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Hecke Transformations of Conformal Blocks in WZW Theory. I. KZB Equations for Non-Trivial Bundles

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Abstract. We describe new families of the Knizhnik–Zamolodchikov–Bernard (KZB) equations related to the WZW-theory corresponding to the adjoint G-bundles of different topological types over complex curves $\Sigma_{g,n}$ of genus g with n marked points. The bundles are defined by their characteristic classes – elements of $H^2(\Sigma_{g,n}, \mathcal{Z}(G))$, where $\mathcal{Z}(G)$ is a center of the simple complex Lie group G. The KZB equations are the horizontality condition for the projectively flat connection (the KZB connection) defined on the bundle of conformal blocks over the moduli space of curves. The space of conformal blocks has been known to be decomposed into a few sectors corresponding to the characteristic classes of the underlying bundles. The KZB connection preserves these sectors. In this paper we construct the connection explicitly for elliptic curves with marked points and prove its flatness.

Key words: integrable system; KZB equation; Hitchin system; characteristic class

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1 Introduction

The Knizhnik–Zamolodchikov–Bernard (KZB) equations [8, 9, 40] are a system of differential equations for conformal blocks in a conformal field theory. Here we consider the WZW theory of the level k, related to a simple complex Lie group G and defined on a Riemann surface $\Sigma_{g,n}$ of genus g with n marked points $(z_1, z_2, \dots, z_n)$. To describe this model, one should define a G-bundle over $\Sigma_{g,n}$. Topologically, the G-bundles are defined by their characteristic classes. Let $\mathcal{Z}(G)$ be a center of G and $G^{\mathrm{ad}} = G/\mathcal{Z}(G)$. The characteristic classes are obstructions to lift the G^{ad}-bundles to the G-bundles. They are elements of the cohomology group $H^2(\Sigma_g, \mathcal{Z}(G)) \sim \mathcal{Z}(G)$ [46][1]. If $\bar{G}$ is the corresponding simply connected group (the universal covering with the natural group structure) and $G = \bar{G}/\mathcal{Z}^\vee(G)$, then elements from $H^2(\Sigma_g, \mathcal{Z}^\vee(G))$ are obstruction to lift the G-bundles to the $\bar{G}$-bundles. In particular, consider $G = \mathrm{Spin}(N)$ and $\mathrm{SO}(N) = \mathrm{Spin}(N)/\mathbb{Z}_2$. Then $H^2(\Sigma_g, \mathbb{Z}_2) \sim \mathbb{Z}_2$ defines the Stiefel–Whitney classes of the $\mathrm{SO}(N)$-bundles over Σ_g.

For generic bundles the WZW theories were studied in [23, 35]. The aim of this paper is to define the KZB equations in these theories. The KZB equations have a large range of applications in mathematics. In particular, on the critical level they produce Hamiltonians of the quantum Hitchin system [30, 34, 43, 57], while in the classical limit they lead to the

[1]See (4.1)–(4.3) in [46].

monodromy-preserving equations [32, 41, 44, 62, 66]. In this way, we obtain new classes of these systems.

The KZB equations are described in the following way. Consider the highest weight representations V_{μ_a} (μ_a are the highest weights) of G attached to the marked points. For a positive integer k define the integrable module $\hat{V}_{\mu_a}$ of level k of the centrally extended loop group $D^\times \to G$, where $D^\times = D \setminus z_a$ is a punctured disk around the marked point z_a. The conformal blocks are linear functionals $\hat{V}^{[n]} \equiv \hat{V}_{\mu_1} \otimes \cdots \otimes \hat{V}_{\mu_n} \to \mathbb{C}$ satisfying some additional conditions (the Ward identities). Let $\mathcal{C}_G(\hat{V}^{[n]})$ be a space of conformal blocks. This space depends on parameters – the complex structure of $\Sigma_{g,n}$, and in this way forms a bundle over the moduli space $\mathfrak{M}_{g,n}$ of complex structures. There exists a projectively flat connection in this bundle (the KZB connection). Then the meaning of the KZB equations is that the conformal blocks are the horizontal sections of the KZB connection. The KZB equations were derived originally for the genus zero case by Knizhnik and Zamolodchikov [40] and were generalized later to arbitrary genus by Bernard [8, 9]. In subsequent years the KZB equations was studied in a number of works [4, 16, 22, 29, 33, 36].

If the cocenter $\mathcal{Z}^\vee(G) = \operatorname{Ker} \bar{G} \to G$ is non-trivial then the integrable module is a sum of sectors, corresponding to the characteristic classes of the underlying bundles

$$\hat{V}_\mu = \bigoplus_{j=0}^{N-1} V_\mu^{(j)}, \qquad N = \operatorname{ord} \mathcal{Z}(G).$$

In terms of the spectra the WZW theory this was studied essentially in [23]. Similarly, the conformal blocks are also a sum of different sectors. In each sector one can define the KZB connection.

The aim of this paper is to construct explicitly the KZB connections in all sectors of conformal blocks for the WZW theory defined on elliptic curves. The compatibility conditions (horizontality of the KZB connection) are verified explicitly.

The KZB connection in the trivial sector was studied in [24]. This construction is based on the classical dynamical r-matrix with the spectral parameter living on the elliptic curve. The r-matrices of this type related to the trivial sector were classified by Etingof and Varchenko [19]. Recently, we have classified the dynamical elliptic r-matrices as sections of some bundles of an arbitrary topological type over elliptic curves [46]. It turned out that the dynamical parameters of the r-matrices are elements of the moduli spaces of the bundles. It allows us to define the KZB connection in these cases.

Different approach to classification of elliptic r-matrices was proposed in [17, 18, 21] and the corresponding KZB connection was also constructed in [17, 18]. The staring point of last approach is an automorphism of the extended Dynkin diagram. In our construction we considered only those automorphisms that isomorphic to elements of the center $\mathcal{Z}$. In this case we come to the same r-matrices and the KZB equations as in [17, 18]. For A_n, D_n and E_6 algebras there exists another type automorphisms. So far the underlying vector bundle structure is unclear. It should be noted that in [17, 18] the derivation of the KZB equation is based on the representations of conformal blocks as twisted traces of intertwiners. We will come to this representation in the forthcoming paper where the Hecke transformation of conformal blocks will be considered (see below).

For the $\mathrm{SL}(N,\mathbb{C})$ WZW model on elliptic curves the KZB equation in the similar to our form was described in [42]. The authors considered a particular type of bundles that lead to the Belavin–Drinfeld classical r-matrix. In this case the corresponding KZB equation has not dynamical parameter and similar to the KZ equation. However, if N is not a prime number there exist r-matrices and the corresponding KZB equations intermediate between Felder and Belavin–Drinfeld cases.

In the subsequent paper we will describe the transformation operators that intertwine the different sectors (the Hecke transformations). It is worthwhile to notice that in the classical case these transformations provide a passage from the elliptic Calogero–Moser system to integrable Euler–Arnold top [48, 69, 70] (see also [47, 59, 70, 73]). For arbitrary characteristic classes these type of models were described in [45]. Different aspects and applications of the Hecke transformations to integrable systems and related topics (such as Painlevé–Schlesinger equations [2, 15, 50, 51, 60, 67, 68, 71], monopoles [14, 25, 31, 37, 39, 49, 58], quadratic Poisson structures [13, 74], applications to AGT conjecture [53, 54, 55] etc.) can be found in wide range of literature.

The paper has the following structure. In Section 2 we consider a general setting of the KZB equations related to arbitrary curves $\Sigma_{g,n}$ and arbitrary characteristic class of the bundles. In Section 3 the space of conformal blocks is described. In Section 4 we consider the genus one case in detail. The proofs of main relations (Propositions 1 and 2) and information about the special basis in simple Lie algebras as well as the elliptic functions identities are given in the appendices.

2 Loop algebras, loop groups and integrable modules

2.1 Loop algebras and loop groups

Let $\bar{G}$ be a simply-connected simple complex Lie group and $\mathcal{Z} = \mathcal{Z}(\bar{G})$ is the center of $\bar{G}$. For all simply-connected groups ($\mathrm{SL}(N,\mathbb{C})$, Sp_N, E_6, E_7 and Spin_N except $N = 4n$), the center is a cyclic group. For Spin_{4N} $\mathcal{Z} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The adjoint group is the quotient group $G^{\mathrm{ad}} = \bar{G}/\mathcal{Z}$. Assume for simplicity that $\mathcal{Z}$ is a cyclic group $\mathbb{Z}_l$ of order l.

Let K be a maximal compact subgroup of $\bar{G}$ and T is the Cartan torus of K. Consider the homomorphisms of $S^1 \to T$

$$\mathbf{e}(\varphi) \to \big(\mathbf{e}(\gamma_1\varphi), \mathbf{e}(\gamma_2\varphi), \dots, \mathbf{e}(\gamma_l\varphi)\big) \in T, \qquad \mathbf{e}(\varphi) = \exp(2\pi\imath\varphi).$$

$P^\vee = \{\gamma = (\gamma_1, \dots, \gamma_l)\}$ is a coweight lattice in the Cartan subalgebra $\mathfrak{h}^K = \mathrm{Lie}(T)$ and in $\mathfrak{h} \subset \mathfrak{g} = \mathrm{Lie}\,(\bar{G})$. Let $Q^\vee$ be the coroot lattice ($Q^\vee \subseteq P^\vee$). The center $\mathcal{Z}(\bar{G})$ is isomorphic to the quotient group $\mathcal{Z} \sim P^\vee/Q^\vee$. In particular, if $\varpi^\vee \in P^\vee$ is a coweight such that $l\varpi^\vee \in Q^\vee$, then the $\mathcal{Z} \sim \mathbb{Z}_l$. It is generated by the element $\mathbf{e}(\varpi^\vee) = \exp(2\pi\imath\varpi^\vee) \in T$. For Spin_{4N} the center is generated by two coweights, corresponding to the left and right spinor representations.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\{\alpha\} = R \in \mathfrak{h}^*$ is the root system [12]. There is the root decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in R} \mathfrak{g}^\alpha, \qquad \mathrm{ad}_X\, \mathfrak{g}^\alpha = \langle X, \alpha\rangle \mathfrak{g}^\alpha, \qquad X \in \mathfrak{h}.$$

R is an union of positive and negative roots $R = R_+ \cup R_-$ with respect to some ordering in $\mathfrak{h}^*$. Let $\Pi = \{\alpha_1, \dots, \alpha_l\}$ be a basis of simple roots in R. The dual system $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_l^\vee\}$ ($\langle \alpha_j, \alpha_k^\vee\rangle = \delta_{jk}$) forms a basis in $\mathfrak{h}$.

Let t be a coordinate in $\mathbb{C}$. Define the loop group $L(G) = \{\mathbb{C}^* \to G\} = \{g(t)\}$ such that $g(t)$ has a finite order poles when $t \to 0$. In other words, $L(G)$ is the group of Laurent polynomials $L(G) = G \otimes \mathbb{C}[[t, t^{-1}]$. There is a central extension $\hat{L}(\mathfrak{g})$ of $L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}]$

$$\hat{L}(\mathfrak{g}) = L(\mathfrak{g}) \oplus \mathbb{C}K, \tag{2.1}$$

defined by a two-cocycle $c(X \otimes f, Y \otimes g) = (X, Y)\,\mathrm{Res}(gf'dt)$.

The set of the affine roots if of the form: $R^{\mathrm{aff}} = \{\hat{\alpha} = \alpha + n,\ n \in \mathbb{Z},\ n \neq 0\}$. Let $\{\mathfrak{h}_\alpha\}$ be the basis of simple coroots in $\mathfrak{h}$. Then the analog of the root decomposition for the loop algebra has the following form

$$L(\mathfrak{g}) = \mathfrak{g} + \sum_{n\neq 0}\sum_{\alpha\in\Pi} \mathfrak{h}_\alpha t^n + \sum_{\tilde{\alpha}\in R^{\rm aff}} \mathfrak{g}^{\hat{\alpha}}, \qquad \mathfrak{g}^{\hat{\alpha}} = x_\alpha e_{\hat{\alpha}} = x_\alpha e_\alpha t^n.$$

Let $-\alpha_0$ be the highest root $-\alpha_0 \in R_+$. The system of simple affine roots is $\hat{\Pi} = \Pi \cup (-\alpha_0 + 1)$. It is a basis in $R^{\rm aff}$. Consider the positive loop subalgebra

$$L^+(\mathfrak{g}) = \big(\mathfrak{b} + \mathfrak{g}\otimes t\mathbb{C}[[t]]\big), \tag{2.2}$$

where $\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha\in R^+} \mathfrak{g}^\alpha$ is the positive Borel subalgebra. Let also

$$L^-(\mathfrak{g}) = \big(\mathfrak{n}_- + \mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]\big), \qquad \mathfrak{n}_- = \sum_{\alpha\in R^-} \mathfrak{g}^\alpha. \tag{2.3}$$

Then $\hat{L}(\mathfrak{g})$ (2.1) is the direct sum

$$\hat{L}(\mathfrak{g}) = L^-(\mathfrak{g}) \oplus L^+(\mathfrak{g}) \oplus \mathbb{C}K. \tag{2.4}$$

Each summand is a Lie subalgebra of $\hat{L}(\mathfrak{g})$. There are two types of the affine Weyl groups: $W_P = W \ltimes P^\vee$ and $W_Q = W \ltimes Q^\vee$, where W is the Weyl group of $\mathfrak{g}$,

$$W_P = \left\{\hat{w} = wt^\gamma,\ w\in W,\ \gamma\in P^\vee\right\}, \qquad W_Q = \left\{\hat{w} = wt^\gamma,\ w\in W,\ \gamma\in Q^\vee\right\}. \tag{2.5}$$

They act on the root vectors as $e_{\hat{\alpha}} = e_\alpha t^n \to e_{\hat{w}(\hat{\alpha})} = e_{w(\alpha)} t^{n+\langle\gamma,\alpha\rangle}$. The loop groups $L(G) = G\otimes\mathbb{C}[[t,t^{-1}]]$ have the Bruhat decomposition [61]. Define subgroups

$$L^+(G) = \big\{g_0 + g_1 t + \cdots\big\}, \quad g_j\in G,\ \ g_0 = b\in B, \quad \text{is the positive Borel subgroup}, \tag{2.6}$$

$$N^-(G) = \big\{n_- + g_1 t^{-1} + \cdots\big\}, \quad n_-\in N_-, \quad \text{is the negative nilpotent subgroup}, \tag{2.7}$$

$$N^+(G) = \big\{n_+ + g_1 t + \cdots\big\}, \quad n_+\in N_+, \quad \text{is the positive nilpotent subgroup}. \tag{2.8}$$

The Bruhat decomposition takes the form

$$L\big(G^{\rm ad}\big) = \bigcup_{\hat{w}\in W_P} N^-\big(G^{\rm ad}\big)\hat{w}L^+\big(G^{\rm ad}\big), \qquad L\big(\bar{G}\big) = \bigcup_{\hat{w}\in W_Q} N^-\big(\bar{G}\big)\hat{w}L^+\big(\bar{G}\big). \tag{2.9}$$

For a loop $g(t)$ in $G^{\rm ad}$ denote by $\bar{g}$ its lift to a map from S^1 to $\bar{G}$. This map can be multivalued, after turning along the circle the value can be multiplied by some element of the center which we call the monodromy: $g(e^{2\pi\imath}t) = \mathbf{e}(\gamma)g(t)$, $(\mathbf{e}(x) = e^{2\pi\imath x})$. If $\gamma\notin Q^\vee$ then $\zeta = \mathbf{e}(\gamma)$ is a non-trivial element of the center $\mathcal{Z}$ and the map $g(t)$ is well defined for $G = G^{\rm ad}$, but not for $\bar{G}$. In this way we have the representations

$$L\big(G^{\rm ad}\big) = \bigcup_{\gamma\in P^\vee} L_\gamma\big(G^{\rm ad}\big), \qquad L_\gamma\big(G^{\rm ad}\big) = \big\{g\big(e^{2\pi\imath}t\big) = \mathbf{e}(\gamma)g(t)\big\}. \tag{2.10}$$

If $\gamma_1 = \gamma_2 + \delta$ for any $\delta\in Q^\vee$ then γ_1 and γ_2 lead to the same monodromies. We say in this case that $L_{\gamma_1}\big(\bar{G}\big)$ and $L_{\gamma_2}\big(\bar{G}\big)$ are equivalent. Then from (2.10) we have

$$L\big(G^{\rm ad}\big) = \bigcup_{\zeta\in\mathcal{Z}} L_\zeta\big(G^{\rm ad}\big).$$

In particular, if the center $\mathcal{Z} \sim \mathbb{Z}_l$ is generated by a fundamental coweight $\varpi^\vee$, then

$$L\big(G^{\rm ad}\big) = \bigcup_{j=0}^{l-1} L_j\big(G^{\rm ad}\big), \qquad L_j\big(G^{\rm ad}\big) = \big\{g\big(e^{2\pi\imath}t\big) = \mathbf{e}(j\varpi^\vee)g(t)\big\}, \tag{2.11}$$

and $L_j\big(G^{\rm ad}\big) = \mathbf{e}(j\varpi^\vee)\big(L\big(\bar{G}\big)/\mathcal{Z}\big)$.

Consider the quotient $\mathrm{Fl}^{\rm aff} = L\big(G^{\rm ad}\big)/L^+\big(G^{\rm ad}\big)$ [61]. It is called the affine flag variety. Let $\Sigma_{\hat{w}}$ be an $N^-\big(G^{\rm ad}\big)$-orbit of $\hat{w}$ in $\mathrm{Fl}^{\rm aff}$. This orbit is dipheomorphic to the intersection $N^-\big(G^{\rm ad}\big)_{\hat{w}} = N^-\big(G^{\rm ad}\big) \cap \hat{w}N^-\big(G^{\rm ad}\big)\hat{w}^{-1}$. Therefore, its codimension in $Fl^{\rm aff}$ is the length $l(\hat{w})$ of $\hat{w}$. It is the number of negative affine roots which $\hat{w}$ transforms to positive ones (Theorem 8.7.2 in [61]). The Bruhat decomposition (2.9) defines the stratification of $\mathrm{Fl}^{\rm aff}$:

$$\mathrm{Fl}^{\rm aff} = L\big(G^{\rm ad}\big)/L^+\big(G^{\rm ad}\big) = \bigcup_{\hat{w}\in W_P} \Sigma_{\hat{w}}. \tag{2.12}$$

2.2 Integrable modules

Consider a subset of dominant weights $P^+ = \{\mu \in P | \langle \mu, \alpha^\vee\rangle \geq 0 \text{ for } \alpha^\vee \in \Pi^\vee\}$. Each dominate weights define a $\mathfrak{g}$-module V_μ. It contains the highest weight vector (HWV) v_μ such that

$$Xv_\mu = \langle X, \mu\rangle v_\mu \quad \text{for} \quad X \in \mathfrak{h}, \qquad \mathfrak{g}^\alpha v_\mu = 0 \quad \text{for} \quad \alpha \in R^+.$$

Define the Verma module $\mathcal{V}_\mu$ of $\hat{\mathcal{L}}(\mathfrak{g})$ associated with V_μ [38]. Let $I_k = \{\mu \in P^+ | \langle \mu, \alpha_0^\vee\rangle \leq k\}$ be a subset of dominant weights. Define the action of $L^+(\mathfrak{g})$ (2.2) on V_μ: $(\mathfrak{g} \otimes t\mathbb{C}[[t]])V_\mu = 0$, $KV_\mu = k\,\mathrm{Id}$, and $\mathfrak{b}$ acts on V_μ as described above. Then $\mathcal{V}_\mu = U(\hat{L}(\mathfrak{g})) \otimes_{U(L(\mathfrak{g})^+)} V_\mu$ is induced, where $\mu \in I_k$. There is the isomorphism

$$\mathcal{V}_\mu \sim U(L^-(\mathfrak{g})) \otimes_{\mathbb{C}} v_\mu.$$

Let E_{α_0} be the root subspace in $\mathfrak{g}$ corresponding to α_0. Consider the maximal submodule $\mathcal{S}_\mu$ of $\mathcal{V}_\mu$ generated by the singular vector

$$\big(E_{\alpha_0} \otimes t^{-1}\big)^{k-\langle\mu,\alpha_0\rangle+1} v_\mu. \tag{2.13}$$

The irreducible *integrable module* $\hat{V}_\mu$ is the quotient

$$\hat{V}_\mu = \mathcal{V}_\mu/\mathcal{S}_\mu. \tag{2.14}$$

We identify the module V_μ with a submodule $V_\mu \otimes 1 \hookrightarrow \mathcal{V}_\mu$. The integrable module $\hat{V}_\mu$ can be characterized in the following way: *the subspace of* $\hat{V}_\mu$ *annihilated by the positive subalgebra* $\mathfrak{g} \otimes \mathbb{C}[[t]]$ *is isomorphic to the finite-dimensional* $\mathfrak{g}$*-module* V_μ

$$V_\mu \sim \big\{v \in \hat{V}_\mu | (\mathfrak{g} \otimes t\mathbb{C}[[t]]) \cdot v = 0\big\}.$$

The group $L(G)$ has a central extension $1 \to \mathbb{C}^* \to \widehat{LG} \to LG \to 1$ corresponding to (2.1). The integrable module can be described in terms of $\widehat{LG}$. The action of $L^+(G)$ on the HWV has the form

$$L^+(G)v_\mu = \chi_\mu(b)v_\mu, \qquad \lambda v_\mu = \mathbf{e}(k)v_\mu, \qquad \lambda \in \text{the center } \mathbb{C}^*, \tag{2.15}$$

where $\chi_\mu(b)$ is the character of the Borel subgroup B. Then $\hat{V}_\mu$ is generated by the action of $N^-(G)$ (2.7) on V_μ.

