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Orthogonal Basic Hypergeometric Laurent Polynomials*

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Abstract. The Askey–Wilson polynomials are orthogonal polynomials in $x = \cos\theta$, which are given as a terminating ${}_4\phi_3$ basic hypergeometric series. The non-symmetric Askey–Wilson polynomials are Laurent polynomials in $z = e^{i\theta}$, which are given as a sum of two terminating ${}_4\phi_3$'s. They satisfy a biorthogonality relation. In this paper new orthogonality relations for single ${}_4\phi_3$'s which are Laurent polynomials in z are given, which imply the non-symmetric Askey–Wilson biorthogonality. These results include discrete orthogonality relations. They can be considered as a classical analytic study of the results for non-symmetric Askey–Wilson polynomials which were previously obtained by affine Hecke algebra techniques.

Key words: Askey–Wilson polynomials; orthogonality

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1 Introduction

The Askey–Wilson polynomials $p_n(x; \mathbf{t}\,|\,q)$ are polynomials in $x = \cos\theta$, and depend upon parameters q, t_1, t_2, t_3, and t_4. They may be defined by the terminating basic hypergeometric series [1], [8, § 15.2]

$$p_n(x; \mathbf{t}\,|\,q) = t_1^{-n}(t_1t_2, t_1t_3, t_1t_4; q)_n\, {}_4\phi_3\left(\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1e^{i\theta}, t_1e^{-i\theta} \\ t_1t_2, t_1t_3, t_1t_4 \end{matrix}\,\middle|\, q, q\right). \tag{1.1}$$

They are orthogonal polynomials in x, and have a known orthogonality relation (see (2.1)).

Clearly the Askey–Wilson polynomials are also Laurent polynomials in $z = e^{i\theta}$, since $2x = z + 1/z$. One may ask if there is a natural basis for Laurent polynomials in z which also satisfy orthogonality relations. The non-symmetric Askey–Wilson polynomials [16, 17] are Laurent polynomials, which are biorthogonal with respect to a modified Askey–Wilson weight, are one answer to this question. They may be explicitly given as a sum of two terminating basic hypergeometric series.

The purpose of this paper is to use classical analytic methods to study the orthogonality relations for single ${}_4\phi_3$'s which imply the non-symmetric Askey–Wilson polynomial biorthogonality. The main orthogonality results are Theorems 3.1, 4.1, 4.4, and 4.5. We show that special function techniques, explicit summations and self-adjoint operators, may be applied to derive

*This paper is a contribution to the Special Issue "Superintegrability, Exact Solvability, and Special Functions". The full collection is available at http://www.emis.de/journals/SIGMA/SESSF2012.html these orthogonalities. We explicitly find three and four term relations for Laurent polynomials in Section 4. As a corollary of our computations we rederive the L^2-norm of the non-symmetric Askey–Wilson polynomials in Section 6, and reprove the creation type formulas for the Laurent polynomials in Section 5. Asymptotics are given in Theorem 9.1, and Racah orthogonality in Theorem 10.2. This work may be considered as a classical study of the one variable case in Cherednik [6], Macdonald [16], carefully studied by Noumi and Stokman [17], who considered many of these results from the point of view of affine Hecke algebras.

There are different theories of orthogonal polynomials, biorthogonal polynomials and rational functions which originate from different types of continued fractions. The J-fractions lead to orthogonal polynomials on the line while the PC fractions lead to orthogonal polynomials on the unit circle [8, 11, 14, 15]. The theory of T-fractions leads to orthogonal Laurent polynomials and is explained in the books [14, 15], and the excellent survey article [5]. Ismail and Masson [9] introduced the more general R_{I} and R_{II} fractions which naturally lead to biorthogonal orthogonal rational functions. A. Zhedanov [21] pointed out the connection of the latter continued fractions and the generalized eigenvalue problem [19]. For applications of the R_{I} and R_{II} fractions to integrable systems, see [18]. There is a more abstract approach to biorthogonality presented in Brezinski's monograph [4] with many applications to numerical analysis and computational mathematics.

2 Notation

Let V_n be the $(2n+1)$-dimensional real vector space spanned by the Laurent polynomials $\{z^{-n}, z^{1-n}, \dots, z^{n-1}, z^n\}$. Most of this section is devoted to notation for specific spanning sets in V_n. These sets have orthogonality relations given in later sections, we define the appropriate measures here.

We assume throughout the paper that $x = \cos\theta$ and $z = e^{i\theta}$, and $\mathbf{t}$ stands for the vector (t_1, t_2, t_3, t_4). We shall always assume that $0 < q < 1$ and each t_i is real with $|t_i| < 1$.

The weight function for the Askey–Wilson polynomials is given by

$$w_{\mathrm{aw}}(x; \mathbf{t} \,|\, q) = \frac{\left(q, e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{\prod_{j=1}^{4} \left(t_j e^{i\theta}, t_j e^{-i\theta}; q\right)_\infty}.$$

The Askey–Wilson orthogonality [8, (15.2.4)] is

$$\begin{aligned} &\frac{1}{2\pi} \int_{-1}^{1} p_m(x; \mathbf{t} \,|\, q) p_n(x; \mathbf{t} \,|\, q) w_{\mathrm{aw}}(x; \mathbf{t} \,|\, q) \frac{dx}{\sqrt{1-x^2}} \\ &\qquad = \prod_{1 \le j < k \le 4} (t_j t_k; q)_n \frac{\left(q, t_1 t_2 t_3 t_4 q^{n-1}; q\right)_n}{(t_1 t_2 t_3 t_4; q)_{2n}} \mu(\mathbf{t} \,|\, q) \delta_{m,n}, \end{aligned} \tag{2.1}$$

while the Askey–Wilson integral evaluation is [3, 8]

$$\frac{1}{2\pi} \int_0^\pi \frac{\left(q, e^{2i\theta}, e^{-2i\theta}; q\right)_\infty}{\prod_{j=1}^{4} \left(t_j e^{i\theta}, t_j e^{-i\theta}; q\right)_\infty} d\theta = \frac{(t_1 t_2 t_3 t_4; q)_\infty}{\prod_{1 \le j < k \le 4} (t_j t_k; q)_\infty} = \mu(\mathbf{t} \,|\, q).$$

The orthogonality relations for Laurent polynomials in V_n uses a small variation of the Askey–Wilson weight, which is given by

$$w_{\mathrm{cher}}(z; \mathbf{t} \,|\, q) = w_{\mathrm{aw}}(x; \mathbf{t} \,|\, q) \frac{(1 - t_1/z)(1 - t_2/z)}{1 - z^{-2}} = \frac{\left(q, z^2, q z^{-2}; q\right)_\infty}{\prod_{j=1}^{4} (t_j z; q)_\infty \prod_{k=3}^{4} (t_k/z, q t_{k-2}/z; q)_\infty}.$$

Even though this weight is not positive on the unit circle, and does not lead to a positive definite quadratic form, we will define a bilinear form using $w_{\rm cher}(z;\mathbf{t}\,|\,q)$, and find orthogonal bases for V_n in Sections 3 and 4.

The orthogonality relations will involve Laurent polynomials in z which may be defined in terms of ${}_4\phi_3$ functions. We shall need four of these functions: R_n, S_n, T_n, and U_n.

The first polynomial $R_n(z,\mathbf{t}\,|\,q)$ is an unscaled version of the Askey–Wilson polynomials

$$R_n(z;\mathbf{t}\,|\,q) = {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1z, t_1/z \\ t_1t_2, t_1t_3, t_1t_4 \end{matrix}\right| q, q\right). \tag{2.2}$$

Put $S_0(z)=0$, and for $n\geq 1$,

$$S_n(z;\mathbf{t}\,|\,q) = z(1-t_3/z)(1-t_4/z)\,{}_4\phi_3\left(\left.\begin{matrix} q^{1-n}, t_1t_2t_3t_4q^{n}, t_1z, qt_1/z \\ qt_1t_2, qt_1t_3, qt_1t_4 \end{matrix}\right| q, q\right), \tag{2.3}$$

$$T_n(z;\mathbf{t}\,|\,q) = {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, t_1t_2t_3t_4q^{n-1}, t_1z, qt_1/z \\ qt_1t_2, t_1t_3, t_1t_4 \end{matrix}\right| q, q\right), \tag{2.4}$$

and $U_0(z)=0$, and for $n\geq 1$,

$$U_n(z;\mathbf{t}\,|\,q) = \frac{1}{z}(1-t_1z)(1-t_2z)R_{n-1}(z;qt_1,qt_2,t_3,t_4\,|\,q). \tag{2.5}$$

The above representation (2.5) expresses U_n as a multiple of a scaled R_n. The S_n and T_n in (2.3) and (2.4) can also written in terms of scaled R_n's as follows

$$S_n(z;\mathbf{t}\,|\,q) = z(1-t_3/z)(1-t_4/z)T_{n-1}(z;t_1,t_2,qt_3,qt_4),$$
$$T_n(z;\mathbf{t}\,|\,q) = R_n\big(q^{-1/2}z;q^{1/2}t_1,q^{1/2}t_2,q^{-1/2}t_3,q^{-1/2}t_4\big).$$

In other words

$$S_n(z;\mathbf{t}\,|\,q) = z(1-t_3/z)(1-t_4/z)R_{n-1}\big(q^{-1/2}z;q^{1/2}t_1,q^{1/2}t_2,q^{1/2}t_3,q^{1/2}t_4\big).$$

Clearly

$$\{R_0,\dots,R_n,S_1,\dots,S_n,T_0,\dots,T_n,U_1,\dots,U_n\}\subset V_n,$$

but this set could not form a basis for V_n for $n\geq 0$. There are very specific dependencies amongst these Laurent polynomials which follow from contiguous relations. It follows from Proposition 2.1 that for $n\geq 1$ the subspace $W_n=\operatorname{span}\{R_n,S_n,T_n,U_n\}$ of V_n is 2-dimensional. Any two of these four Laurent polynomials are linearly independent.

