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The Integrability of New Two-Component KdV Equation*

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Abstract. We consider the bi-Hamiltonian representation of the two-component coupled KdV equations discovered by Drinfel'd and Sokolov and rediscovered by Sakovich and Foursov. Connection of this equation with the supersymmetric Kadomtsev–Petviashvilli–Radul–Manin hierarchy is presented. For this new supersymmetric equation the Lax representation and odd Hamiltonian structure is given.

Key words: KdV equation; Lax representation; integrability; supersymmetry

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1 Introduction

The scalar KdV equation admits various generalizations to the multifield case and have been often considered in the literature [1, 2, 3, 4, 5]. However, the present classification of such systems is not complete and depends on the assumption which we made on the very beginning.

Svinolupov [1] has introduced the class of equations

$$u^i_t = u^i_{xxx} + a^i_{j,k} u^j u^k_x, \tag{1}$$

where $i, j, k = 1, 2, \dots, N$, u_i are functions depending on the variables x and t and $a^i_{j,k}$ are constants. The $a^i_{j,k}$ satisfy the same relations as the structural constants of Jordan algebra

$$a^n_{j,k}\big(a^j_{n,r} a^r_{m,s} - a^i_{m,r} a^r_{n,s}\big) + \mathrm{cyclic}(j, k, m) = 0.$$

These equations possess infinitely many higher generalized symmetries.

Gürses and Karasu [2] extended the Svinolupov construction to the system of equations

$$u^j_t = b^i_j u^i_{xxx} + s^i_{j,k} u^j u^k_x, \tag{2}$$

where b^i_j, $s^i_{j,k}$ are constants.

In general, existence of infinitely many conserved quantities is admitted as the definition of integrability. This implies existence of infinitely many generalized symmetries. Gürses and Karasu, in order to check the integrability of the system of equations (2), assumed that the system is integrable if it admits a recursion operator. Assuming the general form of the second and fourth order recursion operator they found the conditions on the coefficients b^i_j, $s^i_{j,k}$ so that the equations in (2) are integrable.

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A quite different generalization of multicomponent KdV system has been found by Antonowicz and Fordy [3] considering the energy dependent Schrödinger operator. Ma [4] also presented a multicomponent KdV system considering decomposable hereditary operators.

Several years ago Foursov [5] found the conditions on the coefficients b^i_j, $s^i_{j,k}$ under which the two-component system (2) possesses at least 5 generalized symmetries and conserved quantities. He carried out computer algebra computations and found that there are five such systems which are not symmetrical and not triangular. Three of them are known to be integrable, while two of them are new. Foursov conjectured that these two new systems should be integrable. However, it appeared that one of these new systems is not new and has been known for many years. Drinfel'd and Sokolov in 1981 [6] presented the Lax pair for one of these new equation and hence this equation is integrable.

In this paper we present the bi-Hamiltonian formulation and recursion operator for the *new* equation. These results have been obtained during the study of the so called supersymmetric Manin–Radul hierarchy. The application of the supersymmetry to the construction of new integrable systems appeared almost in parallel to the use of this symmetry in the quantum field theory. The quantum field theories with exact correspondences between bosonic and fermionic helicity states are not the only basic ingredients for superstring theories, but have been utilised both in theoretical and experimental research in particle physics. The first results, concerned the construction of classical field theories with fermionic and bosonic fields depending on time and one space variable, can be found in [7, 8, 9, 10]. In many cases, addition of fermion fields does not guarantee that the final theory becomes supersymmetric invariant. Therefore this method was named as a fermionic extension in order to distinguish it from the fully supersymmetrical method which was developed later [11, 12, 13, 14, 15]. There are many recipes how some classical models could be embedded in fully supersymmetric superspace. The main idea is simple: in order to get such generalization we should construct a supermultiplet containing the classical functions. It means that we have to add to a system of k bosonic equations kN fermions and $k(N-1)$ bosons ($k = 1, 2, \dots,\ N = 1, 2, \dots$) in such a way that they create superfields. Now working with this supermultiplet we can step by step apply integrable Hamiltonians methods to our considerations depending on what we would like to construct.

Manin and Radul in 1985 [10], introduced a new system of equations for an infinite set of even and odd functions, depending on an even-odd pair of space variables and even-odd times. This system of equations now called the Manin–Radul supersymmetric Kadomtsev–Petviashvili hierarchy (MR-SKP). It appeared that this hierarchy contains the supersymmetric generalization of the Korteweg–de Vries equation, the Sawada–Kotera equation and as we show in this paper two-component coupled KdV equations discovered by Drinfel'd–Sokolov.

2 Two-component KdV systems

Let us consider a system of two equations

$$u_t = F[u, v], \qquad v_t = G[u, v],$$

where $F[u, v] = F(u, v, u_x, v_x, \dots)$ denotes a differential polynomial function of u and v.

