcal{C}^\infty(X)$.

A *Poisson differential space* is a triple $(X, \mathcal{C}^\infty(X), \{\ ,\ \})$, where $(X, \mathcal{C}^\infty(X))$ is a differentiable space and $\{\ ,\ \}\colon \mathcal{C}^\infty(X) \times \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X)$ is a Poisson bracket.

Definition 6. A *smooth map* from the differential space $(X, \mathcal{C}^\infty(X))$ to the differential space $(Y, \mathcal{C}^\infty(Y))$ is a continuous map $\varphi\colon X \to Y$ such that the pullback $\varphi^*\colon f \mapsto f\circ\varphi$ sends smooth functions on Y to smooth functions on X. If in addition $\varphi^*\colon \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$ preserves the Poisson structures, φ is called a *Poisson map*.

If $(X, \mathcal{C}^\infty(X))$ is a differential space, then a maximal ideal $\mathfrak{m} \subset \mathcal{C}^\infty(X)$ is called a *real maximal ideal* if the residue field $\mathcal{C}^\infty(X)/\mathfrak{m}$ is isomorphic to $\mathbb{R}$. The set $\operatorname{Spec}_{\mathbb{R}}(\mathcal{C}^\infty(X))$ of real maximal ideals in $\mathcal{C}^\infty(X)$ is called the *real spectrum* of $\mathcal{C}^\infty(X)$.

To make the definition of a Poisson differential space $(X, \mathcal{C}^\infty(X), \{\ ,\ \})$ workable, we have to impose some additional assumptions, namely:

(A) The real spectrum consists of points, i.e., every real maximal ideal in $\mathcal{C}^\infty(X)$ is of the form $\mathfrak{m}_\xi := \{f \in \mathcal{C}^\infty(X) \mid f(\xi) = 0\}$. Moreover, we require $\bigcap_{\xi\in X} \mathfrak{m}_\xi = 0$.

(B) All Hamiltonian vector fields D (i.e., those of the form $D := \{h, \ \}$ for some $h \in \mathcal{C}^\infty(X)$) fulfill the chain rule. This means that, if we pick some $\varphi_1, \dots, \varphi_k \in \mathcal{C}^\infty(X)$ and put $\varphi := (\varphi_1, \dots, \varphi_k) \colon X \to \mathbb{R}^k$, then for any $F \in \mathcal{C}^\infty(\mathbb{R}^k)$ we have

$$D(F \circ \varphi) = \sum_{i=1}^{k} \left(\frac{\partial F}{\partial x^i} \circ \varphi \right) D(\varphi_i).$$

Note that condition (A) is always satisfied if X is a closed subset or $\mathbb{R}^n$ and $\mathcal{C}^\infty(X)$ is the quotient algebra $\mathcal{C}^\infty(\mathbb{R}^n)/I$ where I is the closed ideal of $\mathcal{C}^\infty(\mathbb{R}^n)$ consisting of functions that vanish on X; see [24, Proposition 2.13]. It is not clear to the authors if condition (A) remains true, e.g. if X is not paracompact. In addition, it is not known to the authors whether a symplectic stratified space is automatically a Poisson differential space fulfilling conditions (A) and (B). However, using results from [7], it is easy to prove the following.

Proposition 2. *With the notation of Theorem 5, if M has a finite number of orbit types as a G-manifold, then the symplectic quotient $(M_0, \mathcal{C}^\infty(M_0), \{ \ , \ \})$ is a Poisson differential space satisfying conditions (A) and (B).*

Proof. In [7] it is proven that for any action of a compact Lie group G on a manifold M, the space of G-orbits M/G is a differential space. The smooth structure here is given by the the algebra $\mathcal{C}^\infty(M)^G$ of G-invariant functions on M. Every subspace of a differential space is a differential space, and hence the symplectic quotient is a differential space. Using a system of differentiable invariants (see Theorem 1), M_0 can be realized as a closed differential subspace of $\mathbb{R}^k$ so that [24, Proposition 2.13] applies. Property (A) follows. Property (B) is obvious. ∎

4.2 Global charts and the lifting theorem

In this subsection, we impose a more rigid structure on our Poisson differentiable spaces. The terminology chosen stems from the observation that complete sets of differentiable invariants (cf. the Schwarz–Mather Theorem 1) have much in common with linear coordinates on a vector space. In fact, those coordinates can be seen as a Hilbert basis of a trivial group representation.

Definition 7. A *global chart* on a Poisson differential space $(X, \mathcal{C}^\infty(X), \{ \ , \ \})$ is an algebra homomorphism

$$\varphi\colon \ \mathbb{R}[\boldsymbol{x}] := \mathbb{R}[x_1, \dots, x_k] \to \mathcal{C}^\infty(X), \qquad x_i \mapsto \varphi_i, \qquad i \in \{1, \dots, k\},$$

such that

1. The image of φ, denoted $\mathbb{R}[X]$, is a Poisson subalgebra of $\mathcal{C}^\infty(X)$, called *Poisson subalgebra of regular functions* on X.
2. $\mathcal{C}^\infty(X)$ is $\mathcal{C}^\infty$-integral over $\mathbb{R}[\boldsymbol{x}]$, that is, for any $f \in \mathcal{C}^\infty(X)$ there is a $F \in \mathcal{C}^\infty(\mathbb{R}^k)$ such that $f = F \circ \varphi$. Abusing language slightly, here φ denotes the vector valued map $X \to \mathbb{R}^k$, $\xi \mapsto (\varphi_1(\xi), \dots, \varphi_k(\xi))$.
3. The image of φ in $\mathcal{C}^\infty(X)$ separates points.

For a global chart $\varphi\colon \mathbb{R}[\boldsymbol{x}] \to \mathcal{C}^\infty(X)$ we use property (1) to transfer the Poisson structure from $\mathcal{C}^\infty(X)$ to $\mathbb{R}[\boldsymbol{x}]/\ker(\varphi)$. In this way we obtain an embedding of Poisson algebras

$$\overline{\varphi}\colon \ \mathbb{R}[\boldsymbol{x}]/\ker(\varphi) \hookrightarrow \mathcal{C}^\infty(X).$$

Of course, $\mathbb{R}[X]$ is isomorphic to $\mathbb{R}[\boldsymbol{x}]/\ker(\varphi)$. If for the global chart $\varphi\colon \mathbb{R}[\boldsymbol{x}] \to \mathcal{C}^\infty(X)$, the algebra $\mathbb{R}[\boldsymbol{x}]$ carries a $\mathbb{Z}$-grading such that the ideal $\ker(\varphi)$ is homogeneous, we call $\varphi\colon \mathbb{R}[\boldsymbol{x}] \to \mathcal{C}^\infty(X)$ a *$\mathbb{Z}$-graded global chart.*

Our favorite examples are, of course, the symplectic quotients $M_0 = Z/G$ (cf. Theorem 5). In fact, if we pick a complete system $\rho_1, \dots, \rho_k$ of differentiable invariants for the G-action on M, see Theorem 1, then we can make out of it a global chart by defining φ_i to be the class of ρ_i in $\mathcal{C}^\infty(M_0) = \mathcal{C}^\infty(M)^G/(\mathcal{C}^\infty(M)^G \cap I_Z)$ for each $i \in \{1, \dots, k\}$ (confer Theorem 5). Similarly, if we consider an orbit space of a Poisson G-space with finitely many orbit types, we can take a complete system of invariants itself to form a global chart.

Let us also comment on the linear case, which is our main concern in this paper. If we examine the symplectic quotient coming from a unitary representation $G \to \mathrm{U}_n$, then we should of course choose a minimal homogeneous system of polynomial invariants $\rho_1, \dots, \rho_k$. If we assign to the variables x_i in the above definition the degree of ρ_i, the result is a $\mathbb{Z}$-graded global chart. Similar comments apply to the orbit space of a symplectic representation $G \to \mathrm{Sp}(\mathbb{R}^{2n})$. Note that the choice of a complete system of polynomial invariants, and therefore of a global chart, is not unique. This choice turns out not to be essential; see Remark 1 below. A more severe problem is that it might be practically impossible to compute a complete system of invariants. As well, the determination of $\ker(\varphi)$ can be tricky.

Lemma 4. *With the notation of Definition 7, the map $\varphi\colon X \to \mathbb{R}^k$, $\xi \mapsto (\varphi_1(\xi), \dots, \varphi_k(\xi))$ is injective.*

Proof. By definition, any regular function $f \in \mathbb{R}[X] \subset \mathcal{C}^\infty(X)$ can be written as the composition $f = p \circ \varphi = p(\varphi_1, \dots, \varphi_k)$ of a polynomial $p \in \mathbb{R}[\boldsymbol{x}]$. So if $\varphi(\xi_1) = \varphi(\xi_2)$ for some $\xi_1, \xi_2 \in X$, then $f(\xi_1) = f(\xi_2)$ for all $f \in \mathbb{R}[X]$. By Definition 7(3), $\mathbb{R}[X]$ separated points, and hence it follows that $\xi_1 = \xi_2$. ∎

Lemma 5. *Assume that the Poisson differential space in Definition 7 has property (A). Then if $\epsilon\colon \mathcal{C}^\infty(X) \to \mathbb{R}$ is a morphism of $\mathbb{R}$-algebras such that $\epsilon_{|\mathbb{R}[X]} = 0$ it follows that $\epsilon = 0$.*

Proof. Assume that ϵ is nonzero. Then $\ker(\epsilon)$ is a real maximal ideal and hence, by property (A), of the form $\mathfrak{m}_\xi = \{f \in \mathcal{C}^\infty(X) \mid f(\xi) = 0\}$. On the other hand, as $\mathbb{R}[X]$ separates points, there is an $f \in \mathbb{R}[X]$ such that $f(\xi) \neq 0$. This contradicts our assumption that $f \in \ker(\epsilon)$. ∎

We now define morphisms for Poisson differential spaces with global charts.

Definition 8. An *arrow* from a Poisson differential space $(X, \mathcal{C}^\infty(X), \{\ ,\ \})$ with global chart $\varphi\colon \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, \dots, x_k] \to \mathcal{C}^\infty(X)$ to a Poisson differential space $(Y, \mathcal{C}^\infty(Y), \{\ ,\ \})$ with global chart $\psi\colon \mathbb{R}[\boldsymbol{y}] = \mathbb{R}[y_1, \dots, y_m] \to \mathcal{C}^\infty(Y)$ is a morphism of algebras $\lambda\colon \mathbb{R}[\boldsymbol{y}] \to \mathbb{R}[\boldsymbol{x}]$, such that

(*i*) We have $\lambda(\ker(\psi)) \subset \ker(\varphi)$, and the induced morphism of algebras

$$\overline{\lambda}\colon\ \mathbb{R}[\boldsymbol{y}]/\ker(\psi) \to \mathbb{R}[\boldsymbol{x}]/\ker(\varphi)$$

is compatible with the Poisson bracket.

(*ii*) Setting $\lambda_i := \lambda(y_i) \in \mathbb{R}[\boldsymbol{x}]$, $i = 1, \dots, m$, and defining

$$\vartheta\colon\ X \to \mathbb{R}^m, \qquad \vartheta(\xi) := \big((\varphi(\lambda_1))(\xi), \dots, (\varphi(\lambda_m))(\xi)\big),$$

the image $\mathrm{im}(\psi)$ of the map $\psi\colon Y \to \mathbb{R}^m$ contains $\mathrm{im}(\vartheta)$.