In this way we describe only "the trivial sector" of the $L(G)$-module. Consider the Bruhat representation for $L\big(G^{\rm ad}\big)$ (2.9), and let $\hat{w} = t^\gamma$, $\gamma \in P^\vee$. Define the Verma modules with the HWV $t^\gamma v_\mu$,

$$\mathcal{V}_\mu(\gamma) = U(L^-(\mathfrak{g})) \otimes_{\mathbb{C}} t^\gamma v_\mu. \tag{2.16}$$

They have the singular vectors $\big(E_{\alpha_0} \otimes t^{-1}\big)^{k-\langle\mu\alpha_0\rangle+1} t^\gamma v_\mu$ (compare with (2.13)). Let $\mathcal{S}_{\mu,\gamma}$ be the maximal submodules generated by these singular vectors. Consider the quotient spaces

$$\hat{V}_\mu(\gamma) = \mathcal{V}_\mu(\gamma)/\mathcal{S}_{\mu,\gamma}, \tag{2.17}$$

and define their direct sum

$$\hat{\mathbf{V}}_\mu = \bigoplus_{\gamma\in P^\vee} \hat{V}_\mu(\gamma). \tag{2.18}$$

We say that two subspaces $\hat{V}_\mu(\gamma_1)$ and $\hat{V}_\mu(\gamma_2)$ are equivalent if $\gamma_1 = \gamma_2 + \delta$, where $\delta \in Q^\vee$. This equivalence leads to the decomposition of $\hat{\mathbf{V}}_\mu$ (as a $L(G^{\rm ad})$-module) into a sum of $l = {\rm ord}(\mathcal{Z}(\bar{G}))$ sectors,

$$\hat{\mathbf{V}}_\mu = \bigoplus_{\zeta\in\mathcal{Z}} \hat{V}_\mu(\zeta). \tag{2.19}$$

Notice that $(t^\gamma v_\mu)$ is not the HWV with respect to $L^+(\mathfrak{g})$. However, it was proved in [23] that there exists a unique element $\hat{w} = \hat{w}(\gamma) = t^\delta w \in W_Q$ such that $t^\gamma \hat{w} v_\mu$ is the HWV. We demonstrate it below for $L({\rm SL}(2,\mathbb{C}))$. The elements $\hat{w}$ and γ represent the same element $\zeta \in \mathcal{Z}$. Then we define the Verma module

$$\mathcal{V}_\mu(\zeta) \sim U(L(\mathfrak{g})^-) \otimes_{\mathbb{C}} (t^\gamma \hat{w} v_\mu). \tag{2.20}$$

The vector $\big(E_{\alpha_0} \otimes t^{-1}\big)^{k-\langle\mu,w\alpha_0\rangle+1} (\gamma\hat{w}v_\mu)$ is singular and corresponds to the submodule $\mathcal{S}_{\mu,\gamma}$. As in (2.14) we identify the integrable modules $\mathcal{V}_\mu(\zeta)/\mathcal{S}_{\mu,\gamma}$ with $\hat{V}_\mu(\zeta)$ (2.16).

Let $\hat{V}^*_\mu$ be the dual module. The Borel–Weil–Bott theorem for the loop group [61] states that $\hat{V}^*_\mu$ can be realized as the space of sections of a line bundle $\mathcal{L}_\mu$ over the affine flag variety (2.12). The line bundle is determined by the action $L^+(G) \times \mathbb{C}^*$ on its sections as in (2.15),

$$\mathcal{L}_\mu = \big\{(g,\xi) \sim \big(gb, \chi_\mu\big(b^{-1}\big)\,\xi\big),\ g \in L(G),\ b \in L^+(G)\big\}. \tag{2.21}$$

3 Conformal blocks and KZB equation in general case

3.1 Moduli space of holomorphic G-bundles

Let $\mathcal{P}$ be a principle G-bundle over a curve $\Sigma_{g,n}$ of genus g with n marked points $\vec{z} = (z_1,\dots,z_n)$, $(n > 0)$, V is a G-module and $E_G = \mathcal{P} \times_G V$ is the associated bundle. We consider the set of isomorphism classes of holomorphic G-bundles $\mathcal{M}_{G,g,n}$ over $\Sigma_{g,n}$ with *the quasi-parabolic structures at the marked points* [64]. They are defined in the following way. A G-bundle can be trivialized over small disjoint disks $D = \bigcup\limits_{a=1}^{n} D_a$ around the marked points and over $\Sigma_{g,n} \setminus \vec{z}$. Therefore, $\mathcal{P}$ is defined by the transition holomorphic functions on $D^\times = \bigcup\limits_{a=1}^{n} (D_a^\times)$ and $D_a^\times = D_a \setminus z_a$. If $G(X)$ are the holomorphic maps from $X \subset \Sigma_g$ to G, then the isomorphism classes are defined as the double coset space

$$\mathrm{Bun}_G = G(\Sigma_{g,n} \setminus \vec{z}) \setminus G(D^\times)/G(D) \sim \mathcal{M}_{G,g,n}. \tag{3.1}$$

Let t_a be a local coordinate in the disks D_a. Then $G(D) = \prod_{a=1}^{n} G(D_a) = \prod_{a=1}^{n} G \otimes \mathbb{C}[[t_a]]$ and

$$G(D^\times) = \prod_{a=1}^{n} L_a(G), \qquad L_a(G) = G \otimes \mathbb{C}[[t_a, t_a^{-1}]. \tag{3.2}$$

Let us fix G-flags at fibers over the marked points. The quasi-parabolic structure of the G-bundle means that $G(D)$ preserves these G-flags. In other words, $G(D_a) = L_a^+(G)$ (2.6). At the level of the Lie algebra $\mathrm{Lie}(G(D)) = \bigoplus_{a=1}^{n} L_a^+(\mathfrak{g})$ (2.2). We discuss the Lie algebra $\mathfrak{g}_{\mathrm{out}} = \mathrm{Lie}(G(\Sigma_{g,n} \setminus \vec{z}))$ below.

Consider the one-point case $\vec{z} = z_0$ in (3.2). Let $g(t) \in G[[t, t^{-1}] = G(D_{z_0}^\times)$ be the transition function on the punctured disc $D_{z_0}^\times$ with the local coordinate t. This transition function defines a G-bundle. Its Lie algebra $\mathrm{Lie}(G(D^\times)) = \mathfrak{g} \otimes \mathbb{C}[[t, t^{-1}]$ assumes the form (see (3.1))

$$\mathrm{Lie}(G(D^\times)) = \mathfrak{g}_{\mathrm{out}} \oplus T\,\mathrm{Bun}_G \oplus L^+(\mathfrak{g}). \tag{3.3}$$

Introduce a new transition matrix $\tilde{g}(t) = t^\gamma g(t)$, where $\gamma \in P^\vee$ is an element of the coweight lattice. It defines a new bundle $\tilde{E}_G$. The passage from E_G to $\tilde{E}_G$ is called *the modification* of the bundle E_G at the point z_0. The modification amounts to the passage between different sectors of the integrable module attached at z_0 (see (2.16), (2.17), (2.18)). Since $t^\gamma \in B$, where B is the Borel subgroup ($\mathfrak{b} = \mathrm{Lie}(B) \subset L^+(\mathfrak{g})$) (2.2), we say that modification is performed in the "direction", consistent with the quasi-parabolic structure at z_0. In general, it can have an arbitrary direction. It means that t^γ may be replaced by $\mathrm{Ad}_f(t^\gamma)$, where $f \in G$. As it was mentioned in Section 2.2 there is a unique modification that preserves the HWV of the integrable module $\hat{V}_\mu$ attached at z_0.

To be a $\bar{G}$-bundle over Σ_g the transition matrix g should have a trivial monodromy $g\big(te^{2\pi i}\big) = g(t)$ around w. If $g(t)$ has a trivial monodromy and γ belongs to the coroot sublattice $Q^\vee$, then $\tilde{g}(t)$ also has a trivial monodromy. Otherwise, the monodromy is an element of the center $\mathcal{Z}\big(\bar{G}\big)$. For example, let $\gamma = j\varpi^\vee$, where $\varpi^\vee$ generate the group $\mathbb{Z}_l$, i.e. $l\varpi^\vee \in Q^\vee$, while $j\varpi^\vee \notin Q^\vee$ for $j \neq 0$, $\mathrm{mod}(l)$. In this case

$$g\big(te^{2\pi i}\big) = \zeta^j g(t), \qquad \zeta = \mathbf{e}(\varpi^\vee). \tag{3.4}$$

If $j \neq 0$ then $g(t)$ is not a transition matrix for the $\bar{G}$-bundle. But it can be considered as a transition matrix for the G^{ad}-bundle, since $G^{\mathrm{ad}} = G/\mathcal{Z}$. In this case the G-bundle is topologically non-trivial and ζ represents *the characteristic class* of E_G. The characteristic class is an obstruction to lift G^{ad}-bundle to G-bundle. It is represented by an element $H^2(\Sigma_g, \mathcal{Z})$ [46]. Let $\tilde{g}(t) = g_j(t) = t^{j\varpi^\vee}$. Then the multiplication by $g_j(t)$ provides a passage in (2.11) from the trivial sector to the non-trivial sectors

$$g_j(t) \cdot L_0\big(G^{\mathrm{ad}}\big) = L_j\big(G^{\mathrm{ad}}\big).$$

In general, we have a decomposition of the moduli space (3.1) into sectors

$$\begin{aligned}
&\mathcal{M}_{G^{\mathrm{ad}},g,1} = \bigcup_{\gamma \in P^\vee} \mathcal{M}^{(\gamma)}_{G^{\mathrm{ad}},g,1},\\
&\mathcal{M}^{(\gamma)}_{G^{\mathrm{ad}},g,1} = G\big(\Sigma_{g,n} \setminus w\big) \setminus G^{\mathrm{ad}}_\gamma \otimes \mathbb{C}[[t, t^{-1}]/G \otimes \mathbb{C}[[t]],\\
&G^{\mathrm{ad}}_\gamma \otimes \mathbb{C}[[t, t^{-1}] = G^{\mathrm{ad}} \otimes t^\gamma \mathbb{C}[[t, t^{-1}].
\end{aligned} \tag{3.5}$$

In particular, for $\Sigma_{0,1}$ ($\mathbb{C}P^1 \sim \mathbb{C} \cup \infty$) and the marked point $z_1 = 0$ this representation is related to the Grothendieck description of the vector bundles over $\mathbb{C}P^1$. Let $g_- \in G \otimes \mathbb{C}[z^{-1}]$, $g_+ \in G \otimes \mathbb{C}[z]$. Then $g(z) \in L(G) = (\mathbb{C}^* \to G) = \{g(z)\}$ has the Birkhoff decomposition [61]

$$g(z) = g_- z^\gamma g_+, \qquad \gamma \in P^\vee. \tag{3.6}$$

It means that any vector bundle E_G over $\mathbb{C}P^1$ is isomorphic to the direct sum of the line bundles $\oplus_{i=1}^{l}\mathcal{L}_{\gamma_i}$, where $\mathcal{L}_{\gamma_i}$ is defined by the transition function z^{γ_i}, $\gamma=(\gamma_1,\dots,\gamma_l)$. If $\gamma\notin Q^\vee E_G$ then has a non-trivial characteristic class. In fact, the bundle with $\gamma\neq 0$ are unstable.

Two subsets $\mathcal{M}_{G,g,1}^{(\gamma_1)}$ and $\mathcal{M}_{G,g,1}^{(\gamma_2)}$ of the moduli space correspond to the vector bundles with the same characteristic class if $\gamma_1=\gamma_2+\beta$, $\beta\in Q^\vee$. Then the topological classification of the moduli spaces of the vector bundles by their characteristic classes follows from (3.5)

$$\begin{aligned}
&\mathcal{M}_{G,g,1}=\bigcup_{\zeta\in\mathcal{Z}}\mathcal{M}_{G,g,1}^{(\zeta)},\\
&\mathcal{M}_{G,g,1}^{(\zeta)}=G\big(\Sigma_{g,n}\setminus w\big)\setminus G^{\zeta}\otimes\mathbb{C}[[t,t^{-1}]/G\otimes\mathbb{C}[[t]],\\
&G^{(\zeta)}\otimes\mathbb{C}[[t,t^{-1}]=G\otimes t^{j\varpi^\vee}\mathbb{C}[[t,t^{-1}],\qquad \zeta=\mathbf{e}\big(j\varpi^\vee\big).
\end{aligned}\tag{3.7}$$

Similar representation exists for the space $\mathcal{M}_{G,g,n}$.

3.2 Moduli of complex structures of curves

Let $\mathfrak{M}_g$ be the moduli space of complex structures of compact curves Σ_g of genus g. The moduli space $\mathfrak{M}_{g,n}$ of the complex structures of curves with marked points is foliated over $\mathfrak{M}_g$ with fibers $\mathcal{U}\subset\mathbb{C}^n$ corresponding to the moving marked points.

An infinitesimal deformation of the complex structures is represented by the Beltrami $(-1,1)$ differential $\mu(z,\bar z)=\mu\frac{\partial}{dz}\otimes d\bar z$ on $\Sigma_{g,n}$. In this way μ is $(0,1)$ form on Σ taking values in $T^{(1,0)}(\mathcal{M}_{g,n})$ and vanishing at the marked points. The basis in the tangent space $T(\mathfrak{M}_{g,n})$ is represented by the Dolbeault cohomology group $H^1(\Sigma_g,\Gamma(\Sigma_g\setminus\vec z)\otimes\bar K)$, where $\bar K$ is the anti-canonical class.

Let us compare it with the Čech like construction of $T\mathfrak{M}_{g,n}$ as a double coset space. As above, consider small disks D_a around marked points with local coordinates t_a. Let $\mathbb{C}[[t_a,t_a^{-1}]\partial_{t_a}$, $\mathbb{C}[[t_a]]\partial_{t_a}$ be vector fields on $D_a^\times$ while D_a and $\Gamma_{(\Sigma_g\setminus\vec z)}$ is a space of vector fields on $\Sigma_g\setminus\vec z$. The vector fields from the latter space can have poles of finite orders at the marked points. Then

$$T\mathfrak{M}_{g,n}=\Gamma_{(\Sigma_g\setminus\vec z)}\Big\backslash\bigoplus_{a=1}^{n}\mathbb{C}[[t_a,t_a^{-1}]\partial_{t_a}\Big/\bigoplus_{a=1}^{n}\mathbb{C}[[t_a]]\partial_{t_a}.\tag{3.8}$$

This construction has the following relation to the Dolbeault description. We establish correspondence between $\varsigma\in\oplus_{a=1}^{n}\mathbb{C}[[t_a,t_a^{-1}]\partial_{t_a}$ on $\bigcup_{a=1}^{n}D_a^\times$ and the Beltrami differential μ. Let $\varsigma_{\rm out}\in\Gamma_{(\Sigma_g\setminus\vec z)}$, $\varsigma_{\rm int}\in\oplus_{a=1}^{n}\mathbb{C}[[t_a]]\partial_{t_a}$. Consider two equations on $\bigcup_{a=1}^{n}D_a^\times$

$$\bar\partial\varsigma_{\rm out}=\mu,\qquad \bar\partial\varsigma_{\rm int}=\mu,$$

where $\bar\partial|_{D_a^\times}=\partial_{\bar t_a}$. On $D_a^\times\bar\partial(\varsigma_{\rm out}-\varsigma_{\rm int})=0$ and, therefore, $\varsigma_{\rm out}-\varsigma_{\rm int}$ represents a Dolbeault cocycle. The first equation has solutions that can be continued on $\Sigma_g\setminus\vec z$ and the second – on $\bigcup_{a=1}^{n}D_a$. If $\varsigma\in\oplus_{a=1}^{n}\mathbb{C}[[t_a,t_a^{-1}]\partial_{t_a}$ has continuations $\varsigma_{\rm out}$ and $\varsigma_{\rm int}$ then it corresponds to a trivial element of $T\mathfrak{M}_{g,n}$. On the other hand, $\bar\partial\varsigma=\mu$ globally and, therefore, μ represents an exact Dolbeault cocycle. In this way the non-trivial vector fields $\varsigma\in\oplus_{a=1}^{n}\mathbb{C}[[t_a,t_a^{-1}]\partial_{t_a}$ correspond to elements of $H^1(\Sigma_g,\Gamma(\Sigma_g\setminus\vec z)\otimes\bar K)$.

3.3 Definition of conformal blocks and coinvariants

Let us associate with $\Sigma_{g,n}$ the following set: integer k and the weights $\vec{\mu} = (\mu_1, \ldots, \mu_n,\, \mu_a \in I_k)$ attached to the marked points $\vec{z} = (z_1, \ldots, z_n)$. The $\hat{L}(\mathfrak{g})$-module (2.4)

$$\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}} = \bigotimes_{a=1}^{n} \hat{\mathbf{V}}_{\mu_a}, \qquad \vec{z} = (z_1, \ldots, z_n). \tag{3.9}$$

According to (2.19)

$$\hat{\mathbf{V}}_{\mu_a} = \bigoplus_{\zeta_a \in \mathcal{Z}} \hat{V}_{\mu_a}(\zeta_a). \tag{3.10}$$

Coming back to (3.1) we define a Lie algebra $\mathfrak{g}_{\rm out} = {\rm Lie}(G(\Sigma_g \setminus D)$ as a Lie algebra of meromorphic functions on $\Sigma_{g,n}$ with poles at $\vec{z} = (z_1, \ldots, z_n)$ taking values in $\mathfrak{g}$. Let $(t_1, \ldots, t_n)$ are local coordinates in D. There is a homomorphism $\mathcal{O}(\Sigma_g \setminus \vec{z}) \to \mathbb{C}[[t_a, t_a^{-1}]$ for each z_a providing the homomorphism of the Lie algebras $\mathfrak{g}_{\rm out} \to \mathfrak{g} \otimes \mathbb{C}[[t_a, t_a^{-1}]$. In this way $\mathfrak{g}_{\rm out}$ acts on $\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}$ as

$$(X \otimes f) \cdot (v_1 \otimes \cdots \otimes v_n) = \sum_a v_1 \otimes \cdots \otimes (X \otimes f(t_a)) \cdot v_a \otimes \cdots \otimes v_n.$$

This is a Lie algebra action. Due to the residue theorem this homomorphism is lifted to the diagonal central extension

$$\mathfrak{g}_{\rm out} \hookrightarrow \bigoplus_{a=1}^{n} \hat{L}_a(\mathfrak{g}), \qquad \hat{L}_a(\mathfrak{g}) = \big(\mathfrak{g} \otimes \mathbb{C}[[t_a, t_a^{-1}]\big) \oplus K, \qquad K \to k.$$

In what follows we need a relation of $\hat{V}_\mu$ with the space of coinvariants. In general setting the coinvariants are defined in the following way. Let $\mathcal{W}$ be a module of a Lie algebra $\mathfrak{k}$. The space of coinvariants $[\mathcal{W}]_{\mathfrak{k}}$ is the quotient-space $[W]_{\mathfrak{k}} = W/\mathfrak{k} \cdot W$. In the case at hand we define *the space of coinvariants* with respect to the action of $\mathfrak{g}_{\rm out}$,

$$\mathcal{H}(\vec{z}, \vec{\mu}) = \big[\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}\big]_{\mathfrak{g}_{\rm out}}, \qquad \big([V]_{\mathfrak{g}} = V/\mathfrak{g} \cdot V\big).$$

The space of conformal blocks $\mathcal{C}\big(\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}\big)$ *is the dual space to the coinvariants.* In other words, $\mathcal{C}\big(\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}\big)$ is the space of linear functionals on $\hat{V}^{[n]}_{\vec{z},\vec{\mu}}$, invariant under $\mathfrak{g}_{\rm out}$:

$$F: \ \hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}} \to \mathbb{C}, \qquad F(X \cdot v) = 0 \quad \text{for any} \quad X \in \mathfrak{g}_{\rm out}.$$

Put it differently, the conformal blocks are $\mathfrak{g}_{\rm out}$-invariant elements of the contragradient module $\hat{\mathbf{V}}^{*[n]}_{\vec{z},\vec{\mu}}$. For a single marked point case the conformal blocks are $\mathfrak{g}_{\rm out}$ invariant sections of the line bundle $\mathcal{L}_\mu$ over the affine flag variety (2.21).

According to (3.9) and (3.10) the space $\hat{\mathbf{V}}^{[n]}_{\vec{\mu}}$ has the representation

$$\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}} = \bigotimes_{a=1}^{n} \bigoplus_{\zeta_a \in \mathcal{Z}} \hat{V}_{\mu_a}(\zeta_a).$$

In a similar way the conformal blocks are decomposed in subspaces corresponding to the characteristic classes of the bundles

$$\mathfrak{C}\big(\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}\big) = \bigotimes_{a=1}^{n} \bigoplus_{\zeta_a \in \mathcal{Z}} \mathfrak{C}_a\big(\hat{V}_{\mu_a}(\zeta_a)\big),$$

where

$$\mathfrak{C}_a = \big\{F(\zeta_a) : \hat{V}_{\mu_a}(\zeta_a) \to \mathbb{C}\big\}. \tag{3.11}$$

3.4 Variation of the moduli space of complex structures

The space of conformal blocks $\mathfrak{C}\big(\hat{\mathbf{V}}^{[n]}_{\vec{z},\vec{\mu}}\big)$ is a bundle over $\mathfrak{M}_{g,n}$. This bundle is equipped with the KZB connection that can be described as follows.

A stress-tensor $T(z,\bar{z})$ in general theories, defined on a surface $\Sigma_{g,n}$, generates vector fields on $\Sigma_{g,n}$. A dual object to $T(z,\bar{z})$ is the Beltrami differential $\mu(z,\bar{z})$. It means that there is a connection on the bundle of fields over $\mathcal{M}_{g,n}$ (the Friedan–Shenker connection)

$$\nabla_\mu F = \delta_\mu F + \int_\Sigma \mu T F. \tag{3.12}$$

In conformal field theories the stress-tensor is a meromorphic projective structure on $\Sigma_{g,n}$. The connection acting on the space of conformal blocks is projectively flat. The conformal blocks are horizontal sections of this bundle. The horizontality conditions are nothing else but the KZB equations for the conformal blocks. In general setting these equations are discussed in [33] (for the smooth curves) and in [22]. They have the form of non-stationary Schrödinger equations [36].