We need some contiguous relations for balanced ${}_4\phi_3$'s. Let

$$\phi(a,b,c,d;e,f,g) = {}_4\phi_3\left(\left.\begin{matrix} a,b,c,d \\ e,f,g \end{matrix}\right| q, q\right)$$

denote a ${}_4\phi_3$ which is terminating and balanced. By ϕ_+ we mean change $a,b,\dots,g$ by aq,bq, $\dots,gq$, respectively, so ϕ_+ is no longer balanced. When a parameter α is changed to $\alpha q^{\pm1}$ we write $\phi(\alpha\pm)$. Wilson [20] proved contiguous relations which imply

$$\phi(a+,e+)-\phi = \frac{q(a-e)(1-b)(1-c)(1-d)}{(1-e)(1-eq)(1-f)(1-g)}\phi_+(e+), \tag{2.6}$$

$$\frac{a(1-f/a)(1-g/a)}{(1-f)(1-g)}\phi_+(a-) + \frac{(b-e)(1-c/e)}{(1-b)(1-c)}\phi(d+,e+) = \frac{(1-e)(1-bc/e)}{(1-b)(1-c)}\phi. \tag{2.7}$$

Proposition 2.1. *The following connection relations hold for $n \ge 1$,*

$$T_n(z;\mathbf{t}\,|\,q) - R_n(z;\mathbf{t}\,|\,q) = \frac{qt_1\big(1-q^{-n}\big)\big(1-t_1t_2t_3t_4q^{n-1}\big)}{(1-t_1t_2)(1-qt_1t_2)(1-t_1t_3)(1-t_1t_4)}U_n(z;\mathbf{t}\,|\,q),$$

$$(1-t_1t_3)(1-t_1t_4)R_n(z;\mathbf{t}\,|\,q) - t_1S_n(z;\mathbf{t}\,|\,q) = \frac{t_1\big(1-t_1t_2q^n\big)\big(1-t_3t_4q^{n-1}\big)}{q^{n-1}(1-t_1t_2)(1-qt_1t_2)}U_n(z;\mathbf{t}\,|\,q),$$

$$\frac{t_1}{(1-t_1t_3)(1-t_1t_4)}S_n(z;\mathbf{t}\,|\,q) - \frac{\big(1-q^nt_1t_2\big)\big(1-q^{n-1}t_3t_4\big)}{\big(1-q^n\big)\big(1-t_1t_2t_3t_4q^{n-1}\big)}T_n(z;\mathbf{t}\,|\,q)$$
$$+ \frac{q^n(1-t_1t_2)(1-t_3t_4/q)}{\big(1-q^n\big)\big(1-t_1t_2t_3t_4q^{n-1}\big)}R_n(z;\mathbf{t}\,|\,q) = 0.$$

Proof. The first statement follows from (2.6) with $a=t_1/z$, $b=q^{-n}$, $c=t_1t_2t_3t_4q^{n-1}$, $d=t_1z$, $e=t_1t_2$, $f=t_1t_3$, and $g=t_1t_4$. The third statement follows from from (2.7) with $a=t_1z$, $b=q^{-n}$, $c=t_1t_2t_3t_4q^{n-1}$, $d=t_1/z$, $e=t_1t_2$, $f=t_1t_3$, and $g=t_1t_4$. The second statement follows from the other two. ∎

We will also use the Sears transformation [7, (III.15)],

$$ {}_4\phi_3\left(\left.\begin{matrix} q^{-n}, A, B, C \\ D, E, F \end{matrix}\,\right|\, q, q\right) = \frac{(E/A, F/A;q)_n}{(E,F;q)_n}A^n\,{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, A, D/B, D/C \\ D, Aq^{1-n}/E, Aq^{1-n}/F \end{matrix}\,\right|\, q, q\right), \tag{2.8}$$

where $DEF = ABCq^{1-n}$. The Sears transformation (2.8) applied to the R_n and T_n functions give

$$R_n(z;t_1,t_2,t_3,t_4\,|\,q) = \frac{t_1^n(t_2t_3,t_3t_4;q)_n}{t_3^n(t_1t_2,t_1t_4;q)_n}R_n(z;t_3,t_2,t_1,t_4\,|\,q) \tag{2.9}$$

and

$$T_n(z;t_1,t_2,t_3,t_4\,|\,q) = \frac{t_1^n(t_2t_3,t_2t_4;q)_n}{t_2^n(t_1t_3,t_1t_4;q)_n}T_n(z;t_2,t_1,t_3,t_4\,|\,q). \tag{2.10}$$

One may also easily see from the definition of R_n and S_n that

$$R_n(1/z;1/\mathbf{t}\,|\,1/q) = R_n(z;\mathbf{t}\,|\,q), \qquad S_n(1/z;1/\mathbf{t}\,|\,1/q) = S_n(z;\mathbf{t}\,|\,q)/t_3t_4. \tag{2.11}$$

3 Orthogonality relations

The main result of this section is Theorem 3.1.

For Laurent polynomials $f(z)$ and $g(z)$ define a bilinear form by

$$\langle f,g\rangle_{\text{cher}} = \frac{1}{2\pi i}\oint_{|z|=1} f(z)g(z)w_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}.$$

Theorem 3.1. *The set $\{R_0,\dots,R_n,U_1,\dots,U_n\}$ is an orthogonal basis for V_n with respect to the symmetric bilinear form $\langle\,,\,\rangle_{\text{cher}}$. Moreover, $\langle R_m,R_m\rangle_{\text{cher}}$ and $\langle U_m,U_m\rangle_{\text{cher}}$ are non-zero.*

The proof of Theorem 3.1 will be accomplished in three stages. We get $R_n - R_m$ orthogonality for "free". We establish in Propositions 3.2 and 3.4 orthogonalities between R_n and T_m, which changes into $R_n - U_m$ and $U_n - U_m$ orthogonality.

First we establish $R_n - R_m$ orthogonality from Askey–Wilson orthogonality. It is a restatement of (2.1), which was given in [17, Lemma 6.4].

Proposition 3.2. *The orthogonality relation for the R_n functions is*

$$\langle R_m, R_n\rangle_{\text{cher}} = \frac{(1-t_1t_2)t_1^{2n}\big(q, t_2t_3, t_2t_4, t_3t_4, t_1t_2t_3t_4q^{n-1}; q\big)_n}{(t_1t_2, t_1t_3, t_1t_4; q)_n(t_1t_2t_3t_4; q)_{2n}}\mu(\mathbf{t}\,|\,q)\delta_{m,n}.$$

Proof. Rewrite the left-hand side of Proposition 3.2 as

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\pi}^{\pi} R_m\big(e^{i\theta};\mathbf{t}\,|\,q\big)R_n\big(e^{i\theta};\mathbf{t}\,|\,q\big)w_{\text{aw}}(\cos\theta;\mathbf{t}\,|\,q)\frac{e^{i\theta}-(t_1+t_2)+t_1t_2e^{-i\theta}}{2\sin\theta}d\theta\\ &\quad= \frac{1-t_1t_2}{2\pi}\int_0^{\pi} R_m\big(e^{i\theta};\mathbf{t}\,|\,q\big)R_n\big(e^{i\theta};\mathbf{t}\,|\,q\big)w_{\text{aw}}(\cos\theta;\mathbf{t}\,|\,q)d\theta,\end{aligned}$$

since $R_n\big(e^{i\theta};\mathbf{t}\,|\,q\big)$ and w_{aw} are even functions of θ. ■

To establish the $U_n - R_m$ orthogonality we first give an integral evaluation for "gluing" a specific Laurent polynomial onto the measure.

Lemma 3.3. *For non-negative integers k and j we have*

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{|z|=1}(t_1z, qt_1/z; q)_j(t_3z, t_3/z; q)_k w_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}\\ &\quad= \frac{\big(t_1t_2t_3t_4q^{j+k}; q\big)_\infty}{\big(t_1t_2q^{j+1}, t_1t_3q^{j+k}, t_1t_4q^j, t_2t_3q^k, t_2t_4, t_3t_4q^k; q\big)_\infty}.\end{aligned}$$

Proof. The integral is

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{\big(q, z^2, z^{-2}; q\big)_\infty}{\big(t_1q^jz, t_2z, t_3q^kz, t_4z, q^jt_1/z, t_2/z, q^kt_3/z, t_4/z; q\big)_\infty}\frac{\big(1-t_2/z\big)\big(1-q^jt_1/z\big)}{\big(1-z^{-2}\big)}\frac{dz}{z}.$$

This time the rational function in $z = e^{i\theta}$ in the integrand is

$$\frac{e^{i\theta}-\big(q^jt_1+t_2\big)+t_1t_2q^je^{-i\theta}}{2i\sin(\theta)}. \tag{3.1}$$

The remaining part of the integrand is symmetric in θ, so only the odd part of the numerator in (3.1) contributes. The Askey–Wilson integral evaluation gives the result

$$\begin{aligned}&= \frac{1}{2}\big(1-t_1t_2q^j\big)\frac{1}{2\pi}\int_{-\pi}^{\pi} w_{\text{aw}}\big(\cos\theta; q^jt_1, t_2, q^kt_3, t_4\,|\,q\big)d\theta\\ &= \frac{\big(t_1t_2t_3t_4q^{j+k}; q\big)_\infty}{\big(t_1t_2q^{j+1}, t_1t_3q^{j+k}, t_1t_4q^j, t_2t_3q^k, t_2t_4, t_3t_4q^k; q\big)_\infty}.\end{aligned}$$ ■

Proposition 3.4. *We have the orthogonality relation*

$$\langle R_m, T_n\rangle_{\text{cher}} = \langle R_m, R_n\rangle_{\text{cher}}\delta_{m,n}.$$

Proof. The proof is done in three stages: $m < n$, $m = n$, and $m > n$.