If both charts are $\mathbb{Z}$-graded and the algebra morphism λ is compatible with the grading we say that the arrow is *$\mathbb{Z}$-graded*.

Clearly, an arrow contains redundant information – what is really important is $\overline{\lambda}$. We say that two arrows λ and λ' are *equivalent* if they induce the same $\overline{\lambda}$.

Theorem 6 (lifting theorem)**.** *With the notation of Definition 8 and choice of an arrow λ, let us assume that both Poisson differential spaces have property (B) and $(X, \mathcal{C}^\infty(X))$ has property (A). Then there exists unique morphism of Poisson algebras $\widetilde{\lambda}\colon \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$, such that $\varphi \circ \lambda = \widetilde{\lambda} \circ \psi$.*

$$\begin{tikzcd} \mathcal{C}^\infty(Y) \arrow[r, "\widetilde{\lambda}"] & \mathcal{C}^\infty(X) \\ \mathbb{R}[\boldsymbol{y}] \arrow[u, "\psi"] \arrow[r, "\lambda"'] & \mathbb{R}[\boldsymbol{x}] \arrow[u, "\varphi"'] \end{tikzcd}$$

Moreover, the lift *$\widetilde{\lambda}$ depends only on the equivalence class of λ and can be understood as the pullback of the continuous map*

$$\chi\colon\ X \to Y, \qquad \xi \mapsto \psi^{-1}(\vartheta(\xi)).$$

For two arrows λ_1 and λ_2 we have $\widetilde{\lambda_1 \circ \lambda_2} = \widetilde{\lambda_1} \circ \widetilde{\lambda_2}$.

Note that by construction, the map χ is smooth in the sense of Definition 6.

Proof. Take a function $f \in \mathcal{C}^\infty(Y)$ and write it as a composite with ψ,

$$f(\eta) = F\big(\psi_1(\eta), \ldots, \psi_m(\eta)\big) \qquad \forall\, \eta \in Y,$$

for some $F \in \mathcal{C}^\infty(\mathbb{R}^m)$. The function $\widetilde{\lambda}(f) \in \mathcal{C}^\infty(X)$ is defined to be

$$\big(\widetilde{\lambda}(f)\big)(\xi) := F\big((\varphi(\lambda_i))(\xi), \ldots, (\varphi(\lambda_m))(\xi)\big) = F(\vartheta(\xi)) \qquad \forall\, \xi \in X,$$

where $\lambda_i := \lambda(y_i) \in \mathbb{R}[\boldsymbol{x}]$ for $i = 1, \ldots, m$. Clearly $\widetilde{\lambda}$ does not depend the choice of λ within its equivalence class and fulfills $\varphi \circ \lambda = \widetilde{\lambda} \circ \psi$. By assumption (ii) of Definition 8, $\widetilde{\lambda}$ does not depend on the choice of F.

The uniqueness of $\widetilde{\lambda}$ is a consequence of Lemma 5. In fact, given another algebra morphism $\widehat{\lambda}\colon \mathcal{C}^\infty(Y) \to \mathcal{C}^\infty(X)$ such that $\varphi \circ \lambda = \widehat{\lambda} \circ \psi$, then $\epsilon_\xi := (\widetilde{\lambda}(f))(\xi) - (\widehat{\lambda}(f))(\xi)$ is an algebra morphism $\epsilon_\xi\colon \mathcal{C}^\infty(X) \to \mathbb{R}$ whose restriction to $\mathbb{R}[X]$ vanishes. So Lemma 5 implies $\epsilon_\xi = 0$ for all $\xi \in X$. But this means that the function $\widetilde{\lambda}(f) - \widehat{\lambda}(f)$ vanishes everywhere on X, and is hence zero by property (A).

By Definition 8 and the injectivity of $\psi\colon Y \to \mathbb{R}^k$ (cf. Lemma 4), the map $\chi\colon \xi \mapsto \psi^{-1}(\vartheta(\xi))$ is well-defined. The verification of the claim $\chi^* = \widetilde{\lambda}$ is straightforward,

$$\big(\chi^* f\big)(\xi) = f(\chi(\xi)) = f\big(\psi^{-1}(\vartheta(\xi))\big) = (F \circ \psi)\big(\psi^{-1}(\vartheta(\xi))\big) = F(\vartheta(\xi)) = \widetilde{\lambda}(f).$$

Finally, let us show that $\widetilde{\lambda}$ is compatible with the bracket. By construction, for all $i, j \in \{1, \ldots, m\}$ there is a polynomial $\gamma_{ij} = \gamma_{ij}(y_1, \ldots, y_m) \in \mathbb{R}[\boldsymbol{y}]$ representing the class of $\{y_i, y_j\}$ in $\mathbb{R}[\boldsymbol{y}]/\ker(\psi)$. We observe that

$$\{\psi_i, \psi_j\} = \{\psi(y_i), \psi(y_j)\} = \psi(\gamma_{ij}(y_1, \ldots, y_m)) = \gamma_{ij}(\psi_1, \ldots, \psi_m) \in \mathcal{C}^\infty(Y),$$

because ψ is by definition compatible with the bracket. Since, by assumption, $\overline{\lambda}$ is compatible with the bracket, we see that $\{\lambda_i, \lambda_j\} = \{\lambda(y_i), \lambda(y_j)\} \in \mathbb{R}[\boldsymbol{x}]$ coincides with $\lambda(\gamma_{ij}(y_1, \ldots, y_m)) = \gamma_{ij}(\lambda_1, \ldots, \lambda_m) \in \mathbb{R}[\boldsymbol{x}]$ up to $\ker(\varphi)$. It follows that

$$\{\varphi(\lambda_i), \varphi(\lambda_j)\} = \varphi(\{\lambda_i, \lambda_j\}) = \gamma_{ij}(\varphi(\lambda_1), \ldots, \varphi(\lambda_m)) = \gamma_{ij} \circ \vartheta \in \mathcal{C}^\infty(X). \tag{4.1}$$

With these preparations, we compute for $\eta \in Y$, and $f = F \circ \psi$ and $g = G \circ \psi$, making use of property (B):

$$\begin{aligned}\{f,g\}(\eta) &= \sum_{i,j=1}^{m} \frac{\partial F}{\partial x_i}\big(\psi_1(\eta),\dots,\psi_m(\eta)\big)\frac{\partial G}{\partial x_j}\big(\psi_1(\eta),\dots,\psi_m(\eta)\big)\{\psi_i,\psi_j\}(\eta)\\ &= \sum_{i,j=1}^{m} \frac{\partial F}{\partial x_i}\big(\psi_1(\eta),\dots,\psi_m(\eta)\big)\frac{\partial G}{\partial x_j}\big(\psi_1(\eta),\dots,\psi_m(\eta)\big)\gamma_{ij}\big(\psi_1(\eta),\dots,\psi_m(\eta)\big),\end{aligned}$$

which yields for $\xi \in X$:

$$\begin{aligned}\widetilde{\lambda}\left(\{f,g\}\right)(\xi) &= \sum_{i,j=1}^{m} \frac{\partial F}{\partial x_i}\big((\varphi(\lambda_1))(\xi),\dots,(\varphi(\lambda_m))(\xi)\big)\\ &\quad\times \frac{\partial G}{\partial x_j}\big((\varphi(\lambda_1))(\xi),\dots,(\varphi(\lambda_m))(\xi)\big)\gamma_{ij}\big((\varphi(\lambda_1))(\xi),\dots,(\varphi(\lambda_m))(\xi)\big)\\ &= \sum_{i,j=1}^{m} \frac{\partial F}{\partial x_i}(\vartheta(\xi))\frac{\partial G}{\partial x_j}(\vartheta(\xi))\,\gamma_{ij}(\vartheta(\xi)).\end{aligned}$$

On the other hand we have

$$\begin{aligned}\{\widetilde{\lambda}(f),\widetilde{\lambda}(g)\}(\xi) &= \sum_{i,j=1}^{m} \frac{\partial F}{\partial x_i}\big((\varphi(\lambda_1))(\xi),\dots,(\varphi(\lambda_m))(\xi)\big)\\ &\quad\times \frac{\partial G}{\partial x_j}\big((\varphi(\lambda_1))(\xi),\dots,(\varphi(\lambda_m))(\xi)\big)\{\varphi(\lambda_i),\varphi(\lambda_j)\}(\xi),\end{aligned}$$

which in view of equation (4.1) implies that $\widetilde{\lambda}\left(\{f,g\}\right) = \{\widetilde{\lambda}(f),\widetilde{\lambda}(g)\}$. ■

Definition 9. A Poisson map χ between Poisson differential spaces with global charts that is obtained as a lift of an arrow λ as in Theorem 6 is called a *regular Poisson map*. If the arrow λ is such that

1) $\overline{\lambda}$ is an isomorphism, and
2) (in the notation of Definition 8) $\operatorname{im}(\vartheta) = \operatorname{im}(\psi)$,

then χ is called a *regular Poisson diffeomorphism*. If the arrow is in addition $\mathbb{Z}$-graded we say that the regular Poisson map (resp. regular Poisson diffeomorphism) is $\mathbb{Z}$-*graded*.

Regular Poisson diffeomorphisms between symplectic stratified spaces are examples of symplectomorphisms; see Definition 4.

Remark 1. Consider a unitary representation $G \to \mathrm{U}_n$, let M_0 denote the associated symplectic quotient, and let $\rho_1,\dots,\rho_r$ and $\sigma_1,\dots,\sigma_s$ denote two choices of minimal homogeneous systems of polynomial invariants. Let

$$\varphi\colon\ \mathbb{R}[x_1,\dots,x_r] \to \mathcal{C}^\infty(M_0), \qquad x_i \mapsto \rho_i, \qquad i \in \{1,\dots,r\},$$

and

$$\psi\colon\ \mathbb{R}[y_1,\dots,y_s] \to \mathcal{C}^\infty(M_0), \qquad y_i \mapsto \sigma_i, \qquad i \in \{1,\dots,s\},$$

denote the corresponding global charts for M_0. Expressing each σ_i in terms of the polynomials $\rho_1,\dots,\rho_r$ defines an arrow $\lambda\colon \mathbb{R}[\boldsymbol{y}] \to \mathbb{R}[\boldsymbol{x}]$, and one checks that this is a $\mathbb{Z}$-graded regular Poisson diffeomorphism.

In Section 5, we will argue that certain Poisson differential spaces with global chart are not ($\mathbb{Z}$-graded) regularly diffeomorphic, because their rings of regular functions are not isomorphic as ($\mathbb{Z}$-graded) commutative $\mathbb{R}$-algebras. This type of problem is entirely in the realm of the theory of commutative Noetherian rings.

There are many potential applications of the theory presented above, which we will indicate elsewhere. Here, we will return to the consideration of toric symplectic quotients.

4.3 Orbifold cases in dimension 2

The purpose of this subsection is twofold. First of all, we will illustrate the machinery introduced in the last subsection by presenting a concrete example of a $\mathbb{Z}$-graded regular symplectomorphism. Secondly, we will show that the simplicial condition is actually sufficient for a two-dimensional symplectic quotient to be symplectomorphic to an orbifold. Note that by Theorem 4, any two-dimensional symplectic quotient corresponds to a simplicial representation.

