The Group of Quasisymmetric Homeomorphisms of the Circle and Quantization of the Universal Teichmüller Space*

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Abstract. In the first part of the paper we describe the complex geometry of the universal Teichmüller space $\mathcal{T}$, which may be realized as an open subset in the complex Banach space of holomorphic quadratic differentials in the unit disc. The quotient $\mathcal{S}$ of the diffeomorphism group of the circle modulo Möbius transformations may be treated as a smooth part of $\mathcal{T}$. In the second part we consider the quantization of universal Teichmüller space $\mathcal{T}$. We explain first how to quantize the smooth part $\mathcal{S}$ by embedding it into a Hilbert–Schmidt Siegel disc. This quantization method, however, does not apply to the whole universal Teichmüller space $\mathcal{T}$, for its quantization we use an approach, due to Connes.

Key words: universal Teichmüller space; quasisymmetric homeomorphisms; Connes quantization

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1 Introduction

The universal Teichmüller space $\mathcal{T}$, introduced by Ahlfors and Bers, plays a key role in the theory of quasiconformal maps and Riemann surfaces. It can be defined as the space of quasisymmetric homeomorphisms of the unit circle S^1 (i.e. homeomorphisms of S^1, extending to quasiconformal maps of the unit disc Δ) modulo Möbius transformations. The space $\mathcal{T}$ has a natural complex structure, induced by its realization as an open subset in the complex Banach space $B_2(\Delta)$ of holomorphic quadratic differentials in the unit disc Δ. The space $\mathcal{T}$ contains all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group, as complex submanifolds. The space $\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of normalized diffeomorphisms of the circle may be considered as a "smooth" part of $\mathcal{T}$.

Our motivation to study $\mathcal{T}$ comes from the string theory. Physicists have noticed (cf. [15, 3]) that the space $\Omega_d := C_0^\infty(S^1, \mathbb{R}^d)$ of smooth loops in the d-dimensional vector space $\mathbb{R}^d$ may be identified with the phase space of bosonic closed string theory. By looking at a natural symplectic form ω on Ω_d, induced by the standard symplectic form (of type "$dp \wedge dq$") on the phase space, one sees that this form can be, in fact, extended to the Sobolev completion of Ω_d, coinciding with the space $V_d := H_0^{1/2}(S^1, \mathbb{R}^d)$ of half-differentiable vector-functions. Moreover, the latter space is the largest in the scale of Sobolev spaces $H_0^s(S^1, \mathbb{R}^d)$, on which ω is correctly defined. So the form ω itself chooses the "right" space to be defined on. From that point of view, it seems more natural to consider V_d as the phase space of bosonic string theory, rather than Ω_d. In this paper we set $d = 1$ to simplify the formulas and study the space $V := V_1 = H_0^{1/2}(S^1, \mathbb{R})$.

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According to Nag–Sullivan [12], there is a natural group, attached to the space $V = H_0^{1/2}(S^1, \mathbb{R})$, and this is precisely the group $\mathrm{QS}(S^1)$ of quasisymmetric homeomorphisms of the circle. Again one can say that the space V itself chooses the "right" group to be acted on. The group $\mathrm{QS}(S^1)$ acts on V by reparametrization of loops and this action is symplectic with respect to the form ω. The universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ can be identified by this action with a space of complex structures on V, compatible with ω.

The second half of the paper is devoted to the quantization of the universal Teichmüller space $\mathcal{T}$. We start from the Dirac quantization of the smooth part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$. This is achieved by embedding of $\mathcal{S}$ into the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Under this embedding the diffeomorphism group $\mathrm{Diff}_+(S^1)$ is realized as a subgroup of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$, acting on the Siegel disc by operator fractional-linear transformations. There is a holomorphic Fock bundle $\mathcal{F}$ over $\mathcal{D}_{\mathrm{HS}}$, provided with a projective action of $\mathrm{Sp}_{\mathrm{HS}}(V)$, covering its action on $\mathcal{D}_{\mathrm{HS}}$. The infinitesimal version of this action is a projective representation of the Hilbert–Schmidt symplectic Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in a fibre F_0 of the Fock bundle $\mathcal{F}$. This defines the Dirac quantization of the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Its restriction to $\mathcal{S}$ gives a projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ of the group $\mathrm{Diff}_+(S^1)$ in the Fock space F_0, which defines the Dirac quantization of the space $\mathcal{S}$.

However, the described quantization procedure does not apply to the whole universal Teichmüller space $\mathcal{T}$. By this reason we choose another approach to this problem, based on Connes quantization. (We are grateful to Alain Connes for drawing our attention to this approach, presented in [5].) Briefly, the idea is the following. The $\mathrm{QS}(S^1)$-action on $\mathcal{T}$, mentioned above, cannot be differentiated in classical sense (in particular, there is no Lie algebra, associated to $\mathrm{QS}(S^1)$). However, one can define a quantized infinitesimal version of this action by associating with any quasisymmetric homeomorphism $f \in \mathrm{QS}(S^1)$ a quantum differential $d^q f$, being an integral operator on V with kernel, given essentially by the finite-difference derivative of f. In these terms the quantization of $\mathcal{T}$ is given by a representation of the algebra of derivations of V, generated by quantum differentials $d^q f$, in the Fock space F_0.

I. Universal Teichmüller space

2 Group of quasisymmetric homeomorphisms of S^1

2.1 Definition of quasisymmetric homeomorphisms

Definition 1. A homeomorphism $h : S^1 \to S^1$ is called *quasisymmetric* if it can be extended to a quasiconformal homeomorphism w of the unit disc Δ.

Recall that a homeomorphism $w : \Delta \to w(\Delta)$, having locally L^1-integrable derivatives (in generalized sense), is called *quasiconformal* if there exists a measurable complex-valued function $\mu \in L^\infty(\Delta)$ with $\|\mu\|_\infty := \mathrm{ess\ sup}_{z\in\Delta}|\mu(z)| =: k < 1$ such that the following *Beltrami equation*

$$w_{\bar{z}} = \mu w_z \tag{1}$$

holds for almost all $z \in \Delta$. The function μ is called a *Beltrami differential* or *Beltrami potential* of w and the constant k is often indicated in the name of the k-quasiconformal maps.

