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A Connection Formula of the Hahn–Exton q-Bessel Function

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Abstract. We show a connection formula of the Hahn–Exton q-Bessel function around the origin and the infinity. We introduce the q-Borel transformation and the q-Laplace transformation following C. Zhang to obtain the connection formula. We consider the limit $p \to 1^-$ of the connection formula.

Key words: Hahn–Exton q-Bessel function; q-Borel transformation; connection problems

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1 Introduction

In this paper, we show a connection formula of the Hahn–Exton q-Bessel function $J_\nu^{(3)}(x;q)$. At first, we review the Bessel function and q-analogues of the Bessel function. The Bessel equation

$$\frac{d^2u}{dz^2} + \frac{1}{z}\frac{du}{dz} + \left(1 - \frac{\nu^2}{z^2}\right)u = 0$$

has a solution $u(z) = J_\nu(z)$, $J_{-\nu}(z)$. Here, the Bessel function $J_\nu(z)$ is

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^\nu {}_0F_1\left(-, \nu+1, -\frac{z^2}{4}\right).$$

The degenerated confluent hypergeometric function ${}_0F_1(-,\alpha,z)$ is defined by

$${}_0F_1(-,\alpha,z) = \sum_{n\geq 0} \frac{1}{(\alpha)_n n!} z^n, \qquad (\alpha)_n = \alpha\{\alpha+1\}\cdots\{\alpha+(n-1)\}.$$

Both $J_\nu(z)$ and $J_{-\nu}(z)$ are linearly independent if $\nu \notin \mathbb{Z}$.

It is known that there exists three different q-analogues of the Bessel function.

$$J_\nu^{(1)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_2\varphi_1\left(0,0;q^{\nu+1};q,-\frac{x^2}{4}\right), \qquad |x| < 2,$$
$$J_\nu^{(2)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu {}_0\varphi_1\left(-;q^{\nu+1};q,-\frac{q^{\nu-1}x^2}{4}\right), \qquad x \in \mathbb{C},$$
$$J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu {}_1\varphi_1\left(0;q^{\nu+1};q,qx^2\right), \qquad x \in \mathbb{C}.$$

Here,

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n \geq 1, \end{cases}$$

$$(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$$

and

$$(a_1,a_2,\ldots,a_m;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_m;q)_\infty.$$

Moreover, the basic hypergeometric series ${}_r\varphi_s$ is

$${}_r\varphi_s(a_1,\ldots,a_r;b_1,\ldots,b_s;q,x) := \sum_{n\ge 0}\frac{(a_1,\ldots,a_r;q)_n}{(b_1,\ldots,b_s;q)_n(q;q)_n}\left[(-1)^n q^{\frac{n(n-1)}{2}}\right]^{1+s-r} x^n.$$

The first and the second one are called Jackson's first and second q-Bessel function and the third one is called the Hahn–Exton q-Bessel function. They satisfy the following q-difference equations:

$$\begin{aligned}
J_\nu^{(1)}:&\quad u(xq)-\left(q^{\nu/2}+q^{-\nu/2}\right)u(xq^{1/2})+\left(1+\frac{x^2}{4}\right)u(x)=0,\\
J_\nu^{(2)}:&\quad \left(1+\frac{qx^2}{4}\right)u(xq)-\left(q^{\nu/2}+q^{-\nu/2}\right)u\left(xq^{1/2}\right)+u(x)=0,\\
J_\nu^{(3)}:&\quad u(xq)-\left\{\left(q^{\nu/2}+q^{-\nu/2}\right)-q^{-\nu/2+1}x^2\right\}u\left(xq^{1/2}\right)+u(x)=0.
\end{aligned}\tag{1}$$

The limits of these q-analogues of the Bessel function are the Bessel function when $q\to 1^-$:

$$\lim_{q\to 1^-} J_\nu^{(k)}\left((1-q)x;q\right) = J_\nu(x), \qquad k=1,2$$

and

$$\lim_{q\to 1^-} J_\nu^{(3)}\left((1-q)x;q\right) = J_\nu(2x).$$

The relation between $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$ was found by Hahn [3] as follows:

$$J_\nu^{(2)}(x;q) = \left(-\frac{x^2}{4};q\right)_\infty J_\nu^{(1)}(x;q). \tag{2}$$

Connection problems of the q-difference equation between the origin and the infinity are studied by G.D. Birkhoff [1]. We review connection formulae for several q-difference functions.