Before doing so, we would like to comment on a subtle point that one faces when determining the kernel of a global chart. For example, if we are interested in the ideal of smooth functions on $\mathbb{C}$ that vanish on the zero set of the function $J = z\overline{z}$, it turns out that is generated not by J itself, but rather by the linear monomials z and $\overline{z}$. In the next proposition, we indicate that we do not have to worry about this kind of problems in the situation at hand.

Proposition 3. *Let $A \in \mathbb{Z}^{\ell\times n}$ be a weight matrix that can, by elementary row operations and permutation of the column indices, be brought into the form $A = [D \mid C]$, where $D \in \mathbb{Z}^{\ell\times\ell}$ is a diagonal matrix with strictly negative entries and $C \in \mathbb{Z}^{\ell\times(n-\ell)}$ has non-negative entries and no rows that are identically zero. Then the $G = \mathbb{T}^\ell$-invariant part $I_Z^G = I_Z \cap C^\infty(\mathbb{C}^n)^G$ of the vanishing ideal $I_Z \subset \mathcal{C}^\infty(\mathbb{C}^n)$ is generated by the components $J_1, \dots, J_\ell \in C^\infty(\mathbb{C}^n)^G$ of the moment map. Here we view I_Z^G as an ideal in $\mathcal{C}^\infty(\mathbb{C}^n)^G$.*

Proof. Based on the signs of the entries of A, condition (i) of [17, Proposition 2.2] is fulfilled, and the result follows. ∎

Given the $G = \mathbb{T}^\ell$-action on $\mathbb{C}^n$ encoded by our weight matrix $A \in \mathbb{Z}^{\ell\times n}$, we can find a real Hilbert basis (i.e., complete set of real polynomial invariants) $\rho_1, \dots, \rho_k \in \mathbb{R}[\mathbb{C}^n]^G$ such that $\rho_i = z_i\overline{z}_i$ for $i = 1, \dots, n$. Because our group is abelian, the moment map itself is invariant. We can express the components of the moment map in terms of the ρ's using

$$J_a = J_{e_a} = \frac{1}{2}\sum_{i=1}^{n} A_{ai}\rho_i, \qquad a = 1, \dots, \ell. \tag{4.2}$$

We will occasionally refer to the relations of the form $J_a = 0$ as the *shell relations.* Furthermore, let us denote by $f_1, \dots, f_r \in \mathbb{R}[x_1, \dots, x_k]$ a complete set of algebraic relations among the $\rho_1, \dots \rho_k$. Using this data, we construct a global chart

$$\varphi\colon\ \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, \dots, x_k] \to \mathcal{C}^\infty(M_0), \qquad x_i \mapsto \varphi_i$$

for the symplectic quotient $M_0 = J^{-1}(0)/\mathbb{T}^\ell$, where φ_i is the image of ρ_i in $\mathcal{C}^\infty(M_0)$. Proposition 3 enables us to determine the kernel of φ. Then we have the following, which we expect remains true for an arbitrary weight matrix A and will pursue this elsewhere.

Corollary 1. *Under the assumptions of Proposition 3, the homogeneous ideal* $\ker(\varphi) \subset \mathbb{R}[\boldsymbol{x}]$ *is the ideal generated by $f_1, \dots, f_r$ and the linear forms $g_a := \sum_{i=1}^{n} A_{ai}x_i$, $a = 1, \dots, \ell$.*

Proof. From Proposition 3 it follows that the map $\mathbb{R}[\mathbb{C}^n]^G \to \mathcal{C}^\infty(\mathbb{C}^n)^G$ gives rise to an injection

$$\mathbb{R}[\mathbb{C}^n]^G/\langle J_1, \dots, J_\ell\rangle \to \mathcal{C}^\infty(\mathbb{C}^n)^G/I_Z^G.$$

Since φ factors through this injection

$$\begin{tikzcd} \mathbb{R}[\boldsymbol{x}] \arrow[rr, "\varphi"] \arrow[dr] & & \mathcal{C}^\infty(\mathbb{C}^n)^G/I_Z^G \\ & \mathbb{R}[\mathbb{C}^n]^G/\langle J_1, \dots, J_\ell\rangle \arrow[ur] & \end{tikzcd}$$

we have that $\ker(\varphi)$ is the kernel of the substitution homomorphism $\mathbb{R}[\boldsymbol{x}] \to \mathbb{R}[\mathbb{C}^n]^G/\langle J_1, \dots, J_\ell\rangle$. The claim now easily follows from the isomorphism theorems. ■

We can assume without loss of generality that the weight matrix is of the form $A = [D \mid \boldsymbol{n}]$ where $D = \operatorname{diag}(-a_1, \dots, -a_\ell)$ is an $\ell \times \ell$ diagonal matrix with $a_1, \dots, a_\ell > 0$ and $\boldsymbol{n}$ is a single column with entries $n_1, \dots, n_\ell \geq 0$. We assume as well that the $G = \mathbb{T}^\ell$-action is effective, which implies that $\gcd(a_i, n_i) = 1$ for each $i \in \{1, 2, \dots, \ell\}$. Let us introduce the shorthand notation:

$$\mathcal{A} := \operatorname{lcm}(a_1, \dots, a_\ell), \qquad m_i := \frac{n_i \mathcal{A}}{a_i} \qquad \text{for} \quad i = 1, \dots, \ell, \qquad \mathcal{M} := \sum_{i=1}^{\ell} m_i.$$

It is not difficult to show that

$$\rho_1 = \operatorname{Re}\left(z_{\ell+1}^{\mathcal{A}} \prod_{i=1}^{\ell} z_i^{m_i}\right), \qquad \rho_2 = \operatorname{Im}\left(z_{\ell+1}^{\mathcal{A}} \prod_{i=1}^{\ell} z_i^{m_i}\right), \qquad \rho_3 = z_{\ell+1}\overline{z}_{\ell+1},$$
$$\rho_4 = z_1\overline{z}_1, \qquad \dots, \qquad \rho_{\ell+3} = z_\ell\overline{z}_\ell.$$

constitutes a minimal real Hilbert basis of our $\mathbb{T}^\ell$-action on $\mathbb{C}^n$. The degree of ρ_1 and ρ_2 is $\mathcal{A} + \mathcal{M}$, while the degree of $\rho_3, \dots, \rho_{\ell+3}$ is two. Using the language of the previous section, this leads to a $\mathbb{Z}$-graded global chart

$$\psi\colon\ \mathbb{R}[\boldsymbol{y}] = \mathbb{R}[y_1, \dots, y_{\ell+3}] \to \mathcal{C}^\infty(M_0), \qquad y_i \mapsto \psi_i$$

for our symplectic quotient $M_0 = J^{-1}(0)/\mathbb{T}^\ell$, where ψ_i is ρ_i regarded as an element of $\mathcal{C}^\infty(M_0)$. The kernel $\ker(\psi)$ of the algebra morphism ψ is generated by the polynomials

$$y_1^2 + y_2^2 - \frac{\prod_{i=1}^{\ell} m_i^{m_i}}{\mathcal{A}^{\mathcal{M}}} y_3^{\mathcal{A}+\mathcal{M}} \qquad \text{and} \qquad y_{3+i} - \frac{m_i}{\mathcal{A}} y_3 \qquad \text{for} \quad i = 1, \dots, \ell,$$

the latter coming from the shell relations (see equation (4.2)). The image of the vector valued map

$$\psi\colon\ M_0 \to \mathbb{R}^{\ell+3}, \qquad m \mapsto (\psi_1(m), \dots, \psi_{\ell+3}(m))$$

is determined by the semialgebraic condition

$$y_1^2 + y_2^2 - \frac{\prod_{i=1}^{\ell} m_i^{m_i}}{\mathcal{A}^{\mathcal{M}}} y_3^{\mathcal{A}+\mathcal{M}} = 0,$$
$$y_{3+i} - \frac{m_i}{\mathcal{A}} y_3 = 0 \qquad \text{for} \quad i = 1, \dots, \ell, \qquad y_3 \geq 0.$$

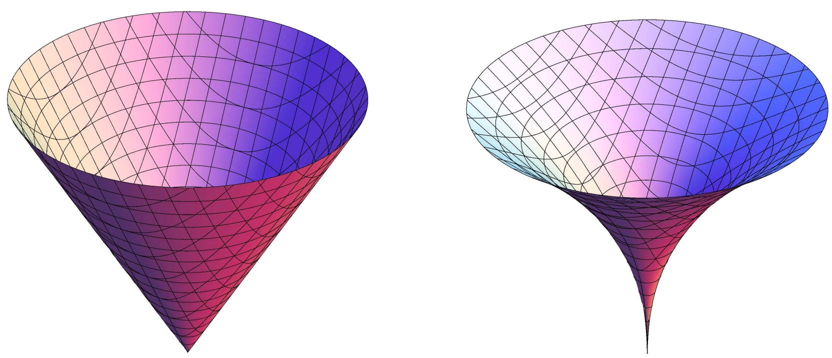

Figure 1. The symplectic orbifold $\mathbb{C}/\mathbb{Z}_N$ for $N = 2$ (left) and $N = 5$ (right).

On the other hand, let us consider the canonical action of the cyclic group $\mathbb{Z}_N$, for $N \geq 2$, on $\mathbb{C}$. In other words, we let $g \in \mathbb{Z}_N \subset \mathbb{S}^1 \subset \mathbb{C}$ act on $z \in \mathbb{C}$ by multiplication. Recall that the action on the complex conjugate variable $\overline{z}$ is given by $g^{-1}\overline{z}$. As the $\mathbb{Z}_N$-action preserves the Kähler structure of $\mathbb{C}$, the quotient space $\big(X_N := \mathbb{C}/\mathbb{Z}_N, \mathcal{C}^\infty(X_N) = \mathcal{C}^\infty(\mathbb{C})^{\mathbb{Z}_N}\big)$ is a Poisson differential space of (real) dimension two. It is easy to determine the real Hilbert basis consisting of $\varphi_1 = \mathrm{Re}(z^N)$, $\varphi_2 = \mathrm{Im}(z^N)$, and $\varphi_3 = z\overline{z}$. Assigning to the variables x_1 and x_2 the degree N and to x_3 the degree 2, we obtain a $\mathbb{Z}$-graded global chart

$$\varphi\colon\ \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, x_2, x_3] \to \mathcal{C}^\infty(X_N), \qquad x_i \mapsto \varphi_i.$$

The kernel $\ker(\varphi)$ of the algebra morphism φ is generated by the polynomial $x_1^2 + x_2^2 - x_3^N$. The image of the map $\varphi\colon X_N \to \mathbb{R}^3$, $z \mapsto (\varphi_1(z), \varphi_2(z), \varphi_3(z))$, is given by the semialgebraic set of solutions of the system

$$x_1^2 + x_2^2 = x_3^N, \qquad x_3 \geq 0,$$

see Fig. 1. With these preparations we are ready for the main result of this subsection.

Theorem 7. *With the above notation, if $N = \mathcal{A} + \mathcal{M}$, then the algebra homomorphism $\lambda\colon \mathbb{R}[y_1, \dots, y_{\ell+3}] \longrightarrow \mathbb{R}[x_1, x_2, x_3]$ given by*

$$y_i \longmapsto \sqrt{\frac{\mathcal{A}^{\mathcal{A}} \prod_{j=1}^{\ell} m_j^{m_j}}{N^N}}\, x_i, \qquad \textit{for} \quad i = 1, 2,$$
$$y_3 \longmapsto \frac{\mathcal{A}}{N} x_3, \qquad y_{3+i} \longmapsto \frac{m_i}{N} x_3, \qquad \textit{for} \quad i = 1, \dots, \ell$$

is a $\mathbb{Z}$-graded arrow lifting to a $\mathbb{Z}$-graded symplectomorphism $X_N \to M_0$.