In the case when $k = 0$ the homeomorphism w, satisfying (1), coincides with a conformal map from D onto $w(D)$. For a diffeomorphism w its quasiconformality means that w transforms infinitesimal circles into infinitesimal ellipses, whose eccentricities (the ratio of the large axis to the small one) are bounded by a common constant $K < \infty$, related to the above constant $k = \|\mu\|_\infty$ by the formula

$$K = \frac{1+k}{1-k}.$$

The least possible constant K is called the *maximal dilatation* of w and is also sometimes indicated in the name of K-quasiconformal maps.

The inverse of a quasiconformal map is again quasiconformal and the same is true for the composition of quasiconformal maps. This implies that orientation-preserving quasisymmetric homeomorphisms of S^1 form a *group of quasisymmetric homeomorphisms of the circle* $\mathrm{QS}(S^1)$ with respect to composition.

Any orientation-preserving diffeomorphism $h \in \mathrm{Diff}_+(S^1)$ extends to a diffeomorphism of the closed unit disc $\overline{\Delta}$, which is evidently quasiconformal, according to the above criterion. So $\mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1)$, and we have the following chain of embeddings

$$\mathrm{M\ddot{o}b}(S^1) \subset \mathrm{Diff}_+(S^1) \subset \mathrm{QS}(S^1) \subset \mathrm{Homeo}_+(S^1).$$

Here, $\mathrm{M\ddot{o}b}(S^1)$ denotes the Möbius group of fractional-linear automorphisms of the unit disc Δ, restricted to S^1.

2.2 Beurling–Ahlfors criterion

There is an intrinsic description of quasisymmetric homeomorphisms of S^1 in terms of cross ratios. Recall that the *cross ratio* of four different points z_1, z_2, z_3, z_4 on the complex plane is given by the quantity

$$\rho = \rho(z_1, z_2, z_3, z_4) := \frac{z_4 - z_1}{z_4 - z_2} : \frac{z_3 - z_1}{z_3 - z_2}.$$

The equality of two cross ratios $\rho(z_1, z_2, z_3, z_4) = \rho(\zeta_1, \zeta_2, \zeta_3, \zeta_4)$ is a necessary and sufficient condition for the existence of a fractional-linear map of the complex plane, transforming the quadruple z_1, z_2, z_3, z_4 into the quadruple ζ_1, ζ_2, ζ_3, ζ_4. In the case of quasiconformal maps the cross ratios of quadruples may change but in a controlled way. This property, reformulated in the right way for orientation-preserving homeomorphisms of S^1, yields a criterion of quasisymmetricity, due to Ahlfors and Beurling.

The required property reads as follows: for an orientation-preserving homeomorphism $h : S^1 \to S^1$ it should exist a constant $0 < \epsilon < 1$ such that the following inequality holds

$$\frac{1}{2}(1-\epsilon) \le \rho(h(z_1), h(z_2), h(z_3), h(z_4)) \le \frac{1}{2}(1+\epsilon) \tag{2}$$

for any quadruple $z_1, z_2, z_3, z_4 \in S^1$ with cross ratio $\rho(z_1, z_2, z_3, z_4) = \frac{1}{2}$.

Theorem 1 (Beurling–Ahlfors, cf. [1, 9]). *Suppose that $h : S^1 \to S^1$ is an orientation-preserving homeomorphism of S^1. Then it can be extended to a quasiconformal homeomorphism $w : \Delta \to \Delta$ if and only if it satisfies condition (2).*

Douady and Earle (cf. [6]) have found an explicit extension operator E, assigning to a quasisymmetric homeomorphism h its extension to a quasiconformal homeomorphism w of Δ, which is conformally invariant in the sense that $g(w \circ h) = w \circ g(h)$ for any fractional-linear automorphism of Δ.

Though quasisymmetric homeomorphisms of S^1, in general, are not smooth, they enjoy certain Hölder continuity, provided by the following

Theorem 2 (Mori, cf. [1]). *Let $w : \Delta \to \Delta$ be a K-quasiconformal homeomorphism of the unit disc onto itself, normalized by the condition: $w(0) = 0$. Then the following sharp estimate*

$$|w(z_1) - w(z_2)| < 16|z_1 - z_2|^{1/K}$$

holds for any $z_1 \neq z_2 \in \Delta$. In other words, the homeomorphism w satisfies the Hölder condition of order $1/K$ in the disc Δ.

3 Universal Teichmüller space

3.1 Definition of universal Teichmüller space

Definition 2. The quotient space

$$\mathcal{T} := \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$$

is called the *universal Teichmüller space*. It can be identified with the space of *normalized* quasisymmetric homeomorphisms of S^1, fixing the points ± 1 and $-i$.

As we have pointed out earlier, there is an inclusion

$$\mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1) \hookrightarrow \mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1).$$

We consider the homogeneous space

$$\mathcal{S} := \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$$

as a "smooth" part of $\mathcal{T}$.

The space $\mathcal{T}$ can be provided with the *Teichmüller distance function*, defined by

$$\mathrm{dist}(g,h) = \frac{1}{2}\log K(h\circ g^{-1})$$

for any quasisymmetric homeomorphisms $g, h \in \mathcal{T}$, extended to quasiconformal homeomorphisms of the disc Δ. Here, $K(h\circ g^{-1})$ denotes the maximal dilatation of the quasiconformal map $h\circ g^{-1}$. This definition does not depend on the extensions of g, h to Δ and defines a metric on $\mathcal{T}$. The universal Teichmüller space is a complete connected contractible metric space with respect to the introduced distance function (cf. [9]). Unfortunately, this metric is not compatible with the group structure on $\mathcal{T}$, given by composition of quasisymmetric homeomorphisms (cf. [9, Theorem 3.3]).