1. Watson's formula. In 1910 [6], Watson showed the connection formula of the basic hypergeometric function ${}_2\varphi_1$ as follows:

$$\begin{aligned}
{}_2\varphi_1(a,b;c;q;x) &= \frac{(b,c/a;q)_\infty(ax,q/ax;q)_\infty}{(c,b/a;q)_\infty(x,q/x;q)_\infty}{}_2\varphi_1(a,aq/c;aq/b;q;cq/abx)\\
&\quad+\frac{(a,c/b;q)_\infty(bx,q/bx;q)_\infty}{(c,a/b;q)_\infty(x,q/x;q)_\infty}{}_2\varphi_1(b,bq/c;bq/a;q;cq/abx).
\end{aligned}$$

2. Connection formula of $J_\nu^{(1)}(x;q)$. C. Zhang has given some connection formulae for the solutions of the q-difference equations of confluent type [7, 8] and [9]. In [8], Zhang has shown connection formulae for $J_\nu^{(1)}(x;q)$ and $J_\nu^{(2)}(x;q)$. The connection formula of $J_\nu^{(1)}(x;q)$ is given by

$$\frac{\left(\frac{\alpha}{\sqrt{p}x};p\right)_\infty}{\theta_p\left(-\frac{\alpha}{x}\right)}{}_2\varphi_1\left(p^{\nu+\frac12},p^{-\nu+\frac12};-p;p,\frac{\alpha}{\sqrt{p}x}\right)$$

$$= \frac{1}{\theta_p\left(-\frac{\alpha}{x}\right)} \left\{ \frac{\theta_p\left(-\frac{\alpha q^{\frac{\nu}{2}}}{x}\right)}{(q, q^{-\nu}; q)_\infty} {}_2\varphi_1\left(0, 0; q^{\nu+1}; q, -\frac{x^2}{4}\right) \right.$$
$$\left. + \frac{\theta_p\left(-\frac{\alpha q^{-\frac{\nu}{2}}}{x}\right)}{(q, q^{\nu}; q)_\infty} {}_2\varphi_1\left(0, 0; q^{-\nu+1}; q, -\frac{x^2}{4}\right) \right\}, \tag{3}$$

where $q = p^2$ and $\alpha^2 = -4q^{3/2}$.

The connection formula of $J_\nu^{(2)}(x;q)$ is obtained by (3) and (2). But it is not known the connection formula of the Hahn–Exton q-Bessel function.

The Hahn–Exton q-Bessel equations (1) has two analytic solutions $u(x) = J_\nu^{(3)}(x)$, $J_{-\nu}^{(3)}(xp^{-\nu})$ around $x = 0$ and has one analytic solution $z(1/x) = \frac{1}{\theta_p(-p^{\nu+2}/x)} \sum_{n\geq 0} a_n x^{-n}$, $a_0 = 1$. We show a connection formula of $J_\nu^{(3)}(x;q)$ in Section 2 as follows:

Theorem 1. *For any $x \in \mathbb{C}^* \setminus [p^{\nu+2}; p]$,*

$$z\left(\frac{1}{x}\right) = \frac{1}{(p^{-2\nu}, p; p)_\infty} \frac{\theta_p\left(-\frac{p^{2\nu+2}}{x}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x}\right)} {}_1\varphi_1\left(0, p^{1+2\nu}; p, x\right)$$
$$+ \frac{1}{(p^{2\nu}, p; p)_\infty} \frac{\theta_p\left(-\frac{p^2}{x}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x}\right)} {}_1\varphi_1\left(0, p^{1-2\nu}; p, p^{-2\nu}x\right). \tag{4}$$

Here, $\theta_p(\cdot)$ is the theta function of Jacobi and $[\lambda; q]$ is the q-spiral (see Section 2). We use the q-Borel transformation and the q-Laplace transformation which is defined by C. Zhang in [8].

In Section 3, we consider the limit $p \to 1^-$ of the connection formula. If we take a suitable limit $p \to 1^-$ of (4), we obtain

$$H_\nu^{(2)}\left(\sqrt{z}\right) = \frac{-ie^{\nu\pi i}}{\sin \nu\pi} \left\{ J_\nu\left(\sqrt{z}\right) - e^{-\nu\pi i} J_{-\nu}\left(\sqrt{z}\right) \right\}.$$

Here, $H_\nu^{(2)}(z)$ is the Hankel function of the second kind. Thus we obtain a connection formula of the Bessel function as a limit $p \to 1^-$ of (4).

2 The connection formula

In this section, we give a connection formula of the Hahn–Exton q-Bessel function. We introduce the p-Borel transformation and the p-Laplace transformation to obtain the connection formula between the origin and the infinity. These transformations are useful to consider connection problems. We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$ and $q = p^2$. The q-difference operator σ_q is given by $\sigma_q f(x) = f(qx)$.

