Separation of Variables and Contractions on Two-Dimensional Hyperboloid[⋆]

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Abstract. In this paper analytic contractions have been established in the $R \to \infty$ contraction limit for exactly solvable basis functions of the Helmholtz equation on the two-dimensional two-sheeted hyperboloid. As a consequence we present some new asymptotic formulae.

Key words: analytic contraction; separation of variables; Lie group; Helmholtz equation

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1 Introduction

In the recent series of papers [5, 6, 7, 9, 12, 13, 14, 15, 16] a special class of Inönu–Wigner contractions was introduced, namely "analytic contractions". For an analytic contraction the contraction parameter R, which is the radius of an n-dimensional sphere or pseudosphere $u_0^2 + \varepsilon\vec{u}^2 = R^2$, $(\varepsilon = \pm 1)$ figures in the separated coordinate systems, and in the eigenfunctions and the eigenvalues for the Laplace–Beltrami operator (or Helmholtz equation). With the help of analytic contractions we have established the connection between the procedure of separation of variables for homogeneous spaces with constant (positive or negative) curvature and flat spaces. For instance it has been indicated how the systems of coordinates on the sphere S_2 and hyperboloid H_2 transform to four systems of coordinates on Euclidean space E_2.

The goal of this note is to establish the contraction limit $R \to \infty$ for eigenfunctions (or basis functions) of the two-dimensional Helmholtz equation on the two-sheeted hyperboloid $H_2 : u_0^2 - u_1^2 - u_2^2 = R^2$,

$$\Delta_{\rm LB}\Psi = -\frac{\sigma(\sigma+1)}{R^2}\Psi, \qquad \sigma = -1/2 + i\rho, \tag{1}$$

where the Laplace–Beltrami operator in the curvilinear coordinates (ξ^1, ξ^2) has the form

$$\Delta_{\rm LB} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi^i}\sqrt{g}g^{ik}\frac{\partial}{\partial\xi^k}, \qquad g = |\det(g_{ik})|, \qquad g_{ik}g^{k\mu} = \delta_i^\mu,$$

[⋆]This paper is a contribution to the Special Issue "Superintegrability, Exact Solvability, and Special Functions". The full collection is available at http://www.emis.de/journals/SIGMA/SESSF2012.html with the following relation between local metric tensor $g_{ik}(\xi)$, $(i,k=1,2)$ and the ambient space metric $G_{\mu\nu} = \mathrm{diag}(1,-1,-1)$,

$$g_{ik}(\xi) = G_{\mu\nu} \frac{\partial u^\mu}{\partial \xi^i} \frac{\partial u^\nu}{\partial \xi^k}, \qquad i,k=1,2, \qquad \mu,\nu = 0,1,2.$$

It is well known that the Helmholtz equation (1) on the two-sheeted hyperboloid admits separation of variables in nine orthogonal systems of coordinates [10, 18]. These nine systems of coordinates can be separated into three classes. The first class includes three systems of coordinates which are of subgroup type: viz. pseudo-spherical, equidistant and horicyclic coordinates, the second class includes three non subgroup type of coordinates: semi circular parabolic, elliptic parabolic and hyperbolic parabolic. The last two coordinate systems in the general case contain the dimensionless parameter γ and in the limiting cases when $\gamma \to 0$ or $\gamma \to \infty$ transform to horicyclic or equidistant systems of coordinates. These systems of coordinates have an important property; the Helmholtz equation (1) admits the exact solution (which we call exactly solvable) in terms of "classical" special functions, namely Bessel and Legendre functions or hypergeometric one. The third class of coordinates consists of elliptic, hyperbolic and semi hyperbolic coordinates each of which belong to the class of non subgroup type. Two of them, elliptic and hyperbolic coordinates, also contain a dimensionless parameter which is included in the metric tensor and Laplace–Beltrami operator. The Helmholtz equation (1) after the separation of variables in these coordinates leads to the two Heun type differential equations with a four singular points (which we call non-exactly solvable equations) and whose solutions appear as Lame or Lame–Wangerin functions [3, 4, 8]. In contrast to the classical special functions which can be appear in terms of hypergeometric functions, the Lame or Lame–Wangerin functions are defined in terms of infinite series where the expansion coefficients can not be written in explicit form because they are the subject the third or higher order recurrence relations.