Proof. Using the relation $\{z_i, \overline{z}_j\} = -2\sqrt{-1}\delta_{ij}$, a straightforward calculation yields

$$\{\rho_1, \rho_2\} = (z_{\ell+1}\overline{z}_{\ell+1})^{\mathcal{A}} \left(\prod_i (z_i\overline{z}_i)^{m_i}\right) \left(\frac{\mathcal{A}^2}{z_{\ell+1}\overline{z}_{\ell+1}} + \sum_i \frac{m_i^2}{z_i\overline{z}_i}\right).$$

Writing $\overline{y}_i$ for the class of y_i in $\mathbb{R}[\boldsymbol{y}]/\ker(\psi)$ and using $\overline{y}_{i+3} = \frac{m_i}{\mathcal{A}}\overline{y}_3$ this leads to the relation

$$\{\overline{y}_1, \overline{y}_2\} = \frac{(\mathcal{A} + \mathcal{M}) \prod_i m_i^{m_i}}{\mathcal{A}^{\mathcal{M}-1}} \overline{y}_3^{\mathcal{A}+\mathcal{M}-1} =: \mathcal{B}\overline{y}_3^{\mathcal{A}+\mathcal{M}-1}.$$

Moreover, one can check that $\{\rho_1,\rho_3\} = 2\mathcal{A}\rho_2$ and $\{\rho_2,\rho_3\} = -2\mathcal{A}\rho_1$. We record our commutation relations $\{\overline{y}_i,\overline{y}_j\}$ in the table:

$$\begin{array}{c|ccc} & \overline{y}_1 & \overline{y}_2 & \overline{y}_3 \\ \hline \overline{y}_1 & 0 & \mathcal{B}\overline{y}_3^{\mathcal{A}+\mathcal{M}-1} & 2\mathcal{A}\overline{y}_2 \\ \overline{y}_2 & & 0 & -2\mathcal{A}\overline{y}_1 \\ \overline{y}_3 & & & 0 \end{array}$$

where we have omitted all $\overline{y}_{3+i}$, $i = 1,\dots,\ell$ for the sake of brevity.

Similarly, we write $\overline{x}_i$ for the class of x_i in $\mathbb{R}[\boldsymbol{x}]/\ker(\varphi)$. We leave it to the reader to verify the multiplication table for the commutation relations $\{\overline{x}_i,\overline{x}_j\}$:

$$\begin{array}{c|ccc} & \overline{x}_1 & \overline{x}_2 & \overline{x}_3 \\ \hline \overline{x}_1 & 0 & N^2\overline{x}_3^{N-1} & 2N\overline{x}_2 \\ \overline{x}_2 & & 0 & -2N\overline{x}_1 \\ \overline{x}_3 & & & 0 \end{array} \tag{4.3}$$

In order to construct the arrow λ we make the definitions

$$y_1 \mapsto \alpha x_1, \qquad y_2 \mapsto \alpha x_2, \qquad y_3 \mapsto \beta x_3,$$

where α, β are determined from the multiplication tables, i.e.,

$$2\alpha\beta N = 2\mathcal{A}\alpha \qquad \Rightarrow \quad \beta = \mathcal{A}/N,$$

$$\alpha^2 N^2 = \beta^{N-1}\mathcal{B} \qquad \Rightarrow \quad \alpha = \sqrt{\frac{(\mathcal{A}/N)^{N-1}\mathcal{B}}{N^2}} = \sqrt{\frac{\mathcal{A}^{\mathcal{A}}\prod_{j=1}^{\ell} m_j^{m_j}}{N^N}}.$$

Due to the identity $\alpha^2 = \frac{\beta^N\mathcal{B}}{N\mathcal{A}}$, the generator

$$y_1^2 + y_2^2 - \frac{\prod_{i=1}^{\ell} m_i^{m_i}}{\mathcal{A}^{\mathcal{M}}} y_3^{\mathcal{A}+\mathcal{M}} = y_1^2 + y_2^2 - \frac{\mathcal{B}}{N\mathcal{A}} y_3^N$$

of $\ker(\psi)$ is sent to the generator $x_1^2 + x_2^2 - x_3^N$ of $\ker(\varphi)$, which proves that λ is an arrow (the semialgebraic condition being clearly fulfilled). By the lifting theorem, λ lifts to a Poisson map $X_N \to M_0$. The inverse Poisson map can be constructed by lifting the arrow $x_1 \mapsto \alpha^{-1}y_1$, $x_2 \mapsto \alpha^{-1}y_2$ and $x_3 \mapsto \beta^{-1}y_3$. ■

Let us finish this section by mentioning a simple application of Theorem 7. It is easy to see that if the weight matrix is of block form

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

then there is a $\mathbb{Z}$-graded regular symplectomorphism from the symplectic quotient M_A to $M_{A_1} \times M_{A_2}$. So if we consider for example (cf. [17, p. 108]) the weight matrix $A \in \mathbb{Z}^{\ell\times 2\ell}$ whose columns are given by $\pm e_i$, where the e_i are the standard basis vectors in $\mathbb{R}^\ell$, it follows that the reduced space M_A is $\mathbb{Z}$-graded regular symplectomorphic to the ℓ-fold cartesian product of $\mathbb{C}/\mathbb{Z}_2$.

5 Counterexamples in dimension 4

In this section, we prove that for certain unitary simplicial circle representations, there cannot exist a $\mathbb{Z}$-graded regular symplectomorphism from the symplectic quotient to a quotient of a linear symplectic action of a finite group. Before explaining our strategy, we note that a natural idea is to examine ring theoretic features to distinguish symplectic quotients from finite quotients. For example, it is known that invariant rings of unimodular representations of finite groups are Gorenstein. Unfortunately, cotangent lifted torus representations lead also to Gorenstein rings, because the representation matrices are unimodular. Since, provided the weight matrix A has full rank, the shell relations cut out a complete intersection in $\mathbb{R}[\mathbb{C}^n]^G$, the rings of regular functions $\mathbb{R}[M_A]$ on the symplectic quotient space M_A are Gorenstein as well.

The only invariant we found useful in telling our symplectic quotients apart from finite quotients as Poisson differential spaces is the *Hilbert series* (also called the *Poincaré series*) of the $\mathbb{Z}$-graded ring of regular functions. This is an invariant under $\mathbb{Z}$-graded regular symplectomorphism; whether it is an invariant under symplectomorphism is not yet clear. Let $V = \oplus_{i\geq 0} V_i$ be a positively graded, locally finite-dimensional vector space over the field $\mathbb{K}$. Then the Hilbert series of V is defined as the formal power series

$$\mathrm{Hilb}_{V|\mathbb{K}}(t) = \sum_{i\geq 0} \dim_{\mathbb{K}}(V_i) t^i \in \mathbb{Z}[\![t]\!].$$

The Hilbert series of an invariant ring of a compact groups can be calculated using Molien's formula (see e.g. [31]). It behaves well under cutting out complete intersections, which can be seen easily using the minimal free resolution. Because our algebras of invariants are finitely generated, their Hilbert series can be written as $Q(t)/P(t)$, where $Q(t) \in \mathbb{Z}[t]$ and $P(t)$ is of the form $\prod_{i=1}^{r} (1 - t^{n_i})^{k_i}$ with k_i the number of generators in degree n_i. The Gorensteinness is reflected by the fact that $Q(t)$ is palindromic.

Computations were performed using Singular[1] and Mathematica[2].

5.1 Weight matrices of type $[-1, 1, m]$

Our first task here is to determine a real Hilbert basis (i.e., complete systems of real homogeneous polynomial invariants) for linear $G = \mathbb{S}^1$-actions on $\mathbb{C}^3$ corresponding to weight matrices of the form $[-1, 1, m]$ for $m = 1, 2, \ldots$. To this end we use the algorithm of Sturmfels [31] together with the Groebner basis facilities of the computer algebra system Singular. There is one pitfall here, namely that we regard $\mathbb{C}^3$ as a 6-dimensional real vector space. This means that G-operates also on the $\overline{z}$'s, so that we actually work with the weight matrix $[-1, 1, m, 1, -1, -m]$, where the coordinates are ordered as $(z_1, z_2, z_3, \overline{z}_1, \overline{z}_2, \overline{z}_3)$.

The computer calculations indicate that the real Hilbert basis consists of the obvious real polynomials $z_1\overline{z}_1$, $z_2\overline{z}_2$, and $z_3\overline{z}_3$, as well as the real and imaginary parts of

$$z_1 z_2 \qquad \text{and} \qquad z_1^{m-i}\overline{z}_2^{i} z_3 \qquad \text{for} \quad i = 0, 1, \ldots, m,$$

giving altogether $7 + 2m$ polynomials. This pattern has been verified for $m = 1, \ldots, 10$ using Singular. The next task is to determine the algebraic relations among these generators. We restrict to the cases $m = 1$ and 2; the rest of this subsection is devoted to a more detailed presentation of these two cases.

[1]Decker W., Greuel G.M., Pfister G., Schönemann H., SINGULAR 3-1-3 – A computer algebra system for polynomial computations, 2011, http://www.singular.uni-kl.de.

[2]Wolfram Research Inc., MATHEMATICA Edition: Version 7.0, Champaign, Illinois, 2008, http://www.wolfram.com/mathematica/

Table 1. The Betti table for the symplectic quotient associated to $[-1, 1, 1]$.