The connection (3.12) can be rewritten in a local form based on the representation (3.8). Let $\bigcup\limits_{a=1}^{n} D_a^\times \subset \Sigma_g$ and $\gamma_a \subset D_a^\times$ is a small contour and ς_a is a vector field in $D_a^\times$. Then (3.12) can be written as

$$\nabla_{\varsigma_a} F = \partial_{\varsigma_a} F + \oint_{\gamma_a} \varsigma_a T F \tag{3.13}$$

and the KZB equation assumes the form

$$\nabla_\varsigma F = 0. \tag{3.14}$$

At the marked points T has the second order poles, while $\varsigma_a \in \mathbb{C}[[t_a]]\partial_{t_a}$ (3.8). Thereby, this integral produces $\partial_{z_a} F$. On the other hand, the product TF is non-singular outside the disks D_a. Then for $\varsigma_a \in \Gamma_{(\Sigma_g\setminus\vec{z})}$ the integrals vanish. It means that the conformal blocks F are defined on $\mathfrak{M}_{g,n}$.

Consider a one point case and let t be a local coordinate on a punctured disk $D^\times$. The stress-tensor in the local coordinate has the Fourier expansion $T(t) = \sum\limits_{n\in\mathbb{Z}} L_n t^{-n-2}$. The coefficients obey the Virasoro commutation relations $[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)$.

In the WZW model the stress-tensor is obtained from the currents by means of the Sugawara construction (see [7]). Let $\{\mathfrak{t}_\alpha\}$ be a basis in $\mathfrak{g}$, $\{\mathfrak{t}^\beta\}$ is the dual basis, and $I_\alpha(t) = \sum_m \mathfrak{t}_{\alpha,m} t^{-m-1} \in \mathfrak{g}\otimes\mathbb{C}[[t,t^{-1}]$. Then

$$T(t) = \frac{1}{2(k+h^\vee)} \sum_\alpha {:}I_\alpha(t) I^\alpha(t){:},$$

where $h^\vee$ is the dual Coxeter number. The Fourier coefficients of $T(t)$ take the form

$$L_m = \frac{1}{2(k+h^\vee)} \sum_{p\in\mathbb{Z}} {:}\mathfrak{t}_{\alpha,-p}\mathfrak{t}^\alpha_{p+m}{:}. \tag{3.15}$$

The normal ordering means placing to the right $\mathfrak{t}^\alpha_n$ $(\mathfrak{t}_{\alpha,n})$ with $n > 0$. The Virasoro central charge is

$$c = \frac{\dim\mathfrak{g}}{k+h^\vee}.$$

The Virasoro algebra acts on $\mathfrak{g}\otimes\mathbb{C}[[t,t^{-1}]$ as

$$L_n \mapsto t^{n+1}\frac{d}{dt}. \tag{3.16}$$

This action is well defined because the action of the Sugawara tensor is well defined on the integrable modules. In particular, it follows from (3.16) that for the moving points equation (3.14) assumes the form

$$\left(\partial_{z_a} - L^a_{-1}\right)F = 0. \tag{3.17}$$

The restriction of ∇_ς on $\mathfrak{C}_a$ (3.11) yields a family of the KZB equations

$$\sum_a \nabla_a F(\zeta_a) = 0. \tag{3.18}$$

In next section we construct these equations explicitly for the bundles over elliptic curves.

3.5 Variation of the moduli space of holomorphic bundles

3.5.1 General construction

The moduli space of holomorphic bundles $\mathcal{M}_{G,g,n} = \mathrm{Bun}_G$ (3.1) is foliated over the moduli space of complex structures $\mathfrak{M}_{g,n}$. Let us consider the dependence of the space of coinvariants $\mathcal{H}(\vec{z},\vec{\mu})$ (conformal blocks $\mathfrak{C}(V^{[n]})$) on the variations of the moduli of the bundles Bun_G. For simplicity consider the one-point case. Let t_a be a local coordinate in $D^\times_a$, and $G_{\rm out} = G(\Sigma_g \setminus z_a)$. Define the quotient

$$M_G = G_{\rm out}\setminus G(D^\times_a), \qquad G(D^\times_a) = G[[t_a,t_a^{-1}]. \tag{3.19}$$

This space is the moduli space of G-bundles with a trivialization around z_a (see (3.1)).

Let $\hat{V}_{\mu_a}$ be an integrable module (2.14) attached to z_a. Recall that $\hat{V}^*_{\mu_a}$ is the space of holomorphic sections $\Gamma(\mathcal{L}_{\mu_a})$ of the line bundle (2.21) over the affine flag variety (2.12). In these terms the space of conformal blocks has the following interpretation [6, 20]. Since $\hat{V}_{\mu_a}$ is the integrable representation, the group $G(D^\times)$ acts on $\hat{V}_{\mu_a}$. Thereby, the subgroup $G_{\rm out}$ acts on $\hat{V}_{\mu_a}$ also. Due to (3.19) $G(D^\times_a)$ acts on $\mathcal{M}_G$ from the right. Therefore, $G(D^\times_a)$ acts on the sections $\hat{V}_{\mu_a}\otimes\mathcal{O}(M_G)$ of the trivial vector bundle $\hat{V}_{\mu_a}\times\mathcal{M}_G$

$$g\cdot v(x) = (gv)(xg), \qquad g\in G[[t_a,t_a^{-1}], \qquad v\in\hat{V}_{\mu_a}. \tag{3.20}$$

Consider the space of the coinvariants

$$\hat{V}_{\mu_a}\otimes\mathcal{O}(M_G)/\big(\hat{V}_{\mu_a}\otimes\mathcal{O}(M_G)\big)\cdot\mathrm{Stab}_x,$$

where Stab_x is $\mathrm{Lie}(G_x(D^\times)$, $G_x(D^\times) = \{g|x\cdot g = x,\, x\in\mathcal{M}_G\}$. In particular, $\mathrm{Stab}_x = \mathfrak{g}_{\rm out}$ for x corresponding to $G_{\rm out}$. The spaces of coinvariants are isomorphic for different choices of x. The dual space $\Gamma(\mathcal{L}_\mu)/G_{\rm out}$ is the space of conformal blocks. The quotient $\Gamma(\mathcal{L}_\mu)/G_{\rm out}$ is a space of sections of the line bundle over Bun_G (3.1). It means that the space of conformal blocks is a non-Abelian generalization of the theta line bundles over the Jacobians.

3.5.2 SL$(2,\mathbb{C})$-bundles over $\mathbb{C}P^1$

It is instructive to consider this construction for $\Sigma = \mathbb{C}P^1 = \mathbb{C}\cup\infty$. This case was analyzed in details in [5] for the trivial G-bundles and $\gamma = 0$ in (3.6). Here we consider $G = \mathrm{SL}(2,\mathbb{C})$-bundles with $\gamma\in P^\vee$. Let $\mathfrak{t}_\alpha = h, e, f$ be the Cartan–Chevalley basis in the Lie algebra $\mathrm{sl}(2,\mathbb{C})$

$$[h,e] = 2e, \qquad [h,f] = 2f, \qquad [e,f] = h,$$

and $\mathfrak{t}_\alpha(n) = \mathfrak{t}_\alpha t^n$. The Verma module $\mathcal{V}_\mu$ is generated by $L^-(\mathrm{sl}(2,\mathbb{C})) = c\cdot f + \mathfrak{g}\otimes t^{-1}\mathbb{C}[t^{-1}]$ (2.20), (2.3). The HV v_μ with the weight $\mu \in P(\mathrm{sl}(2,\mathbb{C}))$ is defined by the conditions $hv_\mu = 2sv_\mu$, $s = \frac{1}{2}\langle\mu, h\rangle \in \frac{1}{2}\mathbb{Z}$, $ev_\mu = 0$, $\mathfrak{t}_\alpha(n)v_\mu = 0$ for $n > 0$. The singular vector $(et^{-1})^{k+1-2s}$ generates the submodule $\mathcal{S}_\mu \subset \mathcal{V}_\mu$, and the integrable module is the quotient $\hat{V}_\mu = \mathcal{V}_\mu/\mathcal{S}_\mu$.

This form of $\hat{V}_\mu$ defines a trivial sector in (2.19). Note that $\mathcal{Z}(\mathrm{SL}(2,\mathbb{C})) = \mathbb{Z}_2 = \{\zeta = (0,1)\}$. Therefore, there are two sectors in the integrable module (2.19). Consider the non-trivial sector corresponding to $\zeta = 1$. Let $W_P = \{\hat{w}\}$ be the Weyl group (2.5), $\hat{w} = \mathbb{Z}_2 \ltimes t^\gamma$, where γ belongs to the weight lattice $\gamma \in P^\vee(\mathrm{sl}(2,\mathbb{C})) = P(\mathrm{sl}(2,\mathbb{C})) = \frac{1}{2}\mathbb{Z}$. Since $Q^\vee(\mathrm{sl}(2,\mathbb{C})) = Q(\mathrm{sl}(2,\mathbb{C}))$, $\mathcal{Z} = P/Q \sim \mathbb{Z}_2$, and $\zeta = 1$ corresponds to $\gamma \notin Q$. It means that $\langle\gamma, h\rangle$ is odd. Then according to (2.19)

$$\mathcal{V}_\mu(\zeta = 1) = \mathcal{U}\big(L^-(\mathrm{sl}(2,\mathbb{C}))\big)(t^\gamma v_\mu), \quad \gamma \notin Q, \qquad \hat{V}_\mu(\zeta = 1) = \mathcal{V}_\mu(\zeta = 1)/\mathcal{S}_\mu.$$

As it was mentioned above, $t^\gamma v_\mu$ is not the HWV. In other words, it is not annihilated by the positive nilpotent loop subalgebra (2.8) $\mathrm{Lie}(N^+(\mathrm{SL}(2,\mathbb{C}))) = \{n(t) = b\cdot e + \mathfrak{g}\otimes t\mathbb{C}[[t]],\ b \in \mathbb{C}\}$. In fact, we have

$$n(t)t^\gamma v_\mu = t^\gamma\,\mathrm{Ad}^{-1}_{t^\gamma}(n(t))v_\mu.$$

Let $\langle\gamma, h\rangle = 2s > 0$, and $s \in \frac{1}{2} + \mathbb{Z}$. Then

$$\mathrm{Ad}^{-1}_{t^\gamma}(n(t)) = \sum_{m\geq 0}\left(a_{m+1}\cdot ht^{m+1} + b_m\cdot et^{2s+m} + c_{m+1}\cdot ft^{-2s+m+1}\right).$$

The terms $c_{m+1}\cdot ft^{-2s+m+1})$ for $m \leq 2s-1$ do not belong to $\mathrm{Lie}(N^+(\mathrm{SL}(2,\mathbb{C})))$. Multiply t^γ by $wt^{\gamma_1} \in W_Q$, where $\langle\gamma_1, h\rangle = -2s+1$ and $w : e \leftrightarrow f$. This transformation preserves the sector. Now $n(t)$ annihilates the vector $wt^{\gamma_1}t^\gamma v_\mu$. Note (wt^{γ_1}) is uniquely defined by γ. Thus, for any vector $t^\gamma v_\mu$ we define a unique HWV from the same sector. It is a particular case of general theorem proved in [23].

Consider the trivial G-bundles over $\mathbb{C}P^1$. It was proved in [5] that the conformal blocks are G-invariant functionals on the module $\hat{V}^{[n]}_\mu$ satisfying some additional conditions. In particular, for $n = 1$

$$\dim(\mathfrak{C}(G)) = \begin{cases} 0, & \mu \neq 0,\\ 1, & \mu = 0,\end{cases} \tag{3.21}$$

and for $\hat{V}^{[2]}_{(0,\infty)(\mu_0,\mu_\infty)}$

$$\dim(\mathfrak{C}(G)) = \begin{cases} 0, & \mu_0 \neq \mu^*_\infty,\\ 1, & \mu_0 = \mu^*_\infty.\end{cases}$$

Let us analyze the case of $\mathrm{SL}(2,\mathbb{C})$-bundles. It follows from the Bruhat decomposition (2.9) that there are two types of the $\mathrm{SL}(2,\mathbb{C})$-bundles over $\mathbb{C}P^1$ – the trivial, when $\gamma \in Q$ in (3.5) is an element of the root lattice $\gamma \in Q$, and non-trivial, when $\gamma \notin Q$. Note that the stable bundles correspond $\gamma = 0$. In the first case we deal with the adjoint bundles that can be lifted to the $\mathrm{SL}(2,\mathbb{C})$-bundles. In the second case there is an obstruction to lift these bundles to the $\mathrm{SL}(2,\mathbb{C})$-bundles.

Let z^{-1} be a local coordinate in a neighborhood of ∞. The Lie algebra $\mathfrak{g}_{\mathrm{out}}$ assumes the form $\mathfrak{g}_{\mathrm{out}} = \mathrm{sl}(2,\mathbb{C}) + z^{-1}\mathrm{sl}(2,\mathbb{C})\otimes\mathbb{C}[z^{-1}]$. Let $n = 1$ and $z = 0$ is the marked point with the attached integrable $L(\mathrm{sl}(2,\mathbb{C}))$-module $\hat{V}_\mu$. We have $\mathfrak{g}_{\mathrm{out}}(v_\mu) = \hat{V}_\mu$ for $\mu \neq 0$. But $v_0 \notin \mathfrak{g}_{\mathrm{out}}(v_0)$ and v_0 is the coinvariant confirming (3.21).

Consider the integrable module generated by $z^\gamma v_\mu$, where, as above, $\langle\gamma, h\rangle = 2s > 0$, and $s \in \frac{1}{2} + \mathbb{Z}$. Then

$$\mathfrak{g}_{\rm out} z^\gamma v_\mu = \mathrm{Ad}^{-1}_{z^\gamma}(n(t)) = z^\gamma \sum_{m\geq 0} \left(a_{-m} z^{-m} \cdot h + b_{-m} z^{-2s-m} \cdot e + c_{-m} z^{2s-m}\right) \cdot f v_\mu.$$

Then the elements $b_{-m} z^{-2s-m} \cdot e(v_\mu)$ for $0 \leq m < 2s$ are not generated by $\mathfrak{g}_{\rm out}$. Thus, if $\gamma \neq 0$, the space of coinvariants (and the space of conformal blocks) is non-empty for an arbitrary weights μ. Its dimension depends on γ: $\dim(\mathfrak{C}(\mathrm{SL}(2,\mathbb{C}))) = 2s$ (compare with (3.21)).

3.5.3 The form of connection

For conformal blocks we have (see (3.20))

$$F(x) = (gF)(xg), \qquad g \in \mathrm{Stab}_x\,.$$

Define the current $J(t_a) = (g^{-1}dg)(t_a) \in \mathfrak{g} \otimes \mathbb{C}[[t_a, t_a^{-1}]) \otimes \Omega^1(D_a^\times)$ for $g(t_a) \in G \otimes \mathbb{C}[[t_a, t_a^{-1}]$. A local version of (3.20) is defined by the operator

$$\nabla_{\mathbf{u}_\alpha} = \partial_{\mathbf{u}_\alpha} + \oint_{\gamma_a} \langle (g^{-1}dg), \mathfrak{t}_\alpha\rangle,$$

where $\mathfrak{t}_\alpha$ is a generator of $\mathfrak{g}$ and $\mathbf{u}_\alpha$ is a coordinate of the tangent vector to Bun_G. The action of $\nabla_{\mathbf{u}_\alpha}$ on the conformal blocks is well defined because the conformal blocks are $\mathfrak{g}_{\rm out}$-invariant. Therefore, they are horizontal with respect to this connection

$$\partial_{\mathbf{u}_\alpha} F + \oint_{\gamma_a} \langle (g^{-1}dg)(t_a), \mathfrak{t}_\alpha\rangle F = 0. \tag{3.22}$$

If one takes $\mathbf{u}$ from $\mathcal{M}^{(\zeta)}_{G,g,n}$ (3.7) then (3.22) takes the form

$$\partial_{\mathbf{u}_\alpha(\zeta_a)} F(\zeta_a) + \oint_{\gamma_a} \langle (g^{-1}dg)(t_a), \mathfrak{t}_\alpha\rangle F(\zeta_a) = 0,$$

where $F(\zeta_a) \in \mathfrak{C}_a$ (3.11).

4 KZB equations related to elliptic curves and non-trivial bundles

4.1 Moduli space of elliptic curves

We consider in details the genus one case $\Sigma_{1,n}$. Let $\Sigma_\tau = \mathbb{C}/\langle\tau, 1\rangle$ be the elliptic curve with the modular parameter in the upper half-plane $\mathcal{H} = \{\mathrm{Im}\, m\tau > 0\}$. For $n \in \mathbb{Z}$, $n \geq 1$ define the set of marked points $\vec{z} = (z_1, \ldots, z_n)$. Due to the $\mathbb{C}$ action on $\Sigma_\tau (z \to z + c)$, we assume that $\sum_a z_a = 0$. A big cell $\mathfrak{M}^0_{1,n}$ in the Teichmüller space $\mathfrak{M}_{1,n}$ is defined as

$$\mathfrak{M}^0_{1,n} = \left\{ (z_1, \ldots, z_n),\ \sum_a z_a = 0,\ z_k \neq z_j,\ \mathrm{mod}(\langle\tau, 1\rangle) \right\} \times \mathcal{H}.$$

4.2 Moduli space of holomorphic G-bundles over elliptic curves

For $G = GL_N$ the moduli space of holomorphic bundles was described by M. Atiyah [3]. For the trivial G-bundles, where G is a complex simple group, it was done in [10, 11, 52]. Non-trivial G-bundles and their moduli spaces were considered in [26, 27, 28, 63]. We describe the moduli space of stable non-trivial holomorphic bundles over Σ_τ using an approach of [46].

Let G be a complex simple Lie group. An universal cover $\bar{G}$ of G in all cases apart G_2, F_4 and E_8 has a non-trivial center $\mathcal{Z}(\bar{G})$. The adjoint group is the quotient $G^{\rm ad} = \bar{G}/\mathcal{Z}(\bar{G})$. For the cases A_{n-1} (when $n = pl$ is non-prime) and D_n the center $\mathcal{Z}(\bar{G})$ has non-trivial subgroups $\mathcal{Z}_l \sim \mu_l = \mathbb{Z}/l\mathbb{Z}$. Assume that (p, l) are co-prime. There exists the quotient-groups

$$G_l = \bar{G}/\mathcal{Z}_l, \qquad G_p = G_l/\mathcal{Z}_p, \qquad G^{\rm ad} = G_l/\mathcal{Z}(G_l), \tag{4.1}$$

where $\mathcal{Z}(G_l)$ is the center of G_l and $\mathcal{Z}(G_l) \sim \mu_p = \mathcal{Z}(\bar{G})/\mathcal{Z}_l$.

Following [56] we define a G-bundle $E_G = \mathcal{P} \times_G V$ by the transition operators $\mathcal{Q}$ and Λ_j acting on the sections of $s \in \Gamma(E_G)$ as

$$\mathbf{s}(z+1) = \mathcal{Q}(z)\mathbf{s}(z), \qquad \mathbf{s}(z+\tau) = \Lambda(z)\mathbf{s}(z), \tag{4.2}$$

where $\mathcal{Q}(z)$ and $\Lambda(z)$ take values in ${\rm End}(V)$. Going around the basic cycles of Σ_τ we come to the equation

$$\mathcal{Q}(z+\tau)\Lambda(z)\mathcal{Q}(z)^{-1}\Lambda^{-1}(z+1) = {\rm Id}\,. \tag{4.3}$$

It follows from [56] that it is possible to choose the constant transition operators. Then we come to the equation

$$\mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1} = {\rm Id}\,. \tag{4.4}$$

Replace (4.4) by the equation

$$\mathcal{Q}\Lambda\mathcal{Q}^{-1}\Lambda^{-1} = \zeta\,{\rm Id},$$

where ζ is a generator of the center $\mathcal{Z}\big(\bar{G}\big)$. In this case $(\mathcal{Q}, \Lambda)$ are the clutching operators for $G^{\rm ad}$-bundles, but not for $\bar{G}$-bundles, and ζ plays the role of obstruction to lift the $G^{\rm ad}$-bundle to the $\bar{G}$-bundle. Here $\zeta = \mathbf{e}(\varpi^\vee)$ is a generator of the center $\mathcal{Z}\big(\bar{G}\big)$, where $\varpi^\vee \in P^\vee$ is a fundamental coweight such that $N\varpi^\vee \in Q^\vee$ and $N = {\rm ord}\big(\mathcal{Z}\big(\bar{G}\big)\big)$.[2]

Let $0 < j \le N$. Consider a bundle with the space of sections with the quasi-periodicities

$$s(z+1) = \mathcal{Q}s(z), \qquad s(z+\tau) = \Lambda_j s(z) \tag{4.5}$$

such that

$$\mathcal{Q}\Lambda_j\mathcal{Q}^{-1}\Lambda_j^{-1} = \zeta^j\,{\rm Id}\,. \tag{4.6}$$

If j and N are co-prime numbers then ζ^j generates $\mathcal{Z}\big(\bar{G}\big)$. In this case $\mathcal{Q}$ and Λ_j can serve as transition operators only for a $G^{\rm ad} = \bar{G}/\mathcal{Z}$-bundle, but not for $\bar{G}$-bundle and ζ^j is an obstruction to lift $G^{\rm ad}$-bundle to $\bar{G}$-bundle.

The element ζ has a cohomological interpretation. It is called *the characteristic class* of E_G. It can be identified with elements of the group $H^2\big(\Sigma_{g,n}, \mathcal{Z}\big(\bar{G}\big)\big)$. This group classifies the of the characteristic classes of the bundles [46].

[2]For the simplicity we assume here and in what follows that $\mathcal{Z} \sim \mathbb{Z}_l$. The case $\mathcal{Z}({\rm Spin}(4n)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ can be considered in a similar way.

Consider, as above, a non-prime $N = pl$ and put $j = p$. Then ζ^j is a generator of the group $\mathbb{Z}_l$. In this case $\mathcal{Q}$ and Λ_j are transition operators for $G_l = \bar{G}/\mathcal{Z}_l$-bundles (see (4.1)) and ζ^j is an obstruction to lift a G_l-bundle to a $\bar{G}$-bundle.