Let $I_{j,k}$ be the integral in Lemma 3.3. Using the explicit definition of $T_n(z;\mathbf{t}\,|\,q)$ we have

$$\begin{aligned}&\frac{1}{2\pi i}\oint_{|z|=1}(t_3z, t_3/z; q)_k T_n(z;\mathbf{t}\,|\,q)w_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z} = \sum_{j=0}^{n}\frac{\big(q^{-n}, t_1t_2t_3t_4q^{n-1}; q\big)_j}{(q, qt_1t_2, t_1t_3, t_1t_4; q)_j}q^jI_{j,k}\\ &\quad= \frac{(t_1t_2t_3t_4q^k; q)_\infty}{\big(q, qt_1t_2, q^kt_1t_3, t_1t_4, q^kt_2t_3, t_2t_4, q^kt_3t_4; q\big)_\infty}\,{}_3\phi_2\left(\left.\begin{matrix}q^{-n}, t_1t_2t_3t_4q^{n-1}, q^kt_1t_3\\ t_1t_3, t_1t_2t_3t_4q^k\end{matrix}\,\right|\, q, q\right).\end{aligned}$$

From the q-Pfaff–Saalschütz theorem [7, (II.12)], the sum of the above ${}_3\phi_2$ contains a factor $(q^{-k};q)_n$ which vanishes for $k<n$. Thus if $m<n$, then $\langle R_m, T_n\rangle_{\text{cher}}=0$.

When $k=n$ the the q-Pfaff–Saalschütz theorem [7, (II.12)] evaluates the above integral to

$$\frac{1}{2\pi i}\oint_{|z|=1}(t_3z,t_3/z;q)_nT_n(z;\mathbf{t}\,|\,q)\,w_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}$$
$$=\frac{\left(t_1t_2t_3t_4q^{2n};q\right)_\infty(-t_1t_3)^nq^{\binom{n}{2}}}{\left(q^{n+1},qt_1t_2,t_1t_3,t_1t_4,t_2t_3q^n,t_2t_4q^n,t_3t_4q^n;q\right)_\infty}.$$

Using (2.9) an explicit computation shows that Proposition 3.4 holds for $m=n$.

Next we consider the case $m>n$. In this case consider the integral

$$I=\frac{1}{2\pi i}\oint_{|z|=1}(t_1z,qt_1/z;q)_jR_m(z;\mathbf{t}\,|\,q)w_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}.$$

After using (2.9) for R_m, I is a constant multiple of

$$\sum_{k=0}^{m}\frac{\left(q^{-m},t_1t_2t_3t_4q^{m-1};q\right)_kq^k}{(q,t_1t_3,t_2t_3,t_3t_4;q)_k}I_{j,k}.$$

Applying Lemma 3.3 we see that the integral I is a constant multiple of

$${}_3\phi_2\left(\begin{matrix}q^{-m},t_1t_2t_3t_4q^{m-1},q^jt_1t_3\\ t_1t_3,t_1t_2t_3t_4q^j\end{matrix}\,\middle|\,q,q\right),$$

which again by the q-Pfaff–Saalschütz theorem [7, (II.12)] produces a factor $(q^{-j};q)_m$ and therefore vanishes for $j<m$. ∎

Finally we come to the T_n-T_m orthogonality, and finding the L^2-norms.

Proposition 3.5. *We have the orthogonality relation*

$$\langle T_m,T_n\rangle_{\text{cher}}=\frac{(1-t_1t_2)(1-t_3t_4/q)}{(1-t_1t_2q^n)(1-t_3t_4q^{n-1})}q^n\langle R_m,R_n\rangle_{\text{cher}}\delta_{m,n}.$$

Proof. We will use (2.4) and (2.10) to expand the T's, so consider the integral

$$\frac{1}{2\pi i}\oint_{|z|=1}(t_1z,qt_1/z;q)_j(t_2z,qt_2/z;q)_kw_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}$$
$$=\frac{1}{2\pi i}\oint_{|z|=1}(t_1z,t_1/z;q)_j(t_2z,t_2/z;q)_k\frac{\left(z-q^kt_2\right)\left(1-q^jt_1/z\right)}{z-1/z}w_{\text{aw}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}$$
$$=\frac{\left(1-t_1t_2q^{k+j}\right)}{2\pi}\int_0^\pi w_{\text{aw}}\left(\cos\theta;t_1q^j,t_2q^k,t_3,t_4\,|\,q\right)d\theta$$
$$=\frac{\left(t_1t_2t_3t_4q^{k+j};q\right)_\infty}{\left(q,t_1t_2q^{j+k+1},t_1t_3q^j,t_1t_4q^j,t_2t_3q^k,t_2t_4q^k,t_3t_4;q\right)_\infty}.$$

Using (2.4) to sum on j we find

$$\frac{1}{2\pi i}\oint_{|z|=1}T_n(z;\mathbf{t}\,|\,q)(t_2z,qt_2/z;q)_kw_{\text{cher}}(z;\mathbf{t}\,|\,q)\frac{dz}{z}$$
$$=\frac{\left(t_1t_2t_3t_4q^k;q\right)_\infty}{\left(t_1t_2q^{k+1},t_1t_3,t_1t_4,t_2t_3q^k,t_2t_4q^k,t_3t_4;q\right)_\infty}{}_3\phi_2\left(\begin{matrix}q^{-n},t_1t_2t_3t_4q^{n-1},t_1t_2q^{k+1}\\ qt_1t_2,t_1t_2t_3t_4q^k\end{matrix}\,\middle|\,q,q\right).$$

Again the q-Pfaff–Saalschütz theorem [7, (II.12)] evaluates the ${}_3\phi_2$ as

$$\frac{\left(q^{-k},q^{2-n}/t_3t_4;q\right)_n}{\left(qt_1t_2,q^{1-k-n}/t_1t_2t_3t_4;q\right)_n},$$

which vanishes when $k<n$. This establishes orthogonality for $m<n$, and the $k=n$ case establishes Proposition 3.5 for $m=n$ using (2.10). ■

To complete the proof of Theorem 3.1, we switch the orthogonality results for R_n and T_n to results for R_n and U_n using Proposition 2.1,

$$c_nU_n=T_n-R_n,\qquad c_n=\frac{qt_1\left(1-q^{-n}\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)}{(1-t_1t_2)(1-qt_1t_2)(1-t_1t_3)(1-t_1t_4)}.$$

The only remaining orthogonality relation we must check is

$$\langle U_n,R_n\rangle_{\text{cher}}=c_n^{-1}\langle T_n-R_n,R_n\rangle_{\text{cher}}=c_n^{-1}(\langle T_n,R_n\rangle_{\text{cher}}-\langle R_n,R_n\rangle_{\text{cher}})=0.$$

The value of $\langle U_n,U_n\rangle_{\text{cher}}$, which is non-zero, is found using Proposition 3.5,

$$\begin{aligned}\langle U_n,U_n\rangle_{\text{cher}}&=c_n^{-2}\langle T_n-R_n,T_n-R_n\rangle_{\text{cher}}=c_n^{-2}(\langle T_n,T_n\rangle_{\text{cher}}-\langle R_n,R_n\rangle_{\text{cher}})\\&=c_n^{-2}\left(\frac{(1-t_1t_2)(1-t_3t_4/q)}{\left(1-t_1t_2q^n\right)\left(1-t_3t_4q^{n-1}\right)}q^n-1\right)\langle R_n,R_n\rangle\\&=c_n^{-2}\frac{\left(1-q^n\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)}{\left(1-t_1t_2q^n\right)\left(1-t_3t_4q^{n-1}\right)}\langle R_n,R_n\rangle_{\text{cher}}.\end{aligned}$$

4 More orthogonal bases for V_n

In this section we find additional orthogonal bases for V_n, see Theorems 4.1, 4.4 and 4.5. We also explicitly give three and four term recurrence relations for some of these bases, which are the analogues of the recurrence relation for orthogonal polynomials.

The first result of this section is

Theorem 4.1. *The set $\{T_0,\dots,T_n,S_1,\dots,S_n\}$ is an orthogonal basis for V_n with respect to the symmetric bilinear from $\langle\ ,\ \rangle_{\text{cher}}$. Moreover, $\langle T_m,T_m\rangle_{\text{cher}}$ and $\langle S_m,S_m\rangle_{\text{cher}}$ are non-zero.*

Proof. Proposition 3.5 gives orthogonality of the T_n's. By Proposition 2.1 there are linear dependencies amongst $\{R_n,U_n,T_n\}$, and $\{R_n,U_n,S_n\}$. So $\langle S_m,S_m\rangle$, $\langle T_m,T_m\rangle$, and $\langle S_m,T_m\rangle$ may all be changed to the appropriate linear combinations of R_n's and T_m's. The results are given in the next proposition. ■

Proposition 4.2. *For all $n\geq 1$ and $m\geq 0$ we have the orthogonality relations*

$$\begin{aligned}&\langle S_n,T_m\rangle_{\text{cher}}=0,\\&t_1\langle S_n,R_m\rangle_{\text{cher}}=(1-t_1t_3)(1-t_1t_4)\langle R_n,R_m\rangle_{\text{cher}}.\end{aligned}$$

For all $n\geq 1$ and $m\geq 1$ we have the orthogonality relations

$$\langle S_n,S_m\rangle_{\text{cher}}=\frac{q^n(1-t_1t_3)^2(1-t_1t_4)^2(1-t_1t_2)(1-t_3t_4/q)}{\left(q^n-1\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)t_1^2}\langle R_n,R_m\rangle_{\text{cher}}.$$

Proof. This follows by direct calculation using Propositions 2.1, 3.4 and 3.5. ■

Corollary 4.3. *If $n \geq 1$, then for any $f \in V_{n-1}$ we have*

$$\langle R_n, f\rangle_{\text{cher}} = \langle S_n, f\rangle_{\text{cher}} = \langle T_n, f\rangle_{\text{cher}} = \langle U_n, f\rangle_{\text{cher}} = 0.$$

Using Corollary 4.3 one may explicitly give many orthogonal bases using the Askey–Wilson polynomials R_n. For example, for the basis

$$\{R_0, \dots, R_n, T_1, \dots, T_n\}$$

of V_n, the only pairwise non-orthogonalities involve T_m and R_m. One can replace these two polynomials by orthogonal linear combinations $R_m + c_m T_m$ and $R_m + d_m T_m$. The values of the constants c_m and d_m yielding orthogonality can be chosen using Propositions 3.4 and 3.5. The non-symmetric Askey–Wilson polynomial of Section 6 are one way of accomplishing this.