By the triangular system we understand such system which involves either an equation depending only on u or an equation depending only on v while by the symmetrical we understand such system in which $G[u, v] = F[v, u]$.

Definition 1. A system of t-independent evolution equations

$$u_t = Q_1[u, v], \qquad v_t = Q_2[u, v]$$

is said to be a generalized symmetry of (1) if their flows formally commute

$$\mathbf{D_K}(\mathbf{Q}) - \mathbf{D_Q}(\mathbf{K}) = \mathbf{0}.$$

Here $\mathbf{Q} = (Q_1, Q_2)$, $\mathbf{K}[u,v] = (F[u,v], G[u,v])$, and $\mathbf{D_K}$ denotes the Fréchet derivative.

The first three systems in the Foursov classification are known to be integrable equations and are

$$\begin{aligned}
u_t &= u_{xxx} + 6uu_x - 12vv_x,\\
v_t &= -2v_{xxx} - 6uv_x;\\
u_t &= u_{xxx} + 3uu_x + 3vv_x,\\
v_t &= u_xv + uv_x;\\
u_t &= u_{xxx} + 2vu_x + uv_x,\\
v_t &= uu_x.
\end{aligned}$$

The first pair of equations is the Hirota–Satsuma system [16], second is the Ito system [17], third is the rescaled Drinfel'd–Sokolov equation [5].

The fourth system of equations is a new one founded by Foursov

$$\begin{aligned}
u_t &= u_{xxx} + v_{xxx} + 2vu_x + 2uv_x,\\
v_t &= v_{xxx} - 9uu_x + 6vu_x + 3uv_x + 2vv_x.
\end{aligned}$$

Foursov showed that this system possesses generalized symmetries of weights 7, 9, 11, 13, 15, 17 and 19, as well as conserved densities of weights 2, 4, 6, 8, 10, 12 and 14, and conjectured that this system is integrable and should possess infinitely many generalized symmetries.

The last system in this classification is

$$\begin{aligned}
u_t &= 4u_{xxx} + 3v_{xxx} + 4uu_x + vu_x + 2uv_x,\\
v_t &= 3u_{xxx} + v_{xxx} - 4vu_x - 2uv_x - 2vv_x,
\end{aligned} \tag{3}$$

and has been first considered many years ago by Drinfel'd and Sokolov [6] and rediscovered by S.Yu. Sakovich [18].

Let us notice that the integrable Hirota–Satsuma equation has the following Lax representation [19]

$$L = \big(\partial^2 + u + v\big)\big(\partial^2 + u - v\big), \qquad \frac{\partial L}{\partial t} = 4\big[L_+^{3/4}, L\big],$$

while the integrable Drinfel'd–Sokolov equation possesses the following Lax representation [2, 20]

$$L = \big(\partial^3 + (u-v)\partial + (u_x - v_x)/2\big)\big(\partial^3 + (u+v)\partial + (u_x + v_x)/2\big), \qquad \frac{\partial L}{\partial t} = 4\big[L_+^{3/4}, L\big].$$

On the other side, the Lax operators of the Hirota–Satsuma equation and of the Drinfel'd–Sokolov equation could be considered as special reduced Lax operators of the fourth and sixth order respectively. Indeed, the Hirota–Satsuma Lax operator could be rewritten as

$$L = \partial^4 + g_2\partial^2 + g_1\partial + g_0, \tag{4}$$

where

$$g_2 = 2u, \qquad g_1 = 2(u_x - v_x), \qquad g_0 = u_{xx} + u^2 - v_{xx} - v^2.$$

In this context, one can ask what kind of the equations follows from the fifth-order Lax operator which is parametrised by two functions of same weight. Let us therefore consider the following Lax operator

$$L = \partial^5 + h_2\partial^3 + h_3\partial^2 + h_4\partial + h_5,$$

where h_i, $i = 2, 3, 4, 5$ are polynomials in u and v and their derivatives of the dimension i. Computing the Lax representation for this operator

$$\frac{\partial L}{\partial t} = 5\big[L, L_+^{3/5}\big]$$

we obtained then

$$L = \left(\partial^3 + \tfrac{2}{3}u\partial + \tfrac{1}{3}u_x\right)\left(\partial^2 - \tfrac{1}{3}v\right) \tag{5}$$

produces the system of equation (3).

Let us notice that the Lax operator (4) is factorized as the product of two Lax operators. The first one is the Lax operator of the Kaup–Kupershmidt equation while the second is the Lax operator of the Korteweg–de Vries equation. It is exactly the same Lax operator which has been found by Drinfel'd and Sokolov [6].