	0	1	2	3	4	5
0	1	–	–	–	–	–
1	–	1	–	–	–	–
2	–	–	–	–	–	–
3	–	9	–	–	–	–
4	–	–	25	–	–	–
5	–	–	–	25	–	–
6	–	–	–	–	9	–
7	–	–	–	–	–	–
8	–	–	–	–	1	–
9	–	–	–	–	–	1
total	1	10	25	25	10	1

$\boldsymbol{m = 1.}$ Here, the Hilbert basis consists of 9 polynomials of degree 2:

$$\rho_1 = z_1\overline{z}_1, \qquad \rho_2 = z_2\overline{z}_2, \qquad \rho_3 = z_3\overline{z}_3, \qquad \rho_4 = \operatorname{Re}(z_1z_2), \qquad \rho_5 = \operatorname{Im}(z_1z_2),$$
$$\rho_6 = \operatorname{Re}(z_1z_3), \qquad \rho_7 = \operatorname{Im}(z_1z_3), \qquad \rho_8 = \operatorname{Re}(z_2\overline{z}_3), \qquad \rho_9 = \operatorname{Im}(z_2\overline{z}_3).$$

Among them we have 9 quadratic relations (seen, of course, as being of degree 4):

$$\begin{aligned} &\rho_4^2 + \rho_5^2 - \rho_1\rho_2 = 0, \qquad &&\rho_3\rho_4 + \rho_7\rho_9 - \rho_6\rho_8 = 0, \qquad &&\rho_6\rho_9 + \rho_7\rho_8 - \rho_3\rho_5 = 0,\\ &\rho_6^2 + \rho_7^2 - \rho_1\rho_3 = 0, &&\rho_4\rho_8 + \rho_5\rho_9 - \rho_2\rho_6 = 0, &&\rho_2\rho_7 + \rho_4\rho_9 - \rho_5\rho_8 = 0,\\ &\rho_8^2 + \rho_9^2 - \rho_2\rho_3 = 0, &&\rho_4\rho_6 + \rho_5\rho_7 - \rho_1\rho_8 = 0, &&\rho_4\rho_7 + \rho_5\rho_6 - \rho_1\rho_9 = 0. \end{aligned}$$

That the numbers of generators and relations coincide here seems to be an accident. Using the language of Subsection 6, we construct a $\mathbb{Z}$-graded global chart $\varphi : \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, x_2, \dots, x_9] \to \mathcal{C}^\infty(M_0)$, of the form $x_i \mapsto \varphi_i$, where φ_i is simply ρ_i seen as an element of $\mathcal{C}^\infty(M_0)$, and we assign to all x_i the degree 2. In order to determine $\ker(\varphi)$, we use the fact that the representation is simplicial and therefore, by Proposition 1, the only additional relation comes from the shell relation $J = (\rho_2+\rho_3-\rho_1)/2 = 0$. Summing up, we conclude that the $\mathbb{Z}$-graded ring $\mathbb{R}[M_0]$ of regular functions on the symplectic quotient $M_0 = Z/G$ is isomorphic to the ring $\mathbb{R}[x_1, x_2, \dots, x_9]/I$, where $I = \ker(\varphi)$ is the homogeneous ideal

$$\begin{aligned} I = \langle &x_4^2 + x_5^2 - x_1x_2,\ x_6^2 + x_7^2 - x_1x_3,\ x_8^2 + x_9^2 - x_2x_3,\ x_3x_4 + x_7x_9 - x_6x_8,\\ &x_4x_8 + x_5x_9 - x_2x_6,\ x_2x_7 + x_4x_9 - x_5x_8,\ x_6x_9 + x_7x_8 - x_3x_5,\\ &x_4x_6 + x_5x_7 - x_1x_8,\ x_4x_7 + x_5x_6 - x_1x_9,\ x_2 + x_3 - x_1\rangle. \end{aligned}$$

It is convenient to record some of the information contained in the minimal free resolution of the $\mathbb{Z}$-graded $\mathbb{R}[x_1, x_2, \dots, x_9]$-module $\mathbb{R}[x_1, x_2, \dots, x_9]/I$ in the so-called *Betti table* (for more details see, e.g., [10]). For the example at hand, we have computed the Betti table using Singular; see Table 1.

It is easy to read off from the Betti table the Hilbert series of $\mathbb{R}[M_0]$:

$$\begin{aligned} \operatorname{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) &= \frac{1 - t^2 - 9t^4 + 25t^6 - 25t^8 + 9t^{10} + t^{12} - t^{14}}{(1-t^2)^9} = \frac{1 + 4t^2 + t^4}{(1-t^2)^4}\\ &= 1 + 8t^2 + 27t^4 + 64t^6 + 125t^8 + \cdots \overset{*}{=} \sum_{n=0}^{\infty} (n+1)^3 t^{2n}. \end{aligned} \tag{5.1}$$

The step $(*)$ follows easily from the identity $\sum_i (-1)^i \binom{4}{i}(k+i)^3 = 0$, $k \geq 0$, which in turn can be proved by induction. △

$\boldsymbol{m = 2.}$ Here the real Hilbert basis consists of eleven elements:

$$\text{degree2:} \begin{cases} \rho_1 = z_1\overline{z}_1, \qquad \rho_2 = z_2\overline{z}_2, \qquad \rho_3 = z_3\overline{z}_3,\\ \rho_4 = \operatorname{Re}(z_1z_2), \quad \rho_5 = \operatorname{Im}(z_1z_2), \end{cases}$$

$$\text{degree3: } \begin{cases} \rho_6 = \mathrm{Re}\left(z_1^2 z_3\right), & \rho_8 = \mathrm{Re}(z_1\overline{z}_2 z_3), \quad \rho_{10} = \mathrm{Re}(z_2^2\overline{z}_3), \\ \rho_7 = \mathrm{Im}\left(z_1^2 z_3\right), & \rho_9 = \mathrm{Im}(z_1\overline{z}_2 z_3), \quad \rho_{11} = \mathrm{Im}(z_2^2\overline{z}_3). \end{cases}$$

According to Singular, we have altogether 24 relations:

$$\text{degree4: } \left\{\rho_4^2 + \rho_5^2 - \rho_1\rho_2 = 0,\right.$$

$$\text{degree5: } \begin{cases} \rho_4\rho_{10} + \rho_5\rho_{11} - \rho_2\rho_8 = 0, & \rho_2\rho_9 + \rho_4\rho_{11} - \rho_5\rho_{10} = 0, \\ \rho_5\rho_9 + \rho_2\rho_6 - \rho_4\rho_8 = 0, & \rho_4\rho_9 + \rho_5\rho_8 - \rho_2\rho_7 = 0, \\ \rho_4\rho_8 + \rho_5\rho_9 - \rho_1\rho_{10} = 0, & \rho_4\rho_9 + \rho_1\rho_{11} - \rho_5\rho_8 = 0, \\ \rho_4\rho_6 + \rho_5\rho_7 - \rho_1\rho_8 = 0, & \rho_1\rho_9 + \rho_5\rho_6 - \rho_4\rho_7 = 0, \end{cases}$$

$$\text{degree6: } \begin{cases} \rho_9\rho_{11} - \rho_8\rho_{10} + \rho_2\rho_3\rho_4 = 0, & \rho_8\rho_{11} + \rho_9\rho_{10} - \rho_2\rho_3\rho_5 = 0, \\ \rho_6\rho_8 + \rho_7\rho_9 - \rho_1\rho_3\rho_4 = 0, & \rho_6\rho_9 - \rho_7\rho_8 + \rho_1\rho_3\rho_5 = 0, \\ \rho_6^2 + \rho_7^2 - \rho_1^2\rho_3 = 0, & \rho_3(\rho_4^2 - \rho_5^2) + \rho_7\rho_{11} - \rho_6\rho_{10} = 0, \\ \rho_8^2 + \rho_9^2 - \rho_1\rho_2\rho_3 = 0, & \rho_9^2 - \rho_8^2 + \rho_6\rho_{10} + \rho_7\rho_{11} = 0, \\ 2\rho_8\rho_9 + \rho_6\rho_{11} - \rho_7\rho_{10} = 0, & \rho_6\rho_{11} + \rho_7\rho_{10} - 2\rho_3\rho_4\rho_5 = 0, \\ \rho_{10}^2 + \rho_{11}^2 - \rho_2^2\rho_3 = 0, & \end{cases}$$

$$\text{degree8: } \begin{cases} \rho_1\left(\rho_{10}^2 + \rho_{11}^2\right) - \rho_2\left(\rho_8^2 + \rho_9^2\right) = 0, \\ \rho_1\left(\rho_8^2 + \rho_9^2\right) - \rho_2\left(\rho_6^2 + \rho_7^2\right) = 0, \\ \rho_1(\rho_8\rho_{10} - \rho_9\rho_{11}) - \rho_2(\rho_6\rho_8 + \rho_7\rho_9) = 0, \\ \rho_1(\rho_8\rho_{11} + \rho_9\rho_{10}) + \rho_2(\rho_6\rho_9 - \rho_7\rho_8) = 0. \end{cases}$$

We would like to point out that it is, in principle, often more convenient to work with the complexification of our base ring $\mathbb{R}[\mathbb{C}^n]$. Doing so, we can find a Hilbert basis consisting of monomials, e.g., $z_1z_2 = \rho_4 + \sqrt{-1}\rho_5$, $\overline{z}_1\overline{z}_2 = \rho_4 - \sqrt{-1}\rho_5$ etc. As a consequence of complexification, the relations can be written as binomials.

Analogous to the case $m = 1$, we obtain a $\mathbb{Z}$-graded global chart for the symplectic quotient $M_0 = Z/G$,

$$\varphi: \ \mathbb{R}[\boldsymbol{x}] = \mathbb{R}[x_1, x_2, \ldots, x_{11}] \to \mathcal{C}^\infty(M_0), \qquad x_i \mapsto \varphi_i,$$

where φ_i is ρ_i seen as an element in $\mathcal{C}^\infty(M_0)$, and we assign to $x_1, \ldots, x_5$ the degree 2 and to $x_6, \ldots, x_{11}$ the degree 3. Taking into account the shell relation $J = (\rho_2 + 2\rho_3 - \rho_1)/2 = 0$, we see that the homogeneous ideal $I = \ker(\varphi) \subset \mathbb{R}[\boldsymbol{x}]$ is generated by 25 polynomials. For the sake of brevity we leave it to the reader to write them down. Once again, Singular is able to compute the minimal free resolution of the $\mathbb{R}[\boldsymbol{x}]$-module $\mathbb{R}[\boldsymbol{x}]/I$. The Betti table given in Table 2. From this it is easy to derive a formula for the Hilbert series $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) = Q(t)/P(t)$:

$$\begin{aligned} Q(t) &= 1 - t^2 - t^4 - 8t^5 - 10t^6 + 16t^7 + 43t^8 + 16t^9 - 53t^{10} - 72t^{11} + 72t^{13} \\ &\quad + 53t^{14} - 16t^{15} - 43t^{16} - 16t^{17} + 10t^{18} + 8t^{19} + t^{20} + t^{22} - t^{24}, \\ P(t) &= \left(1 - t^2\right)^5\left(1 - t^3\right)^6. \end{aligned}$$

This can be reduced to

$$\frac{1 + 2t^2 + 4t^3 + 2t^4 + t^6}{(1 - t^2)^2(1 - t^3)^2} = 1 + 4t^2 + 6t^3 + 9t^4 + 16t^5 + 26t^6 + 30t^7 + \cdots . \qquad \triangle$$

Let us close this subsection with the simple observation that the GIT-quotient $\mathbb{C}^3/\!/\mathbb{T}^1_{\mathbb{C}}$ corresponding to weight matrix $A = [-1, 1, m]$ is the affine space $\mathbb{C}^2$. In fact, $t \in \mathbb{T}^1_{\mathbb{C}}$ acts on $(q_1, q_2, q_3) \in \mathbb{C}^3$ by $t.q_1 = t^{-1}q_1$, $t.q_2 = tq_2$ and $t.q_3 = t^m q_3$. A complex Hilbert basis is provided by the algebraically independent polynomials $q_1q_2, q_1^m q_3 \in \mathbb{C}[q_1, q_2, q_3]^{\mathbb{T}^1_{\mathbb{C}}} = \mathbb{C}[\mathbb{C}^3]^{\mathbb{T}^1_{\mathbb{C}}}$.

Table 2. The Betti table for the symplectic quotient associated to $[-1,1,2]$.