The moduli space of stable holomorphic over Σ_τ with the sections (4.5) is defined as

$$\mathcal{M}^{(j)}_{G,1} = (\text{solutions of } (4.6))/(\text{conjugation}). \tag{4.7}$$

For the stable bundles this description of the moduli space is equivalent to (3.1). In fact, the monodromy of $\mathbf{s}(z)$ around $z = 0$ is the same as in (3.4). Similar to (3.7) we have

$$\mathcal{M}_{G,1} = \bigcup_{j=1}^{N} \mathcal{M}^{(j)}_{G,1}. \tag{4.8}$$

Assume that $\mathcal{Q}$ is a semi-simple element and $\mathcal{Q} \in \mathcal{H}_{\bar{G}}$ is a fixed Cartan subgroup of $\bar{G}$. It means that we consider an open subset

$$\mathcal{M}^{(j)}_{G,1} \supset \left(\mathcal{M}^{(j)}_{G,1}\right)^0 \equiv \mathcal{M}^{(j)}_0(G) = \big\{(\mathcal{Q} \in \mathcal{H}_{\bar{G}}, \Lambda_j)/(\text{conjugation})\big\}.$$

In this case the solutions of (4.4) have the form [46]

$$\mathcal{Q} = \exp\left(2\pi i \frac{\rho^\vee}{h}\right), \qquad \Lambda_j = \Lambda_0 V_j, \tag{4.9}$$

where $\rho^\vee$ is a half-sum of positive coroots, h is the Coxeter number, Λ_0 is an element of the Weyl group defined by ζ^j:

$$\zeta^j \to \Lambda_0, \qquad (\Lambda_0)^l = \mathrm{Id}\,.$$

The element Λ_0 preserves the extended system of simple roots $\Pi^{\text{ext}} = \Pi \cup (\alpha_0)$, where $-\alpha_0$ is a maximal root [46, Proposition 3.1]. In this way Λ_0 is a symmetry of the extended Dynkin diagram of $\mathfrak{g} = \mathrm{Lie}\left(\bar{G}\right)$, generated by $\varpi^\vee$ [12].

Let $\tilde{\mathcal{H}}_0 \subset \mathcal{H}_{\bar{G}}$ be the Cartan subgroup commuting with Λ_0. To describe V_j consider the adjoint action $\lambda = \mathrm{Ad}(\Lambda_0)$ on the Cartan subalgebra $\mathfrak{h} = \mathrm{Lie}(\mathcal{H}_{\bar{G}})$. Let $\tilde{\mathfrak{h}}_0 = \mathrm{Lie}(\tilde{\mathcal{H}}_0)$ be the invariant subalgebra $(\lambda(\tilde{\mathfrak{h}}_0) = \tilde{\mathfrak{h}}_0)$. Then $V_j = \exp(2\pi\imath\mathbf{u})(\mathbf{u} \in \tilde{\mathfrak{h}}_0)$ is an arbitrary element from $\tilde{\mathcal{H}}_0$ defining the moduli space $\mathcal{M}^{(j)}_0(G)$.

There exists a basis $\tilde{\Pi}^\vee_j$ in $\tilde{\mathfrak{h}}_0$ such that $\tilde{\Pi}$ is a system of simple roots for a simple Lie subalgebra $\tilde{\mathfrak{g}}_0 \subset \mathfrak{g}$. For the list of these subalgebras see [46]. If $j = N$, we come to the trivial bundles (4.4). In this case $\Lambda_0 = \mathrm{Id}$, $\tilde{\mathfrak{h}}_0 = \mathfrak{h}$ and $\tilde{\mathfrak{g}}_0 = \mathfrak{g}$.

Let $\tilde{Q}^\vee$ and $\tilde{P}^\vee$ be the coroot and the coweight lattices in $\tilde{\mathfrak{h}}_0$, and $\tilde{W}$ is the Weyl group corresponding to $\tilde{\Pi}$. Define the Bernstein–Schwarzman type groups [10, 11]. They are constructed by means of the lattices $\tilde{Q}^\vee$ or $\tilde{P}^\vee$. In the first case it is the semidirect products

$$\tilde{W}_{\mathrm{BS}} = \tilde{W} \ltimes \left(\tau\tilde{Q}^\vee \oplus \tilde{Q}^\vee\right). \tag{4.10}$$

Then the moduli space of non-trivial $\bar{G}$-bundles with the characteristic class ζ^j is the fundamental domain in $\tilde{\mathfrak{h}}^{(j)}_0$ under the action of $\tilde{W}_{\mathrm{BS}}$

$$\mathcal{M}^{(j)}_0(\bar{G}) = C^{\mathrm{sc}}_j = \tilde{\mathfrak{h}}^{(j)}_0/\tilde{W}_{\mathrm{BS}} \tag{4.11}$$

is the moduli space of non-trivial $\bar{G}$-bundles.

Consider G^{ad}-bundles. Define the semidirect product

$$W^{\mathrm{ad}}_{\mathrm{BS}} = \tilde{W} \ltimes \left(\tau\tilde{P}^\vee \oplus \tilde{P}^\vee\right). \tag{4.12}$$

A fundamental domain of this group in $\tilde{\mathfrak{h}}_0^{(j)}$ is $C^{\rm ad} = \tilde{\mathfrak{h}}_0^{(j)}/\tilde{W}_{\rm BS}^{\rm ad}$ and

$$\mathcal{M}_0^{(j)}(G^{\rm ad}) = C_j^{\rm ad} = \tilde{\mathfrak{h}}_0^{(j)}/\tilde{W}_{\rm BS}^{\rm ad} \tag{4.13}$$

is the moduli space of the non-trivial $G^{\rm ad}$-bundles. It is the moduli space of $E_{G^{\rm ad}}$-bundles with characteristic class defined by ζ^j. In other words

$$\mathbf{u} \in \begin{cases} C_j^{\rm sc} & \text{for } E_{\bar{G}}\text{-bundles}, \\ C_j^{\rm ad} & \text{for } E_{G^{\rm ad}}\text{-bundles}. \end{cases}$$

4.3 The gauge Lie algebra for elliptic curves

Here we define the moduli space of holomorphic G-bundles coming back to the double coset construction (3.1). Recall, that the Lie algebra $\mathfrak{g}_{\rm out} = {\rm Lie}(G(\Sigma_{\tau,n}\backslash\vec{z}))$ is a Lie algebra of meromorphic functions on $\Sigma_{\tau,n}$ with poles at $\vec{z} = (z_1,\dots,z_n)$ and the quasi-periodicities (4.2), (4.3).

Let us take for simplicity the case (4.4) and apply the decomposition (A.1) corresponding the characteristic class defined by ζ to the Lie algebra $\mathfrak{g}_{\rm out}$:

$$\mathfrak{g}_{\rm out} = \oplus_{k=0}^{l-1}\mathfrak{g}_k, \qquad \mathfrak{g}_0 = \mathfrak{g}_0' \oplus \tilde{\mathfrak{g}}_0, \qquad {\rm Ad}_{\Lambda_0}(\mathfrak{g}_k(z)) = \mathbf{e}\left(\frac{k}{l}\right)\mathfrak{g}_k(z).$$

Consider the quasi-periodicity conditions (4.2). The GS-basis is diagonal under ${\rm Ad}_\Lambda$ and ${\rm Ad}_{\mathcal{Q}}$ actions (A.6)–(A.8). We should find functions on $\Sigma_\tau \setminus D$ that have the same phase-factors and pole singularities at $\vec{z}$. To define $\mathfrak{g}_k$ and $\mathfrak{g}_0'$ we use the functions ϕ (B.1), and $\varphi_\alpha^{k,m}$ (B.4). They have the needed quasi-periodicities (B.16), (B.17) and poles at $z=0$ (B.7), (B.9). Then we find

$$\mathfrak{g}_k = \left\{\sum_{a=1}^{n}\sum_{m=0}^{K(a,k)}\left(\sum_{\alpha\in\Pi} x_{\alpha,m,a}^k \partial_z^m\phi\left(\frac{k}{l}, z-z_a\right)\mathfrak{h}_\alpha^k + \sum_{\alpha\in R} y_{\alpha,m,a}^k\varphi_\alpha^{k,m}(\mathbf{u}, z-z_a)\mathfrak{t}_\alpha^k\right)\right\}, \tag{4.14}$$

$$\mathfrak{g}_0' = \left\{\sum_{a=1}^{n}\sum_{\alpha\in R}\sum_{m=0}^{K(a,\alpha)} y_{\alpha,m,a}'\varphi_\alpha^{0,m}(\mathbf{u}, z-z_a)\mathfrak{t}_\alpha^0\right\}. \tag{4.15}$$

Similarly, from (B.10), (B.11), (B.12), (B.14), (B.15) we have

$$\tilde{\mathfrak{g}}_0 = \left\{\sum_{a=1}^{n}\left(\sum_{\alpha\in\tilde{\Pi}}\left(x_{\alpha,0}^0 + \sum_{m=1}^{K(a,\alpha)} x_{\alpha,m,a}^0 E_m(z-z_a)\right)h_\alpha \right.\right.$$
$$\left.\left. + \sum_{\alpha\in\tilde{R}}\sum_{m=0}^{K(a,\alpha)} y_{\alpha,m,a}^0\varphi_\alpha^{0,m}(\mathbf{u}, z-z_a)E_\alpha\right)\right\}. \tag{4.16}$$

Then $\mathfrak{g}_{\rm out}$ has the correct quasi-periodicities and has poles of orders $K(a,m)$, $K(a,\alpha)$ at z_a, $a=1,\dots,n$. In this last expression (due to the residue theorem) from (B.14) we assume that

$$\sum_{a=1}^{n} x_{\alpha,1,a}^0 = 0. \tag{4.17}$$

Let us unify the last two expression (4.15) and (4.16) in a single formula,

$$\mathfrak{g}_0 = \left\{\sum_{a=1}^{n}\left(\sum_{\alpha\in\tilde{\Pi}}\left(x_{\alpha,0}^0 + \sum_{m=1}^{K(a,\alpha)} x_{\alpha,m,a}^0 E_m(z-z_a)\right)h_\alpha\right.\right.$$

$$+\sum_{\alpha\in R}\sum_{m=0}^{K(a,\alpha)} y^0_{\alpha,m,a}\varphi^{0,m}_\alpha(\mathbf{u},z-z_a)\mathfrak{t}^0_\alpha\Bigg)\Bigg\}. \tag{4.18}$$

We will act on the coinvariants by $\mathfrak{g}_{\rm out}$. In what follows we need the limit $z\to z_a$ of these expressions. Notice that $\mathfrak{g}_{\rm out}$ is the filtered Lie algebra. The filtration is defined by the orders of poles. The behavior of $\mathfrak{g}_{\rm out}$ is defined by the asymptotics (B.7)–(B.9), (B.12). As it will become clear below we need the least singular terms in $\mathfrak{g}_{\rm out}$. In this way we take $m=0$ in (4.14), (4.18) and $m=1$ ($E_1(z-z_a)$) in (4.18):

$$\begin{aligned}\mathfrak{g}_k &\sim \sum_{a=1}^n\sum_{\alpha\in\Pi} x^k_{\alpha,0,a}\left(\cdots+\frac{1}{z-z_a}+E_1\left(\frac{k}{l}\right)+\sum_{b\neq a}\phi\left(\frac{k}{l},z_a-z_b\right)+\right)\mathfrak{H}^k_\alpha\\ &\quad + y^k_{\alpha,0,a}\Bigg(\cdots+\frac{1}{z-z_a}+E_1\left(\langle\mathbf{u}+\kappa\tau,\alpha\rangle+\frac{k}{l}\right)+2\pi\imath\langle\kappa,\alpha\rangle\\ &\quad +\sum_{b\neq a}\varphi^{k,0}_\alpha(\mathbf{u},z_b-z_a)+\cdots\Bigg)\mathfrak{t}^k_\alpha,\end{aligned} \tag{4.19}$$

$$\begin{aligned}\mathfrak{g}_0 &\sim \sum_{\alpha\in\tilde\Pi}\left(\cdots+x^0_{\alpha,1,a}\left(\frac{1}{z-z_a}+\sum_{b\neq a}E_1(z_b-z_a)\right)+x^0_{\alpha,0}+\cdots\right)h_\alpha\\ &\quad +\sum_{\alpha\in\tilde R} y^0_{\alpha,0,a}\Bigg(\cdots+\frac{1}{z-z_a}+E_1\big(\langle\mathbf{u}+\kappa\tau,\alpha\rangle\big)+2\pi\imath\langle\kappa,\alpha\rangle\\ &\quad +\sum_{b\neq a}\varphi^0_\alpha(\mathbf{u},z_b-z_a)+\cdots\Bigg)\mathfrak{t}^0_\alpha.\end{aligned} \tag{4.20}$$

Here "$\cdots$" means the terms of order $o\big(z-z_a\big)^{-1}$ and $o(1)$. For $\mathfrak{g}_{\rm int}={\rm Lie}(G(U_D))$ we have local expansions in neighborhoods of the marked points

$$\mathfrak{g}_{\rm int}=\left\{X=\sum_{a=1}^n\left(b_a+\sum_{j>0}y^a_j(z-z_a)^j\right),\ b_a\in\mathfrak{b}_0,\ y^a_j\in\mathfrak{g}\right\}.$$

Define the Lie algebra with the loose condition (4.17))

$$\mathfrak{g}'_{\rm out}=\mathfrak{g}_{\rm out}\qquad\text{with}\quad\sum_{a=1}^n x^0_{\alpha,1,a}\in\mathbb{C}$$

and let $\mathfrak{n}^-=\sum\limits_{\alpha\in R^+}\mathfrak{g}^{-\alpha}$. Then the Lie algebra ${\rm Lie}(G(D^\times))$ has the form (compare with the general case (3.3))

$$\begin{aligned}{\rm Lie}\big(G(D^\times)\big)&=\mathfrak{g}'_{\rm out}\oplus\big(\oplus_{a=1}^n\mathfrak{n}^-_a\big)\oplus\mathfrak{g}_{\rm int}\\ &=\mathfrak{g}_{\rm out}\oplus\left(\sum_{a=1}^n\sum_{\alpha\in\tilde\Pi}x^0_{\alpha,1,a}h_\alpha\right)\oplus\big(\oplus_{a=1}^n\mathfrak{n}^-_a\big)\oplus\mathfrak{g}_{\rm int}.\end{aligned} \tag{4.21}$$

Notice that the constant terms $\mathfrak{n}^-_a$ come from the constant terms $c(m,k)$ in (B.9). We can conclude from (4.21) that locally the action on $G(D^\times)$ by $G_{\rm out}=G(\Sigma_{1,n}\setminus\vec z)$ from the left and by $G_{\rm int}=\prod\limits_{a=1}^n G(D_a)$ from the right absorbs almost all negative and positive modes of $G(D^\times)$ except the two types of modes describing the moduli space:

- The vector $\mathbf{u} = \sum_{a=1}^{n} \sum_{\alpha\in\tilde{\Pi}} x^0_{\alpha,1,a} h_\alpha \in \tilde{\mathfrak{h}}_0$. It defines an element of the moduli space $\mathcal{M}_{G,1}$ (4.8).
- The Lie algebras $\mathfrak{n}_a^-$, $a = 1,\dots n$. They are the tangent spaces to the flag varieties attached the marked points coming from the quasi-parabolic structure of the bundle.

4.4 Conformal blocks

In this section we define connections on the space of conformal blocks and derive the KZB equations in a similar way as it was done for the trivial characteristic classes in [24]. The derivation is based on the representation of the moduli space of bundles as the double coset space (3.1) in a given sector of the decomposition (3.5). In other words, the characteristic class (defined by $j = 0,\dots,l-1$ in (4.7)) is fixed and we deal with $G_{\rm out} \setminus G^{\rm ad}_\gamma \otimes \mathbb{C}[[t_a, t_a^{-1}]/G_{\rm int}$, where $G_{\rm out} = G(\Sigma_{\tau,n} \setminus \vec{z})$ and $G_{\rm int} = G \otimes \mathbb{C}[[t_a]]$ were described above.

Let us write down the Virasoro generators (3.15) using the GS-basis (A.3), (A.4), (A.5) $\mathfrak{t}^k_\alpha \otimes t^m_a \equiv \mathfrak{t}^k_\alpha(m)$ ($\mathfrak{t}^k_\alpha(0) \equiv \mathfrak{t}^k_\alpha$) for the generators of the loop algebra

$$L^a_m = \frac{1}{2(k+h^\vee)} \sum_{p\in\mathbb{Z}} \sum_{q=0}^{l-1} \left(\sum_{\alpha\in R} {:}|\alpha|^2 \mathfrak{t}^q_\alpha(-p)\mathfrak{t}^{-q}_{-\alpha}(p+m){:} + \sum_{\alpha\in\tilde{\Pi}} {:}\mathfrak{H}^q_\alpha(-p)\mathfrak{h}^{-q}_\alpha(p+m){:} \right). \tag{4.22}$$

Consider the integrable modules attached to the marked points $\hat{V}^{[n]}_{\vec{z}\vec{\mu}}$ (3.9) and the corresponding conformal blocks. They satisfy the equations (3.14), (3.18), (3.22). For elliptic curve they assume the form:

- The moving points (3.17):

$$\left(\partial_a - L^a_{-1}\right) F = 0, \qquad \partial_a = \frac{1}{k+h^\vee}\partial_{z_a}, \qquad t_a = z - z_a. \tag{4.23}$$

- The vector field corresponding to the deformation of the moduli τ of the elliptic curve $\Sigma_{\tau,n}$:

$$\left(\partial_\tau - \frac{1}{2\pi i} E_1(z)\partial_z\right) F = 0. \tag{4.24}$$

  This action follows from (3.13) and the operator algebra

$$T(z')F(z) = E_1(z'-z)\partial_z F(z) + \text{analitic part}.$$

- The invariance with respect to the action of $\mathfrak{g}_{\rm out}$ (3.22):

$$\left(l\partial_{\mathbf{u}_{-\alpha}} + E_1(z)\mathfrak{H}^0_\alpha\right) F = 0, \quad \alpha\in\tilde{\Pi}, \qquad \mathbf{u} = \{\mathbf{u}_\alpha\}, \quad \alpha\in\tilde{\Pi}, \tag{4.25}$$

  where $\mathfrak{H}^0_\alpha$ are the Cartan generators (A.5). Notice that this operator is well defined on M_G (3.19).

The vector field (4.24) is defined on the universal curve $\mathcal{H}\times\mathbb{C}/\langle\tau,1\rangle\setminus\mathcal{H}\times 0$, since it is invariant under the lattice shifts $\langle\tau,1\rangle$. The τ deformation can be defined in the non-holomorphic form as $\partial_\tau + \frac{z-\bar{z}}{\tau-\bar{\tau}}\partial_z$.

The invariance with respect to Lie$(G_{\rm out})$ (4.14), (4.18) means that

$$\begin{aligned} &\varphi^k_\alpha(\mathbf{u}_\alpha, z-z_a)\mathfrak{t}^k_\alpha F = 0, \qquad \alpha\in R, \quad \forall\, k,\\ &\varphi^k_\alpha(0, z-z_a)\mathfrak{H}^k_\alpha F = 0, \qquad \alpha\in\tilde{\Pi}, \quad k\neq 0. \end{aligned} \tag{4.26}$$

Now using (4.19), (4.20) and (4.26) we write down the annihilation condition $\mathfrak{g}_{\mathrm{out}}F = 0$ in in the basis $\mathfrak{t}^{k,c}_\alpha(m) = 1 \otimes \cdots \otimes 1 \otimes \mathfrak{t}^{k,c}_\alpha(m) \otimes 1 \otimes \cdots \otimes 1$ (on the c-th place):

$$\Big(\mathfrak{t}^{k,a}_\alpha(-1) + \Big(E_1\Big(\mathbf{u}_\alpha + \langle\kappa,\alpha\rangle\tau + \frac{k}{l}\Big) + 2\pi i\langle\kappa,\alpha\rangle\Big)\mathfrak{t}^{k,a}_\alpha(0)$$
$$+ \sum_{c\neq a}\varphi^k_\alpha(\mathbf{u}_\alpha, z_c - z_a)\mathfrak{t}^{k,c}_\alpha(0)\Big)F = 0,$$
$$\Big(\mathfrak{H}^{k,a}_\alpha(-1) + \Big(E_1\Big(\langle\kappa,\alpha\rangle\tau + \frac{k}{l}\Big) + 2\pi i\langle\kappa,\alpha\rangle\Big)\mathfrak{H}^{k,a}_\alpha(0)$$
$$+ \sum_{c\neq a}\varphi^k_\alpha(0, z_c - z_a)\mathfrak{H}^{k,c}_\alpha(0)\Big)F = 0 \tag{4.27}$$

for $\alpha \in R$, $\forall\, k$ and $\alpha \in \tilde{\Pi}$, $k \neq 0$ correspondingly. In the same way (4.24) and (4.25) assume the form

$$\Big(\partial_\tau + \frac{1}{2\pi i}L^a_{-2} + \frac{1}{2\pi i}\sum_{c\neq a}E_1(z_c - z_a)L^a_{-1}\Big)F = 0,$$
$$\Big(l\partial_{\mathbf{u}_{-\alpha}} + \mathfrak{H}^{0,a}_\alpha(-1) + \sum_{c\neq a}E_1(z_c - z_a)\mathfrak{H}^{0,c}_\alpha(0)\Big)F = 0, \qquad \alpha \in \tilde{\Pi}. \tag{4.28}$$