The term "universal" in the name of the universal Teichmüller space is due to the fact that $\mathcal{T}$ contains, as complex submanifolds, all classical Teichmüller spaces $T(G)$, where G is a Fuchsian group (cf. [10]). If a Riemann surface X is uniformized by the unit disc Δ, so that $X = \Delta/G$, then the corresponding Techmüller space $T(G)$ may be identified with the quotient

$$T(G) = \mathrm{QS}(S^1)^G/\mathrm{Möb}(S^1),$$

where $\mathrm{QS}(S^1)^G$ is the subset of G-invariant quasisymmetric homeomorphisms in $\mathrm{QS}(S^1)$. The universal Teichmüller space $\mathcal{T}$ itself corresponds to the Fuchsian group $G = \{1\}$.

Since quasisymmetric homeomorphisms of S^1 are defined in terms of quasiconformal maps of Δ, i.e. in terms of solutions of Beltrami equation in Δ, one can expect that there is a definition of $\mathcal{T}$ directly in terms of Beltrami differentials. Denote by $B(\Delta)$ the set of Beltrami differentials in the unit disc Δ. It follows from above that it can be identified (as a set) with the unit ball in the complex Banach space $L^\infty(\Delta)$.

Given a Beltrami differential $\mu \in B(\Delta)$, we can extend it to a Beltrami differential $\check{\mu}$ on the extended complex plane $\overline{\mathbb{C}}$ by setting $\check{\mu}$ equal to zero outside the unit disc Δ. Then, applying the existence theorem for quasiconformal maps on the extended complex plane $\overline{\mathbb{C}}$ (cf. [1]), we get a normalized quasiconformal homeomorphism w^μ, satisfying Beltrami equation (1) on $\overline{\mathbb{C}}$ with potential $\check{\mu}$. This homeomorphism is conformal on the exterior Δ_- of the closed unit disc $\overline{\Delta}$ on $\overline{\mathbb{C}}$ and fixes the points ± 1, $-i$. The image $\Delta^\mu := w^\mu(\Delta)$ of Δ under the quasiconformal map w^μ is called a *quasidisc*. We associate with Beltrami differential $\mu \in B(\Delta)$ the normalized quasidisc Δ^μ. Introduce an equivalence relation between Beltrami differentials in Δ by saying that two Beltrami differentials μ and ν are equivalent if $w^\mu|_{\Delta_-} \equiv w^\nu|_{\Delta_-}$. Then the universal Teichmüller space $\mathcal{T}$ will coincide with the quotient

$$\mathcal{T} = B(\Delta)/\sim$$

of the space $B(\Delta)$ of Beltrami differentials modulo introduced equivalence relation. In other words, it coincides with *the space of normalized quasidiscs* in $\overline{\mathbb{C}}$.

3.2 Complex structure of the universal Teichmüller space

We introduce a complex structure on the universal Teichmüller space $\mathcal{T}$, using its embedding into the space of quadratic differentials.

Given an arbitrary point $[\mu]$ of $\mathcal{T}$, represented by a normalized quasidisc $w^\mu(\Delta)$, consider a map

$$\mu \longmapsto S(w^\mu|_{\Delta_-}),$$

assigning to a Beltrami differential $\mu \in [\mu]$ the Schwarz derivative of the conformal map w^μ on Δ. Due to the invariance of Schwarzian under Möbius transformations, the image of μ under the above map depends only on the class $[\mu]$ of μ in $\mathcal{T}$. Moreover, it is a holomorphic quadratic differentials in Δ_-. The latter fact follows from the transformation properties of Beltrami differentials, prescribed by Beltrami equation (according to (1), Beltrami differential behaves as a $(-1, 1)$-differential with respect to conformal changes of variable). Composing the above map with a fractional-linear biholomorphism of Δ_- onto the unit disc Δ, we obtain a map

$$\Psi : \mathcal{T} \longrightarrow B_2(\Delta), \qquad [\mu] \longmapsto \psi(\mu),$$

associating a holomorphic quadratic differential $\psi(\mu)$ in Δ with a point $[\mu]$ of the universal Teichmüller space $\mathcal{T}$.

The space $B_2(\Delta)$ of holomorphic quadratic differentials in Δ is a complex Banach space, provided with a natural hyperbolic norm, given by

$$\|\psi\|_2 := \sup_{z\in\Delta} \left(1 - |z|^2\right)^2 |\psi(z)|$$

for a quadratic differential ψ. It can be proved (cf. [9]) that $\|\psi[\mu]\|_2 \le 6$ for any Beltrami differential $\mu \in B(\Delta)$.

The constructed map $\Psi : \mathcal{T} \to B_2(\Delta)$, called a *Bers embedding*, is a homeomorphism of $\mathcal{T}$ onto an open bounded connected contractible subset in $B_2(\Delta)$, containing the ball of radius $1/2$, centered at the origin (cf. [9]).

Using the constructed embedding, we can introduce a complex structure on the universal Teichmüller space $\mathcal{T}$ by pulling it back from the complex Banach space $B_2(\Delta)$. It provides $\mathcal{T}$ with the structure of a complex Banach manifold. (Note that the topology on $\mathcal{T}$, induced by the map Ψ, is equivalent to the one, determined by the Teichmüller distance function.)

Moreover, the composition of the natural projection

$$B(\Delta) \longrightarrow \mathcal{T} = B(\Delta)/\sim$$

with the constructed map Ψ yields a holomorphic map

$$F : B(\Delta) \longrightarrow B_2(\Delta)$$

with respect to the natural complex structure on $B(\Delta)$ (cf. [10]).