Another such choice corresponds to applying the Gram–Schmidt process to the ordered basis $\{1, z^{-1}, z^1, \dots, z^{-n}, z^n\}$ on V_n with $\langle\, ,\, \rangle_{\text{cher}}$.

Theorem 4.4. *An orthogonal basis for V_n is $\{X_0, X_{-1}, X_1, \dots, X_{-n}, X_n\}$, where*

$$X_{-n} = (1-t_1t_2)(1-t_1t_3)(1-t_1t_4)R_n - t_1\big(1-t_1t_2t_3t_4q^{n-1}\big)S_n, \qquad n>0,$$
$$X_n = (1-t_1t_2)(1-t_1t_3)(1-t_1t_4)R_n + t_1^2t_2\big(1-q^n\big)S_n, \qquad n \geq 0.$$

Proof. The choice of the coefficients for X_{-n} allows $X_{-n} \in V_{n-1} \oplus sp\{z^{-n}\}$, while the coefficients for X_n force $\langle X_n, X_{-n}\rangle_{\text{cher}} = 0$ via Proposition 4.2. An explicit computation shows that all of the L^2-norms are non-zero. ■

For the ordered basis $\{1, z^1, z^{-1}, \dots, z^n, z^{-n}\}$ on V_n we have an analogous result.

Theorem 4.5. *An orthogonal basis for V_n is $\{Y_0, Y_{-1}, Y_1, \dots, Y_{-n}, Y_n\}$, where*

$$Y_{-n} = q^{n-1}t_3t_4(1-t_1t_2)(1-t_1t_3)(1-t_1t_4)R_n - t_1\big(1-t_1t_2t_3t_4q^{n-1}\big)S_n, \qquad n>0,$$
$$Y_n = q^n(1-t_1t_2)(1-t_1t_3)(1-t_1t_4)R_n + t_1\big(1-q^n\big)S_n, \qquad n \geq 0.$$

Orthogonal bases from Theorems 4.4 and 4.5 have a three term recurrence relation.

Proposition 4.6. *For $n \geq 1$, there exist constants a_n, b_n and c_n such that*

$$zX_{-n}(z) = a_nX_{-n}(z) + b_nX_n(z) + c_nX_{n-1}(z).$$

Proof. Because X_{-n} has no z^n term, $zX_{-n}(z) \in V_n$ and has an expansion

$$zX_{-n}(z) = c_0X_0(z) = \sum_{i=1}^{n}\big(c_iX_i(z) + c_{-i}X_{-i}(z)\big).$$

By orthogonality, we can find the coefficients

$$c_k = \frac{\langle zX_{-n}, X_k\rangle_{\text{cher}}}{\langle X_k, X_k\rangle_{\text{cher}}} = \frac{\langle X_{-n}, zX_k\rangle_{\text{cher}}}{\langle X_k, X_k\rangle_{\text{cher}}},$$

as long as $\langle X_k, X_k\rangle_{\text{cher}} \neq 0$ (which is true here). The only terms to survive the orthogonality are X_n, X_{-n}, and X_{n-1}. The term X_{1-n} does not survive because $zX_{1-n} \in V_{n-1}$:

$$\langle zX_{-n}, X_{1-n}\rangle_{\text{cher}} = \langle X_{-n}, zX_{1-n}\rangle_{\text{cher}} = 0. \qquad ■$$

We record two propositions for these recurrences. The coefficients may be found by considering specific powers of z on each side, we do not give the details of the computation.

Proposition 4.7. *The constants in Proposition 4.6 are*

$$a_n = \frac{\big(-t_1-t_2+t_1t_2(t_3+t_4)q^{n-1}\big)\big(1-t_1t_2t_3t_4q^{2n-1}\big)}{\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)\big(1-t_1t_2t_3t_4q^{2n-2}\big)},$$
$$b_n = -\frac{\big(-t_1-t_2+t_1t_2(t_3+t_4)q^{n-1}\big)\big(1-t_1t_2t_3t_4q^{n-1}\big)\big(1-t_3t_4q^{n-1}\big)}{\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)\big(1-t_1t_2t_3t_4q^{2n-2}\big)},$$
$$c_n = \frac{t_1\big(1-t_3t_4q^{n-1}\big)\big(1-t_2t_4q^{n-1}\big)\big(1-t_2t_3q^{n-1}\big)}{\big(1-t_1t_2q^{n-1}\big)\big(1-t_1t_2t_3t_4q^{2n-2}\big)}.$$

Proposition 4.8. *For $n \geq 1$ we have*

$$\frac{1}{z}X_n = a_nX_{-n-1}+b_nX_n+c_nX_{-n},$$
$$a_n = \frac{\big(1-t_1t_2q^n\big)\big(1-t_1t_3q^n\big)\big(1-t_1t_4q^n\big)}{t_1\big(1-t_3t_4q^n\big)\big(1-t_1t_2t_3t_4q^{2n}\big)},$$
$$b_n = \frac{\big(-t_1-t_2+t_1t_2(t_3+t_4)q^n\big)\big(1-t_1t_2t_3t_4q^{2n-1}\big)}{\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)\big(1-t_1t_2t_3t_4q^{2n}\big)},$$
$$c_n = -\frac{\big(1-q^n\big)\big(1-t_1t_2q^n\big)\big(-t_1-t_2+t_1t_2(t_3+t_4)q^n\big)}{\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)\big(1-t_1t_2t_3t_4q^{2n}\big)}.$$

There are four term relations for zX_n and $\frac{1}{z}X_{-n}$.

Proposition 4.9. *For $n \geq 1$, there exist constants a_n, b_n, c_n and d_n such that*

$$zX_n(z) = a_nY_{-n-1}(z)+b_nX_n(z)+c_nY_{-n}(z)+d_nX_{n-1}(z).$$

Proof. By orthogonality we have

$$zX_n(z) = a'_nX_{-n-1}(z)+b'_nX_{n+1}(z)+c'_nX_{-n}(z)+d'_nX_n(z)+e'_nX_{n-1}(z).$$

The sixth term $X_{1-n}(z)$ does not appear because $zX_{1-n} \in V_{n-1}$:

$$\langle zX_n, X_{1-n}\rangle_{\text{cher}} = \langle X_n, zX_{1-n}\rangle_{\text{cher}} = 0.$$

Since $\operatorname{span}\{X_{-n-1}, X_{n+1}\} = \operatorname{span}\{Y_{-n-1}, Y_{n+1}\}$, the first two terms may be written as linear combination of Y_{-n-1} and Y_{n+1}, But $zX_{-n}(z)$ has no z^{-n-1} term, thus only Y_{-n-1} appears. Because $\operatorname{span}\{X_{-n}, X_n\} = \operatorname{span}\{Y_{-n}, X_n\}$ we can replace X_n by Y_{-n}. ■

Proposition 4.10. *The constants in Proposition 4.9 are*

$$a_n = \frac{\big(1-t_1t_2q^n\big)\big(1-t_1t_3q^n\big)\big(1-t_1t_4q^n\big)\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)}{t_1\big(1-t_3t_4q^n\big)\big(1-t_1t_2t_3t_4q^{2n-1}\big)\big(1-t_1t_2t_3t_4q^{2n}\big)},$$
$$b_n = \frac{q^n\big(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\big)\big(-t_3-t_4+t_2t_3t_4q^n+t_1t_3t_4q^n\big)}{\big(1-t_1t_2t_3t_4q^{2n-1}\big)\big(1-t_1t_2t_3t_4q^{2n}\big)},$$
$$c_n = -\frac{\big(1-q^n\big)\big(1-t_1t_2q^n\big)\big(-t_1-t_2+t_1t_2(t_3+t_4)q^{n-1}\big)}{\big(1-t_1t_2t_3t_4q^{2n-2}\big)\big(1-t_1t_2t_3t_4q^{2n-1}\big)},$$
$$d_n = \frac{t_1\big(1-q^n\big)\big(1-t_1t_2q^n\big)\big(1-t_2t_3q^{n-1}\big)\big(1-t_2t_4q^{n-1}\big)\big(1-t_3t_4q^{n-1}\big)}{\big(1-t_1t_2q^{n-1}\big)\big(1-t_1t_2t_3t_4q^{2n-2}\big)\big(1-t_1t_2t_3t_4q^{2n-1}\big)}.$$

Proposition 4.11. *For $n \geq 1$ we have*

$$\begin{aligned}
&\frac{1}{z}X_{-n} = a_n X_{-n-1} + b_n X_{-n} + c_n Y_n + d_n Y_{n-1},\\
&a_n = \frac{\left(1-t_1t_2q^n\right)\left(1-t_1t_3q^n\right)\left(1-t_1t_4q^n\right)\left(1-t_3t_4q^{n-1}\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)}{t_1\left(1-t_3t_4q^n\right)\left(1-t_1t_2t_3t_4q^{2n-1}\right)\left(1-t_1t_2t_3t_4q^{2n}\right)},\\
&b_n = \frac{q^{n-1}\left(-t_3-t_4+(t_1+t_2)t_3t_4q^{n-1}\right)\left(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\right)}{\left(1-t_1t_2t_3t_4q^{2n-2}\right)\left(1-t_1t_2t_3t_4q^{2n-1}\right)},\\
&c_n = -\frac{\left(1-t_3t_4q^{n-1}\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)\left(-t_1-t_2+t_1t_2(t_3+t_4)q^n\right)}{\left(1-t_1t_2t_3t_4q^{2n-1}\right)\left(1-t_1t_2t_3t_4q^{2n}\right)},\\
&d_n = \frac{t_1\left(1-t_2t_3q^{n-1}\right)\left(1-t_2t_4q^{n-1}\right)\left(1-t_3t_4q^{n-1}\right)\left(-1-t_1t_2+t_1t_2t_3t_4q^{n-1}+t_1t_2q^n\right)}{\left(1-t_1t_2q^{n-1}\right)\left(1-t_1t_2t_3t_4q^{2n-2}\right)\left(1-t_1t_2t_3t_4q^{2n-1}\right)}.
\end{aligned}$$

The two bases of Theorems 4.4 and 4.5 can be related, using (2.11). To explicitly show the parameter dependence we write $X_n(z;\mathbf{t}\,|\,q)$ for X_n and $Y_n(z;\mathbf{t}\,|\,q)$ for Y_n.