Hence we encounter the situation in which the Lax operator of the Korteweg–de Vries and the Kaup–Kupershmidt equations can be used for construction of additional equations. This could be schematically presented as:

	$\tilde{L}_{\rm KdV}$	$\tilde{L}_{\rm KK}$
$L_{\rm KdV}$	Hirota–Satsuma	equations (3)
$L_{\rm KK}$	equations (3)	Drinfel'd–Sokolov

where $L_{\rm KdV}$, $\tilde{L}_{\rm Kdv}$ are two different Lax operators of the Korteweg–de Vries equation while $L_{\rm KK}$ and $\tilde{L}_{\rm KK}$ are two different Lax operators of the Kaup–Kupershmidt equation.

3 The recursion operator and bi-Hamiltonian structure

From the knowledge of the Lax operator for evolution equations one can infer a lot of properties of these equations. The generalized symmetries are obtained by computing the higher flow of the Lax representation while the conserved charges follow from the trace formula [21] of the Lax operator.

Using this technique we found first three conserved quantities for the equation (3)

$$\begin{aligned}
H_1 &= \int dx\big(v^2 + 4u^2 + 6uv\big),\\
H_2 &= \int dx\big(495u_{4x}u - 510u_x^2u + 32u^4 + 2v^4 + 630v_{4x}u + 180v_{4x}v - 210v_{xx}vu - 210v_x^2u\\
&\quad + 75v_x^2v - 525v_xu_xu + 14v^3u + 28v^2u^2 - 105vu_{xx}u + 56vu^3\big),\\
H_3 &= \int dx\big(182250u_{8x}u + 769500u_{4x}u_{xx}u + 445500u_{xxx}^2u + 259200u_{xx}^2u^2 + 223425u_{xx}u_x^2u\\
&\quad - 104400u_x^2u^3 + 1344u^6 + 222750v_{8x}u + 70875v_{8x}v - 148500v_{6x}vu - 160875v_{5x}u_xu\\
&\quad - 594000v_{5x}v_xu - 1113750v_{5x}v_{xx}u - 128250v_{5x}v_{xx}v + 54450v_{5x}v^2u - 825v_{5x}vu^2\\
&\quad - 742500v_{xxx}^2u - 74250v_{xxx}^2v - 61875v_{xxx}u_{xxx}u - 217800v_{xxx}u_xu^2 + 267300v_{xxx}v_xvu\\
&\quad + 70125v_{xx}^2u^2 + 19575v_{xx}^2v^2 + 163350v_{xx}^22vu - 193050v_{xx}u_x^2u + 297000v_{xx}v_x^2u
\end{aligned}$$

$$+ 17550v_{xx}v_x^2v + 199650v_{xx}v_xu_xu - 9900v_{xx}v^3u + 15400v_{xx}vu^3 - 32175v_x^2u_{xx}u$$
$$- 4400v_x^2u^3 + 3600v_x^2v^3 - 19800v_x^2v^2u + 13200v_x^2vu^2 - 185625v_xu_{5x}u$$
$$+ 188100v_xu_{xx}u_xu - 79200v_xu_xu^3 + 825v_xv^2u_xu + 57750v_xvu_{xxx}u + 21v^6$$
$$+ 198v^5u + 660v^4u^2 - 7425v^3u_{xx}u + 440v^3u^3 + 44550v^2u_{4x}u - 11550v^2u_x^2u$$
$$+ 2640v^2u^4 - 111375vu_{6x}u - 4950vu_{xx}^2u - 59400vu_x^2u^2 + 3168vu^5\big).$$

Taking into the account a simple form of the first Hamiltonian it is possible to guess the first Hamiltonian structure

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = P\begin{pmatrix} \dfrac{\delta H_1}{\delta u} \\ \dfrac{\delta H_1}{\delta v} \end{pmatrix} = \begin{pmatrix} 3\partial^3 + \partial u + u\partial & 0 \\ 0 & 3\partial^3 - 2(\partial v + v\partial) \end{pmatrix}\begin{pmatrix} \dfrac{\delta H_1}{\delta u} \\ \dfrac{\delta H_1}{\delta v} \end{pmatrix}.$$

In order to define the second Hamiltonian structure we first found the recursion operator. We used the technique described in [22] and we found the following tenth-order recursion operator

$$R = \begin{pmatrix} -\frac{18}{125}\partial^{10} + 268 \text{ terms} & -\frac{11}{375}\partial^{10} + 268 \text{ terms} \\ -\frac{11}{365}\partial^{10} + 268 \text{ terms} & -\frac{7}{375}\partial^{10} + 268 \text{ terms} \end{pmatrix}.$$

Next we assumed that this operator could be factorized as $R = J^{-1}P$ where J^{-1} is the inverse Hamiltonian operator. Due to the diagonal form of the first Hamiltonian structure it is easy to carry out such procedure and as a result we obtained the second Hamiltonian structure

$$J^{-1}\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{\delta H_4}{\delta u} \\ \frac{\delta H_4}{\delta v} \end{pmatrix},$$

where

$$H_4 = \int dx\big(21u_{10x}u + 26v_{10x}u + 95 \text{ terms}\big)$$

and the explicit form of H_4 and J^{-1} is given in the appendix.