The contraction limit of a basis function is not a trivial task in the case of exactly or non-exactly solvable equations. Some calculations come from the papers [5, 6, 7], but many of them are not known till now.

In this paper we restrict ourselves to the contraction limit $R \to \infty$ for four kinds of orthogonal basis functions: horocyclic, semi circular parabolic, elliptic parabolic and hyperbolic parabolic, which is new. We shall present normalizable eigenfunctions but do not give their normalization constants explicitly, except in the case of horocyclic coordinates which is particularly simple. Here we have also included for completeness the contraction for pseudo spherical and equidistant wave functions (see [7, 12]).

We hope that our results beside possible application to the theory of special functions will be useful also in the investigation of super integrable systems which admit separation of variables on the two-dimensional hyperboloid. We think that they can be generalized for the three and higher dimensional hyperbolic space when these six systems of coordinates are the sub systems of more complicated systems of coordinates and where the procedure of variable separation leads to similar differential equations.

2 Contractions

2.1 Pseudo-spherical basis to polar

The first coordinate system is the pseudo-spherical system $(\tau > 0,\ \varphi \in [0, 2\pi))$

$$u_0 = R\cosh\tau, \qquad u_1 = R\sinh\tau\cos\varphi, \qquad u_2 = R\sinh\tau\sin\varphi,$$

and the orthogonal basis functions of equation (1) are [7, 8, 12, 13]

$$\Psi^{\mathrm{S}}_{\rho m}(\tau,\varphi) = P^{|m|}_{i\rho-1/2}(\cosh\tau)e^{im\varphi},$$

where $P^{\mu}_{\nu}(z)$ is the Legendre function.

In the contraction limit $R\to\infty$ we have

$$\sinh\tau \sim \tau \sim \frac{r}{R}, \qquad \varphi\to\varphi, \qquad \rho\sim kR,$$

where (r,φ) are the polar coordinates in Euclidean plane. There is now the matter of how this contraction affects the basic eigenfunctions that can be computed on the hyperboloid. Using the well-known representations of the Legendre function in terms of the hypergeometric function [1],

$$P^{|m|}_{i\rho-1/2}(\cosh\tau) = \frac{\Gamma(1/2+i\rho+|m|)}{\Gamma(1/2+i\rho-|m|)}\left(\sinh\frac{\tau}{2}\right)^{|m|}\left(\cosh\frac{\tau}{2}\right)^{|m|}\frac{1}{|m|!}$$
$$\times\, {}_2F_1\left(1/2+i\rho+|m|, 1/2-i\rho+|m|; 1+|m|; -\sinh^2\frac{\tau}{2}\right),$$

asymptotic formulae for gamma-functions at large z [1],

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx z^{\alpha-\beta},$$

and taking into account that at $R\to\infty$

$$F\left(\frac{1}{2}+|m|+i\rho, \frac{1}{2}+|m|-i\rho; 1+|m|; -\sinh^2\frac{\tau}{2}\right)$$
$$\approx {}_0F_1\left(1+|m|; -\frac{k^2r^2}{4}\right) = \left(\frac{2}{kr}\right)^{|m|}|m|!J_{|m|}(kr),$$

where $J_\nu(z)$ is the Bessel function [2], we get

$$P^{|m|}_{i\rho-1/2}(\cosh\tau) \approx P^{|m|}_{ikR-1/2}\left(1+\frac{r^2}{2R^2}\right) \approx (-kR)^{|m|}J_{|m|}(kr).$$

Finally, the pseudo-spherical functions in the contraction limit $R\to\infty$ take the form

$$\Psi^{\mathrm{S}}_{\rho m}(\tau,\varphi) \approx (-kR)^{|m|}J_{|m|}(kr)e^{im\varphi},$$

i.e., the pseudo-spherical basis up to the constant factor contracts into polar one. The correct correspondence to give limiting orthogonality relations in the polar coordinates r and φ can be obtained using the contraction limit of the normalization constant (see [7]) and the results

$$\sinh\tau d\tau d\varphi \to \frac{1}{R^2}rdrd\varphi, \qquad \delta(\rho-\rho') \to \frac{1}{R}\delta(k-k'),$$

and

$$\int_0^\infty J_{|m|}(kr)J_{|m|}(k'r)rdr = \frac{1}{k}\delta(k-k').$$

2.2 Equidistant basis to Cartesian

The coordinate system is the following one ($\tau_1, \tau_2 \in \mathbb{R}$)

$$u_0 = R\cosh\tau_1\cosh\tau_2, \qquad u_1 = R\cosh\tau_1\sinh\tau_2, \qquad u_2 = R\sinh\tau_1,$$