Now we are ready to evaluate the Virasoro generators, i.e. to express them in terms of zero modes of the loop algebra $\mathfrak{t}^{k,c}_\alpha(0) \equiv \mathfrak{t}^{k,c}_\alpha$ only. As we have found above the positive modes of the loop algebra act on F by zero $\mathfrak{t}^{k,a}_\alpha(m)F = 0$, $m \in \mathbb{Z}_+$. Therefore, from (4.22) we have

$$(k + h^\vee)L^a_{-1} = \sum_{q=0}^{l-1}\Big(\sum_{\alpha\in R}\mathfrak{t}^{q,a}_\alpha(-1)\mathfrak{t}^{-q,a}_{-\alpha}(0) + \sum_{\alpha\in\tilde{\Pi}}\mathfrak{H}^{q,a}_\alpha(-1)\mathfrak{h}^{-q,a}_\alpha(0)\Big) \qquad \text{on } F \tag{4.29}$$

and

$$(k + h^\vee)L^a_{-2} = \sum_{q=0}^{l-1}\Big(\sum_{\alpha\in R}\mathfrak{t}^{q,a}_\alpha(-2)\mathfrak{t}^{-q,a}_{-\alpha}(0) + \sum_{\alpha\in\tilde{\Pi}}\mathfrak{H}^{q,a}_\alpha(-2)\mathfrak{h}^{-q,a}_\alpha(0)\Big)$$
$$+ \frac{1}{2}\sum_{q=0}^{l-1}\Big(\sum_{\alpha\in R}\mathfrak{t}^{q,a}_\alpha(-1)\mathfrak{t}^{-q,a}_{-\alpha}(-1) + \sum_{\alpha\in\tilde{\Pi}}\mathfrak{H}^{q,a}_\alpha(-1)\mathfrak{h}^{-q,a}_\alpha(-1)\Big) \qquad \text{on } F. \tag{4.30}$$

In order to find L^a_{-1} one need to substitute $\mathfrak{t}^{k,a}_\alpha(-1)$, $\mathfrak{H}^{k,a}_\alpha(-1)$ from (4.27) and $\mathfrak{H}^{0,a}_\alpha(-1)$ from (4.28) into (4.29)

$$-(k + h^\vee)L^a_{-1} = l\sum_{\alpha\in\tilde{\Pi}}\mathfrak{h}^{0,a}_\alpha(0)\partial_{\mathbf{u}_\alpha}$$
$$+ \sum_{q=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\Big(\Big(E_1\Big(\mathbf{u}_\alpha + \langle\kappa,\alpha\rangle\tau + \frac{q}{l}\Big) + 2\pi i\langle\kappa,\alpha\rangle\Big)\mathfrak{t}^{q,a}_\alpha(0)\mathfrak{t}^{-q,a}_{-\alpha}(0)$$
$$+ \sum_{c\neq a}\varphi^q_\alpha(\mathbf{u}_\alpha, z_c - z_a)\mathfrak{t}^{q,c}_\alpha(0)\mathfrak{t}^{-q,a}_{-\alpha}(0)\Big)$$

$$+\sum_{q=0}^{l-1}\sum_{\alpha\in\tilde\Pi}\left(\left(E_1\left(\langle\kappa,\alpha\rangle\tau+\frac{q}{l}\right)+2\pi i\langle\kappa,\alpha\rangle\right)\mathfrak{H}_\alpha^{q,a}(0)\mathfrak{h}_{-\alpha}^{-q,a}(0)\right.$$
$$\left.+\sum_{c\neq a}\varphi_\alpha^q(0,z_c-z_a)\mathfrak{H}_\alpha^{q,c}(0)\mathfrak{h}_{-\alpha}^{-q,a}(0)\right),$$

where $\varphi_\alpha^0(0,z_c-z_a)=E_1(z_c-z_a)$. The first term in the last line vanishes due to skew-symmetry with respect to $\alpha,q\to-\alpha,-q$. The similar term in the second line does not vanish because $[\mathfrak{t}_\alpha^{q,a}(0),\mathfrak{t}_{-\alpha}^{-q,a}(0)]=\frac{p_\alpha}{\sqrt{l}}\exp\left(-2\pi i\frac{q}{l}\right)\mathfrak{h}_\alpha^{0,a}$ [46]. Therefore,

$$\sum_{q=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\left(E_1\left(\mathbf{u}_\alpha+\langle\kappa,\alpha\rangle\tau+\frac{q}{l}\right)+2\pi i\langle\kappa,\alpha\rangle\right)\mathfrak{t}_\alpha^{q,a}(0)\mathfrak{t}_{-\alpha}^{-q,a}(0)$$
$$=\frac{1}{2}\sum_{q=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\left(E_1\left(\mathbf{u}_\alpha+\langle\kappa,\alpha\rangle\tau+\frac{q}{l}\right)+2\pi i\langle\kappa,\alpha\rangle\right)\frac{p_\alpha}{\sqrt{l}}\exp\left(-2\pi i\frac{q}{l}\right)\mathfrak{h}_\alpha^{0,a}$$
$$=l\sum_{\alpha\in\tilde\Pi}\mathfrak{h}_\alpha^{0,a}(0)\partial_{\mathbf{u}_\alpha}\left\{\log\prod_{q=0}^{l-1}\prod_{\alpha\in R}\vartheta\left(\mathbf{u}_\alpha+\langle\kappa,\alpha\rangle\tau+\frac{q}{l}\right)^{\frac{p_\alpha|\alpha^2|}{2l\sqrt{l}}\exp\left(-2\pi i\frac{q}{l}\right)}\right\}.$$

The term

$$\frac{1}{2}\sum_{q=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2 2\pi i\langle\kappa,\alpha\rangle\frac{p_\alpha}{\sqrt{l}}\exp\left(-2\pi i\frac{q}{l}\right)\mathfrak{h}_\alpha^{0,a}$$

vanishes because of summation over q. Notice also that the obtained scalar expression does not depend on $\{z_c\}$. Then, the equation (4.23) gives

$$\left(\partial_a+l\sum_{\alpha\in\tilde\Pi}\mathfrak{h}_\alpha^{0,a}(0)\partial_{\mathbf{u}_\alpha}+\sum_{q=0}^{l-1}\sum_{c\neq a}\left(\sum_{\alpha\in R}|\alpha|^2\varphi_\alpha^q(\mathbf{u}_\alpha,z_c-z_a)\mathfrak{t}_\alpha^{q,c}(0)\mathfrak{t}_{-\alpha}^{-q,a}(0)\right.\right.$$
$$\left.\left.+\sum_{\alpha\in\tilde\Pi}\varphi_\alpha^q(0,z_c-z_a)\mathfrak{H}_\alpha^{q,c}(0)\mathfrak{h}_{-\alpha}^{-q,a}(0)\right)\right)\tilde F=0,$$

where

$$\tilde F=F\prod_{q=0}^{l-1}\prod_{\alpha\in R}\vartheta\left(\mathbf{u}_\alpha+\langle\kappa,\alpha\rangle\tau+\frac{q}{l}\right)^{-\frac{p_\alpha|\alpha^2|}{2l\sqrt{l}}\exp\left(-2\pi i\frac{q}{l}\right)}.$$

This is the first set of equations in (4.41). In order to obtain the second one (the KZB connection ∇_τ along τ) one should use (4.30). It is needed to compute L_{-2}^a. The later arises from the local expansion of (B.4) for $k=1$. Then the following identities should be used

$$\partial_z\phi(u,z)=\phi(u,z)\big(E_1(z+u)-E_1(z)\big)=f(u,z)+\big(E_1(u)-E_1(z)\big)\phi(u,z),$$

where $f(u,z)=\partial_u\phi(u,z)$ for $\mathfrak{t}(-2)\mathfrak{t}(0)$-terms and

$$\phi(u,z-z_a)\phi(-u,z-z_b)=-\phi(u,z-z_a)\phi(u,z_b-z)$$
$$=-\phi(u,z_b-z_a)\big(E_1(u)+E_1(z-z_a)+E_1(z_b-z)-E_1(u+z_b-z_a)\big)$$
$$=f(u,z_b-z_a)+\phi(u,z_b-z_a)\big(E_1(z-z_b)-E_1(z-z_a)\big)$$

for $\mathfrak{t}(-1)\mathfrak{t}(-1)$-terms. On the other hand ∇_τ is a unique flat connection for given ∇_a (4.37). The final answer is given below in Section 4.6. This answer is verified in Appendix C.

4.5 Classical r-matrix

The construction of the KZB connection is based on the classical dynamical elliptic r-matrix defined as sections of bundles over elliptic curves [13, 47, 74]. For trivial G-bundles our list coincides with the elliptic r-matrices were defined in [19]. A more general class of elliptic r-matrices was constructed in [17, 18]. The latter classification includes our list though it was derived from different postulates.

4.5.1 Axiomatic description of r-matrices

The classical dynamical r-matrix is a meromorphic one form $r = r(\mathbf{u}, z)dz$, $(\mathbf{u} \in \tilde{\mathfrak{h}}_0)$ on $\mathbb{C}$ taking values in $\mathfrak{g} \otimes \mathfrak{g}$ that satisfies the following conditions:

1. $r(z)$ has a pole at $z = 0$ and

$$\operatorname{Res}\big|_{z=0} r(z) = C_2 = \frac{1}{2}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k} + \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi}\mathfrak{H}_\alpha^k \otimes \mathfrak{h}_\alpha^{-k},$$

   where $\mathfrak{t}_\alpha^k$, $\mathfrak{H}_\alpha^k$, $\mathfrak{h}_\alpha^{-k}$ are generators of the GS basis in $\mathfrak{g}$ (see Appendix A). If V is a $\mathfrak{g}$-module, then C_2 acts by the permutation on $V \otimes V$.

2. Behavior under the shifts by the generators of the lattice $\mathbb{Z} \oplus \tau\mathbb{Z}$:

$$r(z+1) = \operatorname{Ad}_{\mathcal{Q}} r(z), \qquad r(z+\tau) = 2\pi\imath\sum_{\alpha\in\Pi}\mathfrak{H}_\alpha^0 \otimes \mathfrak{h}_\alpha^0 + \operatorname{Ad}_{\Lambda_j} r(z), \tag{4.31}$$

   where the Ad-action is taken with respect to the first factor in $\mathfrak{g} \otimes \mathfrak{g}$. Here $\mathcal{Q} = \mathbf{e}(\kappa)$, $\Lambda^{(j)} = \Lambda_0\mathbf{e}(\mathbf{u})$ (see (4.9)), $\{\mathfrak{h}_\alpha^0\}$ $(\{\mathfrak{H}_\alpha^0\})$ is the simple coroot basis (the dual basis) in the invariant subalgebra $\tilde{\mathfrak{g}}_0$ (see Appendix A). It means that r is a connection in the $\mathfrak{g} \otimes \mathfrak{g}$-bundle over Σ_τ.

3. *The classical dynamical Yang–Baxter equation* (*CDYBE*). It follows from 1 that $r(z)$ can be represented as

$$r(z) = \frac{1}{2}\sum_{k=0}^{l-1}\sum_{\alpha\in R}\Phi_\alpha^k(z)|\alpha|^2\mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k} + \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi}\Psi_\alpha^k(z)\mathfrak{H}_\alpha^k \otimes \mathfrak{h}_\alpha^{-k}.$$

   Then $r(z)$ is a solution of CDYBE:

$$\begin{aligned}
&[r_{12}(z_{12}), r_{13}(z_{13})] + [r_{12}(z_{12}), r_{23}(z_{23})] + [r_{13}(z_{13}), r_{23}(z_{23})]\\
&\quad - \sqrt{l}\sum_{k=0}^{l-1}\sum_{\alpha\in R}\frac{|\alpha|^2}{2}\mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k} \otimes \bar{\mathfrak{h}}_\alpha^0\partial_1\Phi_\alpha^k(\mathbf{u}, z-w)\\
&\quad - \frac{|\alpha|^2}{2}\mathfrak{t}_\alpha^k \otimes \bar{\mathfrak{h}}_\alpha^0 \otimes t_{-\alpha}^{-k}\partial_1\Phi_\alpha^k(\mathbf{u}, z-x)\\
&\quad + \frac{|\alpha|^2}{2}\bar{\mathfrak{h}}_\alpha^0 \otimes \mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k}\partial_1\Phi_\alpha^k(\mathbf{u}, w-x) = 0, \qquad z_{ij} = z_i - z_j,
\end{aligned} \tag{4.32}$$

   where ∂_1 is the differentiation with respect to the first argument.

4. The unitarity

$$r^{12}(\mathbf{u}, z) + r^{21}(\mathbf{u}, -z) = 0.$$

5. The zero weight condition

$$[X \otimes 1 + 1 \otimes X, r(\mathbf{u}, z)] = 0, \qquad X \in \mathfrak{h}.$$

Lemma 1. • *Any r'-matrix satisfying* 1–5 *has the form*

$$r'(\mathbf{u}, z) = r(\mathbf{u}, z) + \delta r(\mathbf{u}),$$

where

$$\begin{aligned} & r(\mathbf{u}, z) = r_{\mathfrak{H}}(\mathbf{u}, z) + r_R(z), \\ & r_R(\mathbf{u}, z) = \frac{1}{2}\sum_{k=0}^{l-1}\sum_{\alpha\in R} r_\alpha^k(\mathbf{u}, z), \qquad r_\alpha^k(\mathbf{u}, z) = |\alpha|^2\varphi_\alpha^k(\mathbf{u}, z)\mathfrak{t}_\alpha^k \otimes \mathfrak{t}_{-\alpha}^{-k}, \\ & r_{\mathfrak{H}}(z) = \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi} r_\alpha^0(z), \qquad r_\alpha^0(z) = \varphi_0^k(z)\mathfrak{H}_\alpha^k \otimes \mathfrak{h}_\alpha^{-k}, \end{aligned} \tag{4.33}$$

satisfies 1–5 *and $\varphi_\beta^k(\mathbf{x}, z)$ is defined in* (B.3), $\varphi_0^k(z) = \phi(k/l, z)$, $\varphi_0^0(z) = E_1(z)$.

• $\delta r(\mathbf{u}) \in \tilde{\mathfrak{h}}_0 \otimes \tilde{\mathfrak{h}}_0$

$$\delta r(\mathbf{u}) = \sum_{\alpha,\beta\in\tilde{\Pi}} A_{\alpha\beta}\mathfrak{H}_\alpha^0 \otimes \mathfrak{h}_\beta^0, \qquad A_{\alpha\beta} = -A_{\beta\alpha}.$$

• *$\delta r(\mathbf{u})$ is generated by the gauge transformation*

$$\delta r(\mathbf{u}) = -l\sum_{\alpha\in\tilde{\Pi}} (\partial_{u_{\hat{\alpha}}} f) f^{-1} \otimes \mathfrak{h}_\alpha^0, \qquad (\partial_{u_{\hat{\alpha}}} f) f^{-1} \in \tilde{\mathfrak{h}}_0, \qquad f = f(\mathbf{u}) \in \tilde{H}_0,$$

where $\tilde{H}_0$ is a Cartan subgroup of the invariant subgroup $\tilde{G}_0 \subset G$ (see Table 1 *in* [46]).

Proof. It follows from the properties of the functions $\varphi_\alpha^k(\mathbf{u}, z)$, $\varphi_0^k(\mathbf{u}, z)$ described in the Appendix B that $r(\mathbf{u}, z)$ satisfy 1 and 2. It was proved in [46] that it is a solution of the CDYBE. This sum is a classical dynamical r-matrix corresponding to a non-trivial characteristic class defined by (4.31). The conditions 4 and 5 can be checked as well. The conditions 1–3, 5 define the r-matrix up to a constant (z-independent) Cartan term δr. Then it follows from 4 that $A_{\alpha\beta}$ is antisymmetric.

Next we wish to prove that locally $A_{\alpha\beta} = -l(\partial_{u_\alpha}(f)f^{-1})_\beta$ for some $f \in \tilde{H}_0$. The twisted r-matrix must satisfy the CDYB equation. Plugging $r + \delta r$ into (4.32) we see that the "commutator" part vanishes identically since $[r^{ab}, \delta r^{ac}] + [r^{ab}, \delta r^{bc}] \equiv 0$ due to

$$\left[\mathfrak{t}_\alpha^{k,a}, \mathfrak{h}_\beta^{0,a}\right] \otimes \mathfrak{t}_{-\alpha}^{-k,a} + \mathfrak{t}_\alpha^{k,a} \otimes \left[\mathfrak{t}_{-\alpha}^{-k,a}, \mathfrak{h}_\beta^{0,a}\right] = 0.$$

The "derivative" part of (4.32) yields $\partial_{u_{\hat{\alpha}}} A_{\beta\gamma} + \partial_{u_{\hat{\gamma}}} A_{\alpha\beta} + \partial_{u_{\hat{\beta}}} A_{\gamma\alpha} = 0$ or

$$dA = 0, \qquad A = \sum_{\alpha,\beta\in\Pi} A_{\alpha\beta} du_\alpha \wedge du_\beta \in \mathcal{M}_G.$$

The term δr is called the *dynamical twist* of the r-matrix. The statement follows from the Poincaré lemma. ∎

4.5.2 r-matrices as sections of bundles over moduli spaces

Consider the behavior of the r-matrix (4.33) under the action of latices $\tau\tilde{Q}^\vee \oplus \tilde{Q}^\vee$ (4.10) and $\tau\tilde{P}^\vee \oplus \tilde{P}^\vee$ (4.12) on the dynamical parameter $\mathbf{u}$. It follows from (B.2), (B.3) and (B.16) that the r-matrices has distinct type of quasi-periodicities with respect $\mathbf{u} \in \tilde{\mathfrak{h}}_0$. Let $\beta^\vee \in \tilde{\Pi}^\vee$ be a simple coroot, corresponding to the invariant algebra $\tilde{\mathfrak{g}}_0$. For $\alpha \in R$ define the integers $n_{\alpha,\beta} = \langle \alpha, \beta^\vee \rangle$. Then we find

$$r^k_\alpha\big(\mathbf{u} + \beta^\vee, z\big) = r^k_\alpha(\mathbf{u}, z), \qquad r^k_\alpha\big(\mathbf{u} + \tau\beta^\vee, z\big) = \mathbf{e}\big(-n_{\alpha,\beta} z\big) r^k_\alpha(\mathbf{u}, z).$$

Let $\Xi^\vee$ be a basis of fundamental co-weights dual to the basis Π, and $\tilde{\varpi}^\vee$ is a fundamental coweight in $\tilde{P}^\vee$. Since $\tilde{P}^\vee$ is a sublattice of $P^\vee$, the weight $\tilde{\varpi}^\vee$ can be decomposed in the basis of the fundamental co-weights $\tilde{\varpi}^\vee = \sum_{\nu^\vee \in \Xi^\vee} n^\varpi_\nu \nu^\vee$, where $n^\varpi_\nu \in \mathbb{Z}$. As above we find

$$r^k_\alpha(\mathbf{u} + \tilde{\varpi}^\vee, z) = r^k_\alpha(\mathbf{u}, z), \qquad r^k_\alpha(\mathbf{u} + \tau\tilde{\varpi}^\vee, z) = \mathbf{e}\big(-n^\varpi_\nu \delta_{\langle \nu^\vee, \alpha\rangle} z\big) r^k_\alpha(\mathbf{u}, z).$$

On the other hand, due to the Λ-invariance of $\tilde{Q}^\vee$, we have $\langle \beta^\vee, \lambda^m(\alpha)\rangle = \langle \beta^\vee, \alpha\rangle$. Therefore, $\mathrm{Ad}_{\exp 2\pi\imath\beta^\vee} E_{\lambda^m(\alpha)} = \mathbf{e}(\langle \alpha, \beta^\vee\rangle) E_{\lambda^m(\alpha)}$. Then from (A.3) we find that

$$\mathrm{Ad}_{\exp(-2\pi\imath\beta^\vee z)} \mathfrak{t}^a_\alpha = \mathbf{e}(-n_{\alpha,\beta} z)\mathfrak{t}^a_\alpha.$$

Similarly, due to Λ-invariance of $\tilde{P}^\vee$, we have also

$$\mathrm{Ad}_{\exp(-2\pi\imath\tilde{\varpi}^\vee z)} \mathfrak{t}^a_\alpha = \mathbf{e}\big(-n^\varpi_\nu \delta_{\langle \nu^\vee, \alpha\rangle} z\big)\mathfrak{t}^a_\alpha.$$

Since the Cartan part $r_{\mathfrak{H}}$ of the r-matrix does not depend on $\mathbf{u}$ we come to the relations

$$r\big(\mathbf{u} + \beta^\vee, z\big) = r(\mathbf{u}, z), \qquad r\big(\mathbf{u} + \tau\beta^\vee, z\big) = \mathrm{Ad}_{\exp(-2\pi\imath\beta^\vee z)} r(\mathbf{u}, z), \tag{4.34}$$

$$r\big(\mathbf{u} + \tilde{\varpi}^\vee, z\big) = r(\mathbf{u}, z), \qquad r\big(\mathbf{u} + \tau\tilde{\varpi}^\vee, z\big) = \mathrm{Ad}_{\exp(-2\pi\imath\tilde{\varpi}^\vee z)} r(\mathbf{u}, z). \tag{4.35}$$

In all cases the adjoint actions Ad_h act on the first component of the tensor product and play the role of the clutching operators.

Let $x(k, \alpha) = \langle \mathbf{u}, \alpha + \kappa\tau\rangle + k/l$. Then $r(\mathbf{u}, z)$ is singular when $x(k, \alpha) \to 0$ (see (B.7) and (B.2))

$$r(\mathbf{u}, z) = |\alpha|^2 \mathbf{e}(\langle \kappa, \alpha\rangle z)\left(\frac{1}{x(k, \alpha)} + O(1)\right) \mathfrak{t}^k_\alpha \otimes \mathfrak{t}^{-k}_{-\alpha}. \tag{4.36}$$

It means that $r(\mathbf{u}, z)$ are sections of the bundles over the moduli spaces C^{sc}_j (4.11), or C^{ad}_j (4.13) with sections taking values in $\mathfrak{g} \otimes \mathfrak{g}$ with the quasi-periodicities (4.34), (4.35) and with the singularities (4.36).