4 The derivation of the Lax representation

The Lax operator of equations (5) has been discovered accidentally during the investigations of the supersymmetric Manin–Radul hierarchy. This hierarchy can be described by the supersymmetric Lax operator

$$L = \mathcal{D} + f_0 + \sum_{j=1}^{\infty} b_j\partial^{-j}\mathcal{D} + \sum_{j=1}^{\infty} f_j\partial^{-j}, \tag{6}$$

where the coefficients b_j, f_j are bosonic and fermionic superfield functions, respectively. We shall use the following notation throughout the paper: ∂ and $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\partial$. As usual, (x, θ) denotes $N = 1$ superspace coordinates. For any super pseudodifferential operator $\mathcal{A} = \sum_j a_{j/2}\mathcal{D}^j$ the subscripts $\pm$ denote its purely differential part $\mathcal{A}_+ = \sum_{j\geq 0} a_{j/2}\mathcal{D}^j$ or its purely pseudo-differential part $\mathcal{A}_- = \sum_{j\geq 1} a_{-j/2}\mathcal{D}^{-j}$ respectively. For any $\mathcal{A}$ the super-residuum is defined as $\operatorname{Res}\mathcal{A} = a_{-1/2}$.

The constrained (r, m) supersymmetric Manin–Radul hierarchy [25] is defined by the following Lax operator

$$L = \mathcal{D}^r + \sum_{j=0}^{r-1} \Psi_{j/2}\mathcal{D}^j + \sum_{j=0}^{m} \Upsilon_{\frac{m-j}{2}}\mathcal{D}^{-1}\Psi_{j/2}.$$

This hierarchy for even r has been widely studied in the literature in contrast to the odd r which is less known. Further we will consider this hierarchy for odd $r = 3, 5$ and $m = 0$, $\Upsilon = \Psi = 0$.

The Lax operator for $r = 3$ and $m = 0$, $\Upsilon = \Psi = 0$ has been considered recently by Tian and Liu [24]

$$L = \mathcal{D}^3 + \Phi,$$

where Φ is a superfermion function. Let us consider the following tower of equations

$$L_{t,k} = 9\big[L, L_+^{k/3}\big].$$

The first four consistent nontrivial equations are

$$\begin{aligned}
&\Phi_{t,2} = \Phi_x,\\
&\Phi_{\tau,7} = \big(\Phi_{1,xx} + \tfrac{1}{2}\Phi_1^2 + 3\Phi\Phi_x\big)_x,\\
&\Phi_{t,10} = \Phi_{5x} + 5\Phi_{xxx}\Phi_1 + 5\Phi_{xx}\Phi_{1,x} + 5\Phi_x\Phi_1^2,\\
&\Phi_{\tau,11} = \Phi_{1,5x} + 3\Phi_{1,xxx}\Phi_1 + 6\Phi_{1,xx}\Phi_{1,x} + 2\Phi_{1,x}\Phi_1^2\\
&\qquad\quad - 3\Phi_{4x}\Phi - 2\Phi_{xxx}\Phi_x - 6\Phi_{xx}\Phi\Phi_1 - 6\Phi_x\Phi\Phi_{1,x},
\end{aligned}$$

where t is a usual time while τ is an odd time.

The third equation in the hierarchy in the component $\Phi = \xi + \theta w$ reads

$$\begin{aligned}
&\xi_t = \xi_{5x} + 5w\xi_{xxx} + 5w_x\xi_{xx} + 5u^2\xi_x,\\
&w_t = w_{5x} + 5ww_{xxx} + 5w_xw_{xx} + 5w^2w_x - 5\xi_{xxx}\xi_x
\end{aligned}$$

and it is a supersymmetric generalization of the Sawada–Kotera equation. This equation is a bi-Hamiltonian system with odd supersymmetric Poisson brackets [23]. The proper Hamiltonian operator which satisfies the Jacobi identity and generates the supersymmetric $N = 1$ Sawada–Kotera equation is

$$\Phi_{t,10} = P\frac{\delta H_1}{\delta\Phi},$$

where $H_1 = \int \Phi\Phi_x\, dxd\theta$ and

$$P = \big(\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}\big)\partial^{-1}\big(\mathcal{D}\partial^2 + 2\partial\Phi + 2\Phi\partial + \mathcal{D}\Phi\mathcal{D}\big).$$