From the above definition we have that

$$\sinh\tau_1 = \frac{u_2}{R}, \qquad \tanh\tau_2 = \frac{u_1}{u_0}$$

and in the limit of $R \to \infty$ we get

$$\sinh\tau_1 \sim \tau_1 \sim \frac{y}{R}, \qquad \sinh\tau_2 \sim \tau_2 \sim \frac{x}{R}, \tag{2}$$

where x and y are the Cartesian coordinates in the Euclidean plane arising from contraction $R \to \infty$. The orthogonal wave function takes the form [7, 8, 12, 13]

$$\Psi^{\mathrm{EQ}}_{\rho\nu}(\tau_1, \tau_2) = (\cosh\tau_1)^{-1/2} P^{i\rho}_{-1/2+i\nu}(-\varepsilon\tanh\tau_1)\exp(i\nu\tau_2),$$

where $\varepsilon = \pm 1$.

To perform the contraction we use the formula [1]

$$\begin{aligned} P^{\mu}_{\nu}(z) = 2^{\mu}\sqrt{\pi}\left(1-z^2\right)^{-\mu/2} & \left[\frac{{}_2F_1\left(-\frac{1}{2}(\mu+\nu), \frac{1}{2}(1-\mu+\nu); \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}(1-(\mu+\nu))\right)\Gamma\left(1+\frac{1}{2}(\nu-\mu)\right)}\right. \\ & \left. - 2z\frac{{}_2F_1\left(\frac{1}{2}(1-\mu-\nu), 1+\frac{1}{2}(\nu-\mu); \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1}{2}(1+\nu-\mu)\right)\Gamma\left(-\frac{1}{2}(\nu+\mu)\right)}\right], \end{aligned} \tag{3}$$

and rewrite the Legendre function in terms of the hypergeometric function

$$\begin{aligned} P^{i\rho}_{i\nu-1/2}(-\varepsilon\tanh\tau_1) = \frac{\sqrt{\pi}2^{i\rho}(\cosh\tau_1)^{i\rho}}{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)} & \left\{ {}_2F_1\left(\frac{1}{4}-a, \frac{1}{4}-b; \frac{1}{2}; \tanh^2\tau_1\right)\right. \\ & \left. + 2\varepsilon\tanh\tau_1\frac{\Gamma\left(\frac{3}{4}-a\right)\Gamma\left(\frac{3}{4}-b\right)}{\Gamma\left(\frac{1}{4}-a\right)\Gamma\left(\frac{1}{4}-b\right)}{}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right)\right\}, \end{aligned}$$

where $a = i(\rho+\nu)/2$, $b = i(\rho-\nu)/2$. In the contraction limit $R \to \infty$ we put

$$\rho \sim kR, \qquad \nu \sim k_1 R.$$

Then using also (2) we have the asymptotic formulae

$$\lim_{R\to\infty} {}_2F_1\left(\frac{1}{4}-a, \frac{1}{4}-b; \frac{1}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{1}{2}; -\frac{y^2k_2^2}{4}\right) = \cos k_2 y, \tag{4}$$

$$\lim_{R\to\infty} {}_2F_1\left(\frac{3}{4}-a, \frac{3}{4}-b; \frac{3}{2}; \tanh^2\tau_1\right) = {}_0F_1\left(\frac{3}{2}; -\frac{y^2k_2^2}{4}\right) = \frac{1}{k_2 y}\sin k_2 y, \tag{5}$$

where $k_1^2 + k_2^2 = k^2$ give us the contraction limit for $P^{i\rho}_{i\nu-1/2}(-\varepsilon\tanh\tau_1)$,

$$P^{i\rho}_{i\nu-1/2}(-\varepsilon\tanh\tau_1) \approx P^{ikR}_{ik_1R-1/2}\left(-\varepsilon\frac{y}{R}\right) \approx \frac{\sqrt{\pi}2^{ikR}\exp(i\varepsilon k_2 y)}{\Gamma\left(\frac{3}{4}-\frac{iR(k+k_1)}{2}\right)\Gamma\left(\frac{3}{4}-\frac{iR(k-k_1)}{2}\right)},$$

and finally for $\Psi^{\mathrm{EQ}}_{\rho\nu}$

$$\Psi^{\mathrm{EQ}}_{\rho\nu}(\tau_1, \tau_2) \approx \frac{\sqrt{\pi}2^{ikR}}{\Gamma\left(\frac{3}{4}-\frac{iR(k+k_1)}{2}\right)\Gamma\left(\frac{3}{4}-\frac{iR(k-k_1)}{2}\right)}\exp(ik_1x + i\varepsilon k_2 y).$$

The last result up to the constant factor coincides with the contraction limit of equidistant wave function to Cartesian one in Euclidean space.