2.1 The theta function of Jacobi

Before we study connection problems, we review the theta function of Jacobi. The theta function of Jacobi is given by the following series:

Definition 1. For any $x \in \mathbb{C}^*$,

$$\theta_q(x) = \theta(x) := \sum_{n\in\mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n.$$

We denote by $\theta_q(x)$ or more shortly $\theta(x)$. The theta function satisfies Jacobi's triple product identity:

$$\theta(x) = \left(q, -x, -\frac{q}{x}; q\right)_\infty .$$

The theta function satisfies the q-difference equation as follows

$$\theta(q^k x) = q^{-\frac{k(k-1)}{2}} x^{-k} \theta(x), \qquad \forall\, x \in \mathbb{C}^*.$$

The theta function has the inversion formula $x\theta(1/x) = \theta(x)$. For all fixed $\lambda \in \mathbb{C}^*$, we define a q-spiral $[\lambda; q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k : k \in \mathbb{Z}\}$. We remark that $\theta\left(\lambda q^k/x\right) = 0$ if and only if $x \in [-\lambda; q]$.

2.2 The Hahn–Exton q-Bessel function

The Hahn–Exton q-Bessel function is defined by

$$J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n\geq 0} \frac{(-1)^n q^{\frac{n(n-1)}{2}}}{(q^{\nu+1}, q; q)_n} \left(qx^2\right)^n .$$

The function $J_\nu^{(3)}(x;q)$ satisfies the q-difference equation

$$\left[\sigma_p^2 - \left\{(p^\nu + p^{-\nu}) - x^2 p^{2-\nu}\right\}\sigma_p + 1\right] y(x) = 0. \tag{5}$$

If we replace ν by $-\nu$ and x by $xp^{-\nu}$, we obtain $J_{-\nu}^{(3)}(xp^{-\nu};q)$ which is another solution of (5) around the origin. This solution corresponds to the classical Neumann function $Y_\nu(x)$ [5]. We consider the behavior of equation (5) around the infinity. We set $1/t$, formally $t^2 \mapsto t$ and $z(t) = y(1/t)$. Then $z(t)$ satisfies

$$\left[\sigma_p^2 - \left\{(p^\nu + p^{-\nu}) - \frac{p^{-2-\nu}}{t}\right\}\sigma_p + 1\right] z\left(t\right) = 0. \tag{6}$$

We set $\mathcal{E}(t) = 1/\theta_p(-p^{\nu+2}t)$ and $f(t) = \sum_{n\geq 0} a_n t^n$, $a_0 = 1$. We assume that $z(t)$ can be described as

$$z(t) = \mathcal{E}(t) f(t) = \frac{1}{\theta_p(-p^{\nu+2}t)} \left(\sum_{n\geq 0} a_n t^n\right).$$

Since $\mathcal{E}(t)$ satisfies the following q-difference equation

$$\sigma_p \mathcal{E}(t) = -p^{\nu+2} t \mathcal{E}(t), \qquad \sigma_p^2 \mathcal{E}(t) = p^{2\nu+5} t^2 \mathcal{E}(t),$$

we can check out that the function $f(t)$ satisfies the equation

$$\left\{p^{2\nu+5} t^2 \sigma_p^2 + p^{\nu+2}(p^\nu + p^{-\nu}) t\sigma_p - \sigma_p + 1\right\} f(t) = 0. \tag{7}$$

2.3 The p-Borel transformation and the p-Laplace transformation

We define the p-Borel transformation and the p-Laplace transformation to solve the equation (7), following Zhang [8].

Definition 2. For $f(t) = \sum_{n\geq 0} a_n t^n$, the p-Borel transformation is defined by

$$g(\tau) = (\mathcal{B}_p f)(\tau) := \sum_{n\geq 0} a_n p^{-\frac{n(n-1)}{2}} \tau^n,$$

and the p-Laplace transformation is given by

$$(\mathcal{L}_p g)(t) := \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau)\theta_p\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau}.$$

Here, $r_0 > 0$ is enough small number.

The p-Borel transformation is considered as a formal inverse of the p-Laplace transformation.

Lemma 1. *We assume that the function f can be p-Borel transformed to the analytic function $g(\tau)$ around $\tau = 0$. Then,*

$$\mathcal{L}_p \circ \mathcal{B}_p f = f.$$

Proof. We can prove this lemma calculating residues of the p-Laplace transformation around the origin. ■

The p-Borel transformation has the following operational relation.