4.6 KZB connection related to elliptic curves

As it was established the part of connection related to the moving points coincides with the introduced above r-matrix. Here we prove that this connection is flat. Consider the following differential operators

$$\nabla_a = \partial_{z_a} + \hat{\partial}^a + \sum_{c \neq a} r^{ac}, \tag{4.37}$$

$$\nabla_\tau = 2\pi i \partial_\tau + \Delta + \frac{1}{2}\sum_{b,d} f^{bd},$$

with

$$r^{ac} = \sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\varphi_\alpha^k(\mathbf{u}, z_a - z_c)\mathfrak{t}_\alpha^{k,a}\mathfrak{t}_{-\alpha}^{-k,c} + \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi}\varphi_0^k(\mathbf{u}, z_a - z_c)\mathfrak{H}_\alpha^{k,a}\mathfrak{h}_\alpha^{k,c},$$
$$f^{ac} = \sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2 f_\alpha^k(\mathbf{u}, z_a - z_c)\mathfrak{t}_\alpha^{k,a}\mathfrak{t}_{-\alpha}^{-k,c} + \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi} f_0^k(\mathbf{u}, z_a - z_c)\mathfrak{H}_\alpha^{k,a}\mathfrak{h}_\alpha^{k,c},$$

where $\mathfrak{t}_\alpha^{k,a} = 1\otimes\cdots\otimes 1\otimes\mathfrak{t}_\alpha^k\otimes 1\otimes\cdots\otimes 1$ (with $\mathfrak{t}_\alpha^k$ on the a-th place) and similarly for the generators $\mathfrak{H}_\alpha^{k,a}$ and $\mathfrak{h}_\alpha^{k,a}$.[3] The following short notations are used here

$$\hat\partial^a = l\sum_{\alpha\in\Pi}\mathfrak{h}_\alpha^{0,a}\partial_{\hat\alpha}, \qquad \Delta = \frac{l}{2}\sum_{\alpha\in\Pi}\sum_{s=0}^{l-1}\partial_{u_\alpha}\partial_{u_{\lambda^s\hat\alpha}}$$

and

$$\varphi_\alpha^k(\mathbf{u}, z) = e^{2\pi i\langle\kappa,\alpha\rangle z}\phi\left(\langle\mathbf{u}+\kappa\tau,\alpha\rangle + \frac{k}{l}, z\right),$$
$$f_\alpha^k(\mathbf{u}, z) = e^{2\pi i\langle\kappa,\alpha\rangle z} f\left(\langle\mathbf{u}+\kappa\tau,\alpha\rangle + \frac{k}{l}, z\right). \tag{4.38}$$

From the definition it follows that $r^{ac} = -r^{ca}$ and $f^{ac} = f^{ca}$. Following (B.6) and (B.7) we put

$$\varphi_0^0(z) = E_1(z), \tag{4.39}$$
$$f_0^0(z) = \rho(z) = \frac{1}{2}\big(E_1^2(z) - \wp(z)\big). \tag{4.40}$$

Notice that

$$f_\alpha^k(\mathbf{u}, 0) = -E_2\left(\langle u+\kappa\tau,\alpha\rangle + \frac{k}{l}\right) = -\wp\left(\langle u+\kappa\tau,\alpha\rangle + \frac{k}{l}\right) - 2\eta_1$$

and, therefore

$$f^{cc} = -\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\wp_\alpha^k\mathfrak{t}_\alpha^{k,c}\mathfrak{t}_{-\alpha}^{-k,c} - \sum_{k=0}^{l-1}\sum_{\alpha\in\Pi}\mathfrak{H}_\alpha^{k,c}\mathfrak{h}_\alpha^{-k,c} - 2l\eta_1 C_2^c,$$

where C_2^c is the Casimir operator acting on the c-th component. Recall that we study the following system of differential equations

$$\nabla_a F = 0, \quad a = 1,\dots,n, \qquad \nabla_\tau F = 0. \tag{4.41}$$

There are two types of the compatibility conditions of KZB equations (4.41)

$$[\nabla_a, \nabla_b]F = 0, \quad a, b = 1,\dots,n, \qquad [\nabla_a, \nabla_\tau]\,F = 0, \quad a = 1,\dots,n. \tag{4.42}$$

It is important to mention that the solutions of (4.41) F are assumed to satisfy the following condition

$$\left(\sum_{c=1}^n\mathfrak{h}_\alpha^{0,c}\right)F = 0, \qquad \text{for any}\ \ \alpha\in\tilde\Pi. \tag{4.43}$$

[3]For brevity we write $\mathfrak{t}_\alpha^{k,a}$, $\mathfrak{h}_\alpha^{k,a}$ instead of representations of these generators in the spaces V_{μ_a}.

Proposition 1. *The upper equations in* (4.42) $[\nabla_a, \nabla_b] = 0$ *are valid for the r-matrix* (4.33) *on the space of solutions of* (4.41) *satisfying* (4.43)*. They follow from the classical dynamical Yang–Baxter equations*

$$\left[r^{ab}, r^{ac}\right] + \left[r^{ab}, r^{bc}\right] + \left[r^{ac}, r^{bc}\right] + \left[\hat{\partial}^{a}, r^{bc}\right] + \left[\hat{\partial}^{c}, r^{ab}\right] + \left[\hat{\partial}^{b}, r^{ca}\right] = 0. \tag{4.44}$$

Proposition 2. *The lower equations in* (4.42) $\left[\nabla_a, \nabla_\tau\right] = 0$ *are valid for the r-matrix* (4.38) *on the space of solutions of* (4.41) *satisfying* (4.43)*.*

The proofs of these statements are given in the Appendix C.

Let us also remark that the non-trivial trigonometric and rational limits of the above formulae can be obtained via procedures described in [1, 65, 72].

A Generalized Sine (GS) basis in simple Lie algebras

Let $\mathcal{Z}$ be a subgroup of the center $\mathcal{Z}(\bar{G})$ of $\bar{G}$, and consider a quotient group $G = \bar{G}/\mathcal{Z}$. Assume for simplicity that $\mathcal{Z}(\bar{G})$ is cyclic. The case $\mathrm{Spin}(4n)$ where $\mathcal{Z}(G) = \mu_2 \times \mu_2$ can be treated similarly.

Let us take an element $\zeta \in \mathcal{Z}(\bar{G})$ of order l, generating $\mathcal{Z}$. It defines uniquely an element Λ_0 from the Weyl group W (see [12, 46]). It is a symmetry of the corresponding extended Dynkin diagram and $(\Lambda_0)^l = \mathrm{Id}$. Λ_0 generates a cyclic group $\mu_l = \left(\Lambda_0, (\Lambda_0)^2, \ldots, (\Lambda_0)^l = 1\right)$ isomorphic to a subgroup of $\mathcal{Z}(\bar{G})$. Note that l is a divisor of $\mathrm{ord}(\mathcal{Z}(\bar{G}))$. Consider the action of Λ_0 on $\mathfrak{g}$. Since $(\Lambda_0)^l = \mathrm{Id}$ we have a l-periodic gradation

$$\begin{aligned} &\mathfrak{g} = \oplus_{a=0}^{l-1} \mathfrak{g}_a, \qquad \lambda(\mathfrak{g}_a) = \omega^a \mathfrak{g}_a, \qquad \omega = \exp \frac{2\pi i}{l}, \qquad \lambda = \mathrm{Ad}_{\Lambda_0}, \\ &[\mathfrak{g}_a, \mathfrak{g}_b] = \mathfrak{g}_{a+b} \quad \mathrm{mod}\, l, \end{aligned} \tag{A.1}$$

where $\mathfrak{g}_0$ is a subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ and the subspaces $\mathfrak{g}_a$ are its representations. Since $\mathcal{Q}$ and Λ commute in the adjoint representations the root subspaces $\mathfrak{g}_a$ are their common eigenspaces.

GS-basis. Here we shortly reproduce the construction of the GS-basis following [46]. Since $\Lambda_0 \in W$ it preserves the root system R. Define the quotient set $\mathcal{T}_l = R/\mu_l$. Then R is represented as a union of μ_l-orbits $R = \cup_{\mathcal{T}_l} \mathcal{O}$. We denote by $\mathcal{O}(\bar{\beta})$ an orbit starting from the root β

$$\mathcal{O}(\bar{\beta}) = \left\{\beta, \lambda(\beta), \ldots, \lambda^{l-1}(\beta)\right\}, \qquad \bar{\beta} \in \mathcal{T}_l.$$

The number of elements in an orbit $\mathcal{O}$ (the length of $\mathcal{O}$) is $l/p_\alpha = l_\alpha$, where p_α is a divisor of l. Let ν_α be a number of orbits $\mathcal{O}_{\bar{\alpha}}$ of the length l_α. Then $\sharp R = \sum \nu_\alpha l_\alpha$. Notice that if $\mathcal{O}(\bar{\beta})$ has length $l_\beta (l_\beta \neq 1)$, then the elements $\lambda^k \beta$ and $\lambda^{k+l_\beta}\beta$ coincide. First, transform the root basis $\mathcal{E} = \{E_\beta,\, \beta \in R\}$ in $\mathfrak{L}$. Define an orbit in $\mathcal{E}$

$$\mathcal{E}_\beta = \left\{E_\beta, E_{\lambda(\beta)}, \ldots, E_{\lambda^{l-1}(\beta)}\right\}$$

corresponding to $\mathcal{O}(\bar{\beta})$. Again $\mathcal{E} = \cup_{\bar{\beta} \in \mathcal{T}_l} \mathcal{E}_{\bar{\beta}}$. For $\mathcal{O}(\bar{\beta})$ define the set of integers

$$J_{p_\alpha} = \left\{a = m p_\alpha \,\middle|\, m \in \mathbb{Z},\ a \text{ is defined} \,\mathrm{mod}\, l\right\}, \qquad p_\alpha = \frac{l}{l_\alpha}. \tag{A.2}$$

Let E_α ($\alpha \in R$) be the root basis of $\mathfrak{g}$. "The Fourier transform" of the root basis on the orbit $\mathcal{O}(\bar{\beta})$ is defined as

$$\mathfrak{t}^a_\beta = \frac{1}{\sqrt{l}} \sum_{m=0}^{l-1} \omega^{ma} E_{\lambda^m(\beta)}, \qquad \omega = \exp \frac{2\pi i}{l}, \qquad a \in J_\beta. \tag{A.3}$$

Almost the same construction exists in $\mathfrak{H}$. Again let Λ_0 generates the group μ_l. Since Λ_0 preserves the extended Dynkin diagram, its action preserves the extended coroot system $\Pi^{\vee\mathrm{ext}} = \Pi^\vee \cup \alpha_0^\vee$ in $\mathfrak{H}$. Consider the quotient $\mathcal{K}_l = \Pi^{\vee\mathrm{ext}}/\mu_l$. Define an orbit $\mathcal{H}(\bar\alpha)$ of length $l_\alpha = l/p_\alpha$ in $\Pi^{\vee\mathrm{ext}}$ passing through $H_\alpha \in \Pi^{\vee\mathrm{ext}}$

$$\mathcal{H}(\bar\alpha) = \left\{H_\alpha, H_{\lambda(\alpha)}, \dots, H_{\lambda^{l-1}(\alpha)}\right\}, \qquad \bar\alpha \in \mathcal{K}_l = \Pi^{\vee\mathrm{ext}}/\mu_l.$$

The set $\Pi^{\vee\mathrm{ext}}$ is a union of $\mathcal{H}(\bar\alpha)$:

$$(\Pi^\vee)^{\mathrm{ext}} = \cup_{\bar\alpha\in\mathcal{K}_l}\mathcal{H}(\bar\alpha).$$

Define "the Fourier transform"

$$\mathfrak{h}^c_{\bar\alpha} = \frac{1}{\sqrt{l}}\sum_{m=0}^{l-1}\omega^{mc}H_{\lambda^m(\alpha)}, \qquad \omega = \exp\frac{2\pi i}{l}, \qquad c \in J_\alpha \quad (\text{see (A.2)}).$$

The basis $\mathfrak{h}^c_\alpha$ ($c \in J_\alpha$, $\bar\alpha \in \mathcal{K}_l$) is over-complete in $\mathfrak{H}$. Namely, let $\mathcal{H}(\bar\alpha_0)$ be an orbit passing through the minimal coroot $\left\{H_{\alpha_0}, H_{\lambda(\alpha_0)}, \dots, H_{\lambda^{l-1}(\alpha_0)}\right\}$. Then the element $\mathfrak{h}^0_{\bar\alpha_0}$ is a linear combination of elements $\mathfrak{h}^0_{-\bar\alpha}$, ($\alpha \in \Pi$) and we should exclude it from the basis. We replace the basis $\Pi^\vee$ in $\mathfrak{H}$ by

$$\mathfrak{h}^c_{\bar\alpha}, \quad c \in J_\alpha, \qquad \begin{cases} \alpha \in \tilde{\mathcal{K}}_l = \mathcal{K}_l \setminus \mathcal{H}(\bar\alpha_0), & c = 0, \\ \bar\alpha \in \mathcal{K}_l, & c \neq 0. \end{cases} \tag{A.4}$$

As before there is a one-to-one map $\Pi^\vee \leftrightarrow \{\mathfrak{h}^c_{\bar\alpha}\}$. The elements $(\mathfrak{h}^a_{\bar\alpha}, \mathfrak{t}^a_{\bar\alpha})$ form GS basis in $\mathfrak{g}_{(l-a)}$ (A.1). The dual basis is generated by elements $\mathfrak{H}^a_{\bar\alpha}$

$$\left(\mathfrak{H}^a_{\bar\alpha}, \mathfrak{h}^b_{\bar\beta}\right) = \delta^{(a+b,0(\mathrm{mod}\, l))}\delta_{\alpha,\beta}, \qquad \mathfrak{H}^a_{\bar\alpha} = \sum_{\beta\in\Pi}(\mathcal{A}^a_{\alpha,\beta})^{-1}\mathfrak{h}^{-a}_{\bar\beta}, \qquad \mathfrak{h}^a_{\bar\beta} = \sum_{\alpha\in\Pi}(\mathcal{A}^{-a}_{\alpha,\beta})\mathfrak{H}^{-a}_{\bar\alpha}, \tag{A.5}$$

where

$$\mathcal{A}^a_{\alpha,\beta} = \frac{2}{(\beta,\beta)}\sum_{s=0}^{l-1}\omega^{-sa}a_{\beta,\lambda^s(\alpha)}$$

and $a_{\alpha,\beta}$ is the Cartan matrix of $\mathfrak{g}$.

The λ-invariant subalgebra $\mathfrak{g}_0$ contains the subspace

$$V = \left\{\sum_{\bar\beta\in\mathcal{T}'_l} a_{\bar\beta}\mathfrak{t}^0_{\bar\beta}, \ a_{\bar\beta} \in \mathbb{C}\right\}.$$

Then $\mathfrak{g}_0$ is a sum of $\tilde{\mathfrak{g}}_0$ and V

$$\mathfrak{g}_0 = \tilde{\mathfrak{g}}_0 \oplus V.$$

In the invariant simple algebra $\tilde{\mathfrak{g}}_0$ instead of the basis $(\mathfrak{h}^0_{\bar\alpha}, \mathfrak{t}^0_{\bar\beta})$ we can use the Chevalley basis and incorporate it in the GS-basis

$$\left\{\mathfrak{h}^0_{\bar\alpha}, \mathfrak{t}^0_{\bar\beta}\right\} \to \left\{\tilde{\mathfrak{g}}_0 = \left(H_{\tilde\alpha}, \tilde\alpha \in \tilde\Pi, \ E_{\tilde\beta}, \tilde\beta \in \tilde R\right), \ V = \left(\mathfrak{t}^0_{\bar\beta}, \bar\beta \in \mathcal{T}'\right)\right\},$$

where $\tilde\Pi$ is a system of simple roots constructed by the averaging of the λ action on Π^{ext}, and $\tilde R$ is a system of roots of $\tilde{\mathfrak{g}}_0$ generated by $\tilde\Pi$. We have the following action of the adjoint operators on the GS basis:

$$\mathrm{Ad}_\Lambda\left(\mathfrak{t}^c_{\bar\beta}\right) = \mathbf{e}\left(\langle\tilde{\mathbf{u}}, \beta\rangle - \frac{c}{l}\right)\mathfrak{t}^c_{\bar\beta}, \qquad \mathrm{Ad}_\Lambda\left(\mathfrak{h}^c_{\bar\beta}\right) = \mathbf{e}\left(-\frac{c}{l}\right)\mathfrak{h}^c_{\bar\beta}, \qquad \mathbf{e}(x) = \exp(2\pi i x). \tag{A.6}$$

In addition,

$$\mathrm{Ad}_{\mathcal{Q}}\left(\mathfrak{h}^c_{\bar\beta}\right)=\mathfrak{h}^c_{\bar\beta},\qquad \mathrm{Ad}_{\mathcal{Q}}(H_{\tilde\alpha})=H_{\tilde\alpha}, \tag{A.7}$$

$$\mathrm{Ad}_{\mathcal{Q}}\left(\mathfrak{t}^c_{\bar\beta}\right)=\mathbf{e}(\langle\kappa,\beta\rangle)\mathfrak{t}^c_{\bar\beta},\qquad \mathrm{Ad}_{\mathcal{Q}}(E_{\tilde\alpha})=\mathbf{e}\langle\kappa,\tilde\alpha\rangle E_{\tilde\alpha}. \tag{A.8}$$

There are also the evident relations

$$\mathrm{Ad}_\Lambda(E_{\tilde\alpha})=\mathbf{e}(\langle\tilde{\mathbf{u}},\tilde\alpha\rangle)E_{\tilde\alpha},\qquad \mathrm{Ad}_\Lambda(H_{\tilde\alpha})=H_{\tilde\alpha},\qquad \tilde{\mathbf{u}}\in\tilde{\mathfrak{h}}.$$

In particular,

$$\mathrm{Ad}_\Lambda\left(\mathfrak{t}^c_{\bar\beta}\right)=\mathbf{e}\left(\langle\tilde{\mathbf{u}},\beta\rangle-\frac{c}{l}\right)\mathfrak{t}^c_{\bar\beta},\qquad \mathrm{Ad}_\Lambda\left(\mathfrak{h}^c_{\bar\beta}\right)=\mathbf{e}\left(-\frac{c}{l}\right)\mathfrak{h}^c_{\bar\beta},\qquad \mathbf{e}(x)=\exp(2\pi ix).$$

Commutation relations in the GS basis:

$$\left[\mathfrak{t}^a_\alpha,\mathfrak{t}^b_\beta\right]=\begin{cases}\dfrac{1}{\sqrt{l}}\displaystyle\sum_{s=0}^{l-1}\omega^{bs}C_{\alpha,\lambda^s\beta}\mathfrak{t}^{a+b}_{\alpha+\lambda^s\beta}, & \alpha\neq-\lambda^s\beta,\\ \dfrac{p_\alpha}{\sqrt{l}}\omega^{sb}\mathfrak{h}^{a+b}_\alpha, & \alpha=-\lambda^s\beta,\end{cases}$$

$$\left[\mathfrak{h}^k_\alpha,\mathfrak{t}^m_\beta\right]=\frac{1}{\sqrt{l}}\sum_{s=0}^{l-1}\omega^{-ks}\frac{2(\alpha,\lambda^s\beta)}{(\alpha,\alpha)}\mathfrak{t}^{k+m}_\beta,$$

$$\left[\mathfrak{H}^k_\alpha,\mathfrak{t}^m_\beta\right]=\frac{1}{\sqrt{l}}\sum_{s=0}^{l-1}\omega^{-ks}\frac{(\alpha,\alpha)}{2}(\hat\alpha,\lambda^s\beta)\mathfrak{t}^{k+m}_\beta.$$

B Elliptic functions

The basic function is the theta-function

$$\vartheta(z|\tau)=q^{\frac{1}{8}}\sum_{n\in\mathbf{Z}}(-1)^n e^{\pi i(n(n+1)\tau+2nz)}.$$

It is a holomorphic function on $\mathbb{C}$ with simple zeroes at the lattice $\tau\mathbb{Z}+\mathbb{Z}$ and the quasi-periodicities

$$\vartheta(z+1)=-\vartheta(z),\qquad \vartheta(z+\tau)=-q^{-\frac{1}{2}}e^{-2\pi iz}\vartheta(z).$$

Define the ration of the theta-functions

$$\phi(u,z)=\frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \tag{B.1}$$

Then

$$\phi(u,z)=\phi(z,u),\qquad \phi(-u,-z)=-\phi(u,z). \tag{B.2}$$

Related functions:

$$\varphi^m_\beta(\mathbf{u},z)=\mathbf{e}(\langle\kappa,\beta\rangle z)\phi\left(\langle\mathbf{u}+\kappa\tau,\beta\rangle+\frac{m}{l},z\right), \tag{B.3}$$

$$\varphi^{m,k}_\beta(\mathbf{u},z)=\partial^k_z\Big(\mathbf{e}(\langle\kappa,\beta\rangle z)\phi\left(\langle\mathbf{u}+\kappa\tau,\beta\rangle+\frac{m}{l},z\right)\Big),\qquad \varphi^{m,0}_\beta=\varphi^m_\beta, \tag{B.4}$$

$$f(u,z)=\partial_u\phi(u,z), \tag{B.5}$$

$$f(u,z)=\phi(u,z)(E_1(u+z)-E_1(u)). \tag{B.6}$$

$\phi(u,z)$ has a pole at $z=0$ and

$$\phi(u,z)=\frac{1}{z}+E_1(u)+\frac{z}{2}\big(E_1^2(u)-\wp(u)\big)+\cdots. \tag{B.7}$$

Similarly,

$$\varphi^m_\beta(\mathbf{u},z)=\frac{1}{z}+E_1\left(\langle\mathbf{u}+\kappa\tau,\beta\rangle+\frac{m}{l}\right)+2\pi\imath\langle\kappa,\beta\rangle+\frac{z}{2}\big(E_1^2(u)-\wp(u)\big)+\cdots, \tag{B.8}$$

where E_1 is (B.10). It follows from this expansion that

$$\begin{aligned}&\varphi^{m,1}_\beta(\mathbf{u},z)=-\frac{1}{z^2}+\frac{1}{2}\big(E_1^2(u)-\wp(u)\big)+\cdots,\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&\varphi^{m,k}_\beta(\mathbf{u},z)=\frac{(-1)^k}{z^{k+1}}+c(m,k)+\cdots.\end{aligned} \tag{B.9}$$

In other words $\varphi^{m,k}_\beta(\mathbf{u},z)$ has not poles of order less than $k+1$.