2.3 Horocyclic basis to Cartesian in E_2

In the notation of the article mentioned horocyclic coordinates on the hyperboloid can be written ($\bar{x} \in \mathbb{R}, \bar{y} > 0$)

$$u_0 = R\frac{\bar{x}^2 + \bar{y}^2 + 1}{2\bar{y}}, \qquad u_1 = R\frac{\bar{x}^2 + \bar{y}^2 - 1}{2\bar{y}}, \qquad u_2 = R\frac{\bar{x}}{\bar{y}}.$$

From these relations we see that

$$\bar{x} = \frac{u_2}{u_0 - u_1}, \qquad \bar{y} = \frac{R}{u_0 - u_1}.$$

In the limit as $R \to \infty$ we obtain

$$\bar{x} \to \frac{y}{R}, \qquad \bar{y} \to 1 + \frac{x}{R},$$

where x and y are the Cartesian coordinates in the Euclidean plane. The horicyclic basis functions satisfying the orthonormality condition

$$R^2 \int_{-\infty}^{\infty} d\tilde{x} \int_0^{\infty} \Psi_{\rho' s'}^{*\mathrm{HO}}(\tilde{y}, \tilde{x}) \Psi_{\rho s}^{\mathrm{HO}}(\tilde{y}, \tilde{x}) \frac{d\tilde{y}}{\tilde{y}^2} = \delta(\rho - \rho')\delta(s - s'),$$

have the form

$$\Psi_{\rho s}^{\mathrm{HO}}(\bar{x}, \bar{y}) = \sqrt{\frac{\rho \sinh \pi\rho}{2R^2\pi^3}} \sqrt{\bar{y}} K_{i\rho}(|s|\bar{y}) e^{is\bar{x}},$$

where $K_\nu(x)$ is the Macdonald function [2].

To effect the correct contraction we further require that $\rho \to kR$, and $s \to k_2 R$ where $k = \sqrt{k_1^2 + k_2^2}$. Consequently we need the asymptotic formula for

$$K_{ikR}(k_2(R + x))$$

as $R \to \infty$. For this we use the asymptotic formula [2]

$$K_{i\nu}(x) \sim \frac{\sqrt{2\pi}}{(\nu^2 - x^2)^{1/4}} \exp\left(-\frac{\pi\nu}{2}\right) \sin\left(\frac{\pi}{4} - \sqrt{\nu^2 - x^2} + \nu \cosh^{-1}\frac{\nu}{x}\right), \tag{6}$$

valid if $\nu > x \gg 1$ and both ν and x are large and positive. If we use this formula for the Macdonald functions and take into account that

$$\sqrt{\rho \sinh \pi\rho} \sim \sqrt{\frac{kR}{2}} e^{\frac{k\pi R}{2}},$$

then we finally get

$$\Psi_{\rho s}^{\mathrm{HO}}(\bar{y}, \bar{x}) \sim \frac{1}{R\pi}\sqrt{\frac{k}{2k_1}} \sin(k_1 x - M) \exp(ik_2 y), \qquad M = \frac{\pi}{4} + R\left[k \cosh^{-1}\frac{k}{k_2} - k_1\right].$$

The correct correspondence to give the limiting result in the Cartesian coordinates x and y can be obtained using the delta-function contractions

$$\delta(\rho - \rho')\delta(s - s') \sim \frac{k}{k_1 R^2}\delta(k_1 - k_1')\delta(k_2 - k_2').$$

2.4 Semi circular parabolic to Cartesian coordinates

These coordinates are given by the formulae $(\eta, \xi > 0)$

$$u_0 = R\frac{\left(\xi^2+\eta^2\right)^2+4}{8\xi\eta}, \qquad u_1 = R\frac{\left(\xi^2+\eta^2\right)^2-4}{8\xi\eta}, \qquad u_2 = R\frac{\left(\eta^2-\xi^2\right)}{2\xi\eta}.$$