Lemma 2. *For any $l, m \in \mathbb{Z}_{\geq 0}$,*

$$\mathcal{B}_p\big(t^m \sigma_p^l\big) = p^{-\frac{m(m-1)}{2}} \tau^m \sigma_p^{l-m} \mathcal{B}_p.$$

Applying the p-Borel transformation to the equation (7) and using Lemma 2, $g(\tau)$ satisfies the first order difference equation

$$g(p\tau) = \left(1 + p^{2\nu+2}\tau\right)\left(1 + p^2\tau\right) g(\tau).$$

Since $g(0) = 1$, we get an infinite product of $g(\tau)$:

$$g(\tau) = \frac{1}{(-p^{2\nu+2}\tau; p)_\infty (-p^2\tau; p)_\infty}.$$

Then $g(\tau)$ has single poles at

$$\left\{-p^{-2\nu-2-k}, -p^{-2-k}; k \in \mathbb{Z}_{\geq 0}\right\}.$$

We set

$$0 < r < r_0 := \min\left\{\frac{1}{|p^{2\nu+2}|}, \frac{1}{|p^2|}\right\}.$$

and choose the radius $r > 0$ such that $0 < r < r_0$. By Cauchy's residue theorem, the p-Laplace transform of $g(\tau)$ is

$$\begin{aligned} f(t) &= \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau)\theta_p\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\ &= -\sum_{k\geq 0} \operatorname{Res}\left\{g(\tau)\theta_p\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = -p^{-2\nu-2-k}\right\} - \sum_{k\geq 0} \operatorname{Res}\left\{g(\tau)\theta_p\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = -p^{-2-k}\right\}, \end{aligned}$$

where $0 < r < r_0$. To calculate the residue, we use the following lemma.

Lemma 3. *For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, we have*

1. $\mathrm{Res}\left\{\frac{1}{(\tau/\lambda;p)_\infty}\frac{1}{\tau} : \tau = \lambda p^{-k}\right\} = \frac{(-1)^{k+1}p^{\frac{k(k+1)}{2}}}{(p;p)_k(p;p)_\infty}$,
2. $\frac{1}{(\lambda p^{-k};p)_\infty} = \frac{(-\lambda)^{-k}p^{\frac{k(k+1)}{2}}}{(\lambda;p)_\infty\,(p/\lambda;p)_k}, \qquad \lambda \notin p^{\mathbb{Z}}.$

Summing up all of the residues, we obtain the convergent series $f(t)$ as follows

$$f(t) = \frac{\theta_p\left(-p^{2\nu+2}t\right)}{(p^{-2\nu},p;p)_\infty}{}_1\varphi_1\left(0,p^{1+2\nu};p,x\right) + \frac{\theta_p\left(-p^2t\right)}{(p^{2\nu},p;p)_\infty}{}_1\varphi_1\left(0,p^{1-2\nu};p,p^{-2\nu}x\right),$$

where $xt = 1$. Therefore, we acquire the connection formula for $z(t) = \mathcal{E}(t)f(t)$.

3 The limit of the connection formula

In this section, we show that the limit $p \to 1^-$ of the connection formula gives a connection formula of the Bessel function. At first, we assume that $0 < p < 1$ and $0 < \sqrt{p} < 1$. For the Bessel function, we set the Hankel function of the first and the second kind $H_\nu^{(1)}(z)$ and $H_\nu^{(2)}(z)$.

Definition 3. The Hankel function of the first kind is given by

$$H_\nu^{(1)}(z) := \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\pi i\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{1+\infty i}^{(1+)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}}dt, \qquad -\pi < \arg z < 2\pi.$$

The Hankel function of the second kind is defined by

$$H_\nu^{(2)}(z) := \frac{\Gamma\left(\frac{1}{2}-\nu\right)}{\pi i\sqrt{\pi}}\left(\frac{z}{2}\right)^\nu \int_{-1+\infty i}^{(-1-)} e^{izt}\left(t^2-1\right)^{\nu-\frac{1}{2}}dt, \qquad -2\pi < \arg z < \pi.$$

The contour for $H_\nu^{(1)}(z)$ is a path starting from $t = +1+\infty i$, rounding the circle around $t = 1$ counterclockwise, and going back to $t = +1+\infty i$. Moreover, the contour for $H_\nu^{(2)}(z)$ is a path starting from $t = -1+\infty i$, rounding the circle around $t = 1$ clockwise, and going back to $t = -1+\infty i$.