The implectic operator for this equation was defined in [23] as

$$J\Phi_t = \frac{\delta H_3}{\delta\Phi}, \qquad J = \partial_{xx} + (\mathcal{D}\Phi) - \partial^{-1}(\mathcal{D}\Phi)_x + \partial^{-1}\Phi_x\mathcal{D} + \Phi_x\partial^{-1}\mathcal{D}, \tag{7}$$

where

$$\begin{aligned}
H_3 = \int dxd\theta\big(&\Phi_{7x}\Phi + 8\Phi_{xxx}\Phi(\mathcal{D}\Phi)_{xx} + \Phi_x\Phi(4(\mathcal{D}\Phi)_{4x}\\
&+ 20(\mathcal{D}\Phi)_{xx}(\mathcal{D}\Phi) + 10(\mathcal{D}\Phi)_x^2 + \tfrac{8}{3}(\mathcal{D}\Phi)^3)\big).
\end{aligned}$$

This supersymmetric equation possesses an infinite number of conserved charges [24] which are generated by the supertrace formula of the Lax operator. However, these charges are not reduced to the known conserved charges in the bosonic limit. Hence we can not in general conclude that from the supersymmetric integrability follow the integrability of the bosonic sector.

Let us now consider the Lax operator (6) for $r = 5$ and $m = 0$

$$L = \mathcal{D}^5 + \tfrac{1}{3}(\partial U + U\partial) - \tfrac{1}{3}\mathcal{D}V\mathcal{D},$$

where U and V are superfermionic functions $U = \xi + \theta u$, $V = \psi + \theta v$. The first nontrivial equations in the hierarchy generated by this Lax operator is given as

$$\begin{aligned}
&L_t = \left[L_+^{6/5}, L\right],\\
&U_t = 4U_{xxx} + 3V_{xxx} - 2U_x(\mathcal{D}U + \mathcal{D}V) + U(6\mathcal{D}U_x + 2\mathcal{D}V_x) - V_x\mathcal{D}V + V(3\mathcal{D}U_x + \mathcal{D}V_x),\\
&V_t = 3U_{xxx} + V_{xxx} + 8U_x\mathcal{D}U - U(8\mathcal{D}U_x - 6\mathcal{D}V_x) + V_x(4\mathcal{D}U + \mathcal{D}V) - V(4\mathcal{D}U_x + 3\mathcal{D}V_x).
\end{aligned}$$

The bosonic sector of the latter system where $\xi = 0$, $\psi = 0$ gives us the system of two interacted KdV type equations discovered by Drinfel'd–Sokolov.

Interestingly, the Lax operator equations (7) did not reduce in the bosonic sector to our Lax operator (5), however, its second power reduces that one can easy verify. As we checked, this system possesses the same properties as the supersymmetric Sawada–Kotera equation. Namely, this model, due to the Lax representation, has an infinite number of conserved quantities, which are not reduced to the usual conserved charges in the bosonic limit. For example, the first two conserved charges are

$$\begin{aligned}
H_1 &= \int dx d\theta (4U_xU + 6V_xV + V_xV),\\
H_2 &= \int dx d\theta \big(75U_{xxx}U + 32U_xU(\mathcal{D}U) - 24U_xU(\mathcal{D}V) + 90V_{xxx}U + 30V_{xxx}V\\
&\quad + 36V_xU(\mathcal{D}U) - 6V_xU(\mathcal{D}V) - 4V_xV(\mathcal{D}V) - 30VU(\mathcal{D}V_x)\big).
\end{aligned} \tag{8}$$

We found the following odd Hamiltonian structure for our supersymmetric equation (8)

$$\frac{d}{dt}\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} \frac{1}{30} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{2}{15} \end{pmatrix} \begin{pmatrix} \dfrac{\delta H_2}{\delta u} \\ \dfrac{\delta H_2}{\delta v} \end{pmatrix}.$$

Unfortunately, we have been not able to found second Hamiltonian structure for our superequation.