II. QS-action on the Sobolev space of half-differentiable functions

4 Sobolev space of half-differentiable functions on S^1

4.1 Definition

The *Sobolev space of half-differentiable functions* on S^1 is a Hilbert space $V := H_0^{1/2}(S^1,\mathbb{R})$, consisting of functions $f \in L^2(S^1,\mathbb{R})$ with zero average over the circle, having generalized derivatives of order 1/2 again in $L^2(S^1,\mathbb{R})$. In terms of Fourier series, a function $f \in L^2(S^1,\mathbb{R})$ with Fourier series

$$f(z) = \sum_{k\neq 0} f_k z^k, \qquad f_k = \bar{f}_{-k}, \qquad z = e^{i\theta},$$

belongs to $H_0^{1/2}(S^1,\mathbb{R})$ if and only if it has a finite Sobolev norm of order 1/2:

$$\|f\|_{1/2}^2 = \sum_{k\neq 0} |k||f_k|^2 = 2\sum_{k>0} k|f_k|^2 < \infty. \tag{3}$$

The space $H_0^{1/2}(S^1,\mathbb{R})$ is well known and widely used in classical function theory (cf. [18]). However, our motivation to employ this space comes from its relation to string theory (cf. below).

4.2 Kähler structure

A symplectic form on V is given by a 2-form $\omega : V \times V \to \mathbb{R}$, defined in terms of Fourier coefficients of $\xi, \eta \in V$ by

$$\omega(\xi,\eta) = 2\,\mathrm{Im}\sum_{k>0} k\xi_k\bar{\eta}_k. \tag{4}$$

Because of (3), this form is correctly defined on V. Moreover, $H_0^{1/2}(S^1,\mathbb{R})$ is the largest Hilbert space in the scale of Sobolev spaces $H_0^s(S^1,\mathbb{R})$, $s \in \mathbb{R}$, on which this form is defined. It should be also underlined that the form ω is the only natural symplectic form on V (we shall make this point clear in Section 5.1).

We return to our motivation for studying the space V. It is well known to physicists (cf., e.g., [15, 3]) that the space $\Omega_d = C_0^\infty(S^1,\mathbb{R}^d)$ of smooth loops in the d-dimensional vector space $\mathbb{R}^d$ can be identified with the phase space of bosonic closed string theory. The space Ω_d has a natural symplectic form, which coincides with the image of the standard symplectic form (of type "$dp \wedge dq$") on the phase space of closed string theory under the above identification. This form, computed in terms of Fourier decompositions, coincides precisely with the form ω, given by (4). As we have remarked, the latter form may be extended to the Sobolev space $V_d := H_0^{1/2}(S^1,\mathbb{R}^d)$ and this space is the largest in the scale $H_0^s(S^1,\mathbb{R}^d)$ of Sobolev spaces, on which ω is correctly defined. One can say that symplectic form ω "chooses" the Sobolev space V_d. This is in contrast to Ω_d, which was taken for the phase space of string theory simply because it's easier to work with smooth loops. By this reason, we find it more natural to consider V_d as the phase space of string theory, which motivates the study of V_d in more detail. In our analysis we set $d = 1$ for simplicity.

Apart from symplectic form, the Sobolev space V has a complex structure J^0, which can be given in terms of Fourier decompositions by the formula

$$\xi(z) = \sum_{k\neq 0} \xi_k z^k \longmapsto (J^0\xi)(z) = -i\sum_{k>0} \xi_k z^k + i\sum_{k<0} \xi_k z^k.$$

This complex structure is compatible with symplectic form ω and, in particular, defines a Kähler metric g^0 on V by $g^0(\xi,\eta) := \omega(\xi, J^0\eta)$ or, in terms of Fourier decompositions,

$$g^0(\xi,\eta) = 2\mathrm{Re}\sum_{k>0} k\xi_k\bar{\eta}_k.$$

In other words, V has the structure of a Kähler Hilbert space.

The complexification $V^{\mathbb{C}} = H_0^{1/2}(S^1,\mathbb{C})$ of V is a complex Hilbert space and the Kähler metric g^0 on V extends to a Hermitian inner product on $V^{\mathbb{C}}$, given by

$$\langle\xi,\eta\rangle = \sum_{k\neq 0} |k|\xi_k\bar{\eta}_k. \tag{5}$$

We extend the symplectic form ω and complex structure operator J^0 complex linearly to $V^{\mathbb{C}}$.

The space $V^{\mathbb{C}}$ is decomposed into the direct sum of the form

$$V^{\mathbb{C}} = W_+ \oplus W_-,$$

where $W_\pm$ is the $(\mp i)$-eigenspace of the operator $J^0 \in \mathrm{End}\, V^{\mathbb{C}}$. In other words,

$$W_+ = \Big\{f \in V^{\mathbb{C}}:\ f(z) = \sum_{k>0} f_k z^k\Big\}, \qquad W_- = \overline{W}_+ = \Big\{f \in V^{\mathbb{C}}:\ f(z) = \sum_{k<0} f_k z^k\Big\}.$$

The subspaces $W_\pm$ are isotropic with respect to symplectic form ω and the splitting $V^{\mathbb{C}} = W_+ \oplus W_-$ is an orthogonal direct sum with respect to the Hermitian inner product $\langle\cdot,\cdot\rangle$, given by (5).

5 Grassmann realization of $\mathcal{T}$

5.1 QS-action on the Sobolev space

Note that any homeomorphism h of S^1, preserving the orientation, acts on $L^2_0(S^1,\mathbb{R})$ by change of variable. In other words, there is an operator $T_h : L^2_0(S^1,\mathbb{R}) \to L^2_0(S^1,\mathbb{R})$, acting by

$$T_h(\xi) := \xi\circ h - \frac{1}{2\pi}\int_0^{2\pi} \xi\left(h(\theta)\right) d\theta.$$

This operator has the following remarkable property.