The Hankel functions can be written by $J_\nu(z)$:

$$H_\nu^{(1)}(z) = \frac{ie^{-\nu\pi i}}{\sin\nu\pi}\left\{J_\nu(z) - e^{\nu\pi i}J_{-\nu}(z)\right\}, \tag{8}$$

$$H_\nu^{(2)}(z) = -\frac{ie^{\nu\pi i}}{\sin\nu\pi}\left\{J_\nu(z) - e^{-\nu\pi i}J_{-\nu}(z)\right\}. \tag{9}$$

The Hankel functions have asymptotic expansions around $z = 0$ [4]:

$$H_\nu^{(1)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{i\zeta}\sum_{s\geq 0} i^s\frac{A_s(\nu)}{z^s}, \qquad -\pi+\delta \leq \arg z \leq 2\pi-\delta,$$

$$H_\nu^{(2)}(z) \sim \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i\zeta}\sum_{s\geq 0} (-i)^s\frac{A_s(\nu)}{z^s}, \qquad -2\pi+\delta \leq \arg z \leq \pi-\delta,$$

as $z \to \infty$. Here, δ is an any small constant,

$$A_s(\nu) = \frac{(4\nu^2-1^2)(4\nu^2-3^2)\cdots\left\{4\nu^2-(2s-1)^2\right\}}{s!8^s}$$

and

$$\zeta = z - \frac{1}{2}\nu\pi - \frac{1}{4}\pi.$$

In this sense, (8) and (9) considered as connection formula of the Bessel equation.

3.1 Limit of the connection formula

We rewrite the connection formula in Theorem 1 in order to take a limit $p \to 1^-$. We set new functions $h_\nu(t;p)$ and $J_\nu^\pm(x;p)$. We set $h_\nu(t;p) := (p^{1/2}, p^{1/2}; p)_\infty z(t)$. For any $x \in \mathbb{C}^* \setminus [-\lambda; p]$ and $\lambda \in \mathbb{C}^*$, $J_{\nu,\lambda}^+(x;p)$ is

$$J_{\nu,\lambda}^+(x;p) := \frac{(p^{\nu+1};p)_\infty}{(p;p)_\infty} \frac{\theta_p\left(\frac{\lambda p^\nu}{x}\right)}{\theta_p\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0; p^{1+2\nu}; p, x\right).$$

Similarly, $J_{\nu,\lambda}^-(x;p)$ is

$$J_{\nu,\lambda}^-(x;p) := \frac{(p^{\nu+1};p)_\infty}{(p;p)_\infty} \frac{\theta_p\left(\frac{\lambda p^\nu}{x}\right)}{\theta_p\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0; p^{1+2\nu}; p, p^{2\nu}x\right).$$

We remark that the function $\theta_p(\lambda p^\nu/x)/\theta_p(\lambda/x)$ satisfies the following q-difference equation

$$u(px) = p^\nu u(x),$$

which is also satisfied by the function $u(x) = x^\nu$. We remark that the pair $(J_{\nu,\lambda}^+(x;p), J_{-\nu,\lambda}^-(x;p))$ gives a fundamental system of solutions of equation (6) if $\nu \notin \mathbb{Z}$. We set the function $C_\nu^+(\lambda,t;p)$ and $C_\nu^-(\lambda,t;p)$ as follow:

Definition 4. For any $\lambda \in \mathbb{C}^*$, $C_\nu^+(\lambda,t;p)$ is

$$C_\nu^+(\lambda,t;p) := \frac{(p^{\frac12}, p^{\frac12}; p)_\infty}{(p^{\nu+1}, p^{-2\nu}; p)_\infty} \frac{\theta_p(-p^{2\nu+2}t)}{\theta_p(-p^{\nu+2}t)} \frac{\theta_p(\lambda t)}{\theta_p(\lambda p^\nu t)}.$$

Similarly, the function $C_\nu^-(\lambda,t;p)$ is

$$C_\nu^-(\lambda,t;p) := \frac{(p^{\frac12}, p^{\frac12}; p)_\infty}{(p^{-\nu+1}, p^{2\nu}; p)_\infty} \frac{\theta_p(-p^2 t)}{\theta_p(-p^{\nu+2}t)} \frac{\theta_p(\lambda t)}{\theta_p(\lambda p^{-\nu} t)}.$$

Then, $C_\nu^+(\lambda,t;p)$ and $C_\nu^-(\lambda,t;p)$ are single valued as a function of t. The function $C_\nu^+(\lambda,t;p)$ and $C_\nu^-(\lambda,t;p)$ are the p-elliptic functions. By using these new functions, our connection formula is rewritten by

$$h_\nu\left(\frac{1}{x};p\right) = C_\nu^+\left(\lambda, \frac{1}{x}; p\right) J_\nu^+(x;p) + C_\nu^-\left(\lambda, \frac{1}{x}; p\right) J_{-\nu,\lambda}^-(x;p).$$