Proposition 4.12. *If $n \geq 1$, we have*

$$\begin{aligned}
&X_{-n}(1/z;1/\mathbf{t}\,|\,1/q) = -Y_{-n}(z;\mathbf{t}\,|\,q)/\left(t_1^3t_2t_3^2t_4^2q^{n-1}\right),\\
&X_n(1/z;1/\mathbf{t}\,|\,1/q) = -Y_n(z;\mathbf{t}\,|\,q)/\left(t_1^3t_2t_3t_4q^n\right).
\end{aligned}$$

Thus three term relations can be explicitly given for zY_n and $\frac{1}{z}Y_{-n}$, and four term relations for zY_{-n} and $\frac{1}{z}Y_n$.

5 The operators A_0 and A_1

Motivated by operators previously given in Noumi–Stokman [17], in this section we define linear transformations A_0 and A_1 on V_n. We explicitly find their actions on the possible bases R_m, S_m, T_m, and U_m. We use the self-adjointness of these operators to give an alternative proof of Theorems 3.1, 4.1, 4.4, and 4.5. We describe the eigenvalue problem whose solutions are the non-symmetric Askey–Wilson polynomials. We explicitly give the creation operators of Noumi–Stokman [17] via the three and four term recurrence relations of Section 4. These results are in [17], but we include them for completeness.

For a Laurent polynomial $f \in V_n$, define

$$(A_0f)(z) = \frac{(1-t_3/z)(1-t_4/z)}{(1-q/z^2)}(f(q/z)-f(z)),$$

and

$$(A_1f)(z) = \frac{(1-t_1z)(1-t_2z)}{(1-z^2)}(f(1/z)-f(z)).$$

By a direct computation it is easy to see that each A_i is a linear transformation from V_n to V_n.

Proposition 5.1. *Each $A_i : V_n \to V_n$ is self-adjoint, that is for any $f, g \in V_n$, we have*

$$\langle A_0f, g\rangle_{\mathrm{cher}} = \langle f, A_0g\rangle_{\mathrm{cher}}, \qquad \langle A_1f, g\rangle_{\mathrm{cher}} = \langle f, A_1g\rangle_{\mathrm{cher}}.$$

Proof. It is clear that $\langle A_0 f, g\rangle_{\text{cher}}$ equals

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{\left(q,z^2,1/z^2;q\right)_\infty}{\prod_{j=1}^{4}(t_jz,t_j/z;q)_\infty}\frac{(1-t_1/z)(1-t_2/z)}{1-z^{-2}}$$
$$\times\frac{(1-t_3/z)(1-t_4/z)}{\left(1-q/z^2\right)}\left[f(q/z)g(z)-f(z)g(z)\right]\frac{dz}{z}$$
$$=\frac{1}{2\pi i}\oint_{|z|=1}\frac{\left(q,z^2,q^2/z^2;q\right)_\infty}{\prod_{j=1}^{4}(t_jz,qt_j/z;q)_\infty}\left[f(q/z)g(z)-f(z)g(z)\right]\frac{dz}{z}$$

This a difference of two integrals. In the first we let $z\to q/z$. A calculation shows that the first integral becomes

$$\frac{1}{2\pi i}\oint_{|z|=q}\frac{\left(q,z^2,q^2/z^2;q\right)_\infty}{\prod_{j=1}^{4}(t_jz,qt_j/z;q)_\infty}f(z)g(q/z)\frac{dz}{z}.$$

We now deform the above contour to the circular contour $|z|=1$ because the integrand has no poles between the contours since each $|t_i|<1$. The proof of $\langle A_1 f, g\rangle_{\text{cher}}=\langle f, A_1 g\rangle_{\text{cher}}$ is similar and will be omitted. ∎

Note that in the proof of Proposition 5.1 we only used the fact that f and g are analytic in $q\le|z|\le 1/q$.

Proposition 5.2. *The action of the operators A_0 and A_1 on the functions R_n, T_n, S_n, and U_n is given by*

$$A_0(R_n)=\frac{t_1\left(q^{-n}-1\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)}{(1-t_1t_2)(1-t_1t_3)(1-t_1t_4)}S_n=\alpha_nS_n,\qquad A_1(R_n)=0,$$
$$A_0(T_n)=0,\qquad A_1(T_n)=\frac{qt_1\left(q^{-n}-1\right)\left(1-t_1t_2t_3t_4q^{n-1}\right)}{(1-qt_1t_2)(1-t_1t_3)(1-t_1t_4)}U_n=\beta_nU_n,$$
$$A_0(S_n)=(-1+t_3t_4/q)S_n,\qquad A_1(S_n)=\frac{q^{1-n}\left(1-t_1t_2q^n\right)\left(1-t_3t_4q^{n-1}\right)}{(1-qt_1t_2)}U_n,$$
$$A_0(U_n)=q^{-1}(1-qt_1t_2)S_n,\qquad A_1(U_n)=(-1+t_1t_2)U_n,$$

respectively.

Proof. These follow by direct term by term application of A_0 and A_1 to the defining series. ∎

These relations give another proof of the orthogonality relations in Theorems 3.1 and 4.1, for example

$$0=\langle A_0(T_n),R_m\rangle_{\text{cher}}=\langle T_n,A_0(R_m)\rangle_{\text{cher}}=\alpha_m\langle T_n,S_m\rangle_{\text{cher}},$$
$$0=\langle A_1(R_n),U_m\rangle_{\text{cher}}=\langle R_n,A_1(U_m)\rangle_{\text{cher}}=\beta_m\langle R_n,U_m\rangle_{\text{cher}}.$$

From Proposition 5.2 the operators A_0 and A_1 act on the 2-dimensional space W_n spanned by $\{R_n,S_n,U_n,T_n\}$. An elementary computation yields the next two theorems.

Theorem 5.3. *If $n\ge1$, the eigenvalues of $(A_1-t_1t_2I)\left(A_0-q^{-1}t_3t_4I\right)$ on W_n are q^{-n} and $t_1t_2t_3t_4q^{n-1}$, with corresponding eigenvectors $X_{-n}(z;\mathbf{t}\,|\,q)$ and $Y_n(z;\mathbf{t}\,|\,q)$.*

Theorem 5.4. *If $n \geq 1$, the eigenvalues of $\big(A_0 - q^{-1}t_3t_4I\big)(A_1 - t_1t_2I)$ on W_n are q^{-n} and $t_1t_2t_3t_4q^{n-1}$, with corresponding eigenvectors $Y_{-n}(z;\mathbf{t}\,|\,q)$ and $X_n(z;\mathbf{t}\,|\,q)$.*

These two theorems give alternative proofs of Theorems 4.4 and 4.5. For example

$$\begin{aligned} q^{-n}\langle X_{-n}, X_n\rangle_{\mathrm{cher}} &= \langle (A_1 - t_1t_2I)\big(A_0 - q^{-1}t_3t_4I\big)X_{-n}, X_n\rangle_{\mathrm{cher}} \\ &= \langle X_{-n}, \big(A_0 - q^{-1}t_3t_4I\big)(A_1 - t_1t_2I)X_n\rangle_{\mathrm{cher}} \\ &= t_1t_2t_3t_4q^{n-1}\langle X_{-n}, X_n\rangle_{\mathrm{cher}} \end{aligned}$$

implying that $0 = \langle X_{-n}, X_n\rangle_{\mathrm{cher}}$.

Our next goal is to define ladder maps $\mathbb{S}_0$ and $\mathbb{S}_1$ such that $\mathbb{S}_1$ maps Y_n to X_{-n-1}, and $\mathbb{S}_0$ maps X_{-n} to Y_n. If such maps are found, then Y_n is an n^{th} iterate of their composition $\mathbb{S}_0\mathbb{S}_1$ applied to Y_0, see Theorems 5.9 and 5.12.

Maps on W_n that interchange the bases elements $\{X_{-n}, Y_n\}$ and $\{X_n, Y_{-n}\}$ can be defined using the commutators of A_i with

$$Y = (A_1 - t_1t_2I)\big(A_0 - q^{-1}t_3t_4I\big) \qquad \text{and} \qquad \mathbb{Y} = \big(A_0 - q^{-1}t_3t_4I\big)(A_1 - t_1t_2I).$$

From Proposition 5.2 the actions of Y, $\mathbb{Y}$, A_0 and A_1 are explicitly known on the 2-dimensional space $W_n = \operatorname{span}\{R_n, S_n, U_n, T_n\}$. An explicit computation yields the next two propositions.