Proposition 1 (Nag–Sullivan [12]). *The operator T_h acts on V, i.e. $T_h : V \to V$, if and only if $h \in \mathrm{QS}(S^1)$. Moreover, if h extends to a K-quasiconformal homeomorphism of the unit disc Δ, then the operator norm of T_h does not exceed $\sqrt{K+K^{-1}}$, where $K = K(h)$ is the maximal dilatation of h.*

Moreover, transformations T_h with $h \in \mathrm{QS}(S^1)$ generate symplectic transformations of V.

Proposition 2 (Nag–Sullivan [12]). *For any $h \in \mathrm{QS}(S^1)$ we have*

$$\omega(h^*(\xi), h^*(\eta)) = \omega(\xi,\eta)$$

for all $\xi,\eta \in V$. Moreover, the complex-linear extension of QS-action to the complexification $V^{\mathbb{C}}$ preserves the holomorphic subspace W_+ if and only if $h \in \mathrm{Möb}(S^1)$. In the latter case, T_h acts as a unitary operator on W_+.

We have pointed out in Section 4.2 that the Sobolev space V is "chosen" by the symplectic form ω. In the same way, one can say that the space V chooses the reparametrization group $QS(S^1)$. Indeed, this is the biggest reparametrization group, leaving V invariant, according to Proposition 1. On the other hand, it is a group of "canonical transformations", preserving the symplectic form ω, according to Proposition 2. So we have a natural phase space (V, ω) together with a natural group $QS(S^1)$ of its canonical transformations.

Here is an assertion, making clear in what sense ω is a unique natural symplectic form on V.

Proposition 3 (Nag–Sullivan [12]). *Suppose that $\tilde{\omega} : V \times V \to \mathbb{R}$ is a continuous bilinear form on V such that*

$$\tilde{\omega}(h^*(\xi), h^*(\eta)) = \tilde{\omega}(\xi, \eta)$$

for all $\xi, \eta \in V$ and all $h \in \text{Möb}(S^1)$. Then $\tilde{\omega} = \lambda\omega$ for some real constant λ. In particular, $\tilde{\omega}$ is non-degenerate (if it is not identically zero) and invariant under the whole group $\text{QS}(S^1)$.

5.2 Embedding of the universal Teichmüller space into an infinite-dimensional Siegel disc

The Propositions 1 and 2 imply that quasisymmetric homeomorphisms act on the Hilbert space V by bounded symplectic operators. Hence, we have a map

$$\mathcal{T} = \text{QS}(S^1)/\text{Möb}(S^1) \longrightarrow \text{Sp}(V)/\text{U}(W_+). \tag{6}$$

Here, $\text{Sp}(V)$ is the symplectic group of V, consisting of linear bounded symplectic operators on V, and $\text{U}(W_+)$ is its subgroup, consisting of unitary operators (i.e. the operators, whose complex-linear extensions to $V^{\mathbb{C}}$ preserve the subspace W_+).

In terms of the decomposition

$$V^{\mathbb{C}} = W_+ \oplus W_-$$

any linear operator $A : V^{\mathbb{C}} \to V^{\mathbb{C}}$ is written in the block form

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Such an operator belongs to symplectic group $\text{Sp}(V)$, if it has the form

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}$$

with components, satisfying the relations

$$\bar{a}^t a - b^t \bar{b} = 1, \qquad \bar{a}^t b = b^t \bar{a},$$

where a^t, b^t denote the transposed operators $a^t : W_- \to W_-$, $b^t : W_- \to W_+$. The unitary group $\text{U}(W_+)$ is embedded into $\text{Sp}(V)$ as a subgroup, consisting of diagonal block matrices of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & \bar{a} \end{pmatrix}.$$

The space

$$\text{Sp}(V)/\text{U}(W_+),$$

standing on the right hand side of (6), can be regarded as an infinite-dimensional analogue of the Siegel disc, since it may be identified with the space of complex structures on V, compatible with ω. Indeed, any such structure J determines a decomposition

$$V^{\mathbb{C}} = W \oplus \overline{W} \tag{7}$$

of $V^{\mathbb{C}}$ into the direct sum of subspaces, isotropic with respect to ω. This decomposition is orthogonal with respect to the Kähler metric g_J on $V^{\mathbb{C}}$, determined by J and ω. The subspaces W and $\overline{W}$ are identified with the $(-i)$- and $(+i)$-eigenspaces of the operator J on $V^{\mathbb{C}}$ respectively. Conversely, any decomposition (7) of the space $V^{\mathbb{C}}$ into the direct sum of isotropic subspaces determines a complex structure J on $V^{\mathbb{C}}$, which is equal to $-iI$ on W and $+iI$ on $\overline{W}$ and is compatible with ω. This argument shows that symplectic group $\mathrm{Sp}(V)$ acts transitively on the space $\mathcal{J}(V)$ of complex structures J on V, compatible with ω. Moreover, a complex structure J, obtained from a reference complex structure J^0 by the action of an element A of $\mathrm{Sp}(V)$, is equivalent to J^0 if and only if $A \in \mathrm{U}(W_+)$. Hence,

$$\mathrm{Sp}(V)/\mathrm{U}(W_+) = \mathcal{J}(V).$$

The space on the right can be, in its turn, identified with the *Siegel disc* $\mathcal{D}$, defined as the set

$$\mathcal{D} = \{Z : W_+ \to W_- \text{ is a symmetric bounded linear operator with } \bar{Z}Z < I\}.$$

The symmetricity of Z means that $Z^t = Z$ and the condition $\bar{Z}Z < I$ means that symmetric operator $I - \bar{Z}Z$ is positive definite. In order to identify $\mathcal{J}(V)$ with $\mathcal{D}$, consider the action of the group $\mathrm{Sp}(V)$ on $\mathcal{D}$, given by fractional-linear transformations $A : \mathcal{D} \to \mathcal{D}$ of the form

$$Z \longmapsto (\bar{a}Z + \bar{b})(bZ + a)^{-1},$$

where $A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \mathrm{Sp}(V)$. The isotropy subgroup at $Z = 0$ coincides with the set of operators $A \in \mathrm{Sp}(V)$ such that $b = 0$, i.e. with $\mathrm{U}(W_+)$.