In terms of these coordinates we see that

$$\eta^2 = \frac{\sqrt{R^2+u_2^2}+u_2}{u_0-u_1}, \qquad \xi^2 = \frac{\sqrt{R^2+u_2^2}-u_2}{u_0-u_1}.$$

In the limit as $R \to \infty$ we have

$$\eta^2 = 1 + \frac{x+y}{R}, \qquad \xi^2 = 1 + \frac{x-y}{R}.$$

We see that in the contraction limit the Cartesian variables x and y are mixed up. To have a correct limit we can introduce the equivalent semi-circular parabolic system of coordinate connected with previous one by the rotation about axis u_0 through the angle $\pi/4$,

$$u_0 = R\frac{\left(\eta^2+\xi^2\right)^2+4}{8\xi\eta}, \qquad u_1 = R\frac{\sqrt{2}}{2}\left(\frac{\eta^2-\xi^2}{2\xi\eta} + \frac{\left(\eta^2+\xi^2\right)^2-4}{8\xi\eta}\right),$$
$$u_2 = R\frac{\sqrt{2}}{2}\left(\frac{\eta^2-\xi^2}{2\xi\eta} - \frac{\left(\eta^2+\xi^2\right)^2-4}{8\xi\eta}\right).$$

In terms of a new coordinates we have that

$$\eta^2 = \frac{\sqrt{2R^2+(u_2-u_1)^2}-(u_2-u_1)}{\sqrt{2}u_0-(u_2+u_1)}, \qquad \xi^2 = \frac{\sqrt{2R^2+(u_2-u_1)^2}+(u_2-u_1)}{\sqrt{2}u_0-(u_2+u_1)},$$

and in the contraction limit $R \to \infty$ we obtain

$$\eta^2 \to 1 + \sqrt{2}\frac{x}{R}, \qquad \xi^2 \to 1 + \sqrt{2}\frac{y}{R}.$$

The suitable set of basis functions are [4, 8]

$$\Psi_{\rho s}^{\mathrm{SCP}}(\xi, \eta) = \sqrt{\xi\eta} J_{i\rho}\left(\sqrt{s}\xi\right) K_{i\rho}\left(\sqrt{s}\eta\right),$$

for $s > 0$ and

$$\Psi_{\rho s}^{SCP}(\xi, \eta) = \sqrt{\xi\eta} K_{i\rho}\left(\sqrt{-s}\xi\right) J_{i\rho}\left(\sqrt{-s}\eta\right),$$

for $s < 0$. The correct limit is then obtained by choosing $s = R^2(k_2^2 - k_1^2)$ and $\rho = kR$, where $k_1^2 + k_2^2 = k^2$. To find the contraction limit of the basis function let us use the asymptotic relation for the Bessel function of pure imaginary index [17]

$$2\pi J_{ip}(z) \sim \frac{\sqrt{2\pi}}{(p^2+z^2)^{1/4}} \exp\left(i\sqrt{p^2+z^2} - ip\sinh^{-1}\frac{p}{z} - i\frac{\pi}{4}\right)\exp\left(\frac{p\pi}{2}\right),$$

and for the MacDonald function the formula (6). Then

$$J_{i\rho}\left(\sqrt{s}\xi\right) \sim J_{ikR}\left(R\sqrt{k_2^2-k_1^2}\sqrt{1+\sqrt{2}\frac{y}{R}}\right)$$

$$\sim \frac{e^{\frac{\pi}{2}kR}}{2^{\frac{1}{4}}\sqrt{2\pi k_2 R}} \exp\left(ik_2 y + iR\left[\sqrt{2}k_2 - k\sinh^{-1}\frac{k}{\sqrt{k_2^2-k_1^2}}\right] - i\frac{\pi}{4}\right)$$

and

$$\begin{aligned} K_{i\rho}(\sqrt{s}\eta) &\sim K_{ikR}\left(R\sqrt{k_2^2-k_1^2}\sqrt{1+\sqrt{2}\frac{x}{R}}\right) \\ &\sim \frac{2^{\frac{1}{4}}\sqrt{\pi}e^{-\frac{\pi}{2}kR}}{\sqrt{Rk_1}} \sin\left(\frac{\pi}{4} - xk_1 + R\left[\sqrt{2}k\cosh^{-1}\frac{k}{\sqrt{k_2^2-k_1^2}} - k_1\right]\right). \end{aligned}$$