The Eisenstein functions:

$$E_1(z|\tau)=\partial_z\log\vartheta(z|\tau),\qquad E_1(z|\tau)\sim\frac{1}{z}-2\eta_1 z+\cdots, \tag{B.10}$$

where

$$\begin{aligned}&\eta_1(\tau)=\frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty\,\prime}\frac{1}{(m\tau+n)^2}=\frac{24}{2\pi i}\frac{\eta'(\tau)}{\eta(\tau)},\qquad \eta(\tau)=q^{\frac{1}{24}}\prod_{n>0}\big(1-q^n\big),\\&E_2(z|\tau)=-\partial_z E_1(z|\tau)=\partial_z^2\log\vartheta(z|\tau),\qquad E_2(z|\tau)\sim\frac{1}{z^2}+2\eta_1,\end{aligned} \tag{B.11}$$

and more general for $k>2$

$$E_k(z|\tau)=(-\partial_z)^{k+1}\log\vartheta(z|\tau),\qquad E_k(z|\tau)\sim\frac{1}{z^k}+\cdots. \tag{B.12}$$

Relation to the Weierstrass functions:

$$\zeta(z,\tau)=E_1(z,\tau)+2\eta_1(\tau)z,\qquad \wp(z,\tau)=E_2(z,\tau)-2\eta_1(\tau).$$

Quasi-periodicity:

$$\vartheta(z+1)=-\vartheta(z),\qquad \vartheta(z+\tau)=-q^{-\frac{1}{2}}e^{-2\pi iz}\vartheta(z), \tag{B.13}$$

$$E_1(z+1)=E_1(z),\qquad E_1(z+\tau)=E_1(z)-2\pi i, \tag{B.14}$$

$$E_k(z+1)=E_k(z),\qquad E_k(z+\tau)=E_k(z),\qquad k>1, \tag{B.15}$$

$$\phi(u,z+1)=\phi(u,z),\qquad \phi(u,z+\tau)=e^{-2\pi\imath u}\phi(u,z), \tag{B.16}$$

$$\begin{aligned}&\varphi^{m,k}_\beta(\mathbf{u},z+1)=\mathbf{e}(\langle\kappa,\beta\rangle)\varphi^{m,k}_\beta(\mathbf{u},z),\\&\varphi^{m,k}_\beta(\mathbf{u},z+\tau)=\mathbf{e}\left(-\langle\mathbf{u},\beta\rangle-\frac{m}{l}\right)\varphi^{m,k}_\beta(\mathbf{u},z),\\&f(u,z+1)=f(u,z),\qquad f(u,z+\tau)=e^{-2\pi\imath u}f(u,z)-2\pi\imath\phi(u,z).\end{aligned} \tag{B.17}$$

The following identities are also used here

$$2\pi i\partial_\tau\phi(u,z)=\partial_z\partial_u\phi(u,z)=\partial_z f(u,z)$$

and for the functions (4.38) this identity takes the form

$$2\pi i\partial_\tau \varphi^m_\alpha(z) = \partial_z f^k_\alpha(z). \tag{B.18}$$

Fay identity:

$$\phi(u_1,z_1)\phi(u_2,z_2) - \phi(u_1+u_2,z_1)\phi(u_2,z_2-z_1) - \phi(u_1+u_2,z_2)\phi(u_1,z_1-z_2) = 0.$$

Differentiating over u_2 we find

$$\begin{aligned}&\phi(u_1,z_1)f(u_2,z_2) - \phi(u_1+u_2,z_1)f(u_2,z_2-z_1)\\ &\qquad = \phi(u_2,z_2-z_1)f(u_1+u_2,z_1) + \phi(u_1,z_1-z_2)f(u_1+u_2,z_2).\end{aligned}$$

Substituting here

$$\begin{aligned}&u_1 = \langle u+\kappa\tau,\alpha+\beta\rangle + \frac{k+m}{l}, \qquad u_2 = -\langle u+\kappa\tau,\beta\rangle - \frac{m}{l},\\ &z_1 = z_a - z_c = z_{ac}, \qquad z_2 = z_b - z_c = z_{bc},\end{aligned}$$

and multiplying by appropriate exponential factor we can rewrite it in the form

$$\varphi^k_\alpha(z_{ac})f^m_\beta(z_{ab}) - \varphi^m_\beta(z_{ab})f^k_\alpha(z_{ac}) + \varphi^{k+m}_{\alpha+\beta}(z_{ab})f^k_\alpha(z_{cb}) - \varphi^{k+m}_{\alpha+\beta}(z_{ac})f^{-m}_{-\beta}(z_{bc}) = 0. \tag{B.19}$$

Taking the limit $m=0$, $\beta=0$ and using the expansion

$$\phi(z,u) \sim \frac{1}{u} + E_1(z) + u\rho(z) + \cdots,$$

we find

$$\varphi^k_\alpha(z_{ac})\rho(z_{ab}) - E_1(z_{ab})f^k_\alpha(z_{ac}) + \varphi^k_\alpha(z_{ab})f^k_\alpha(z_{bc}) - \varphi^k_\alpha(z_{ac})\rho(z_{cb}) = \frac{1}{2}\partial_u f^k_\alpha(z_{ac}). \tag{B.20}$$

More Fay identities:

$$\varphi^k_\alpha(z_{ac})f^m_\beta(z_{ac}) - \varphi^m_\beta(z_{ac})f^k_\alpha(z_{ac}) = \varphi^{k+m}_{\alpha+\beta}(z_{ac})\big(\wp^k_\alpha - \wp^m_\beta\big), \tag{B.21}$$

$$\varphi^m_\beta(z_{ac})f^{-m}_{-\beta}(z_{ac}) - \varphi^{-m}_{-\beta}(z_{ac})f^m_\beta(z_{ac}) = E'^{\,m}_{2\beta}, \tag{B.22}$$

$$\varphi^k_\beta(z_{ac})\wp^k_\beta - \varphi^k_\beta(z_{ac})\rho(z_{ac}) + E_1(z_{ac})f^k_\beta(z_{ac}) = \frac{1}{2}\partial_u f^k_\beta(z_{ac}). \tag{B.23}$$

The last one follows from

$$\partial_u\phi(u,z) = \phi(u,z)\big(E_1(z+u) - E_1(u)\big)$$

and

$$\big(E_1(z+u) - E_1(u) - E_1(z)\big)^2 = \wp(z) + \wp(u) + \wp(z+u).$$

C Proofs of Propositions 1 and 2

Proof of Proposition 1.

$$\begin{aligned}[\nabla_a,\nabla_b] = {}&\big[\partial_{z_a}, r^{ba}\big] + \big[\hat\partial^a, r^{ba}\big]\\ &+ \sum_{c\neq a,b}\big[\hat\partial^a, r^{bc}\big] - \big[\partial_{z_b}, r^{ab}\big] - \big[\hat\partial^b, r^{ab}\big] - \sum_{c\neq a,b}\big[\hat\partial^b, r^{ac}\big] + \sum_{d\neq b}\sum_{a\neq c}\big[r^{ac}, r^{bd}\big].\end{aligned} \tag{C.1}$$

First, notice that

$$\big[\partial_{z_a}+\partial_{z_b}, r^{ab}\big]=0.$$

Secondly,

$$\big[\hat\partial^a, r^{ba}\big]-\big[\hat\partial^b, r^{ab}\big]=-\big[\hat\partial^a+\hat\partial^b, r^{ab}\big]=\sum_{c\neq a,b}\big[\hat\partial^c, r^{ab}\big]-\left[\sum_c\hat\partial^c, r^{ab}\right].$$

Thirdly,

$$\sum_{d\neq b}\sum_{a\neq c}\big[r^{ac}, r^{bd}\big]=\sum_{c\neq a,b}\big(\big[r^{ac}, r^{ba}\big]+\big[r^{ab}, r^{bc}\big]+\big[r^{ac}, r^{bc}\big]\big).$$

Therefore,

$$\big[\nabla_a,\nabla_b\big]=\sum_{c\neq a,b}\mathrm{CDYB}^{abc}-\left[\sum_c\hat\partial^c, r^{ab}\right], \tag{C.2}$$

where

$$\mathrm{CDYB}^{abc}=\big[r^{ab}, r^{ac}\big]+\big[r^{ab}, r^{bc}\big]+\big[r^{ac}, r^{bc}\big]+\big[\hat\partial^a, r^{bc}\big]+\big[\hat\partial^c, r^{ab}\big]+\big[\hat\partial^b, r^{ca}\big]\overset{(4.44)}{=}0. \tag{C.3}$$

and $\big[\sum_c\hat\partial^c, r^{ab}\big]\overset{(4.43)}{=}0.$ ■

Proof of Proposition 2.

1.

$$\big[\partial_{z_a}f^{ac}\big]-2\pi i\big[\partial_\tau r^{ac}\big]\overset{(B.18)}{=}0.$$

2.

$$\left[\Delta,\sum_{c\neq a}r^{ac}\right]=\frac{l}{2}\sum_{c\neq a}\sum_{\beta\in R}\sum_{m=0}^{l-1}|\beta|^2\partial_u f_\beta^m(z_a-z_c)\big[\mathfrak{h}_\beta^{0,a},\mathfrak{t}_\beta^{m,a}\big]_+\mathfrak{t}_{-\beta}^{-m,c}$$
$$+\,l\sum_{c\neq a}\sum_{\beta\in R}\sum_{\alpha\in\Pi}\sum_{s=0}^{l-1}|\beta|^2\langle\lambda^s\hat\alpha,\beta\rangle f_\beta^m(z_a-z_c)\partial_{u_\alpha}\mathfrak{t}_\beta^{m,a}\mathfrak{t}_{-\beta}^{-m,c}.$$

3. Terms $\big[\hat\partial^a,\frac12 f^{bc}\big]$ for b, $c\neq a$ and $b\neq c$:

$$\left[\hat\partial^a,\frac12\sum_{\substack{b,c\neq a\\ b\neq c}}f^{bc}\right]=\frac{l}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_b-z_c)\mathfrak{h}_\alpha^{0,a}\mathfrak{t}_\alpha^{k,b}\mathfrak{t}_{-\alpha}^{-k,c}. \tag{C.4}$$

3.1. Terms $\big[\hat\partial^a,\frac12 f^{ac}\big]$ for $c\neq a$:

$$\left[\hat\partial^a,\frac12\sum_{c\neq a}(f^{ac}+f^{ca})\right]=\left[\hat\partial^a,\sum_{c\neq a}f^{ac}\right]=l\sum_{c\neq a}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u f_\beta^m(z_a-z_c)\mathfrak{h}_\beta^{0,a}\mathfrak{t}_\beta^{m,a}\mathfrak{t}_{-\beta}^{-m,c}$$
$$+\,l\sum_{c\neq a}\sum_{\beta\in R}\sum_{\alpha\in\Pi}\sum_{s=0}^{l-1}|\beta|^2\langle\lambda^s\hat\alpha,\beta\rangle f_\beta^m(z_a-z_c)\partial_{u_\alpha}\mathfrak{t}_\beta^{m,a}\mathfrak{t}_{-\beta}^{-m,c}.$$

Therefore we get

$$\left[\hat{\partial}^a, \sum_{c\neq a} f^{ac}\right] - \left[\Delta, \sum_{c\neq a} r^{ac}\right] = \frac{l}{2}\sum_{c\neq a}\sum_{\alpha\in R}\sum_{m=0}^{l-1}|\alpha|^2\partial_u f_\alpha^m(z_a - z_c)\left[\mathfrak{h}_\alpha^{0,a}, \mathfrak{t}_\alpha^{m,a}\right]_+ \mathfrak{t}_{-\alpha}^{-m,c}. \tag{C.5}$$

3.2. Terms $\left[\hat{\partial}^a, \frac{1}{2}f^{aa}\right]$:

$$\begin{aligned}\left[\hat{\partial}^a, \frac{1}{2}f^{aa}\right] &= -\frac{l}{2}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u\wp_\beta^m\mathfrak{h}_\beta^{0,a}\mathfrak{t}_\beta^{m,a}\mathfrak{t}_{-\beta}^{-m,a}\\ &= -\frac{l}{4}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2 E'^{\,m}_{2\beta}\mathfrak{h}_\beta^{0,a}\left[\mathfrak{t}_\beta^{m,a}\mathfrak{t}_{-\beta}^{-m,a}\right]_+.\end{aligned} \tag{C.6}$$

3.3. Terms $\left[\hat{\partial}^a, \frac{1}{2}f^{cc}\right]$ for $c\neq a$:

$$\left[\hat{\partial}^a, \sum_{c\neq a}\frac{1}{2}f^{cc}\right] = -\frac{l}{4}\sum_{c\neq a}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2 E'^{\,m}_{2\beta}\mathfrak{h}_\beta^{0,a}\left[\mathfrak{t}_\beta^{m,c}\mathfrak{t}_{-\beta}^{-m,c}\right]_+. \tag{C.7}$$

4. Terms $[r,f]$:

$$\begin{aligned}\left[\sum_{c\neq a} r^{ac}, \frac{1}{2}\sum_{b,d} f^{b,d}\right] &= \frac{1}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\left(\left[r^{ac}, f^{ab}\right] + \left[r^{ac}, f^{bc}\right] + \left[r^{ab}, f^{ac}\right] + \left[r^{ab}, f^{bc}\right]\right)\\ &\quad + \sum_{c\neq a}\left(\left[r^{ac}, f^{ac}\right] + \frac{1}{2}\left[r^{ac}, f^{aa}\right] + \frac{1}{2}\left[r^{ac}, f^{cc}\right]\right).\end{aligned}$$

4.1. Terms $\left[r,f\right]$ for b, $c\neq a$ and $b\neq c$:

$$\begin{aligned}&\frac{1}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\left(\left[r^{ac}, f^{ab}\right] + \left[r^{ac}, f^{bc}\right] + \left[r^{ab}, f^{ac}\right] + \left[r^{ab}, f^{bc}\right]\right)\\
&\quad = \frac{1}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k,m,s=0}^{l-1}\sum_{\alpha,\beta\in R}|\alpha|^2|\beta|^2\omega^{ks}C_{\lambda^s\alpha,\beta}\Big(\varphi_\alpha^k(z_{ac})f_\beta^m(z_{ab}) - \varphi_\beta^m(z_{ab})f_\alpha^k(z_{ac})\\
&\quad + \varphi_{\alpha+\beta}^{k+m}(z_{ab})f_\alpha^k(z_{cb}) - \varphi_{\alpha+\beta}^{k+m}(z_{ac})f_{-\beta}^{-m}(z_{bc})\Big)\mathfrak{t}_{\alpha+\lambda^s\beta}^{k+m,a}\mathfrak{t}_{-\beta}^{-m,b}\mathfrak{t}_{-\alpha}^{-k,c}\\
&\quad - \frac{l}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k,m=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\Big[\Big(\varphi_\alpha^k(z_{ac})f_0^m(z_{ab}) - \varphi_0^m(z_{ab})f_\alpha^k(z_{ac})\\
&\quad + \varphi_\alpha^{k+m}(z_{ab})f_\alpha^k(z_{cb}) - \varphi_\alpha^{k+m}(z_{ac})f_0^{-m}(z_{bc})\Big)\mathfrak{t}_\alpha^{k+m,a}\mathfrak{h}_\alpha^{-m,b}\mathfrak{t}_\alpha^{-k,c}\\
&\quad + \Big(\varphi_\alpha^k(z_{ab})f_0^m(z_{ac}) - \varphi_0^m(z_{ac})f_\alpha^k(z_{ab})\\
&\quad + \varphi_\alpha^{k+m}(z_{ac})f_\alpha^k(z_{bc}) - \varphi_\alpha^{k+m}(z_{ac})f_0^{-m}(z_{cb})\Big)\mathfrak{t}_\alpha^{k+m,a}\mathfrak{t}_\alpha^{-k,b}\mathfrak{h}_\alpha^{-m,c}\\
&\quad + 2\Big(\varphi_\alpha^k(z_{bc})f_0^m(z_{ba}) - \varphi_0^m(z_{ba})f_\alpha^k(z_{bc})\\
&\quad + \varphi_\alpha^{k+m}(z_{ba})f_\alpha^k(z_{ba}) - \varphi_\alpha^{k+m}(z_{ba})f_0^{-m}(z_{ac})\Big)\mathfrak{t}_\alpha^{k+m,b}\mathfrak{h}_\alpha^{-m,a}\mathfrak{t}_\alpha^{-k,c}\Big].\end{aligned} \tag{C.8}$$

Almost all terms in this big sum vanish due to the Fay identity (B.19), except the terms with $m = 0$. Using (B.20) for these terms in the expression (C.8) we get

$$-\frac{l}{4}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_{ac})\mathfrak{t}_\alpha^{k,a}\mathfrak{h}_\alpha^{0,b}\mathfrak{t}_{-\alpha}^{-k,c} - \frac{l}{4}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_{ab})\mathfrak{t}_\alpha^{k,a}\mathfrak{h}_\alpha^{0,c}\mathfrak{t}_{-\alpha}^{-k,b}$$
$$+\frac{l}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_{bc})\mathfrak{t}_\alpha^{k,b}\mathfrak{h}_\alpha^{0,a}\mathfrak{t}_{-\alpha}^{-k,c}.$$

Finally, using the symmetry in summation over c and b we obtain

$$\frac{1}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\left(\left[r^{ac},f^{ab}\right]+\left[r^{ac},f^{bc}\right]+\left[r^{ab},f^{ac}\right]+\left[r^{ab},f^{bc}\right]\right)$$
$$=\frac{l}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_{ac})\mathfrak{t}_\alpha^{k,a}\mathfrak{h}_\alpha^{0,b}\mathfrak{t}_{-\alpha}^{-k,c}$$
$$-\frac{l}{2}\sum_{\substack{b,c\neq a\\ b\neq c}}\sum_{k=0}^{l-1}\sum_{\alpha\in R}|\alpha|^2\partial_u f_\alpha^k(z_{bc})\mathfrak{t}_\alpha^{k,b}\mathfrak{h}_\alpha^{0,a}\mathfrak{t}_{-\alpha}^{-k,c}. \tag{C.9}$$

Notice that the second term here cancels the expression (C.4).

4.2. Terms $\left[r^{ac},f^{ac}\right]$:

$$\sum_{c\neq a}\left[r^{ac},f^{ac}\right] = \frac{1}{4}\sum_{c\neq a}\sum_{k,m,s=0}^{l-1}\sum_{\alpha,\beta\in R}|\alpha|^2|\beta|^2\Big(\varphi_\alpha^k(z_{ac})f_\beta^m(z_{ac})$$
$$-\varphi_\beta^m(z_{ac})f_\alpha^k(z_{ac})\Big)\omega^{ms}C_{\alpha,\lambda^s\beta}\mathfrak{t}_{\alpha+\lambda^s\beta}^{k+m,a}\left[\mathfrak{t}_{-\alpha}^{-k,c},\mathfrak{t}_{-\beta}^{-m,c}\right]_+$$
$$+\frac{1}{4}\sum_{c\neq a}\sum_{k,m,s=0}^{l-1}\sum_{\alpha,\beta\in R}|\alpha|^2|\beta|^2\Big(\varphi_\alpha^k(z_{ac})f_\beta^m(z_{ac})$$
$$-\varphi_\beta^m(z_{ac})f_\alpha^k(z_{ac})\Big)\omega^{-ms}C_{\alpha,\lambda^s\beta}\mathfrak{t}_{-\alpha-\lambda^s\beta}^{-k-m,c}\left[\mathfrak{t}_\alpha^{k,a},\mathfrak{t}_\beta^{m,a}\right]_+$$
$$-\frac{l}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(\varphi_{-\beta}^{-m}(z_{ac})f_\beta^{m+k}(z_{ac})$$
$$-\varphi_\beta^{m+k}(z_{ac})f_{-\beta}^{-m}(z_{ac})\Big)\mathfrak{h}_\beta^{k,a}\left[\mathfrak{t}_{-\beta}^{-k-m,c},\mathfrak{t}_\beta^{m,c}\right]_+$$
$$+\frac{l}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(\varphi_{-\beta}^{-m}(z_{ac})f_\beta^{m+k}(z_{ac})$$
$$-\varphi_\beta^{m+k}(z_{ac})f_{-\beta}^{-m}(z_{ac})\Big)\mathfrak{h}_\beta^{-k,c}\left[\mathfrak{t}_\beta^{k+m,a},\mathfrak{t}_{-\beta}^{-m,a}\right]_+$$
$$+\frac{l}{2}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(-\varphi_0^{-m}(z_{ac})f_\beta^{m+k}(z_{ac})$$
$$+f_0^{-m}(z_{ac})\varphi_\beta^{m+k}(z_{ac})\Big)\left[\mathfrak{h}_\beta^{-m,a},\mathfrak{t}_\beta^{m+k,a}\right]_+\mathfrak{t}_{-\beta}^{-k,c}$$
$$+\frac{l}{2}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(\varphi_0^{-m}(z_{ac})f_\beta^{m+k}(z_{ac})$$

$$- f_0^{-m}(z_{ac})\varphi_\beta^{m+k}(z_{ac})\Big)\left[\mathfrak{h}_\beta^{m,c}, \mathfrak{t}_{-\beta}^{-m-k,a}\right]_+ \mathfrak{t}_\beta^{k,c}. \tag{C.10}$$