	0	1	2	3	4	5	6	7
0	1	–	–	–	–	–	–	–
1	–	1	–	–	–	–	–	–
2	–	–	–	–	–	–	–	–
3	–	1	–	–	–	–	–	–
4	–	8	1	–	–	–	–	–
5	–	11	16	–	–	–	–	–
6	–	–	43	8	–	–	–	–
7	–	–	24	53	–	–	–	–
8	–	–	–	72	21	–	–	–
9	–	–	–	21	72	–	–	–
10	–	–	–	–	53	24	–	–
11	–	–	–	–	8	43	–	–
12	–	–	–	–	–	16	11	–
13	–	–	–	–	–	1	8	–
14	–	–	–	–	–	–	1	–
15	–	–	–	–	–	–	–	–
16	–	–	–	–	–	–	1	–
17	–	–	–	–	–	–	–	1
total	1	21	84	154	154	84	21	1

5.2 Finite subgroups of U_2

In this subsection, we consider the finite subgroups G of U_2. The classification of such subgroups is due to P. du Val [5, 8], and the details of the classification are recalled in Appendix A. Of particular importance is the ADE-classification of finite subgroups of SU_2. Here, the quotients $\mathbb{C}^2/G$ lead to the famous Kleinian singularities, while quotients $(\mathbb{C}^2\times\overline{\mathbb{C}^2})/G$ appear not to be as well-understood.

Rather than an exhaustive computation of the Hilbert series for the nine families of finite subgroups U_2, it will be sufficient for our purposes to determine the Hilbert series for certain subgroups $G<\mathrm{SU}_2$. By the ADE-classification, any such group is cyclic or conjugate to a binary dihedral, tetrahedral, octahedral, or icosahedral group.

Recall the formula of Molien [22], see e.g. [31, Theorem 2.2.1], which expresses the Hilbert series for the G-invariant polynomials on $\mathbb{C}^2$ as

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t)=\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(\mathrm{id}-tg)}. \tag{5.2}$$

In order to evaluate these summations, we will make use of a Dedekind sum formula of I. Gessel [13].

5.2.1 Cyclic groups

Suppose $G\cong\mathbb{Z}_N$ is a cyclic group of order N with generator $\mathrm{diag}(\omega_N,\omega_N^{-1})$ in complex coordinates, where ω_N is a primitive Nth root of unity. The action of this generator on $\mathbb{C}^2\times\overline{\mathbb{C}^2}$ is given by the matrix $\alpha_N=\mathrm{diag}(\omega_N,\omega_N^{-1},\omega_N^{-1},\omega_N)$. We will give explicit computations of the Hilbert series for two cases of particular interest separately followed by the general case.

$\boldsymbol{N=2}$. In this case, as $\omega_2=\omega_2^{-1}$, all quadratic polynomials are invariant, and a Hilbert basis is given by

$$\rho_1=z_1\overline{z}_1,\qquad \rho_2=z_2\overline{z}_2,\qquad \rho_3=\mathrm{Re}(z_1z_2),\qquad \rho_4=\mathrm{Im}(z_1z_2),\qquad \rho_5=\mathrm{Re}(z_1\overline{z}_2),$$
$$\rho_6=\mathrm{Im}(z_1\overline{z}_2),\qquad \rho_7=\mathrm{Re}\left(z_1^2\right),\qquad \rho_8=\mathrm{Im}\left(z_1^2\right),\qquad \rho_9=\mathrm{Re}\left(z_2^2\right),\qquad \rho_{10}=\mathrm{Im}\left(z_2^2\right).$$

The Hilbert series is computed to be

$$\begin{aligned}\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{Z}_2}|\mathbb{R}}(t) &= \frac{1+6t^2+t^4}{(1-t^2)^4} = \sum_{n=0}^{\infty}\binom{3+2n}{3}t^{2n}\\ &= 1+10t^2+35t^4+84t^6+165t^8+286t^{10}+\cdots.\end{aligned}$$ △

***N* = 3.** In this case, the Hilbert series is computed to be

$$\begin{aligned}\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{Z}_3}|\mathbb{R}}(t) &= \frac{1-2t+5t^2-2t^3+t^4}{(1-t)^2(1-t^3)^2}\\ &= 1+4t^2+8t^3+9t^4+20t^5+30t^6+36t^7+57t^8+\cdots.\end{aligned}$$

It is moreover easy to see that the 4 quadratic invariants are given by $z_1\overline{z}_1$, $z_2\overline{z}_2$, and the real and imaginary parts of z_1z_2. △

General *N*. In general, equation (5.2) yields the Hilbert series

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{Z}_N}|\mathbb{R}}(t) = \frac{1}{N}\sum_{g\in\mathbb{Z}_N}\frac{1}{\det(\mathrm{id}-gt)} = \frac{1}{N}\sum_{\zeta^N=1}\frac{1}{(1-\zeta t)^2(1-\zeta^{-1}t)^2}.$$

Applying Gessel's formula [13, Theorem 4.2], it follows that $\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{Z}_N}|\mathbb{R}}(t)$ is given by the x^2y^2-coefficient in the formal power series expansion of

$$\frac{1}{(1-x)(1-y)-t^2}\left(\frac{x\big(1-x-t^2\big)(1-x)^{N-1}}{(1-x)^N-t^N}+\frac{y(1-y-t^2)(1-y)^{N-1}}{(1-y)^N-t^N}-xy\right).$$

The x^2y^2-coefficient is

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{Z}_N}|\mathbb{R}}(t) = \frac{1+t^2+2Nt^N-t^{2N}-2Nt^{N+2}-t^{2N+2}}{(1-t^2)^3(1-t^N)^2}.$$

From this expression, it is easy to compute that if $N>2$, then the t^2-coefficient of the Hilbert series is 4. Similarly, the t^3-coefficient vanishes for $N>3$.

5.2.2 Binary dihedral groups

Suppose now that G is a binary dihedral group $\mathbb{D}_N$ of order $4N$ for $N\geq 1$, which in complex coordinates is generated by the two elements, $\mathrm{diag}(\omega_{2N},\omega_{2N}^{-1})$ and

$$b = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}, \tag{5.3}$$

where ω_{2N} is a primitive $2N$th root of unity. The action of these generators on $\mathbb{C}^2\times\overline{\mathbb{C}^2}$ is given by the matrices $\alpha_{2N} = \mathrm{diag}\left(\omega_{2N},\omega_{2N}^{-1},\omega_{2N}^{-1},\omega_{2N}\right)$ and $\beta = \mathrm{diag}(b,b)$, the latter in 2×2 blocks.

To apply equation (5.2), note that the $4N$ elements of $\mathbb{D}_N$ are given by the $2N$ powers of α_{2N} as well as elements of the form $\alpha^k\beta$ with $0\leq k\leq 2N-1$. A simple computation demonstrates that $\det(\mathrm{id}-\alpha^k\beta t) = (1+t^2)^2$ does not depend on k so that the sum in equation (5.2) can be split into a sum over the cyclic group $\langle\alpha\rangle$ and its complement in $\mathbb{D}_N$. That is,

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{D}_N}|\mathbb{R}}(t) = \frac{1}{4N}\sum_{g\in\mathbb{D}_N}\frac{1}{\det(\mathrm{id}-gt)} = \frac{1}{4N}\left(\sum_{k=1}^{2N}\frac{1}{\det(\mathrm{id}-\alpha^k t)}+\sum_{k=1}^{2N}\frac{1}{\det(\mathrm{id}-\alpha^k\beta t)}\right)$$

$$= \frac{1}{4N}\left(\sum_{\zeta^{2N}=1}\frac{1}{(1-\zeta t)^2(1-\zeta^{-1}t)^2}\right) + \frac{1}{2(1+t^2)^2},$$

where the final sum corresponds to the case of a cyclic group treated above. It follows again by an application of Gessel's formula that the Hilbert series is given by $\operatorname{Hilb}_{\mathbb{R}[\mathbb{C}^2]^{\mathbb{D}_N}|\mathbb{R}}(t) = Q(t)/P(t)$ where

$$\begin{aligned} Q(t) &= 1 + 3t^4 + (2N-1)t^{2N} + (2N+3)t^{2N+2} - (2N+3)t^{2N+4} \\ &\quad - (2N-1)t^{2N+6} - 3t^{4N+2} - t^{4N+6}, \\ P(t) &= \left(1-t^2\right)\left(1-t^4\right)^2\left(1-t^{2N}\right)^2. \end{aligned}$$

In particular, as $Q(t)/P(t)$ are even functions, the Hilbert series coefficients vanish in odd orders. Moreover, one computes that the t^2-coefficient is 4 if $N = 1$ and 1 if $N > 1$.

5.2.3 Binary tetrahedral, octahedral, and icosahedral groups

The three remaining subgroups of SU_2 are the binary tetrahedral group $\mathbb{T}_{24}$, the binary octahedral group $\mathbb{O}_{48}$, and the binary icosahedral group $\mathbb{I}_{120}$. For our purposes, it will be sufficient to note that $\mathbb{T}_{24}$ and $\mathbb{O}_{48}$ both contain a subgroup isomorphic to $\mathbb{Z}_4$ which in complex coordinates is generated by $\operatorname{diag}(\sqrt{-1}, -\sqrt{-1})$, and hence coincides with $\mathbb{Z}_4$ above. Similarly, $\mathbb{I}_{120}$ contains a subgroup isomorphic to $\mathbb{Z}_{10}$ which in complex coordinates is generated by $\operatorname{diag}(-\omega_5^3, -\omega_5^2)$, and hence is conjugate to the action of $\mathbb{Z}_{10}$ given above.

5.3 $[-1,1,1]$ and $[-1,1,2]$ are not orbifolds

Our final aim is to show that $\operatorname{Hilb}_{\mathbb{R}[M_A]|\mathbb{R}}(t)$ for $A = [-1,1,1]$ or $[-1,1,2]$ cannot coincide with $\operatorname{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t)$ for any finite subgroup $G < \mathrm{U}_2$. The argument will be based on the following observation. Suppose $H \leq G$ and consider

$$\operatorname{Hilb}_{\mathbb{R}[\mathbb{C}^2]^H|\mathbb{R}}(t) = \sum_{k=0}^{\infty} a_k t^k, \qquad \operatorname{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t) = \sum_{k=0}^{\infty} b_k t^k.$$

Then any polynomial invariant under G is also invariant under H, implying that $b_k \leq a_k$ for each k. Moreover, note that any finite subgroup of U_1 acting on $\mathbb{C}^2$ as scalar multiplication is cyclic, and in $(z_1, z_2, \overline{z}_1, \overline{z}_2)$-coordinates is generated by $\operatorname{diag}(\omega_N, \omega_N, \omega_N^{-1}, \omega_N^{-1})$ for some N. Permuting coordinates, this action is conjugate to the action generated by $\operatorname{diag}(\omega_N, \omega_N^{-1}, \omega_N, \omega_N^{-1})$ so that the algebra of real invariant polynomials is isomorphic to the algebra of invariants of a cyclic subgroup of SU_2, see Subsection 5.2.1.

We use the notation $(L/L_K; R/R_K)$ to indicate a finite subgroup of U_2, where $L_K \trianglelefteq L$ are finite subgroups of U_1, $R_K \trianglelefteq R$ are finite subgroups of SU_2, and $(L/L_K; R/R_K)$ contains both L_K and R_K as subgroups. Note that $\trianglelefteq$ indicates a normal subgroup.

5.3.1 $[-1,1,1]$ is not an orbifold

Let M_0 denote the reduced space associated to the weight matrix $[-1,1,1]$. Recall (see equation (5.1)) that in this case, $\operatorname{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) = 1 + 8t^2 + 27t^4 + \cdots$. Based on the t^2-coefficient, it follows that if $\operatorname{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) = \operatorname{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t)$ for some finite $G < \mathrm{U}_2$, then G contains no cyclic subgroups of order greater than 2 and no binary dihedral subgroups. This eliminates all of the nine families of finite subgroups of U_2 other than Type 1 and Type 3, see Appendix A.