Using the last asymptotic formulae it is easily to get for a large R

$$\Psi_{\rho s}^{\mathrm{SCP}}(\xi,\eta) \sim \frac{-1}{R\sqrt{2k_1k_2}} \exp\left(ik_2 y + i\delta_1 - i\frac{\pi}{4}\right) \sin\left(k_1 x - \frac{\pi}{4} + \delta_2\right),$$

where

$$\begin{aligned} \delta_1 + \delta_2 &= \sqrt{2}R(k_1+k_2) - Rk\sinh^{-1}\left(\frac{k_2+k_1}{k_2-k_1}\right), \\ \delta_1 - \delta_2 &= \sqrt{2}R(k_2-k_1) - Rk\sinh^{-1}\left(\frac{k_2-k_1}{k_2+k_1}\right). \end{aligned}$$

These expressions are arrived at under the assumption that $k_2^2 > k_1^2$. In case of $k_2^2 < k_1^2$ we can make the interchanges x with y, and k_1 with k_2.

2.5 Elliptic parabolic basis to parabolic

Elliptic parabolic basis contracts into a parabolic one on E_2. In these coordinates the points on the hyperbola are given by $[\theta \in (-\pi/2, \pi/2), a \geq 0]$,

$$u_0 = R\frac{\cosh^2 a + \cos^2\theta}{2\cosh a\ \cos\theta}, \qquad u_1 = R\frac{\sinh^2 a - \sin^2\theta}{2\cosh a\ \cos\theta}, \qquad u_2 = R\tanh a\ \tan\theta.$$

From these relations we see that

$$\cos^2\theta = \frac{u_0 - \sqrt{u_0^2-R^2}}{u_0-u_1}, \qquad \cosh^2 a = \frac{u_0 + \sqrt{u_0^2-R^2}}{u_0-u_1}.$$

In the limit as $R \to \infty$ we obtain

$$\cos^2\theta \to 1 - \frac{\eta^2}{R}, \qquad \cosh^2 a \to 1 + \frac{\xi^2}{R},$$

where the parabolic coordinates (ξ, η) are given by

$$x = \frac{1}{2}(\xi^2 - \eta^2), \qquad y = \xi\eta, \qquad \xi, \eta > 0.$$

The elliptic parabolic wave functions on the hyperboloid have the form [4]

$$\Psi_{\rho s}^{\mathrm{EP}}(a,\theta) = \sqrt{\cos\theta}P_{is-1/2}^{i\rho}(\sin\theta)P_{i\rho-1/2}^{is}(\tanh a).$$

To effect the contraction we take $s \to \kappa R$ and $\rho \to kR$ and

$$\tanh a \sim \frac{\xi}{\sqrt{R}}, \qquad \sin\theta \sim \frac{\eta}{\sqrt{R}}.$$

The separation equation

$$\left(\frac{d^2}{da^2} + s^2 - \frac{\rho^2 + 1/4}{\cosh^2 a}\right) F(a) = 0,$$

becomes

$$\left(\frac{d^2}{d\xi^2} + \lambda + k^2\xi^2\right) F(\xi) = 0,$$

where λ is now the parabolic separation constant and we impose the condition

$$\lambda = R\big(\kappa^2 - k^2\big),$$

or

$$\kappa = k + \frac{\lambda}{2kR} + O\big(R^{-2}\big).$$

This requires taking the limit of a and θ the dependent part of the eigenfunctions $P^{is}_{i\rho-1/2}(\tanh a)$ and $P^{i\rho}_{is-1/2}(\sin\theta)$. The limit can be established from the known representation of the Legendre function (3). This results in the asymptotic formula

$$P^{ikR+\frac{i\lambda}{2k}}_{ikR-\frac{1}{2}}\left(\frac{\xi}{\sqrt{R}}\right) \to \frac{2^{ikR+\frac{1}{4}+\frac{i\lambda}{4k}}}{\Gamma\left(\frac{3}{4} - \frac{i\lambda}{4k} - ikR\right)} D_{\frac{1}{2}\left(i\frac{\lambda}{k}-1\right)}\big(\sqrt{-2ik}\xi\big),$$

where $D_\nu(z)$ is a parabolic cylinder function [1]

$$D_\nu(z) = 2^{\nu/2}\sqrt{\pi}e^{-\frac{z^2}{4}}\left[\frac{1}{\Gamma\left(\frac{1-\nu}{2}\right)} {}_1F_1\left(-\frac{\nu}{2};\frac{1}{2};\frac{z^2}{2}\right) - \frac{z\sqrt{2}}{\Gamma\left(-\frac{\nu}{2}\right)} {}_1F_1\left(\frac{1-\nu}{2};\frac{3}{2};\frac{z^2}{2}\right)\right].$$