Proposition 5.5. *For $n \geq 1$ we have*

$$\begin{aligned} &[Y, A_1](X_{-n}) = t_1t_2q^{-n}\big(1 - t_3t_4q^{n-1}\big)\big(1 - t_1t_2t_3t_4q^{n-1}\big)Y_n, \\ &[\mathbb{Y}, A_0](Y_{-n}) = -q^{-1}t_3t_4\big(1 - t_3t_4q^{n-1}\big)\big(1 - t_1t_2t_3t_4q^{n-1}\big)X_n, \end{aligned}$$

and if $n \geq 0$,

$$\begin{aligned} &[Y, A_1](Y_n) = q^{-n}\big(1 - q^n\big)\big(1 - t_1t_2q^n\big)X_{-n}, \\ &[\mathbb{Y}, A_0](X_n) = -q^{-2n}\big(1 - q^n\big)\big(1 - t_1t_2q^n\big)Y_{-n}. \end{aligned}$$

Proposition 5.6. *For $n \geq 1$ we have*

$$\begin{aligned} [Y, A_1](R_n) &= \frac{t_1q^{1-2n}\big(1 - q^n\big)\big(1 - t_3t_4q^{n-1}\big)\big(1 - t_1t_2q^n\big)\big(1 - t_1t_2t_3t_4q^{n-1}\big)}{(1 - t_1t_2)(1 - qt_1t_2)/(1 - t_1t_3)/(1 - t_1t_4)}U_n, \\ [Y, A_1](U_n) &= q^{-1}t_2(1 - t_1t_4)(1 - t_1t_2)(1 - qt_1t_2)(1 - t_1t_3)R_n, \\ [Y, A_1](S_n) &= -q^{-n}t_2\big(1 - q^nt_1t_2\big)(1 - t_1t_3)(1 - t_1t_4)\big(1 - t_3t_4q^{n-1}\big)R_n \\ &\quad + q^{1-2n}\frac{\big(1 - q^n\big)\big(1 - t_1t_2q^n\big)\big(1 - t_3t_4q^{n-1}\big)\big(1 - t_1t_2t_3t_4q^{n-1}\big)}{(1 - t_1t_2)(1 - qt_1t_2)}U_n, \\ [Y, A_1](T_n) &= -q^{-n}t_1t_2\big(1 - t_1t_2t_3t_4q^{n-1}\big)R_n \\ &\quad + q^{1-2n}t_1\frac{\big(1 - q^n\big)\big(1 - t_1t_2q^n\big)\big(1 - t_3t_4q^{n-1}\big)\big(1 - t_1t_2t_3t_4q^{n-1}\big)}{(1 - t_1t_2)(1 - qt_1t_2)(1 - t_1t_3)(1 - t_1t_4)}U_n. \end{aligned}$$

We see from Proposition 5.5 that the commutator $\mathbb{S}_0 = [Y, A_1]$ does map X_{-n} to Y_n. To define an appropriate $\mathbb{S}_1$, we need another pair of operators.

We define operators $\hat{T}_1, \hat{T}_2 : V_n \to V_{n+1}$ by

$$(\hat{T}_1f)(z) = \frac{(A_1f)(z) + f(z)}{z}, \qquad (\hat{T}_2f)(z) = z((A_0f)(z) + f(z)).$$

Proposition 5.7. *The action of $\hat{T}_1$ on Y_n and X_{-n} is given by*

$$\hat{T}_1(Y_n) = \frac{(1-q^nt_1t_2)(1-q^nt_1t_3)(1-q^nt_1t_4)}{t_1(1-q^nt_3t_4)(1-q^{2n}t_1t_2t_3t_4)}X_{-n-1} + \frac{(t_1+t_2-q^nt_1t_2(t_3+t_4))}{(1-q^{2n}t_1t_2t_3t_4)}Y_n,$$

if $n \geq 0$ and

$$\begin{aligned}\hat{T}_1(X_{-n}) &= q^{n-1}\frac{t_1t_2(-t_4+t_3(-1+q^{n-1}(t_1+t_2)t_4))}{(-1+q^{2(n-1)}t_1t_2t_3t_4)}X_{-n} \\ &\quad + \frac{t_1^2t_2(1-q^{n-1}t_2t_3)(1-q^{n-1}t_2t_4)(1-q^{n-1}t_3t_4)}{(1-q^{n-1}t_1t_2)(-1+q^{2(n-1)}t_1t_2t_3t_4)}Y_{n-1},\end{aligned}$$

if $n \geq 1$, respectively.

Proof (sketch). Note that $\hat{T}_1 = \frac{1}{z}(A_1 + I)$. The action of A_1 on any element of W_n is given by Proposition 5.2. The action of division by z is given by Propositions 4.8, 4.11, and their Y versions. Thus $\hat{T}_1(Y_n)$ and $\hat{T}_1(X_{-n})$ may be given explicitly. ∎

From Theorem 5.3 we know that Y_n and X_{-n} are eigenfunctions of Y, and can obtain Proposition 5.8 from Proposition 5.7.

Proposition 5.8. *For $n \geq 0$ we have*

$$\begin{aligned}[Y, \hat{T}_1](Y_n) &= -\frac{(1-t_1t_2q^n)(1-t_1t_3q^n)(1-t_1t_4q^n)}{t_1q^{n+1}(1-t_3t_4q^n)}X_{-n-1}, \\ [Y, \hat{T}_1](X_{-n-1}) &= -t_1^2t_2\frac{(1-t_2t_3q^n)(1-t_2t_4q^n)(1-t_3t_4q^n)}{q^{n+1}(1-t_1t_2q^n)}Y_n.\end{aligned}$$

Finally we define $\mathbb{S}_1 = [Y, \hat{T}_1]$, which maps Y_n to X_{-n-1}. The next theorem is Proposition 9.3 in [17]. Recall that $Y_0 = (1-t_1t_2)(1-t_1t_3)(1-t_1t_4)$ is a constant.

Theorem 5.9. *We have*

$$c_nY_n = (\mathbb{S}_0\mathbb{S}_1)^n(Y_0), \qquad d_nX_{-n-1} = \mathbb{S}_1(\mathbb{S}_0\mathbb{S}_1)^n(Y_0).$$

where

$$\begin{aligned}c_n &= (-t_2)^nq^{-n(n+1)}(t_1t_2, t_1t_3, t_1t_4, t_1t_2t_3t_4; q)_n, \\ d_n &= -(-t_2)^nq^{-(n+1)^2}(t_1t_2, t_1t_3, t_1t_4; q)_{n+1}(t_1t_2t_3t_4; q)_n/t_1(1-t_3t_4q^n).\end{aligned}$$

Completely analogous results hold for

$$\hat{T}_2, \qquad \mathbb{S}_2 = [\mathbb{Y}, A_0], \qquad \mathbb{S}_3 = [\mathbb{Y}, \hat{T}_2],$$

which we state below. Note that Proposition 5.5 shows that $\mathbb{S}_2$ interchanges Y_{-n} and X_n.

Proposition 5.10. *For $n \geq 0$ we have*

$$\begin{aligned}\hat{T}_2(X_n) &= \frac{t_3+t_4-t_3t_4q^n(t_1+t_2)}{(1-q^{2n}t_1t_2t_3t_4)}X_n + \frac{(1-q^nt_1t_2)(1-q^nt_1t_3)(1-q^nt_1t_4)}{t_1q^n(1-q^nt_3t_4)(1-q^{2n}t_1t_2t_3t_4)}Y_{-n-1}, \\ \hat{T}_2(Y_{-n-1}) &= t_1t_3t_4q^n\frac{(1-q^nt_2t_3)(1-q^nt_2t_4)(1-q^nt_3t_4)}{(1-q^nt_1t_2)(1-q^{2n}t_1t_2t_3t_4)}X_{-n} \\ &\quad + q^nt_3t_4\frac{(t_2+t_1-t_1t_2q^n(t_3+t_4))}{(1-q^{2n}t_1t_2t_3t_4)}Y_{-n-1}.\end{aligned}$$

The map $\mathbb{S}_3 = [\mathbb{Y}, \hat{T}_2]$ interchanges X_n and Y_{-n-1}.

Proposition 5.11. *For $n \geq 0$ we have*

$$[\mathbb{Y}, \hat{T}_2](X_n) = \frac{(1 - t_1t_2q^n)(1 - t_1t_3q^n)(1 - t_1t_4q^n)}{t_1q^{2n+1}(1 - t_3t_4q^n)} Y_{-n-1},$$
$$[\mathbb{Y}, \hat{T}_2](Y_{-n-1}) = q^{-1}t_1t_3t_4 \frac{(1 - t_2t_3q^n)(1 - t_2t_4q^n)(1 - t_3t_4q^n)}{(1 - t_1t_2q^n)} X_n.$$

Recall that $X_0 = (1 - t_1t_2)(1 - t_1t_3)(1 - t_1t_4)$ is a constant.

Theorem 5.12. *We have*

$$e_nX_n = (\mathbb{S}_2\mathbb{S}_3)^n(X_0), \qquad f_nY_{-n-1} = \mathbb{S}_3(\mathbb{S}_2\mathbb{S}_3)^n(X_0),$$

where

$$e_n = (-t_3t_4/t_1)^n q^{-n(n+1)}(t_1t_2, t_1t_3, t_1t_4, t_1t_2t_3t_4; q)_n,$$
$$f_n = (-t_3t_4/t_1)^n q^{-n(n+1)}(t_1t_2, t_1t_3, t_1t_4; q)_{n+1}(t_1t_2t_3t_4; q)_n/t_1q^{2n+1}(1 - t_3t_4q^n).$$

Since any two elements from $\{X_n, X_{-n}, Y_n, Y_{-n}\}$ form a basis for W_n, any element of W_n has several such formulas.

6 Non-symmetric Askey–Wilson polynomials

The non-symmetric Askey–Wilson polynomials may be defined as the eigenfunctions given in Theorem 5.3, namely

$$\{Y_0, Y_1, \dots, Y_n, \dots\} \cup \{X_{-1}, X_{-2}, \dots, X_{-n}, \dots\}.$$

This set of Laurent polynomials also satisfy an biorthogonality relation which follows from Theorems 4.4 and 4.5.

Define a new bilinear form on V_n by

$$\langle f, g\rangle_{\mathrm{cher}'} = \frac{1}{2\pi i}\oint_{|z|=1} f(z)g(1/z)w_{\mathrm{cher}}(z; \mathbf{t} \,|\, q)\frac{dz}{z}.$$

Put

$$P_m(z, \mathbf{t} \,|\, q) = \begin{cases} Y_m(z, \mathbf{t} \,|\, q) & \text{if } m \geq 0, \\ X_m(z, \mathbf{t} \,|\, q) & \text{if } m < 0, \end{cases}$$

and $P'_m(z, \mathbf{t} \,|\, q)) = P_m(z, 1/\mathbf{t} \,|\, 1/q)$.