Suppose G is a Type 1 group of the form $(\mathbb{Z}_{2m}/\mathbb{Z}_f; \mathbb{Z}_{2n}/\mathbb{Z}_g)_d$, and then as G contains $\mathbb{Z}_f < \mathrm{U}_2$ and $\mathbb{Z}_g < \mathrm{SU}_2$, we have that $f \leq 2$ and $g \leq 2$. As $f \equiv g \mod 2$, there are only two cases.

Type 1, $f = g = 2$. In this case, G is of the form $(\mathbb{Z}_{2r}/\mathbb{Z}_2; \mathbb{Z}_{2r}/\mathbb{Z}_2)_d$ where r is a positive integer and $d \leq r$ is relatively prime to r. If $r = 1$, then $d = 1$ and $G = \mathbb{Z}_2 < \mathrm{SU}_2$, whose Molien series was computed in Subsection 5.2 and does not coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. So assume $r \geq 2$. Then G contains a subgroup generated by $\omega_{2r}\,\mathrm{diag}(\omega_{2r}^{d}, \omega_{2r}^{-d}) = \mathrm{diag}(\omega_{2r}^{d+1}, \omega_{2r}^{1-d})$ for a primitive $2r$th root of unity ω_{2r}, which in $(z_1, z_2, \overline{z}_1, \overline{z}_2)$-coordinates is given by the 4×4 matrix $\mathrm{diag}(\omega_{2r}^{d+1}, \omega_{2r}^{1-d}, \omega_{2r}^{-d-1}, \omega_{2r}^{d-1})$.

To show that the dimension of quadratic invariants fixed by $\langle \alpha \rangle$ has dimension strictly less than 8, and hence that no group containing $\langle \alpha \rangle$ can have the same Hilbert series as $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$, consider the action of α on the quadratic polynomials in z_1, z_2, $\overline{z}_2$, $\overline{z}_2$. With respect to the basis

$$\left\{ z_1\overline{z}_1,\ z_2\overline{z}_2,\ z_1z_2,\ \overline{z}_1\overline{z}_2,\ z_1\overline{z}_2,\ \overline{z}_1z_2,\ z_1^2,\ \overline{z}_1^2,\ z_2^2,\ \overline{z}_2^2 \right\}$$

for the quadratic polynomials, α has 10×10 matrix

$$Q_\alpha = \mathrm{diag}\left(1,\ 1,\ \omega_{2r}^{2},\ \omega_{2r}^{-2},\ \omega_{2r}^{2d},\ \omega_{2r}^{-2d},\ \omega_{2r}^{2d+2},\ \omega_{2r}^{-2d-2},\ \omega_{2r}^{-2d+2},\ \omega_{2r}^{2d-2}\right).$$

Using the trace formula [31, Lemma 2.2.2] used to prove Molien's formula, the dimension of the quadratic polynomials invariant under the action of $\langle \alpha \rangle$ is

$$\frac{1}{|Q_\alpha|} \sum_{k=1}^{|Q_\alpha|} \mathrm{trace}\, Q_\alpha^k,$$

where $|Q_\alpha|$ denotes the the order of Q_α. Note that $|Q_\alpha|$ clearly divides r and moreover that the above formula holds if $|Q_\alpha|$ is replaced by any positive multiple of $|Q_\alpha|$. Clearly, ω_{2r}^2 and ω_{2r}^{-2} are primitive rth roots of unity; similarly, as d is relatively prime to r, ω_{2r}^{2d} and ω_{2r}^{-2d} are primitive rth roots of unit as well. With this, as the sum of all rth roots of unity is zero, we have that the dimension of quadratic $\langle \alpha \rangle$-invariants is

$$\frac{1}{r} \sum_{k=1}^{r} 2 + \omega_{2r}^{(2d+2)k} + \omega_{2r}^{-(2d+2)k} + \omega_{2r}^{(-2d+2)k} + \omega_{2r}^{-(2d+2)k} \leq \frac{1}{r}(6r) = 6.$$

Therefore, G cannot contain α. $\triangle$

Type 1, $f = g = 1$. If $G = (\mathbb{Z}_r/1; \mathbb{Z}_r/1)_d$ for r even and $d < r$ relatively prime to r, then G contains $\alpha = \omega\,\mathrm{diag}(\omega_r^{d}, \omega_r^{-d}) = \mathrm{diag}(\omega_r^{d+1}, \omega_r^{-d+1})$. If $r = 2$, then G is trivial, so assume $r \geq 4$. The action on quadratic polynomials is given by

$$Q_\alpha = \mathrm{diag}\left(1,\ 1,\ \omega_r^{2},\ \omega_r^{-2},\ \omega_r^{2d},\ \omega_r^{-2d},\ \omega_r^{2d+2},\ \omega_r^{-2d-2},\ \omega_r^{-2d+2},\ \omega_r^{2d-2}\right).$$

As ω_r^2, ω_r^{-2}, ω_r^{2d}, and ω_r^{-2d} are primitive $r/2$nd roots of unity, the dimension of quadratic $\langle \alpha \rangle$-invariants is

$$\frac{1}{r} \sum_{k=1}^{r} 2 + \omega_r^{(2d+2)k} + \omega_r^{-(2d+2)k} + \omega_r^{(-2d+2)k} + \omega_r^{-(2d+2)k} \leq \frac{1}{r}(6r) = 6.$$

Again, G cannot contain α. It follows that G cannot be a Type 1 group. $\triangle$

Type 3. Suppose G is a Type 3 group of the form either $(\mathbb{Z}_{4m}/\mathbb{Z}_{2m}; \mathbb{D}_l/\mathbb{Z}_{2l})$ or $(\mathbb{Z}_{4m}/\mathbb{Z}_m; \mathbb{D}_l/\mathbb{Z}_l)$ with m and l odd. As G cannot contain cyclic subgroups of U_2 or SU_2 of orders larger than 2, we need only consider the case of $m = l = 1$. Both $(\mathbb{Z}_4/\mathbb{Z}_2; \mathbb{D}_1/\mathbb{Z}_2)$ and $(\mathbb{Z}_4/1; \mathbb{D}_1/1)$ contain the element $\sqrt{-1}b$ where b the 2×2 matrix defined in equation (5.3) above. One computes with Singular that the Hilbert series of invariants for the group generated by this element is

$$\frac{1 + t^2}{(1 - t^2)^2(1 - t)^2} = 1 + 2t + 6t^2 + 10t^3 + 19t^4 + \cdots. \tag{5.4}$$

Hence, this cannot occur as an element of G. $\triangle$

With this, it follows that no such G exists, and $[-1, 1, 1]$ does not admit a $\mathbb{Z}$-graded regular symplectomorphism with an orbifold.

5.3.2 $[-1, 1, 2]$ is not an orbifold

Let M_0 denote the reduced space associated to the weight matrix $[-1, 1, 2]$. Recall (equation (5.1)) that in this case, $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) = 1 + 4t^2 + 6t^3 + 9t^4 + 16t^5 + \cdots$. Based on the t^3-coefficient, it follows that if $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t) = \mathrm{Hilb}_{\mathbb{C}[z_1,\overline{z}_1,z_2,\overline{z}_2]^G}(t)$ for some finite $G < \mathrm{U}_2$, then G contains no cyclic subgroups of order other than 3 and no binary dihedral subgroups. This eliminates all of the nine families of finite subgroups of U_2 other than Type 1 and Type 3.

Suppose G is a Type 1 subgroup of the form $(\mathbb{Z}_{2m}/\mathbb{Z}_f; \mathbb{Z}_{2n}/\mathbb{Z}_g)_d$, and then as G contains $\mathbb{Z}_f < \mathrm{U}_2$ and $\mathbb{Z}_g < \mathrm{SU}_2$, we have that $f = 1$ or 3 and $g = 1$ or 3.

Type 1, $f = g = 3$. In this case, G is of the form $(\mathbb{Z}_{3r}/\mathbb{Z}_3; \mathbb{Z}_{3r}/\mathbb{Z}_3)_d$ where r is a positive even integer and $d < r$ is relatively prime to r. Then G contains $\mathbb{Z}_3 < \mathrm{SU}_2$ as well as the subgroup of U_1 generated by scalar multiplication by a primitive 3rd root of unity ω_3. As the quadratic invariants of $\mathbb{Z}_3 < \mathrm{SU}_2$ are $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1 z_2$, and $\overline{z}_1\overline{z}_2$, we need only note that $z_1 z_2$ is not invariant under scalar multiplication by ω_3, so that the space of quadratic G-invariants has dimension strictly less than 4. It follows that the Hilbert series of G-invariants cannot coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$, and G cannot be of this type. $\triangle$

Type 1, $f = 3$, $g = 1$. In this case, $G = (\mathbb{Z}_{3r}/\mathbb{Z}_3; \mathbb{Z}_r/1)_d$ where r is a positive even integer and $d < r$ is relatively prime to r. Note that $\mathbb{Z}_3 < \mathrm{U}_1$ is a proper subgroup of G. Then $G \ni \omega_{3r}\,\mathrm{diag}(\omega_{3r}^{3d}, \omega_{3r}^{-3d}) = \mathrm{diag}(\omega_{3r}^{3d+1}, \omega_{3r}^{-3d+1})$ for a primitive $3r$th root of unity ω_{3r}, which is given in $(z_1, z_2, \overline{z}_1, \overline{z}_2)$-coordinates by $\mathrm{diag}(\omega_{3r}^{3d+1}, \omega_{3r}^{-3d+1}, \omega_{3r}^{-3d-1}, \omega_{3r}^{3d-1})$. The quadratic invariants of the action of $\mathbb{Z}_3 < \mathrm{U}_1$ are $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1\overline{z}_2$, and $\overline{z}_1 z_2$. It is easy to see that $z_1\overline{z}_2$ is not invariant under the action of α for $r > 2$, implying that the space of quadratic G-invariants has dimension strictly less than 4. So assume $r = 2$, and then the Hilbert series of $G = (\mathbb{Z}_6/\mathbb{Z}_3; \mathbb{Z}_2/1)_1$ is computed on Singular to be

$$\frac{1 - 2t + 5t^2 - 2t^3 + t^4}{(1 - t^3)^2(1 - t)^2} = 1 + 2t^2 + 4t^3 + 3t^4 + 8t^5 + 12 + \cdots,$$

which does not coincide $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. Hence G cannot be of this type. $\triangle$

Type 1, $f = 1$, $g = 3$. In this case, G is of the form $(\mathbb{Z}_r/1; \mathbb{Z}_{3r}/\mathbb{Z}_3)_d$ where r is a positive even integer and $d < r$ is relatively prime to r. Then G contains a subgroup generated by $\omega_{3r}^3\,\mathrm{diag}(\omega_{3r}^{d}, \omega_{3r}^{-d}) = \mathrm{diag}(\omega_{3r}^{d+3}, \omega_{3r}^{-d+3})$ for a primitive $3r$th root of unity ω_{3r}. Note that G contains $\mathbb{Z}_3 < \mathrm{SU}_2$, whose quadratic invariants are again spanned by $z_1\overline{z}_1$, $z_2\overline{z}_2$, $z_1 z_2$, and $\overline{z}_1\overline{z}_2$. If $r > 2$, then $z_1 z_2$ is not invariant under the above, implying that the quadratic invariants have dimension strictly less than 4. If $r = 2$, then $G = (\mathbb{Z}_2/1; \mathbb{Z}_6/\mathbb{Z}_3)_1$ is in fact a subgroup of SU_2 isomorphic to $\mathbb{Z}_3$. We again conclude that G cannot be of this type. $\triangle$

Type 1, $f = g = 1$. If $G = (\mathbb{Z}_r/1; \mathbb{Z}_r/1)_d$ for even r and $d < r$ relatively prime to r, then G is generated by $\alpha = \omega_r\,\mathrm{diag}(\omega_r^{d}, \omega_r^{-d}) = \mathrm{diag}(\omega_r^{d+1}, \omega_r^{-d+1})$ If $r = 2$, then G is trivial, so assume $r \geq 4$. If $d = 1$, then $\alpha = \mathrm{diag}(\omega_r^2, 1)$, and if $d = r - 1$, then $\alpha = \mathrm{diag}(1, \omega_r^2)$. In either case, G has nontrivial linear invariants, so that as $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$ has zero t-coefficient, we may exclude these cases. So assume $1 < d < r - 1$, and then as d must be relatively prime to r, it must be that $r \geq 8$.