A Appendix

The conserved quantity H_4 is

$$\begin{aligned}
H_4 = {} & u_{10x}u - \tfrac{7312}{315}u_{5x}u_{xxx}u - \tfrac{6638}{315}u_{4x}^2u - \tfrac{2032}{945}u_{xxx}^2u^2 - \tfrac{3496}{315}u_{xxx}u_{xx}u_xu - \tfrac{584}{945}u_{xx}^3u\\
& + \tfrac{2416}{2025}u_{xx}^2u^3 - \tfrac{6196}{6075}u_x^4u - \tfrac{448}{1215}u_x^2u^4 + \tfrac{8704}{3189375}u^7 + \tfrac{26}{21}v_{10x}u + \tfrac{8}{21}v_{10x}v - \tfrac{338}{315}v_{8x}vu\\
& - \tfrac{1612}{315}v_{7x}v_xu - \tfrac{832}{63}v_{6x}v_{xx}u + \tfrac{52}{105}v_{6x}v^2u - \tfrac{832}{35}v_{5x}v_{xxx}u + \tfrac{1234}{315}v_{5x}v_{xxx}v + \tfrac{208}{63}v_{5x}v_xvu\\
& + \tfrac{416}{1575}v_{5x}vu_xu - \tfrac{494}{35}v_{4x}^2u + \tfrac{1121}{315}v_{4x}^2v + \tfrac{10127}{315}v_{4x}u_{4x}u + \tfrac{1976}{315}v_{4x}v_{xx}vu + \tfrac{572}{105}v_{4x}v_x^2u\\
& + \tfrac{6136}{2835}v_{4x}v_xu_xu - \tfrac{572}{4725}v_{4x}v^3u - \tfrac{416}{14175}v_{4x}v^2u^2 - \tfrac{416}{945}v_{xxx}^2u^2 - \tfrac{53}{315}v_{xxx}^2v^2 + \tfrac{754}{189}v_{xxx}^2vu\\
& + \tfrac{23582}{315}v_{xxx}u_{5x}u + \tfrac{416}{189}v_{xxx}v_{xx}u_xu + \tfrac{1144}{63}v_{xxx}v_{xx}v_xu - \tfrac{872}{945}v_{xxx}v_{xx}v_xv - \tfrac{3952}{4725}v_{xxx}v_xv^2u
\end{aligned}$$

$$
\begin{aligned}
&-\tfrac{130}{567}v_{xxx}v_xvu^2-\tfrac{52}{945}v_{xxx}vu_{xxx}u+\tfrac{104}{27}v_{xx}^3u-\tfrac{74}{945}v_{xx}^3v-\tfrac{416}{945}v_{xx}^2u_{xx}u+\tfrac{2288}{14175}v_{xx}^2u^3\\
&-\tfrac{88}{2025}v_{xx}^2v^3-\tfrac{7592}{14175}v_{xx}^2v^2u-\tfrac{26}{135}v_{xx}^2vu^2+\tfrac{2782}{45}v_{xx}u_{6x}u+\tfrac{2548}{135}v_{xx}u_{xx}^2u-\tfrac{3484}{2025}v_{xx}v_x^2vu\\
&-\tfrac{2236}{2835}v_{xx}v_xu_{xxx}u+\tfrac{104}{6075}v_{xx}v^4u+\tfrac{208}{42525}v_{xx}v^3u^2+\tfrac{104}{405}v_{xx}v^2u_{xx}u-\tfrac{494}{2025}v_x^4u\\
&+\tfrac{233}{6075}v_x^4v+\tfrac{26}{567}v_x^3u_xu-\tfrac{1976}{4725}v_x^2u_{xx}u^2-\tfrac{1924}{1575}v_x^2u_x^2u-\tfrac{832}{14175}v_x^2u^4-\tfrac{38}{6075}v_x^2v^4\\
&+\tfrac{52}{1215}v_x^2v^3u-\tfrac{208}{8505}v_x^2v^2u^2+\tfrac{286}{567}v_x^2vu_{xx}u-\tfrac{3328}{42525}v_x^2vu^3+\tfrac{728}{45}v_xu_{7x}u-\tfrac{8164}{315}v_xu_{4x}u_xu\\
&-\tfrac{10348}{945}v_xu_{xxx}u_{xx}u-\tfrac{11518}{4725}v_xu_x^3u+\tfrac{3952}{4725}v_xvu_{5x}u-\tfrac{12688}{14175}v_xvu_{xx}u_xu-\tfrac{68}{3189375}v^7\\
&-\tfrac{104}{455625}v^6u-\tfrac{416}{455625}v^5u^2+\tfrac{104}{8505}v^4u_{xx}u-\tfrac{832}{637875}v^4u^3-\tfrac{1144}{14175}v^3u_{4x}u+\tfrac{1664}{42525}v^3u_{xx}u^2\\
&+\tfrac{416}{14175}v^3u_x^2u+\tfrac{1664}{637875}v^3u^4+\tfrac{338}{945}v^2u_{6x}u+\tfrac{208}{4725}v^2u_{4x}u^2+\tfrac{104}{4725}v^2u_{xxx}u_xu\\
&-\tfrac{104}{4725}v^2u_{xx}^2u+\tfrac{1664}{14175}v^2u_{xx}u^3+\tfrac{10816}{42525}v^2u_x^2u^2+\tfrac{3328}{455625}v^2u^5-\tfrac{247}{315}vu_{8x}u+\tfrac{208}{105}vu_{6x}u^2\\
&-\tfrac{10868}{945}vu_{5x}u_xu-\tfrac{14872}{315}vu_{4x}u_{xx}u+\tfrac{4576}{4725}vu_{4x}u^3-\tfrac{29692}{945}vu_{xxx}^2u+\tfrac{102128}{14175}vu_{xxx}u_xu^2\\
&+\tfrac{65416}{14175}vu_{xx}^2u^2+\tfrac{123682}{14175}vu_{xx}u_x^2u+\tfrac{1664}{6075}vu_{xx}u^4+\tfrac{4576}{6075}vu_x^2u^3+\tfrac{3328}{455625}vu^6.
\end{aligned}
$$