The biorthogonality relation with respect to this bilinear form [16], [17, § 6.7] is

Corollary 6.1. *For $m \neq n$, $\langle P_m, P'_n\rangle_{\mathrm{cher}'} = 0$.*

Proof. We use Proposition 4.12 to change the bilinear form $\langle\, , \rangle_{\mathrm{cher}'}$ to $\langle\, , \rangle_{\mathrm{cher}}$. For example, if $m, n < 0$, $m \neq n$, then

$$\langle X_m, X'_n\rangle_{\mathrm{cher}'} = -\langle X_m, Y_n\rangle_{\mathrm{cher}}/(t_1^3t_2t_3^2t_4^2q^{n-1}) = 0,$$

while if $m < 0$ and $n \geq 0$

$$\langle X_m, Y'_n\rangle_{\mathrm{cher}'} = -\langle X_m, X_n\rangle_{\mathrm{cher}}t_1^3t_2t_3t_4q^n = 0,$$

The only term which requires checking is for $n > 0$,

$$\langle Y_n, X'_{-n}\rangle_{\mathrm{cher}'} = -\langle Y_n, Y_{-n}\rangle_{\mathrm{cher}}/\big(t_1^3t_2t_3^2t_4^2q^{n-1}\big) = 0$$

and

$$\langle X_{-n}, Y'_n\rangle_{\mathrm{cher}'} = -\langle X_{-n}, X_n\rangle_{\mathrm{cher}} t_1^3t_2t_3^2t_4^2q^n = 0$$

by Theorems 4.4 and 4.5. ■

The L^2-norms may also be found.

Corollary 6.2. *We have for $n > 0$,*

$$\begin{aligned}\langle Y_n, Y'_n\rangle_{\mathrm{cher}'} &= -q^n\langle R_n, R_n\rangle_{\mathrm{cher}}/\big(t_1^3t_2t_3t_4\big)\\ &\qquad\times\frac{(1-t_1t_2)\big(1-t_1t_2q^n\big)(1-t_1t_3)^2(1-t_1t_4)^2\big(1-t_1t_2t_3t_4q^{2n-1}\big)}{\big(1-t_1t_2t_3t_4q^{n-1}\big)},\\ \langle X_{-n}, X'_{-n}\rangle_{\mathrm{cher}'} &= -t_1^3t_2t_3^2t_4^2q^{1-n}\langle R_n, R_n\rangle_{\mathrm{cher}}\\ &\qquad\times\frac{(1-t_1t_2)\big(1-t_3t_4q^{n-1}\big)(1-t_1t_3)^2(1-t_1t_4)^2\big(1-t_1t_2t_3t_4q^{2n-1}\big)}{\big(1-q^n\big)}.\end{aligned}$$

Proof. Using Proposition 4.12 we have

$$\langle Y_n, Y'_n\rangle_{\mathrm{cher}'} = -q^n\langle Y_n, X_n\rangle_{\mathrm{cher}}/\big(t_1^3t_2t_3t_4\big),$$

and $\langle Y_n, X_n\rangle_{\mathrm{cher}}$ may be found using Theorems 4.4, 4.5 and Proposition 4.2. The proof for $\langle X_{-n}, X'_{-n}\rangle_{\mathrm{cher}'}$ is similar. ■

Koornwinder and Bouzeffour [13] gave a positive definite orthogonality relation for the non-symmetric Askey–Wilson polynomials.

Macdonald [16, (6.6.8), p. 167] gives an expression for $P_m(z;\mathbf{t}\,|\,q)$ as linear combination of $R_m(z;\mathbf{t}\,|\,q)$ and $T_m(z;\mathbf{t}\,|\,q)$. This is equivalent to the Noumi–Stokman expression using Proposition 2.1.

Another [17, § 6.8] biorthogonal pair of bases with respect to $\langle\ ,\ \rangle_{\mathrm{cher}'}$ is given by the bases

$$\{R_0,\dots,R_n,U_1,\dots,U_n\} \qquad \text{and} \qquad \{R'_0,\dots,R'_n,U'_1,\dots,U'_n\}.$$

The $U_n(z;\mathbf{t}\,|\,q)$ are the anti-symmetric Askey–Wilson polynomials. However since

$$R_n(1/z;1/\mathbf{t}\,|\,1/q) = R_n(z;\mathbf{t}\,|\,q), \qquad U_n(1/z;1/\mathbf{t}\,|\,1/q) = U_n(z;\mathbf{t}\,|\,q)/t_1t_2,$$

this orthogonality is equivalent to Theorem 3.1.

7 Askey–Wilson orthogonality

One may ask if there are orthogonality relations for a basis of V_n using the Askey–Wilson weight. By symmetry, one could take

$$\{R_0,\dots,R_n,(z-1/z),(z-1/z)R_1,\dots,(z-1/z)R_{n-1}\}.$$

We include a partial result in this direction.

Theorem 7.1. *We have the orthogonality relation*

$$\frac{1}{2\pi i}\oint_{|z|=1} R_m(z;\mathbf{t}\,|\,q)T_n(z;\mathbf{t}\,|\,q)w_{aw}(x;\mathbf{t}\,|\,q)\frac{dz}{z}$$
$$= \frac{\left(-t_1^2\right)^n q^{\binom{n}{2}}\left(1+q^n\right)\left(t_1t_2t_3t_4q^{2n};q\right)_\infty}{2\prod\limits_{1\le j<k\le 4}(t_jt_k;q)_\infty}\prod_{2\le j<k\le 4}(t_jt_k;q)_n\delta_{m,n},$$

for $m \ge n$.

Proof. Consider the integral

$$\frac{1}{2\pi i}\oint_{|z|=1} R_m(z;\mathbf{t}\,|\,q)(t_1z,qt_1/z;q)_k w_{\mathrm{aw}}(x;\mathbf{t}\,|\,q)\frac{dz}{z},$$

for $0 \le k \le n$. The integral vanishes when $k = 0$ by (2.1), so we assume $k > 0$. Write $(t_1z,qt_1/z;q)_k$ as $(1-t_1z)(1-t_1q^k/z)(qt_1z,qt_1/z;q)_{k-1}$. Therefore the above integral is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} R_m\left(e^{i\theta};\mathbf{t}\,|\,q\right)\left(qt_1e^{i\theta},qt_1e^{-i\theta};q\right)_{k-1}\left[1-t_1\left(1+q^k\right)\cos\theta+t_1^2q^k\right]w_{\mathrm{aw}}(x;\mathbf{t}\,|\,q)d\theta,$$

which vanishes for $k < m$ by (2.1). If $k = m$ the above integral is

$$\frac{1+q^n}{4\pi}\int_{-\pi}^{\pi} R_m\left(e^{i\theta};\mathbf{t}\,|\,q\right)\left(t_1e^{i\theta},t_1e^{-i\theta};q\right)_n w_{\mathrm{aw}}(x;\mathbf{t}\,|\,q)d\theta$$
$$= \frac{1+q^n}{4\pi}\frac{(q,t_1t_2,t_1,t_3,t_1t_4;q)_n}{q^n\left(q^{-n},t_1t_2t_3t_4q^{n-1};q\right)_n}\int_{-\pi}^{\pi} R_m^2\left(e^{i\theta}\right)w_{\mathrm{aw}}(x;\mathbf{t}\,|\,q)d\theta,$$

and the result follows from (1.1), (2.1) and (2.2). ■

8 Recurrences

In this section we record three term recurrence relations satisfied by R_n, S_n, T_n and U_n. The three term recurrence relation for R_n is the Askey–Wilson recurrence relation [12, (3.1.4)]

$$[z+1/z]R_n(z;\mathbf{t}\,|\,q) = A_n^{(r)}R_{n+1}(z;\mathbf{t}\,|\,q) + C_n^{(r)}R_{n-1}(z;\mathbf{t}\,|\,q)$$
$$+\left[-A_n^{(r)} - C_n^{(r)} + t_1 + 1/t_1\right]R_n(z;\mathbf{t}\,|\,q),$$

where

$$A_n^{(r)} = \frac{\left(1-t_1t_2t_3t_4q^{n-1}\right)\prod\limits_{j=2}^{4}\left(1-t_1t_jq^n\right)}{t_1\left(1-t_1t_2t_3t_4q^{2n-1}\right)\left(1-t_1t_2t_3t_4q^{2n}\right)},$$
$$C_n^{(r)} = \frac{t_1\left(1-q^n\right)\prod\limits_{2\le j<k\le 4}\left(1-t_jt_kq^{n-1}\right)}{\left(1-t_1t_2t_3t_4q^{2n-2}\right)\left(1-t_1t_2t_3t_4q^{2n-1}\right)}.$$

The three term recurrence relation for T_n is

$$[z+q/z]T_n(z;\mathbf{t}\,|\,q) = A_n^{(t)}T_{n+1}(z;\mathbf{t}\,|\,q) + C_n^{(t)}T_{n-1}(z;\mathbf{t}\,|\,q)$$
$$+\left[-A_n^{(t)} - C_n^{(t)} + qt_1 + 1/t_1\right]T_n(z;\mathbf{t}\,|\,q),$$

with $n > 0$, where

$$A_n^{(t)} = \frac{\big(1 - t_1t_2t_3t_4q^{n-1}\big)\big(1 - t_1t_2q^{n+1}\big)\prod_{j=3}^{4}\big(1 - t_1t_jq^n\big)}{t_1\big(1 - t_1t_2t_3t_4q^{2n-1}\big)\big(1 - t_1t_2t_3t_4q^{2n}\big)},$$

$$C_n^{(t)} = \frac{qt_1\big(1 - q^n\big)\big(1 - t_3t_4q^{n-2}\big)\prod_{j=3}^{4}\big(1 - t_2t_jq^{n-1}\big)}{\big(1 - t_1t_2t_3t_4q^{2n-2}\big)\big(1 - t_1t_2t_3t_4q^{2n-1}\big)}.$$

On the other hand the three term recurrence relation for U_n is

$$\begin{aligned}[z + q/z]U_n(z;\mathbf{t}\,|\,q) &= A_n^{(u)}U_{n+1}(z;\mathbf{t}\,|\,q) + C_n^{(u)}U_{n-1}(z;\mathbf{t}\,|\,q)\\ &\quad + \big[-A_n^{(u)} - C_n^{(u)} + qt_1 + q^{-1}/t_1\big]U_n(z;\mathbf{t}\,|\,q),\end{aligned}$$

with $n > 1$, where

$$A_n^{(u)} = \frac{\big(1 - t_1t_2t_3t_4q^{n}\big)\big(1 - t_1t_2q^{n+1}\big)\prod_{j=3}^{4}\big(1 - t_1t_jq^n\big)}{qt_1\big(1 - t_1t_2t_3t_4q^{2n-1}\big)\big(1 - t_1t_2t_3t_4q^{2n}\big)},$$

$$C_n^{(u)} = \frac{qt_1\big(1 - q^{n-1}\big)\big(1 - t_1t_2q^n\big)\big(1 - t_3t_4q^{n-2}\big)}{\big(1 - t_1t_2t_3t_4q^{2n-2}\big)\big(1 - t_1t_2t_3t_4q^{2n-1}\big)}\prod_{j=3}^{4}\big(1 - t_1t_jq^{n-1}\big)\big(1 - t_2t_jq^{n-1}\big).$$