If we look at the theta-dependent part of the eigenfunctions and the corresponding limit taking $\sin\theta = \frac{\eta}{\sqrt{R}}$ we then obtain the limit

$$P^{ikR}_{ikR+\frac{i\lambda}{2k}-\frac{1}{2}}\left(\frac{\eta}{\sqrt{R}}\right) \to \frac{2^{ikR+\frac{1}{4}+\frac{i\lambda}{4k}}}{\Gamma\left(\frac{3}{4} - \frac{i\lambda}{4k} - ikR\right)} D_{-\frac{1}{2}\left(i\frac{\lambda}{k}+1\right)}\big(\sqrt{-2ik}\eta\big),$$

and finally

$$\Psi^{\rm EP}_{\rho s}(a,\theta) \sim \frac{2^{2ikR+\frac{1}{2}+\frac{i\lambda}{2k}}}{\left[\Gamma\left(\frac{3}{4} - \frac{i\lambda}{4k} - ikR\right)\right]^2} D_{-\frac{1}{2}\left(i\frac{\lambda}{k}+1\right)}\big(\sqrt{-2ik}\eta\big) D_{\frac{1}{2}\left(i\frac{\lambda}{k}-1\right)}\big(\sqrt{-2ik}\xi\big).$$

From these expressions we see that we do indeed obtain the correct asymptotic limit.

2.6 Hyperbolic parabolic to a Cartesian basis

Hyperbolic parabolic basis contracts into a Cartesian one on E_2. In these coordinates the points on the hyperbola are given by $[\theta \in (0,\pi),\ b > 0]$,

$$u_0 = R\frac{\cosh^2 b + \cos^2\theta}{2\sinh a\ \sin\theta}, \qquad u_1 = R\frac{\sinh^2 b - \sin^2\theta}{2\sinh a\ \sin\theta}, \qquad u_3 = R\cot\theta\coth b.$$

From these relations we see that

$$\cos^2\theta = \frac{u_0 - \sqrt{u_1^2 + R^2}}{u_0 - u_1}, \qquad \cosh^2 b = \frac{u_0 + \sqrt{u_1^2 + R^2}}{u_0 - u_1}.$$

In the limit as $R \to \infty$ we can choose

$$\cos^2\theta \to \frac{y^2}{2R^2}, \qquad \cosh^2 b \to 2\left(1+\frac{x}{R}\right).$$

The hyperbolic parabolic basis function on hyperbolid can be chosen in the form [4]

$$\Psi^{\mathrm{HP}}(b,\theta) = (\sinh b \sin\theta)^{1/2} P^{i\rho}_{is-1/2}(\cosh b) P^{i\rho}_{is-1/2}(\cos\theta). \tag{7}$$

To proceed further with this limit we take $\rho^2 \sim k^2R^2$ and $s^2 \sim \left(k_1^2 - k_2^2\right)R^2$ (the case of $s^2 < 0$, or $k_1^2 < k_2^2$, corresponds to the discrete spectrum of constant s, and we do not consider this case here) where $k_1^2 + k_2^2 = k^2$, then using the relation between Legendre function and hypergeometric functions (3) and formulae (4), and (5), we obtain for θ depending part of basis function

$$\begin{aligned}\sqrt{\sin\theta}P^{i\rho}_{is-1/2}(\cos\theta) &\sim P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\frac{y}{\sqrt{2}R}\right)\\ &\sim \frac{2^{ikR}\sqrt{\pi}\exp(ik_2y)}{\Gamma\left(\frac{3}{4}-\frac{iR}{2}\left(k+\sqrt{k_1^2-k_2^2}\right)\right)\Gamma\left(\frac{3}{4}-\frac{iR}{2}\left(k-\sqrt{k_1^2-k_2^2}\right)\right)}.\end{aligned}$$