4.3. Terms $\left[r^{ac}, f^{aa}\right]$ and $\left[r^{ac}, f^{cc}\right]$ for $c \neq a$:

$$\begin{aligned}\left[\sum_{c\neq a} r^{ac}, f^{aa}\right] &= -\frac{1}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\alpha,\beta\in R}|\alpha|^2|\beta|^2\varphi_{\alpha+\beta}^{k+m}(z_{ac})(\wp_\alpha^k - \wp_\beta^m)\\ &\times \omega^{-ms}C_{\alpha,\lambda^s\beta}\left[\mathfrak{t}_\alpha^{k,a}, \mathfrak{t}_\beta^{m,a}\right]_+ \mathfrak{t}_{-\alpha-\lambda^s\beta}^{-k-m,c}\\ &- \frac{l}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\varphi_0^k(\wp_\beta^m - \wp_\beta^{k+m})\mathfrak{h}_\beta^{-k,c}\left[\mathfrak{t}_\beta^{k+m,a}, \mathfrak{t}_{-\beta}^{-m,a}\right]_+\\ &+ \frac{l}{2}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\varphi_\beta^k(z_{ac})(\wp_0^m - \wp_\beta^{m+k})\left[\mathfrak{h}_\beta^{-m,a}, \mathfrak{t}_\beta^{m+k,a}\right]_+ \mathfrak{t}_{-\beta}^{-k,c},\end{aligned} \tag{C.11}$$

$$\begin{aligned}\left[\sum_{c\neq a} r^{ac}, f^{cc}\right] &= -\frac{1}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\alpha,\beta\in R}|\alpha|^2|\beta|^2\varphi_{\alpha+\beta}^{k+m}(z_{ac})(\wp_\alpha^k - \wp_\beta^m)\\ &\times \omega^{ms}C_{\alpha,\lambda^s\beta}\left[\mathfrak{t}_{-\alpha}^{-k,a}, \mathfrak{t}_{-\beta}^{-m,a}\right]_+ \mathfrak{t}_{\alpha+\lambda^s\beta}^{k+m,c}\\ &+ \frac{l}{4}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\varphi_0^k(\wp_\beta^m - \wp_\beta^{k+m})\mathfrak{h}_\beta^{k,a}\left[\mathfrak{t}_{-\beta}^{-k-m,c}, \mathfrak{t}_\beta^{m,c}\right]_+\\ &- \frac{l}{2}\sum_{c\neq a}\sum_{k,m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\varphi_\beta^k(z_{ac})(\wp_0^m - \wp_\beta^{m+k})\left[\mathfrak{h}_\beta^{m,c}, \mathfrak{t}_{-\beta}^{-m-k,c}\right]_+ \mathfrak{t}_\beta^{k,a}.\end{aligned} \tag{C.12}$$

The first two lines in (C.10) are canceled by first lines in (C.11) and (C.12) due to identity (B.21). Next, the sum of the third line in (C.10), the second line in (C.12), the sum of fourth line in (C.10) and the second line in (C.11) are vanished due to (B.21) for all values of summation parameters except $k = 0$. For $k = 0$ these sums give

$$\begin{aligned}&-\frac{l}{4}\sum_{c\neq a}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(\varphi_{-\beta}^{-m}(z_{ac})f_\beta^m(z_{ac}) - \varphi_\beta^m(z_{ac})f_{-\beta}^{-m}(z_{ac})\Big)\\ &\qquad\times\Big(\mathfrak{h}_\beta^{0,a}\left[\mathfrak{t}_{-\beta}^{-m,c}, \mathfrak{t}_\beta^{m,c}\right]_+ - \mathfrak{h}_\beta^{0,c}\left[\mathfrak{t}_\beta^{m,a}, \mathfrak{t}_{-\beta}^{-m,a}\right]_+\Big)\\ &\qquad\overset{(B.22)}{=}\frac{l}{4}\sum_{c\neq a}\sum_{m=0}^{l-1}\sum_{\beta\in R}|\beta|^2 E'^{\,m}_{2\beta}\Big(\mathfrak{h}_\beta^{0,a}\left[\mathfrak{t}_{-\beta}^{-m,c}, \mathfrak{t}_\beta^{m,c}\right]_+ - \mathfrak{h}_\beta^{0,c}\left[\mathfrak{t}_\beta^{m,a}, \mathfrak{t}_{-\beta}^{-m,a}\right]_+\Big)\end{aligned}$$

and this is exactly what we need to compensate (C.6) and (C.7). (Note that (C.7) cancels by the first term here and (C.6) cancels by the second one due to (4.43).)

Finally, the last two lines in (C.10) are canceled by the list lines in (C.11) and (C.12) for all values of summation parameters except $m = 0$. For $m = 0$ the sum of these terms equals

$$\begin{aligned}&-\frac{l}{2}\sum_{c\neq a}\sum_{k=0}^{l-1}\sum_{\beta\in R}|\beta|^2\Big(\varphi_0^0(z_{ac})f_\beta^k(z_{ac}) - f_0^0(z_{ac})\varphi_\beta^k(z_{ac}) + \varphi_\beta^k(z_{ac})\wp_\beta^k\Big)\\ &\qquad\times\Big(\left[\mathfrak{h}_\beta^{0,a}, \mathfrak{t}_\beta^{k,a}\right]_+ \mathfrak{t}_{-\beta}^{-k,c} - \left[\mathfrak{h}_\beta^{0,c}, \mathfrak{t}_{-\beta}^{-k,c}\right]_+ \mathfrak{t}_\beta^{k,a}\Big).\end{aligned} \tag{C.13}$$

Notice that due to (4.39) and (4.40) $f_0^0(z_{ac}) = \rho(z_{ac})$ and $\varphi_0^0(z_{ac}) = E_1(z_{ac})$. Then using (B.23) we can simplify the expression (C.13)

$$-\frac{l}{4}\sum_{c\neq a}\sum_{k=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u f_\beta^k(z_{ac})\Big(\left[\mathfrak{h}_\beta^{0,a}, \mathfrak{t}_\beta^{k,a}\right]_+ \mathfrak{t}_{-\beta}^{-k,c} - \left[\mathfrak{h}_\beta^{0,c}, \mathfrak{t}_{-\beta}^{-k,c}\right]_+ \mathfrak{t}_\beta^{k,a}\Big). \tag{C.14}$$

Finally, we have nonzero terms from (C.5), first term in (C.9) and (C.14). All these terms are proportional to $\partial_u f^k_\beta$. Summing them up we find

$$
\begin{aligned}
& l\sum_{c\neq a}\sum_{k=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u f^k_\beta(z_{ac})\Bigg(-\frac{1}{4}\big[\mathfrak{h}^{0,a}_\beta,\mathfrak{t}^{k,a}_\beta\big]_+\mathfrak{t}^{-k,c}_{-\beta}+\frac{1}{4}\big[\mathfrak{h}^{0,c}_\beta,\mathfrak{t}^{-k,c}_{-\beta}\big]_+\mathfrak{t}^{k,a}_\beta \\
&\qquad +\frac{1}{2}\sum_{b\neq a,c}\mathfrak{t}^{k,a}_\beta\mathfrak{t}^{-k,c}_{-\beta}\mathfrak{h}^{0,b}_\beta+\frac{1}{2}\big[\mathfrak{h}^{0,a}_\beta,\mathfrak{t}^{k,a}_\beta\big]_+\mathfrak{t}^{-k,c}_{-\beta}\Bigg) \\
&\quad = l\sum_{c\neq a}\sum_{k=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u f^k_\beta(z_{ac})\Bigg(\frac{1}{4}\big[\mathfrak{h}^{0,a}_\beta,\mathfrak{t}^{k,a}_\beta\big]_+\mathfrak{t}^{-k,c}_{-\beta} \\
&\qquad +\frac{1}{4}\big[\mathfrak{h}^{0,c}_\beta,\mathfrak{t}^{-k,c}_{-\beta}\big]_+\mathfrak{t}^{k,a}_\beta+\frac{1}{2}\sum_{b\neq a,c}\mathfrak{t}^{k,a}_\beta\mathfrak{t}^{-k,c}_{-\beta}\mathfrak{h}^{0,b}_\beta\Bigg) \\
&\quad \overset{(4.43)}{=} l\sum_{c\neq a}\sum_{k=0}^{l-1}\sum_{\beta\in R}|\beta|^2\partial_u f^k_\beta(z_{ac})\left(\frac{1}{4}\big[\mathfrak{h}^{0,a}_\beta,\mathfrak{t}^{k,a}_\beta\big]\mathfrak{t}^{-k,c}_{-\beta}+\frac{1}{4}\big[\mathfrak{h}^{0,c}_\beta,\mathfrak{t}^{-k,c}_{-\beta}\big]\mathfrak{t}^{k,a}_\beta\right)=0. \qquad \blacksquare
\end{aligned}
$$

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References

[1] Aminov G., Arthamonov S., Reduction of the elliptic SL$(N,\mathbb{C})$ top, *J. Phys. A: Math. Theor.* **44** (2011), 075201, 34 pages, arXiv:1009.1867.

[2] Aminov G., Arthamonov S., Levin A.M., Olshanetsky M.A., Zotov A.V., Around Painlevé VI field theory, submitted.

[3] Atiyah M.F., Vector bundles over an elliptic curve, *Proc. London Math. Soc.* **7** (1957), 414–452.

[4] Axelrod S., Pietra S.D., Witten E., Geometric quantization of Chern–Simons gauge theory, *J. Differential Geom.* **33** (1991), 787–902.

[5] Beauville A., Conformal blocks, fusion rules and the Verlinde formula, in Proceedings of the Hirzebruch 65 Conference on Algebraic Geometry (Ramat Gan, 1993), *Israel Math. Conf. Proc.*, Vol. 9, Bar-Ilan Univ., Ramat Gan, 1996, 75–96, alg-geom/9405001.

[6] Beauville A., Laszlo Y., Conformal blocks and generalized theta functions, *Comm. Math. Phys.* **164** (1994), 385–419, alg-geom/9309003.

[7] Ben-Zvi D., Frenkel E., Geometric realization of the Segal–Sugawara construction, in Topology, geometry and quantum field theory, *London Math. Soc. Lecture Note Ser.*, Vol. 308, Cambridge University Press, Cambridge, 2004, 46–97, math.AG/0301206.

[8] Bernard D., On the Wess–Zumino–Witten models on Riemann surfaces, *Nuclear Phys. B* **309** (1988), 145–174.

[9] Bernard D., On the Wess–Zumino–Witten models on the torus, *Nuclear Phys. B* **303** (1988), 77–93.

[10] Bernstein J., Schwarzman O., Chevalley's theorem for complex crystallographic Coxeter groups, *Funct. Anal. Appl.* **12** (1978), 308–310.

[11] Bernstein J., Schwarzman O., Complex crystallographic Coxeter groups and affine root systems, *J. Nonlinear Math. Phys.* **13** (2006), 163–182.

[12] Bourbaki N., Lie groups and Lie algebras, Chapters 4–6, *Elements of Mathematics (Berlin)*, Springer-Verlag, Berlin, 2002.

[13] Braden H.W., Dolgushev V.A., Olshanetsky M.A., Zotov A.V., Classical r-matrices and the Feigin–Odesskii algebra via Hamiltonian and Poisson reductions, *J. Phys. A: Math. Gen.* **36** (2003), 6979–7000, hep-th/0301121.

[14] Bulycheva K., Monopole solutions to the Bogomolny equation as three-dimensional generalizations of the Kronecker series, *Theoret. and Math. Phys.* **172** (2012), 1232–1242, arXiv:1203.4674.

[15] Chernyakov Yu.B., Levin A.M., Olshanetsky M.A., Zotov A.V., Elliptic Schlesinger system and Painlevé VI, *J. Phys. A: Math. Gen.* **39** (2006), 12083–12101, nlin.SI/0602043.

[16] Enriquez B., Rubtsov V., Hecke–Tyurin parametrization of the Hitchin and KZB systems, in Moscow Seminar on Mathematical Physics. II, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 221, Amer. Math. Soc., Providence, RI, 2007, 1–31, math.AG/9911087.

[17] Etingof P., Schiffmann O., Twisted traces of intertwiners for Kac–Moody algebras and classical dynamical R-matrices corresponding to generalized Belavin–Drinfeld triples, *Math. Res. Lett.* **6** (1999), 593–612, math.QA/9908115.

[18] Etingof P., Schiffmann O., Varchenko A., Traces of intertwiners for quantum groups and difference equations, *Lett. Math. Phys.* **62** (2002), 143–158, math.QA/0207157.

[19] Etingof P., Varchenko A., Geometry and classification of solutions of the classical dynamical Yang–Baxter equation, *Comm. Math. Phys.* **192** (1998), 77–120, q-alg/9703040.

[20] Faltings G., A proof for the Verlinde formula, *J. Algebraic Geom.* **3** (1994), 347–374.

[21] Fehér L., Pusztai B.G., Generalizations of Felder's elliptic dynamical r-matrices associated with twisted loop algebras of self-dual Lie algebras, *Nuclear Phys. B* **621** (2002), 622–642, math.QA/0109132.

[22] Felder G., The KZB equations on Riemann surfaces, in Symétries Quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, 687–725, hep-th/9609153.

[23] Felder G., Gawedzki K., Kupiainen A., Spectra of Wess–Zumino–Witten models with arbitrary simple groups, *Comm. Math. Phys.* **117** (1988), 127–158.

[24] Felder G., Wieczerkowski C., Conformal blocks on elliptic curves and the Knizhnik–Zamolodchikov–Bernard equations, *Comm. Math. Phys.* **176** (1996), 133–161, hep-th/9411004.

[25] Frenkel E., Lectures on the Langlands program and conformal field theory, in Frontiers in Number Theory, Physics, and Geometry. II, Springer, Berlin, 2007, 387–533, hep-th/0512172.

[26] Friedman R., Morgan J.W., Holomorphic principal bundles over elliptic curves, math.AG/9811130.

[27] Friedman R., Morgan J.W., Holomorphic principal bundles over elliptic curves. II. The parabolic construction, *J. Differential Geom.* **56** (2000), 301–379, math.AG/0006174.

[28] Friedman R., Morgan J.W., Witten E., Principal G-bundles over elliptic curves, *Math. Res. Lett.* **5** (1998), 97–118, alg-geom/9707004.

[29] Fuchs J., Schweigert C., The action of outer automorphisms on bundles of chiral blocks, *Comm. Math. Phys.* **206** (1999), 691–736, hep-th/9805026.

[30] Gorsky A., Nekrasov N., Hamiltonian systems of Calogero-type, and two-dimensional Yang–Mills theory, *Nuclear Phys. B* **414** (1994), 213–238, hep-th/9304047.

[31] Gukov S., Witten E., Branes and quantization, *Adv. Theor. Math. Phys.* **13** (2009), 1445–1518, arXiv:0809.0305.

[32] Harnad J., Quantum isomonodromic deformations and the Knizhnik–Zamolodchikov equations, in Symmetries and Integrability of Difference Equations (Estérel, PQ, 1994), *CRM Proc. Lecture Notes*, Vol. 9, Amer. Math. Soc., Providence, RI, 1996, 155–161, hep-th/9406078.

[33] Hitchin N.J., Flat connections and geometric quantization, *Comm. Math. Phys.* **131** (1990), 347–380.

[34] Hitchin N.J., Stable bundles and integrable systems, *Duke Math. J.* **54** (1987), 91–114.

[35] Hori K., Global aspects of gauged Wess–Zumino–Witten models, *Comm. Math. Phys.* **182** (1996), 1–32, hep-th/9411134.

[36] Ivanov D.A., Knizhnik–Zamolodchikov–Bernard equations on Riemann surfaces, *Internat. J. Modern Phys. A* **10** (1995), 2507–2536, hep-th/9410091.

[37] Ivanova T.A., Lechtenfeld O., Popov A.D., Rahn T., Instantons and Yang–Mills flows on coset spaces, *Lett. Math. Phys.* **89** (2009), 231–247, arXiv:0904.0654.

[38] Kac V.G., Infinite-dimensional Lie algebras, 3rd ed., Cambridge University Press, Cambridge, 1990.

[39] Kapustin A., Witten E., Electric-magnetic duality and the geometric Langlands program, *Commun. Number Theory Phys.* **1** (2007), 1–236, hep-th/0604151.

[40] Knizhnik V.G., Zamolodchikov A.B., Current algebra and Wess–Zumino model in two dimensions, *Nuclear Phys. B* **247** (1984), 83–103.

[41] Korotkin D., Samtleben H., On the quantization of isomonodromic deformations on the torus, *Internat. J. Modern Phys. A* **12** (1997), 2013–2029, hep-th/9511087.

[42] Kuroki G., Takebe T., Twisted Wess–Zumino–Witten models on elliptic curves, *Comm. Math. Phys.* **190** (1997), 1–56, q-alg/9612033.

[43] Levin A.M., Olshanetsky M.A., Double coset construction of moduli space of holomorphic bundles and Hitchin systems, *Comm. Math. Phys.* **188** (1997), 449–466, alg-geom/9605005.

[44] Levin A.M., Olshanetsky M.A., Hierarchies of isomonodromic deformations and Hitchin systems, in Moscow Seminar in Mathematical Physics, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 191, Amer. Math. Soc., Providence, RI, 1999, 223–262.

[45] Levin A.M., Olshanetsky M.A., Smirnov A.V., Zotov A.V., Calogero–Moser systems for simple Lie groups and characteristic classes of bundles, *J. Geom. Phys.* **62** (2012), 1810–1850, arXiv:1007.4127.

[46] Levin A.M., Olshanetsky M.A., Smirnov A.V., Zotov A.V., Characteristic classes and Hitchin systems. General construction, *Comm. Math. Phys.* **316** (2012), 1–44, arXiv:1006.0702.

[47] Levin A.M., Olshanetsky M.A., Smirnov A.V., Zotov A.V., Characteristic classes of SL(N)-bundles and quantum dynamical elliptic R-matrices, *J. Phys. A: Math. Theor.*, to appear, arXiv:1208.5750.

[48] Levin A.M., Olshanetsky M.A., Zotov A.V., Hitchin systems – symplectic Hecke correspondence and two-dimensional version, *Comm. Math. Phys.* **236** (2003), 93–133, nlin.SI/0110045.

[49] Levin A.M., Olshanetsky M.A., Zotov A.V., Monopoles and modifications of bundles over elliptic curves, *SIGMA* **5** (2009), 065, 22 pages, arXiv:0811.3056.

[50] Levin A.M., Olshanetsky M.A., Zotov A.V., Painlevé VI, rigid tops and reflection equation, *Comm. Math. Phys.* **268** (2006), 67–103, math.QA/0508058.

[51] Levin A.M., Zotov A.V., On rational and elliptic forms of Painlevé VI equation, in Moscow Seminar on Mathematical Physics. II, *Amer. Math. Soc. Transl. Ser. 2*, Vol. 221, Amer. Math. Soc., Providence, RI, 2007, 173–183.

[52] Looijenga E., Root systems and elliptic curves, *Invent. Math.* **38** (1976), 17–32.

[53] Mironov A., Morozov A., Runov B., Zenkevich Y., Zotov A., Spectral duality between Heisenberg chain and Gaudin model, *Lett. Math. Phys.*, to appear, arXiv:1206.6349.

[54] Mironov A., Morosov A., Shakirov Sh., Towards a proof of AGT conjecture by methods of matrix models, *Internat. J. Modern Phys. A* **27** (2012), 1230001, 32 pages, arXiv:1011.5629.

[55] Mironov A., Morozov A., Zenkevich Y., Zotov A., Spectral duality in integrable systems from AGT conjecture, *JETP Lett.*, to appear, arXiv:1204.0913.

[56] Narasimhan M.S., Seshadri C.S., Stable and unitary vector bundles on a compact Riemann surface, *Ann. of Math. (2)* **82** (1965), 540–567.

[57] Nekrasov N., Holomorphic bundles and many-body systems, *Comm. Math. Phys.* **180** (1996), 587–603, hep-th/9503157.

[58] Nekrasov N., Pestun V., Seiberg–Witten geometry of four dimensional $N = 2$ quiver gauge theories, arXiv:1211.2240.

[59] Olshanetsky M.A., Three lectures on classical integrable systems and gauge field theories, *Phys. Part. Nuclei* **40** (2009), 93–114, arXiv:0802.3857.

[60] Olshanetsky M.A., Zotov A. V., Isomonodromic problems on elliptic curve, rigid tops and reflection equations, in Elliptic Integrable Systems, *Rokko Lectures in Mathematics*, Vol. 18, Kobe University, 2005, 149–172.

[61] Presley A., Segal G., Loop groups, Clarendon Press, Oxford, 1986.

[62] Reshetikhin N., The Knizhnik–Zamolodchikov system as a deformation of the isomonodromy problem, *Lett. Math. Phys.* **26** (1992), 167–177.

[63] Schweigert C., On moduli spaces of flat connections with non-simply connected structure group, *Nuclear Phys. B* **492** (1997), 743–755, hep-th/9611092.

[64] Simpson C.T., Harmonic bundles on noncompact curves, *J. Amer. Math. Soc.* **3** (1990), 713–770.

[65] Smirnov A.V., Integrable $\mathrm{sl}(N,\mathbb{C})$-tops as Calogero–Moser systems, *Theoret. and Math. Phys.* **158** (2009), 300–312, arXiv:0809.2187.

[66] Takasaki K., Gaudin model, KZ equation and an isomonodromic problem on the torus, *Lett. Math. Phys.* **44** (1998), 143–156, hep-th/9711058.

[67] Zabrodin A.V., Zotov A.V., Quantum Painlevé–Calogero correspondence, *J. Math. Phys.* **53** (2012), 073507, 19 pages, arXiv:1107.5672.

[68] Zabrodin A.V., Zotov A.V., Quantum Painlevé–Calogero correspondence for Painlevé VI, *J. Math. Phys.* **53** (2012), 073508, 19 pages, arXiv:1107.5672.

[69] Zotov A.V., $1+1$ Gaudin model, *SIGMA* **7** (2011), 067, 26 pages, arXiv:1012.1072.

[70] Zotov A.V., Classical integrable systems and their field-theoretical generalizations, *Phys. Part. Nuclei* **37** (2006), 759–842.

[71] Zotov A.V., Elliptic linear problem for the Calogero–Inozemtsev model and Painlevé VI equation, *Lett. Math. Phys.* **67** (2004), 153–165, hep-th/0310260.

[72] Zotov A.V., Chernyakov Yu.B., Integrable multiparticle systems obtained by means of the Inozemtsev limit, *Theoret. and Math. Phys.* **129** (2001), 1526–1542, hep-th/0102069.

[73] Zotov A.V., Levin A.M., An integrable system of interacting elliptic tops, *Theoret. and Math. Phys.* **146** (2006), 45–52.

[74] Zotov A.V., Levin A.M., Olshanetsky M.A., Chernyakov Yu.B., Quadratic algebras associated with elliptic curves, *Theoret. and Math. Phys.* **156** (2008), 1103–1122, arXiv:0710.1072.