So the space

$$\mathcal{J}(V) = \mathrm{Sp}(V)/\mathrm{U}(W_+)$$

can be identified with the Siegel disc $\mathcal{D}$, and we have the following

Proposition 4 (Nag–Sullivan [12]). *The map*

$$\mathcal{T} = QS(S^1)/\text{Möb}(S^1) \hookrightarrow \mathcal{J}(V) = \mathrm{Sp}(V)/U(W_+) = \mathcal{D}$$

is an equivariant holomorphic embedding of Banach manifolds.

For the smooth part $\mathcal{S}$ of the universal Teichmüller space we can obtain a stronger version of this assertion by replacing symplectic group $\mathrm{Sp}(V)$ with its *Hilbert–Schmidt subgroup* $\mathrm{Sp}_{\mathrm{HS}}(V)$. By definition, this subgroup consists of bounded linear operators $A \in \mathrm{Sp}(V)$ with block representations

$$A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix},$$

in which the operator b is Hilbert–Schmidt.

The map $f \mapsto T_f$ defines an embedding

$$\mathcal{S} \hookrightarrow \mathrm{Sp_{HS}}(V)/\mathrm{U}(W_+).$$

We identify, as above, the right hand side with a subspace $\mathcal{J}_{\mathrm{HS}}(V)$ of the space $\mathcal{J}(V)$ of compatible complex structures on V. We call complex structures $J \in \mathcal{J}_{\mathrm{HS}}(V)$ *Hilbert–Schmidt*. As before, the space $\mathcal{J}_{\mathrm{HS}}(V)$ of Hilbert–Schmidt complex structures on V can be realized as a *Hilbert–Schmidt Siegel disc*

$$\mathcal{D}_{\mathrm{HS}} = \{Z : W_+ \to W_- \text{ is a symmetric Hilbert–Schmidt operator with } \bar{Z}Z < I\}.$$

We have

Proposition 5 (Nag [11]). *The map*

$$\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{M\ddot{o}b}(S^1) \hookrightarrow \mathcal{J}_{\mathrm{HS}}(V) = \mathrm{Sp_{HS}}(V)/\mathrm{U}(W_+) = \mathcal{D}_{\mathrm{HS}}$$

is an equivariant holomorphic embedding.

III. Quantization of $\mathcal{S}$

6 Statement of the problem

6.1 Dirac quantization

We start by recalling a general definition of quantization of finite-dimensional classical systems, due to Dirac. A *classical system* is given by a pair $(M, \mathcal{A})$, where M is the phase space and $\mathcal{A}$ is the algebra of observables.

The *phase space* M is a smooth symplectic manifold of even dimension $2n$, provided with a symplectic 2-form ω. Locally, it is equivalent to the standard model, given by symplectic vector space $M_0 := \mathbb{R}^{2n}$ together with standard symplectic form ω_0, given in canonical coordinates (p_i, q_i), $i = 1, \ldots, n$, on $\mathbb{R}^{2n}$ by

$$\omega_0 = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

The *algebra of observables* $\mathcal{A}$ is a Lie subalgebra of the Lie algebra $C^\infty(M, \mathbb{R})$ of smooth real-valued functions on the phase space M, provided with the Poisson bracket, determined by symplectic 2-form ω. In particular, in the case of standard model $M_0 = (\mathbb{R}^{2n}, \omega_0)$ one can take for $\mathcal{A}$ the *Heisenberg algebra* $\mathrm{heis}(\mathbb{R}^{2n})$, which is the Lie algebra, generated by coordinate functions p_i, q_i, $i = 1, \ldots, n$, and 1, satisfying the commutation relations

$$\{p_i, p_j\} = \{q_i, q_j\} = 0,$$
$$\{p_i, q_j\} = \delta_{ij} \qquad \text{for} \quad i, j = 1, \ldots, n.$$

Definition 3. The *Dirac quantization* of a classical system $(M, \mathcal{A})$ is an irreducible Lie-algebra representation

$$r : \ \mathcal{A} \longrightarrow \mathrm{End}^* H$$

of the algebra of observables $\mathcal{A}$ in the algebra of linear self-adjoint operators, acting on a complex Hilbert space H, called the *quantization space*. The algebra $\mathrm{End}^* H$ is provided with the Lie bracket, given by the commutator of linear operators of the form $\frac{1}{i}[A, B]$. In other words, it is required that

$$r(\{f, g\}) = \frac{1}{i}\left(r(f)r(g) - r(g)r(f)\right)$$

for any $f, g \in \mathcal{A}$. We also assume the following normalization condition: $r(1) = I$.

For complexified algebras of observables $\mathcal{A}^{\mathbb{C}}$ or, more generally, complex involutive Lie algebras of observables (i.e. Lie algebras with conjugation) their Dirac quantization is given by an irreducible Lie-algebra representation

$$r: \ \mathcal{A}^{\mathbb{C}} \longrightarrow \operatorname{End} H \ ,$$

satisfying the normalization condition and the conjugation law: $r(\bar{f}) = r(f)^*$ for any $f \in \mathcal{A}$.

We are going to apply this definition of quantization to infinite-dimensional classical systems, in which both the phase space and algebra of observables are infinite-dimensional. For infinite-dimensional algebras of observables it is more natural to look for their projective Lie-algebra representations. The above definition of quantization will apply also to this case if one replaces the original algebra of observables with its suitable central extension.