Finally the three term recurrence relation for S_n is

$$\begin{aligned}\big[q^{-1/2}z + q^{1/2}/z\big]R_n(z;\mathbf{t}\,|\,q) &= A_n^{(s)}R_{n+1}(z;\mathbf{t}\,|\,q) + C_n^{(s)}R_{n-1}(z;\mathbf{t}\,|\,q)\\ &\quad + \big[-A_n^{(s)} - C_n^{(s)} + t_1 + 1/t_1\big]R_n(z;\mathbf{t}\,|\,q),\end{aligned}$$

and the coefficients are given by

$$A_n^{(s)} = \frac{\big(1 - t_1t_2t_3t_4q^{n}\big)\prod_{j=2}^{4}\big(1 - t_1t_jq^n\big)}{t_1\big(1 - t_1t_2t_3t_4q^{2n-1}\big)\big(1 - t_1t_2t_3t_4q^{2n}\big)},$$

$$C_n^{(s)} = \frac{t_1\big(1 - q^n\big)\prod_{2\le j<k\le 4}\big(1 - t_jt_kq^{n-1}\big)}{\big(1 - t_1t_2t_3t_4q^{2n-2}\big)\big(1 - t_1t_2t_3t_4q^{2n-1}\big)}.$$

9 Asymptotics

Consider a general balanced terminating ${}_4\phi_3$,

$${}_4\phi_3\left(\left.\begin{matrix} q^{-n}, Aq^{n-1}, B, C\\ D, E, F\end{matrix}\,\right|\, q, q\right), \qquad \text{with} \quad ABC = DEF.$$

Since the ${}_4\phi_3$ is symmetric in B and C we may assume

$$|B| \le |C|.$$

Ismail and Wilson [10] determined the large degree asymptotics of the Askey–Wilson polynomials. When $|B| < |C|$ their result is

$${}_4\phi_3\left(\left.\begin{matrix} q^{-n}, Aq^{n-1}, B, C\\ D, E, F\end{matrix}\,\right|\, q, q\right) = C^n\frac{(B, D/C, E/C, F/C; q)_\infty}{(B/C, D, E, F; q)_\infty}\big[1 + \mathcal{O}\big(q^{n/2}\big)\big].$$

On the other hand when $|B| = |C|$ we let $B/C = e^{2i\theta}$ and in this case the Ismail–Wilson asymptotic result is

$$
{}_4\phi_3\left(\left.\begin{matrix} q^{-n}, Aq^{n-1}, B, C \\ D, E, F \end{matrix}\right| q, q\right) = \left[1 + \mathcal{O}\big(q^{n/2}\big)\right]
\times\left[\frac{C^n(B, D/C, E/C, F/C; q)_\infty}{(B/C, D, E, F; q)_\infty} + \frac{B^n(C, D/B, E/B, F/B; q)_\infty}{(C/B, D, E, F; q)_\infty}\right].
$$

We now record the asymptotics of R_n, S_n, T_n and U_n.

Theorem 9.1. *For $|z| = 1$ the following asymptotic formulas hold*

$$
\prod_{j=2}^{4}(t_1t_j; q)_\infty R_n(z; \mathbf{t}) = \left(\frac{t_1}{z}\right)^n \frac{\prod\limits_{j=1}^{4}(t_jz; q)_\infty}{\big(z^2; q\big)_\infty}\left[1 + \mathcal{O}\big(q^{n/2}\big)\right]
+ (t_1z)^n \frac{\prod\limits_{j=1}^{4}(t_j/z; q)_\infty}{\big(1/z^2; q\big)_\infty}\left[1 + \mathcal{O}\big(q^{n/2}\big)\right],
$$

$$
\frac{\big(qz^{-2}; q\big)_\infty \prod\limits_{j=2}^{4}(qt_1t_j; q)_\infty}{(qt_1/z, qt_2/z, t_3/z, t_4/z; q)_\infty} S_n(z; \mathbf{t}) = t_1^{n-1}z^n\left[1 + \mathcal{O}\big(q^{n/2}\big)\right],
$$

$$
\frac{\big(qz^{-2}; q\big)_\infty (qt_1t_2; q)_\infty \prod\limits_{j=3}^{4}(t_1t_j; q)_\infty}{\prod\limits_{j=1}^{2}(qt_j/z, t_{j+2}/z; q)_\infty} T_n(z; \mathbf{t}) = (t_1z)^n\left[1 + \mathcal{O}\big(q^{n/2}\big)\right],
$$

$$
\frac{\prod\limits_{j=2}^{4}(qt_1t_j; q)_\infty}{(1 - qt_1t_2)} U_n(z; \mathbf{t}) = \frac{(qt_1)^{n-1}}{z^n} \frac{\prod\limits_{j=1}^{4}(t_jz; q)_\infty}{\big(z^2; q\big)_\infty}\left[1 + \mathcal{O}\big(q^{n/2}\big)\right]
+ (qt_1)^{n-1}z^{n-2} \frac{(1 - t_1z)(1 - t_2z)\prod\limits_{j=1}^{4}(t_j/z; q)_\infty}{\big(1/z^2; q\big)_\infty (1 - t_1/z)(1 - t_2/z)}\left[1 + \mathcal{O}\big(q^{n/2}\big)\right].
$$

10 Discrete orthogonality

The Askey–Wilson polynomials have a discrete orthogonality, the q-Racah orthogonality [2, 12], when $t_1t_j = q^{-N}$, for some $j = 2, 3$ or 4, and for some positive integer N. In this section we give the corresponding discrete orthogonality for the Laurent polynomials. This may be done by deforming the contours, we do not give the details.

The q-Racah measure for the Askey–Wilson polynomials is purely discrete and based upon the very well poised terminating ${}_6\phi_5$ evaluation [7, (II.20)]

$$
\sum_{k=0}^{N} \frac{\big(t_1^2, t_1t_2, t_1t_3, t_1t_4; q\big)_k}{(q, qt_1/t_2, qt_1/t_3, qt_1/t_4; q)_k} \frac{1 - t_1^2q^{2k}}{1 - t_1^2}\left(\frac{q}{t_1t_2t_3t_4}\right)^k
= \frac{\big(qt_1^2, qt_1^2/t_2t_3, qt_1^2/t_2t_4, qt_1^2/t_3t_4; q\big)_\infty}{(qt_1/t_2, qt_1/t_3, qt_1/t_4, q/t_1t_2t_3t_4; q)_\infty}. \tag{10.1}
$$

Using

$$\frac{1-t_1^2q^{2k}}{1-t_1^2} = \frac{1}{(1-t_1^2)(1-t_1t_2)}\big(-t_1t_2\big(1-q^k\big)\big(1-q^kt_1/t_2\big)+\big(1-t_1^2q^k\big)\big(1-t_1t_2q^k\big)\big)$$

one can rewrite the left side of the ${}_6\phi_5$ evaluation (10.1) as a sum of two ${}_4\phi_3$'s

$$\sum_{k=1}^{N}\frac{\big(qt_1^2,qt_1t_2;q\big)_{k-1}(t_1t_3,t_1t_4;q)_k}{(qt_1/t_2,q;q)_{k-1}(qt_1/t_3,qt_1/t_4;q)_k}\left(\frac{q}{t_1t_2t_3t_4}\right)^k(-t_1t_2)$$
$$+\sum_{k=0}^{N}\frac{\big(qt_1^2,qt_1t_2,t_1t_3,t_1t_4;q\big)_k}{(qt_1/t_2,qt_1/t_3,qt_1/t_4,q;q)_k}\left(\frac{q}{t_1t_2t_3t_4}\right)^k.$$

Definition 10.1. If $t_1t_j = q^{-N}$ for some positive integer N, the Racah bilinear form on V_N is defined by

$$\langle f,g\rangle_{\mathrm{Racah}} = \sum_{k=1}^{N}\frac{\big(qt_1^2,qt_1t_2;q\big)_{k-1}(t_1t_3,t_1t_4;q)_k}{(qt_1/t_2,q;q)_{k-1}(qt_1/t_3,qt_1/t_4;q)_k}\left(\frac{q}{t_1t_2t_3t_4}\right)^k(-t_1t_2)f\big(t_1q^k\big)g\big(t_1q^k\big)$$
$$+\sum_{k=0}^{N}\frac{\big(qt_1^2,qt_1t_2,t_1t_3,t_1t_4;q\big)_k}{(qt_1/t_2,qt_1/t_3,qt_1/t_4,q;q)_k}\left(\frac{q}{t_1t_2t_3t_4}\right)^k f\big(q^{-k}/t_1\big)g\big(q^{-k}/t_1\big).$$

Theorem 10.2. *If $t_1t_j = q^{-N}$ for some positive integer N and $j=3$ or 4, then*

$$\{R_0,\dots,R_N,U_1,\dots,U_N\},\qquad \{T_0,\dots,T_N,S_1,\dots,S_N\},$$
$$\{X_{-N},\dots,X_0,X_1,\dots,X_N\}\qquad \text{and}\qquad \{Y_{-N},\dots,Y_0,Y_1,\dots,Y_N\}$$

are Racah-orthogonal bases for V_N.

If $t_1t_2 = q^{-N}$, the Racah bilinear form contains a zero term: the $k=N$ term in the second sum is 0 due to $(qt_1t_2;q)_k$. In this case U_N and T_N are not well-defined. A slight modification of Theorem 10.2 can be formulated, we do not give it here.

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