The inverse Hamiltonian operator J^{-1} has the following form

$$
J^{-1}=\begin{pmatrix} J^{-1}_{1,1} & J^{-1}_{1,2} \\ -\big(J^{-1}_{1,2}\big)^* & J^{-1}_{2,2}\end{pmatrix},
$$

where

$$
\begin{aligned}
J^{-1}_{1,1}&=-\tfrac{3}{125}\partial^7+a_{1,1,5}\partial^5+a_{1,1,3}\partial^3+a_{1,1,1}\partial+b_{1,1}\partial^{-1}+b_{1,1,1}\partial^{-1}b_{1,1,2}-\text{h.c.},\\
J^{-1}_{1,2}&=-\tfrac{11}{375}\partial^7+\sum_{i=0}^{5}a_{1,2,i}\partial^i+b_{1,2}\partial^{-1}+\partial^{-1}c_{1,2}+b_{1,2,1}\partial^{-1}b_{1,2,2},\\
J^{-1}_{2,2}&=-\tfrac{7}{750}\partial^7+a_{2,2,5}\partial^5+a_{2,2,3}\partial^3+a_{2,2,1}\partial+b_{2,2}\partial^{-1}+b_{2,2,1}\partial^{-1}b_{2,2,2}-\text{h.c.},\\
a_{1,1,5}&=(-43u+14v)/1125,\\
a_{1,1,3}&=(225u_{xx}-448u^2-690v_{xx}-77v^2-42vu)/16875,\\
a_{1,1,1}&=\big(-3150u_{4x}-1800u_{xx}u+2397u_x^2-888u^3+720v_{4x}-2520v_{xx}u+420v_{xx}v\\
&\quad+798v_x^2-1512v_xu_x+56v^3-126v^2u-728vu^2\big)/101250,\\
b_{1,1}&=\big(-6750u_{6x}-9540u_{4x}u-19080u_{xxx}u_x-14310u_{xx}^2-5520u_{xx}u^2-5520u_x^2u-352u^4\\
&\quad-4050v_{6x}-3780v_{4x}u+1890v_{4x}v-810v_{3x}u_x+6480v_{3x}v_x+3510v_{xx}^2+2160v_{xx}u_{xx}\\
&\quad-2160v_{xx}u^2-540v_{xx}v^2-1440v_{xx}vu+720v_x^2u-810v_x^2v+5940v_xu_{xxx}-1080v_xu_xu\\
&\quad-2160v_xvu_x+18v^4+96v^3u-1080v^2u_{xx}-288v^2u^2+2970vu_{4x}-1080vu_{xx}u\\
&\quad-540vu_x^2-576vu^3\big)/759375,\\
b_{1,1,1}\partial^{-1}b_{1,1,2}&=\big(-990u_{4x}-1020u_{xx}u-510u_x^2-128u^3-630v_{4x}-420v_{xx}u\\
&\quad+210v_{xx}v+210v_x^2+210v_xu_x-14v^3-56v^2u+210vu_{xx}\\
&\quad-168vu^2\big)\partial^{-1}(4u+3v)/759375,\\
a_{1,2,5}&=(-38u+19v)/1125,\\
a_{1,2,4}&=(-36u+68v)_x/1125,\\
a_{1,2,3}&=\big(-45u_{xx}-124u^2+515v_{xx}-31v^2-36vu\big)/5625,\\
a_{1,2,2}&=\big(885u_{3x}-656u_xu+1680v_{3x}+178v_xu-354v_xv-332vu_x\big)/16875,\\
a_{1,2,1}&=\big(2520u_{4x}-1428u_{xx}u-1599u_x^2-328u^3+2610v_{4x}-156v_{xx}u\\
&\quad-957v_{xx}v-876v_x^2-1716v_xu_x+41v^3+114v^2u-1371vu_{xx}-228vu^2\big)/50625,
\end{aligned}
$$