6.2 Statement of the problem

We start from the Dirac quantization of an infinite-dimensional system $(V, \mathcal{A})$ with the phase space, given by the Sobolev space of half-differentiable functions $V := H_0^{1/2}(S^1, \mathbb{R})$. The role of algebra of observables $\mathcal{A}$ will be played by the semi-direct product

$$\mathcal{A} = \operatorname{heis}(V) \rtimes \operatorname{sp}_{\mathrm{HS}}(V),$$

being the Lie algebra of the Lie group $\mathcal{G} = \operatorname{Heis}(V) \rtimes \operatorname{Sp}_{\mathrm{HS}}(V)$. The symplectic Hilbert–Schmidt group $\operatorname{Sp}_{\mathrm{HS}}(V)$ was introduced in Section 4.2, while the Heisenberg algebra $\operatorname{heis}(V)$ and the corresponding Heisenberg group $\operatorname{Heis}(V)$ are defined, as in finite-dimensional situation. Namely, the *Heisenberg algebra* $\operatorname{heis}(V)$ of V is a central extension of the Abelian Lie algebra V, generated by coordinate functions. In other words, it coincides, as a vector space, with $\operatorname{heis}(V) = V \oplus \mathbb{R}$, provided with the Lie bracket

$$[(x, s), (y, t)] := (0, \omega(x, y)), \qquad x, y \in V, \quad s, t, \in \mathbb{R}.$$

Respectively, the *Heisenberg group* $\operatorname{Heis}(V)$ is a central extension of the Abelian group V, i.e. the direct product $\operatorname{Heis}(V) = V \times S^1$, provided with the group operation, given by

$$(x, \lambda) \cdot (y, \mu) := \left(x + y, \lambda\mu\, e^{i\omega(x,y)}\right).$$

The choice of the introduced Lie algebra $\mathcal{A}$ for the algebra of observables is motivated by the following physical considerations. As we have pointed put, the space V_d is a natural Sobolev completion of the space $\Omega_d := C_0^\infty(S^1, \mathbb{R}^d)$ of smooth loops in $\mathbb{R}^d$. In the same way, the Lie algebra $\mathcal{A} = \operatorname{heis}(V) \rtimes \operatorname{sp}_{\mathrm{HS}}(V)$ is a natural extension of the Lie algebra $\operatorname{heis}(\Omega_d) \rtimes \operatorname{Vect}(S^1)$, where $\operatorname{Vect}(S^1)$ is the Lie algebra of the diffeomorphism group $\operatorname{Diff}_+(S^1)$. The algebra $\operatorname{heis}(\Omega_d)$ can be identified with the Lie algebra of coordinate functions on Ω_d, while the algebra $\operatorname{Vect}(S^1)$ is generated by certain quadratic functions on Ω_d (cf. [3]). One can say that the Lie algebra $\operatorname{heis}(\Omega_d) \rtimes \operatorname{Vect}(S^1)$ is an infinite-dimensional analogue of the Poincarè algebra of the d-dimensional Minkowski space M^d, where $\operatorname{heis}(\Omega_d)$ plays the role of the Lie algebra of translations of M^d, while $\operatorname{Vect}(S^1)$ is an analogue of the Lie algebra of hyperbolic rotations of M^d.

7 Heisenberg representation

In this Section we recall the well known Heisenberg representation of the first component heis(V) of algebra of observables $\mathcal{A}$. A detailed exposition of this subject may be found in [13, 8, 2].

7.1 Fock space

Fix an admissible complex structure $J \in \mathcal{J}(V)$. It defines a polarization of V, i.e. a decomposition of $V^{\mathbb{C}}$ into the direct sum

$$V^{\mathbb{C}} = W \oplus \overline{W}, \tag{8}$$

where W (resp. $\overline{W}$) is the $(-i)$-eigenspace (resp. $(+i)$-eigenspace) of the complex structure operator J. The splitting (8) is the orthogonal direct sum with respect to the Hermitian inner product $\langle z, w\rangle_J := \omega(z, Jw)$, determined by J and sympletic form ω.

The Fock space $F(V^{\mathbb{C}}, J)$ is the completion of the algebra of symmetric polynomials on W with respect to a natural norm, generated by $\langle\cdot,\cdot\rangle_J$. In more detail, denote by $S(W)$ the algebra of symmetric polynomials in variables $z \in W$ and introduce an inner product on $S(W)$, defined in the following way. It is given on monomials of the same degree by the formula

$$\langle z_1 \cdot \ldots \cdot z_n, z_1' \cdot \ldots \cdot z_n'\rangle_J = \sum_{\{i_1,\ldots,i_n\}} \langle z_1, z_{i_1}'\rangle_J \cdot \ldots \cdot \langle z_n, z_{i_n}'\rangle_J,$$

where the summation is taken over all permutations $\{i_1, \ldots, i_n\}$ of the set $\{1, \ldots, n\}$ (the inner product of monomials of different degrees is set to zero), and extended to the whole algebra $S(W)$ by linearity. The completion $\widehat{S(W)}$ of $S(W)$ with respect to the introduced norm is called the *Fock space* of $V^{\mathbb{C}}$ with respect to complex structure J:

$$F_J = F(V^{\mathbb{C}}, J) := \widehat{S(W)}.$$

If $\{w_n\}$, $n = 1, 2, \ldots$, is an orthonormal basis of W, then an orthonormal basis of F_J can be given by the family of polynomials

$$P_K(z) = \frac{1}{\sqrt{k!}} \langle z, w_1\rangle_J^{k_1} \cdot \ldots \cdot \langle z, w_n\rangle_J^{k_n}, \qquad z \in W, \tag{9}$$

where $K = (k_1, \ldots, k_n, 0, \ldots)$, $k_i \in \mathbb{N} \cup 0$, and $k! = k_1! \cdot \ldots \cdot k_n!$.