Theorem 2. *For any $x \in \mathbb{C}^* \setminus (-\infty, 0]$ where $\arg x \in (-\pi,\pi)$, we have*

$$\lim_{p\to 1^-} h_\nu\left(\frac{1}{(1-p)^2 x}; p\right) = -ie^{-\nu\pi i} H_{2\nu}^{(2)}(2\sqrt{x}).$$

Here, $H_{2\nu}^{(2)}(\cdot)$ is the Hankel function of the second kind.

The aim of this section is to give a proof of the theorem above.

By the definition, $h_\nu\left(1/\{(1-p)^2 x\}; p\right)$ can be described as follows

$$h_\nu\left(\frac{1}{(1-p)^2 x}; p\right) = \left\{\frac{(p^{\frac12}, p^{\frac12}; p)_\infty}{(p^{-2\nu}, p; p)_\infty}(1-p)^{2\nu}\right\} \left\{\frac{\theta_p\left(-\frac{p^{2\nu+2}}{x(1-p)^2}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x(1-p)^2}\right)}(1-p)^{-2\nu}\right\}.$$
$$\times \left\{{}_1\varphi_1\left(0; p^{1+2\nu}; p, (1-p)^2 x\right)\right\}$$

$$+\left\{\frac{(p^{\frac{1}{2}},p^{\frac{1}{2}};p)_\infty}{(p^{2\nu},p;p)_\infty}(1-p)^{-2\nu}\right\}\left\{\frac{\theta_p\left(-\frac{p^2}{x(1-p)^2}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x(1-p)^2}\right)}(1-p)^{2\nu}\right\}$$
$$\times\left\{{}_1\varphi_1\left(0;p^{1-2\nu};p,p^{-2\nu}(1-p)^2x\right)\right\}. \tag{10}$$

We consider the limit of each part $\{\cdot\}$.

Lemma 4. *For any $\nu\in\mathbb{C}^*\setminus\mathbb{Z}$, we have*

$$\lim_{p\to 1^-}\frac{(p^{\frac{1}{2}},p^{\frac{1}{2}};p)_\infty}{(p^{-2\nu},p;p)_\infty}(1-p)^{2\nu}=-\frac{1}{\sin(2\nu\pi)\Gamma(2\nu+1)}.$$

Proof. We can check out as follows

$$\frac{(p^{\frac{1}{2}},p^{\frac{1}{2}};p)_\infty}{(p^{-2\nu},p;p)_\infty}(1-p)^{2\nu}=\frac{\frac{(p;p)_\infty}{(p^{-2\nu};p)_\infty}(1-p)^{1+2\nu}}{\left\{\frac{(p;p)_\infty}{(p^{\frac{1}{2}};p)_\infty}(1-p)^{\frac{1}{2}}\right\}\left\{\frac{(p;p)_\infty}{(p^{\frac{1}{2}};p)_\infty}(1-p)^{\frac{1}{2}}\right\}}=\frac{\Gamma_p(-2\nu)}{\Gamma_p\left(\frac{1}{2}\right)\Gamma_p\left(\frac{1}{2}\right)}.$$

Here, $\Gamma_q(\cdot)$ is Jackson's q-gamma function which is defined by

$$\Gamma_q(x):=\frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x},\qquad 0<q<1.$$

This function satisfies $\lim\limits_{q\to 1^-}\Gamma_q(x)=\Gamma(x)$ [2]. Therefore,

$$\lim_{p\to 1^-}\frac{(p^{\frac{1}{2}},p^{\frac{1}{2}};p)_\infty}{(p^{-2\nu},p;p)_\infty}(1-p)^{2\nu}=\frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}.$$

By Euler's reflection formula of the gamma function, we get

$$\frac{\Gamma(-2\nu)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}=-\frac{1}{\sin(2\nu\pi)\Gamma(2\nu+1)}.$$

Therefore, we get the conclusion. ∎

If we replace ν by $-\nu$, we get the limit

$$\lim_{p\to 1^-}\frac{(p^{\frac{1}{2}},p^{\frac{1}{2}};p)_\infty}{(p^{2\nu},p;p)_\infty}(1-p)^{-2\nu}=\frac{1}{\sin(2\nu\pi)\Gamma(1-2\nu)}.$$

In [8], the following proposition can be found:

Proposition 1. *For any $x\in\mathbb{C}^*$ $(-\pi<\arg x<\pi)$, we have*

$$\lim_{p\to 1^-}\frac{\theta_p\left(\frac{p^{\nu_1}}{(1-p^2)x}\right)}{\theta_p\left(\frac{p^{\nu_2}}{(1-p^2)x}\right)}\left(1-p^2\right)^{\nu_2-\nu_1}=x^{\nu_1-\nu_2},$$

and

$$\lim_{p\to 1^-}\frac{\theta_p\left(-\frac{p^{\nu_1}}{(1-p^2)x}\right)}{\theta_p\left(-\frac{p^{\nu_2}}{(1-p^2)x}\right)}\left(1-p^2\right)^{\nu_2-\nu_1}=(-x)^{\nu_1-\nu_2}.$$

Lemma 5. *For any $x \in \mathbb{C}^*$ $(-\pi < \arg x \le \pi)$ and fixed constant K, we have*

$$\theta_p(-\sqrt{p})\theta_p\left(-\frac{K}{x}\right) = \theta_{\sqrt{p}}\left(\sqrt{\frac{K}{x}}\right)\theta_{\sqrt{p}}\left(-\sqrt{\frac{K}{x}}\right).$$

Proof. From Jacobi's triple product identity and $(a^2;q^2)_n = (a,-a;q)_n$, we obtain

$$\frac{(\sqrt{p};\sqrt{p})_\infty}{(-\sqrt{p};\sqrt{p})_\infty}\theta_p\left(-\frac{K}{x}\right) = \theta_{\sqrt{p}}\left(\sqrt{\frac{K}{x}}\right)\theta_{\sqrt{p}}\left(-\sqrt{\frac{K}{x}}\right).$$

We remark that $(\sqrt{p};\sqrt{p})_\infty/(-\sqrt{p};\sqrt{p})_\infty$ can be rewritten as follows [2]:

$$\frac{(\sqrt{p};\sqrt{p})_\infty}{(-\sqrt{p};\sqrt{p})_\infty} = \sum_{n\in\mathbb{Z}}(-1)^n(\sqrt{p})^{n^2} = \theta_p(-\sqrt{p}).$$

We obtain the conclusion. ■

Therefore, we obtain the following relation.

Corollary 1. *For any $x \in \mathbb{C}^*$ $(-\pi < \arg x \le \pi)$, we have*

$$\frac{\theta_p\left(p^{2\nu+2}\frac{-1}{(1-p)^2x}\right)}{\theta_p\left(p^{\nu+2}\frac{-1}{(1-p)^2x}\right)} = \frac{\theta_{\sqrt{p}}\left(p^{\nu+1}\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p^{\nu+1}\frac{-1}{(1-p)\sqrt{x}}\right)}{\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{-1}{(1-p)\sqrt{x}}\right)} \tag{11}$$

and

$$\frac{\theta_p\left(p^{2}\frac{-1}{(1-p)^2x}\right)}{\theta_p\left(p^{\nu+2}\frac{-1}{(1-p)^2x}\right)} = \frac{\theta_{\sqrt{p}}\left(p\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p\frac{-1}{(1-p)\sqrt{x}}\right)}{\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{-1}{(1-p)\sqrt{x}}\right)}. \tag{12}$$

We consider the limit $p \to 1^-$ (i.e., $\sqrt{p} \to 1^-$) of (11) and (12).

Lemma 6. *For any $x \in \mathbb{C}^* \setminus (-\infty, 0]$ $(-\pi < \arg x \le \pi)$, we have*

1. $\displaystyle\lim_{p\to 1^-}\frac{\theta_p\left(-\frac{p^{2\nu+2}}{x(1-p)^2}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x(1-p)^2}\right)}(1-p)^{-2\nu} = e^{\nu\pi i}x^{\nu}$ *and*

2. $\displaystyle\lim_{p\to 1^-}\frac{\theta_p\left(-\frac{p^{2}}{x(1-p)^2}\right)}{\theta_p\left(-\frac{p^{\nu+2}}{x(1-p)^2}\right)}(1-p)^{2\nu} = e^{-\nu\pi i}x^{-\nu}.$