$$
\begin{aligned}
a_{1,2,0} = \big(&1575u_{5x} + 534u_{3x}u + 276u_{xx}u_x - 128u_xu^2 + 1080v_{5x} + 198v_{3x}u\\
&- 534v_{3x}v - 873v_{xx}u_x - 1206v_{xx}v_x - 1338v_xu_{xx} + 84v_xu^2\\
&+ 96v_xv^2 + 166v_xvu + 128v^2u_x - 792vu_{3x} + 8vu_xu\big)/50625,\\
b_{1,2} = \big(&3375u_{6x} + 4770u_{4x}u + 9540u_{3x}u_x + 7155u_{xx}^2 + 2760u_{xx}u^2\\
&+ 2760u_x^2u + 176u^4 + 2025v_{6x} + 1890v_{4x}u - 945v_{4x}v + 405v_{3x}u_x - 3240v_{3x}v_x\\
&- 1755v_{xx}^2 - 1080v_{xx}u_{xx} + 1080v_{xx}u^2 + 270v_{xx}v^2 + 720v_{xx}vu - 360v_x^2u\\
&+ 405v_x^2v - 2970v_xu_{3x} + 540v_xu_xu + 1080v_xvu_x - 9v^4 - 48v^3u\\
&+ 540v^2u_{xx} + 144v^2u^2 - 1485vu_{4x} + 540vu_{xx}u + 270vu_x^2 + 288vu^3\big)/759375,\\
c_{1,2} = \big(&-4050u_{6x} - 3780u_{4x}u - 14310u_{3x}u_x - 9045u_{xx}^2 - 2160u_{xx}u^2 - 3780u_x^2u - 144u^4\\
&- 2700v_{6x} - 2160v_{4x}u + 1530v_{4x}v - 4320v_{3x}u_x + 3060v_{3x}v_x + 2295v_{xx}^2 - 1080v_{xx}u_{xx}\\
&- 2160v_{xx}u^2 - 390v_{xx}v^2 - 540v_{xx}vu - 270v_x^2u - 390v_x^2v + 1080v_xu_{3x} - 4320v_xu_xu\\
&- 540v_xvu_x + 11v^4 + 72v^3u - 540v^2u_{xx} + 144v^2u^2 + 1890vu_{4x} - 1440vu_{xx}u\\
&- 360vu_x^2 - 192vu^3\big)/759375,\\
b_{1,2,1}\partial^{-1}b_{1,2,2} = \big(\big(&-990u_{4x} - 1020u_{xx}u - 510u_x^2 - 128u^3 - 630v_{4x} - 420v_{xx}u + 210v_{xx}v\\
&+ 210v_x^2 + 210v_xu_x + 14v^3 - 56v^2u + 210vu_{xx} - 168vu^2\big)\partial^{-1}(3u+v)\\
&- (4u+3v)\partial^{-1}\big(630u_{4x} + 420u_{xx}u + 525u_x^2 + 56u^3 + 360v_{4x} - 150v_{xx}v\\
&- 75v_x^2 + 8v^3 + 42v^2u - 210vu_{xx} + 56vu^2\big)\big)/759375,\\
a_{2,2,5} = (&-9u + 7v)/1125,\\
a_{2,2,3} = \big(&540u_{xx} - 122u^2 - 30v_{xx} - 33v^2 - 28vu\big)/16875,\\
a_{2,2,1} = \big(&-1170u4x + 900u_{xx}u + 1203u_x^2 - 120u^3 + 540v_{4x} - 150v_{xx}v\\
&+ 162v_x^2 - 108v_xu_x + 30v^3 + 110v^2u - 570vu_{xx} + 120vu^2\big)/101250,\\
b_{2,2} = \big(&4050u_{6x} + 3780u_{4x}u + 14310u_{3x}u_x + 9045u_{xx}^2 + 2160u_{xx}u^2 + 3780u_x^2u\\
&+ 144u^4 + 2700v_{6x} + 2160v_{4x}u - 1530v_{4x}v + 4320v_{3x}u_x - 3060v_{3x}v_x\\
&- 2295v_{xx}^2 + 1080v_{xx}u_{xx} + 2160v_{xx}u^2 + 390v_{xx}v^2 + 540v_{xx}vu + 270v_x^2u + 390v_x^2v\\
&- 1080v_xu_{3x} + 4320v_xu_xu + 540v_xvu_x - 11v^4 - 72v^3u + 540v^2u_{xx}\\
&- 144v^2u^2 - 1890vu_{4x} + 1440vu_{xx}u + 360vu_x^2 + 192vu^3\big)/1518750,\\
b_{2,2,1}\partial^{-1}b_{2,2,2} = \big(&-630u_{4x} - 420u_{xx}u - 525u_x^2 - 56u^3 - 360v_{4x}\\
&+ 150v_{xx}v + 75v_x^2 - 8v^3 - 42v^2u + 210vu_{xx} - 56vu^2\big)\partial^{-1}(3u+v)/759375.
\end{aligned}
$$

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