Now, with respect to the following basis for the cubic monomials,

$$\big\{z_1^3,\, \overline{z}_1^3,\, z_2^3,\, \overline{z}_2^3,\, z_1^2 z_2,\, \overline{z}_1^2\overline{z}_2,\, z_1 z_2^2,\, \overline{z}_1\overline{z}_2^2,\, z_1^2\overline{z}_2,\, \overline{z}_1^2 z_2,\, z_1\overline{z}_2^2,$$
$$\overline{z}_1 z_2^2,\, z_1^2\overline{z}_1,\, z_1\overline{z}_1^2,\, z_2^2\overline{z}_2,\, z_2\overline{z}_2^2,\, z_1\overline{z}_1 z_2,\, z_1\overline{z}_1\overline{z}_2,\, z_1 z_2\overline{z}_2,\, \overline{z}_1 z_2\overline{z}_2\big\}$$

the action of α is given by

$$Q_\alpha = \mathrm{diag}\big(\omega_r^{3d+3},\, \omega_r^{-3d-3},\, \omega_r^{-3d+3},\, \omega_r^{3d-3},\, \omega_r^{d+3},\, \omega_r^{-d-3},\, \omega_r^{-d+3},\, \omega_r^{d-3},\, \omega_r^{3d+1},\, \omega_r^{-3d-1},$$
$$\omega_r^{3d-1},\, \omega_r^{-3d+1},\, \omega_r^{d+1},\, \omega_r^{-d-1},\, \omega_r^{-d+1},\, \omega_r^{d-1},\, \omega_r^{-d+1},\, \omega_r^{d-1},\, \omega_r^{d+1},\, \omega_r^{-d-1}\big).$$

Applying the trace formula [31, Lemma 2.2.2], we have that the dimension of the cubic polynomials invariant under the action of $\langle\alpha\rangle$ is given by

$$\begin{aligned}\frac{1}{r}\sum_{k=1}^{r}\omega_r^{(3d+3)k} &+ \omega_r^{(-3d-3)k} + \omega_r^{(-3d+3)k} + \omega_r^{(3d-3)k} + \omega_r^{(d+3)k} + \omega_r^{(-d-3)k} + \omega_r^{(-d+3)k}\\ &+ \omega_r^{(d-3)k} + \omega_r^{(3d+1)k} + \omega_r^{(-3d-1)k} + \omega_r^{(3d-1)k} + \omega_r^{(-3d+1)k} + \omega_r^{(d+1)k} + \omega_r^{(-d-1)k}\\ &+ \omega_r^{(-d+1)k} + \omega_r^{(d-1)k} + \omega_r^{(-d+1)k} + \omega_r^{(d-1)k} + \omega_r^{(d+1)k} + \omega_r^{(-d-1)k}.\end{aligned} \tag{5.5}$$

For $k = 1$, each term in the above sum is a primitive sth root of unity for some s that divides r so that unless a term is equal to 1, the sum over k of that term vanishes. Recalling that $1 < d < r - 1$ and $r \geq 8$, it is clear that $\omega_r^{\pm d\pm 1} \neq 1$, so that the sum of each $\omega_r^{(\pm d\pm 1)k}$ vanishes. Similarly, as d is relatively prime to r and hence invertible mod r, it is easy to see that at most two of the following congruences can be true mod r:

$$\begin{aligned} 3d+3 &\equiv 0, & 3d-3 &\equiv 0, & d+3 &\equiv 0,\\ d-3 &\equiv 0, & 3d+1 &\equiv 0, & 3d-1 &\equiv 0.\end{aligned}$$

Therefore, when $k = 1$, at most four of the terms in equation (5.5) can be equal to 1, and the dimension of cubic invariants is bounded by $\frac{1}{r}(4r) = 4$. As the dimension of cubic invariants on M_0 is six, we have excluded all groups in this case. △

Type 3, $(\mathbb{Z}_{12}/\mathbb{Z}_3; \mathbb{D}_3/\mathbb{Z}_3)$. As in the case of a Type 1 group with $f = g = 3$, this group contains $\mathbb{Z}_3 < \mathrm{SU}_2$ as well as the subgroup of U_1 generated by scalar multiplication by a primitive 3rd root of unity ω_3; hence the Hilbert series of G-invariants cannot coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. △

Type 3, $(\mathbb{Z}_4/1; \mathbb{D}_3/\mathbb{Z}_3)$. This group has six elements and is generated by $\alpha = \mathrm{diag}(\omega_3, \omega_3^2)$ and $\beta = \sqrt{-1}b$, where b is defined in equation (5.3). The Hilbert series is given by

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t) = \frac{1 + t^2 + 2t^3 + t^4 + t^6}{(1 - t^2)^2(1 - t^3)^2} = 1 + 3t^2 + 4t^3 + 6t^4 + \cdots,$$

which does not coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. △

Type 3, $(\mathbb{Z}_{12}/\mathbb{Z}_3; \mathbb{D}_1/1)$. This group has six elements and is generated by $\omega_{12}b$. The Hilbert series is given by

$$\mathrm{Hilb}_{\mathbb{R}[\mathbb{C}^2]^G|\mathbb{R}}(t) = \frac{1 - t + t^2 + 2t^3 + 2t^5 + t^6 - t^7 + t^8}{(1 - t^6)(1 - t^3)(1 - t^2)(1 - t)} = 1 + 2t^2 + 4t^3 + \cdots,$$

which does not coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. △

Type 3, $(\mathbb{Z}_4/1; \mathbb{D}_1/1)$. The only nontrivial element of this group is $\sqrt{-1}b$. The Hilbert series was computed in equation (5.4) above and does not coincide with $\mathrm{Hilb}_{\mathbb{R}[M_0]|\mathbb{R}}(t)$. △

A Finite subgroups of U_2

For the convenience of the reader, we recall the classification of finite subgroups of U_2 given by [5, 8]. We follow [11]; see also [9].

For $l \in \mathrm{U}_1$ and $r \in \mathrm{SU}_2$, we let (l, r) denote the element of U_2 given by the scalar multiple lr of r. Note that every element of U_2 arises in this way and that the expression is unique up to $(l, r) = (-l, -r)$. Let $L_K \trianglelefteq L < \mathrm{U}_1$ and $R_K \trianglelefteq R < \mathrm{SU}_2$ be finite subgroups of U_1 and SU_2, respectively, such that L/L_K is isomorphic to R/R_K, and let $\phi\colon L/L_K \to R/R_K$ be an isomorphism. Then the group $(L/L_K; R/R_K)_\phi$ is defined as

$$(L/L_K; R/R_K)_\phi = \big\{(l, r) \in L \times R : \phi(lL_K) = rR_K\big\}.$$

Note that ϕ is omitted if it is obvious, and $(L/L_K; R/R_K)_\phi$ has order $|R||L_K|/2$.

Let $\mathbb{Z}_k$ denote a cyclic subgroup of order k. Note that $\mathbb{Z}_k < \mathrm{U}_1$ is generated by a primitive kth root of unity ω_k, while $\mathbb{Z}_k < \mathrm{SU}_2$ is generated by $\mathrm{diag}(\omega_k, \omega_k^{-1})$. The distinction will be clear from the context. Let $\mathbb{D}_p$ denote the binary dihedral group of order $4p$, and let $\mathbb{T}_{24}$, $\mathbb{O}_{48}$, and $\mathbb{I}_{120}$ denote the binary tetrahedral, octahedral, and icosahedral groups, respectively.

The finite subgroups of U_2 are given by the following list.

Type 1. $(\mathbb{Z}_{2m}/\mathbb{Z}_f; \mathbb{Z}_{2n}/\mathbb{Z}_g)_d$ where $f \equiv g \mod 2$, and d is relatively prime to $2m/f = 2n/g$ and indicates the isomorphism $\mathbb{Z}_{2m}/\mathbb{Z}_f \to \mathbb{Z}_{2n}/\mathbb{Z}_g$ sending the class of 1 to the class of d,

Type 2. $(\mathbb{Z}_{2m}/\mathbb{Z}_{2m}; \mathbb{D}_l/\mathbb{D}_l)$,

Type 3. $(\mathbb{Z}_{4m}/\mathbb{Z}_{2m}; \mathbb{D}_l/\mathbb{Z}_{2l})$ and $(\mathbb{Z}_{4m}/\mathbb{Z}_m; \mathbb{D}_l/\mathbb{Z}_l)$ for m and l odd,

Type 4. $(\mathbb{Z}_{4m}/\mathbb{Z}_{2m}; \mathbb{D}_{2l}/\mathbb{D}_l)$,

Type 5. $(\mathbb{Z}_{2m}/\mathbb{Z}_{2m}; \mathbb{T}_{24}/\mathbb{T}_{24})$,

Type 6. $(\mathbb{Z}_{6m}/\mathbb{Z}_{2m}; \mathbb{T}_{24}/\mathbb{D}_2)$,

Type 7. $(\mathbb{Z}_{2m}/\mathbb{Z}_{2m}; \mathbb{O}_{48}/\mathbb{O}_{48})$,

Type 8. $(\mathbb{Z}_{4m}/\mathbb{Z}_{2m}; \mathbb{O}_{48}/\mathbb{T}_{24})$, and

Type 9. $(\mathbb{Z}_{2m}/\mathbb{Z}_{2m}; \mathbb{I}_{120}/\mathbb{I}_{120})$.

Acknowledgements

The authors would like to thank Srikanth Iyengar, Luchezar Avramov, Markus Pflaum, Johan Martens, Karl-Heinz Fieseler, Jedrzej Sniatycki, Gerry Schwarz, Johannes Huebschmann, Michael J. Field and Graeme Wilkin for promptly answering questions, stimulating discussions, and moral support. We would also like to thank the referees for helpful suggestions and comments.

C.F. would like to thank the University of Florence for hospitality during the completion of this manuscript. The research of H.-C. H. has been supported by the Center for the Quantum Geometry of Moduli spaces which is funded by the Danish National Research Foundation, and by the Department of Mathematics of the University of Nebraska at Lincoln. C.S. received support from the Center for the Quantum Geometry of Moduli spaces, a Rhodes College Faculty Development Endowment Grant, and a grant to Rhodes College from the Andrew W. Mellon Foundation.

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