For the limit of the b dependent part of the eigenfunctions we must proceed differently. In fact we need to calculate the limit of

$$P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\sqrt{2\left(1+\frac{x}{R}\right)}\right)$$

as $R \to \infty$. We know that the leading terms of this expansion have the form

$$A\exp(ik_1x) + B\exp(-ik_1x),$$

and we now make use of this fact. By this we mean that

$$\lim_{R\to\infty} P^{ikR}_{i\sqrt{k_1^2-k_2^2}R-1/2}\left(\sqrt{2\left(1+\frac{x}{R}\right)}\right) = A\exp(ik_1x) + B\exp(-ik_1x),$$

where the constants A and B depend on R. It remains to determine A and B. To do this let us consider $x = 0$. We then need to determine the following limit

$$\lim_{R\to\infty} P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}(\sqrt{2}) = A + B. \tag{8}$$

From the integral representation formula [1]

$$\begin{aligned}&\frac{\Gamma(-\nu-\mu)\Gamma(1+\nu-\mu)}{\Gamma(1/2-\mu)}\sqrt{\frac{\pi}{2}}P^{\mu}_{\nu}(z)\\ &\quad = \left(z^2-1\right)^{-\mu/2}\int_0^\infty (z+\cosh t)^{\mu-1/2}\cosh\left([\nu+1/2]\,t\right)\mathrm{d}t,\end{aligned} \tag{9}$$

the above limit requires us to calculate as $R \to \infty$

$$\int_0^\infty \left(\sqrt{2}+\cosh t\right)^{ikR-1/2}\cos\left(\left[R\sqrt{k_1^2-k_2^2}\right]t\right)\mathrm{d}t.$$

It can be done using the method of stationary phase [11]. We obtain

$$P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}(\sqrt{2}) \sim \frac{2^{-\frac{5}{4}+\frac{iR}{2}\left(\sqrt{k_1^2-k_2^2}-k\right)}\Gamma\left(\frac{1}{2}-ikR\right)}{\Gamma\left[\frac{1}{2}-iR\left(\sqrt{k_1^2-k_2^2}+k\right)\right]\Gamma\left[\frac{1}{2}+iR\left(\sqrt{k_1^2-k_2^2}-k\right)\right]}$$

$$\times\left(\frac{i}{Rk_1}\right)^{1/2}\left(\frac{k_1-\sqrt{k_1^2-k_2^2}}{k+\sqrt{k_1^2-k_2^2}}\right)^{iR\sqrt{k_1^2-k_2^2}}\left(\frac{k}{k-k_1}\right)^{ikR}. \tag{10}$$

By considering the expression for the derivatives of the Legendre function (9) at $x=0$, we derive the expression

$$\frac{d}{dx}P_\nu^\mu(z)\bigg|_{x=0}\sim -ik_1P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}(\sqrt{2})\sim ik_1(A-B),$$

then

$$P^{ikR}_{-\frac{1}{2}+iR\sqrt{k_1^2-k_2^2}}(\sqrt{2})\sim -A+B.$$

Comparing the above relation with (8), we obtain that $A=0$ and B is equal to (10), that is

$$P^{i\rho}_{is-\frac{1}{2}}(\cosh b)\to Be^{-ixk_1}.$$

Finally, solution (7) contracts as follows

$$\Psi_{\rho s}(b,\theta)\sim\frac{2^{ikR}\sqrt{\pi}B}{\Gamma\left(\frac{3}{4}-iR\frac{k+\sqrt{k_1^2-k_2^2}}{2}\right)\Gamma\left(\frac{3}{4}-iR\frac{k-\sqrt{k_1^2-k_2^2}}{2}\right)}\exp(ik_2y-ik_1x).$$

3 Conclusion

In this note we have constructed the contraction limit $R\to\infty$ for the unnormalized wave functions which are the solution of the Helmholtz equation on the two dimensional two sheeted hyperboloid in four coordinates systems, namely, horocyclic, semi circular parabolic, elliptic parabolic and hyperbolic parabolic. Of course the complete analysis of the contraction problem must include also the solutions of Helmholtz equation in the three additional systems of coordinates as elliptic, hyperbolic and semi hyperbolic. We will study this in the near future.

We have not presented limits associated with nonsubgroup coordinates. The extension of these ideas to problems in higher dimensions is natural and will be presented in subsequent work.

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