Proof. Combining Proposition 1 and Corollary 1, we consider the limit $\sqrt{p} \to 1^-$ as follows:

$$\begin{aligned}
\frac{\theta_p\left(p^{2\nu+2}\frac{-1}{(1-p)^2x}\right)}{\theta_p\left(p^{\nu+2}\frac{-1}{(1-p)^2x}\right)}(1-p)^{-2\nu} &= \frac{\theta_{\sqrt{p}}\left(p^{\nu+1}\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p^{\nu+1}\frac{-1}{(1-p)\sqrt{x}}\right)}{\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{1}{(1-p)\sqrt{x}}\right)\theta_{\sqrt{p}}\left(p^{\frac{\nu}{2}+1}\frac{-1}{(1-p)\sqrt{x}}\right)}(1-p)^{-2\nu} \\
&= \left\{\frac{\theta_{\sqrt{p}}\left((\sqrt{p})^{2\nu+2}\frac{1}{(1-(\sqrt{p})^2)\sqrt{x}}\right)}{\theta_{\sqrt{p}}\left((\sqrt{p})^{\nu+2}\frac{1}{(1-\{\sqrt{p})^2)\sqrt{x}}\right)}\left\{1-(\sqrt{p})^2\right\}^{-\nu}\right\} \\
&\quad\times\left\{\frac{\theta_{\sqrt{p}}\left(-(\sqrt{p})^{2\nu+2}\frac{1}{(1-(\sqrt{p})^2)\sqrt{x}}\right)}{\theta_{\sqrt{p}}\left(-(\sqrt{p})^{\nu+2}\frac{1}{(1-\{\sqrt{p})^2)\sqrt{x}}\right)}\left\{1-(\sqrt{p})^2\right\}^{-\nu}\right\} \\
&\to (\sqrt{x})^{\nu}\cdot(-\sqrt{x})^{\nu} = (-x)^{\nu} = e^{\nu\pi i}x^{\nu}, \qquad \sqrt{p}\to 1^-.
\end{aligned}$$

Similarly, we can prove the latter one. We obtain the conclusion. ■

We consider the last part.

Lemma 7. *For any $x \in \mathbb{C}^*$, we have*

$$\lim_{p\to 1^-} {}_1\varphi_1\left(0; p^{1+2\nu}; p, (1-p)^2 x\right) = {}_0F_1\left(-, 1+2\nu; -x\right)$$

and

$$\lim_{p\to 1^-} {}_1\varphi_1\left(0; p^{1-2\nu}; p, p^{-2\nu}(1-p)^2 x\right) = {}_0F_1\left(-, 1-2\nu; -x\right).$$

Proof. We check each of the term of

$${}_1\varphi_1\left(0; p^{1+2\nu}; p, (1-p)^2 x\right) = \sum_{n\geq 0} \frac{1}{(p^{1+2\nu}, p; p)_n} (-1)^n p^{\frac{n(n-1)}{2}} \left\{(1-p)^2 x\right\}^n.$$

For any $n \geq 0$,

$$\begin{aligned}
&\frac{1}{(p^{1+2\nu}, p; p)_n} (-1)^n p^{\frac{n(n-1)}{2}} \left\{(1-p)^2 x\right\}^n \\
&\quad = \frac{(1-p)^n (1-p)^n}{(p^{1+2\nu}; p)_n (p; p)_n} p^{\frac{n(n-1)}{2}} (-x)^n \to \frac{1}{(1+2\nu)_n \cdot n!} (-x)^n, \qquad p \to 1^-.
\end{aligned}$$

Summing up all terms, we get

$$\sum_{n\geq 0} \frac{1}{(1+2\nu)_n \cdot n!} (-x)^n = {}_0F_1\left(-, 1+2\nu; -x\right).$$

Therefore, we obtain the conclusion. Similarly, we can prove the latter. ■

We give the proof of Theorem 2.

Proof. Apply Lemma 4, Lemma 6 and Lemma 7 to (10), we obtain

$$\begin{aligned}
h_\nu\left(\frac{1}{(1-p)^2 x}; p\right) &\to \left\{-\frac{1}{\sin(2\nu\pi)\Gamma(1+2\nu)}\right\} e^{\nu\pi i} x^\nu {}_0F_1\left(-, 1+2\nu; -x\right) \\
&\qquad + \left\{\frac{1}{\sin(2\nu\pi)\Gamma(1-2\nu)}\right\} e^{-\nu\pi i} x^{-\nu} {}_0F_1\left(-, 1-2\nu; -x\right) \\
&= \frac{-e^{\nu\pi i} J_{2\nu}\left(2\sqrt{x}\right) + e^{-\nu\pi i} J_{-2\nu}\left(2\sqrt{x}\right)}{\sin(2\nu\pi)} \\
&= \frac{e^{-\nu\pi i}}{i} H^{(2)}_{2\nu}\left(2\sqrt{x}\right), \qquad p \to 1^-.
\end{aligned}$$

Therefore, we acquire the conclusion. ■

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