7.2 Heisenberg representation

There is an irreducible representation of the Heisenberg algebra heis(V) in the Fock space $F_J = F(V^{\mathbb{C}}, J)$, defined in the following way. Elements of $S(W)$ may be considered as holomorphic functions on $\overline{W}$, if we identify $z \in W$ with a holomorphic function $\bar{w} \mapsto \langle w, z\rangle$ on $\overline{W}$. Accordingly, F_J may be considered as a subspace of the space $\mathcal{O}(\overline{W})$ of functions, holomorphic on $\overline{W}$. With this convention the *Heisenberg representation*

$$r_J : \text{ heis}(V) \longrightarrow \text{End}^* F_J$$

of the Heisenberg algebra heis(V) in the Fock space $F_J = F(V^{\mathbb{C}}, J)$ is defined by the formula

$$r_J(v) f(\bar{w}) := -\partial_v f(\bar{w}) + \langle w, v\rangle_J f(\bar{w}), \tag{10}$$

where ∂_v is the derivative in direction of $v \in V$. Extending r_J to the complexified algebra heis$^{\mathbb{C}}(V)$, we obtain

$$r_J(\bar{z}) f(\bar{w}) := -\partial_{\bar{z}} f(\bar{w})$$

for $v = \bar{z} \in \overline{W}$ and

$$r_J(z)f(\bar{w}) := \langle w, z\rangle_J f(\bar{w})$$

for $z \in W$. We set also $r_J(c) := \lambda \cdot I$ for the central element $c \in \mathrm{heis}(V)$, where λ is an arbitrary fixed non-zero constant.

Introduce the *creation* and *annihilation operators* on F_J, defined for $v \in V^{\mathbb{C}}$ by

$$a_J^*(v) := \frac{r_J(v) - ir_J(Jv)}{2}, \qquad a_J(v) := \frac{r_J(v) + ir_J(Jv)}{2}.$$

In particular, for $z \in W$

$$a_J^*(z)f(\bar{w}) = \langle w, z\rangle_J f(\bar{w}),$$

and for $\bar{z} \in \overline{W}$

$$a_J(\bar{z})f(\bar{w}) = -\partial_{\bar{z}} f(\bar{w}).$$

For an orthonormal basis $\{w_n\}$ of W, we define the operators

$$a_n^* := a^*(w_n), \qquad a_n := a(\bar{w}_n), \qquad n = 1, 2, \dots,$$

and $a_0 := \lambda \cdot I$.

A vector $f_J \in F_J \setminus \{0\}$ is called the *vacuum*, if $a_n f_J = 0$ for $n = 1, 2, \dots$. In other words, it is a non-zero vector, annihilated by operators a_n. It is uniquely defined by r_J (up to a multiplicative constant) and in the case of the initial Fock space $F_0 = F(V, J^0)$ we set $f_0 \equiv 1$. Acting on vacuum f_J by creation operators a_n^*, we can define the action of representation r_J on any polynomial, which implies the irreducubility of r_J.

So we have the following

Proposition 6 (cf. [13, 8, 2]). *There is an irredicible Lie algebra representation*

$$r_J : \ \mathrm{heis}(V) \longrightarrow \mathrm{End}^* F_J$$

of the Heisenberg algebra $\mathrm{heis}(V)$ *in the Fock space* $F_J = F(V^{\mathbb{C}}, J)$*, given by the formula* (10).

We shall see in the next Section that this representation is essentially unique.

8 Symplectic group action on the Fock bundle

8.1 Shale theorem

To construct an irreducible representation of the second component $\mathrm{sp}_{\mathrm{HS}}(V)$ of the algebra of observables $\mathcal{A}$, we study an action of the Hilbert–Schmidt symplectic group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on the Fock spaces F_J. This action is provided by the following theorem.

Theorem 3 (Shale). *The representations r_0 in F_0 and r_J in F_J are unitary equivalent if and only if $J \in \mathcal{J}_{\mathrm{HS}}(V)$. In other words, for $J \in \mathcal{J}_{\mathrm{HS}}(V)$ there exists a unitary intertwining operator $U_J : F_0 \to F_J$ such that*

$$r_J(v) = U_J \circ r_0(v) \circ U_J^{-1}.$$

This theorem was proved by Shale [17] in 1962, an independent proof was given in Berezin's book [2], published in Russian in 1965 (Berezin obtained also an explicit formula for the intertwining operator U_J).

The following Proposition gives a description of U_J in terms of the Hilbert–Schmidt Siegel disc $\mathcal{D}_{\mathrm{HS}}$, based on the identification of $\mathcal{J}_{\mathrm{HS}}(V)$ with $\mathcal{D}_{\mathrm{HS}}$.

Proposition 7 (Segal [16]). *There is a projective unitary action of the group* $\mathrm{Sp}_{\mathrm{HS}}(V)$ *on Fock spaces, defined by the unitary operator* U_J*, given by the formula* (11) *below.*

Here is an idea of Segal's construction, details may be found in [16]. Given an admissible complex structure $J \in \mathcal{J}_{\mathrm{HS}}(V)$, we identify it with a point Z in the Siegel disc $\mathcal{D}_{\mathrm{HS}}$. Regarding Z as an element of the symmetric square $\widehat{S}^2(W)$, we can associate with it an element $e^{Z/2}$ of $\widehat{S(W)}$. The inner product of two such elements has a simple expression

$$\langle e^{Z_1/2}, e^{Z_2/2}\rangle = \det(1-\bar{Z}_1 Z_2)^{-1/2}.$$

The normalized elements

$$\epsilon_Z := \det(1-\bar{Z}Z)^{1/4} e^{Z/2}$$

play the role of *coherent states* (cf., e.g., [2]). In terms of these states the action of the group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces, defined by

$$\mathrm{Sp}_{\mathrm{HS}}(V) \ni A = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \longmapsto U_J : F_0 \to F_J \qquad \text{for} \quad J = A \cdot J^0,$$

is given by the formula

$$U_J : \ \epsilon_Z \longmapsto \mu \det(1 + a^{-1}\bar{b}Z)^{1/2} \epsilon_{A\cdot Z}, \tag{11}$$

where $\mu : \mathbb{C}^* \to S^1$ is the radial projection.

8.2 Dirac quantization of V and $\mathcal{S}$

We can unite Fock spaces F_J into a *Fock bundle* over $\mathcal{D}_{\mathrm{HS}}$, having the following properties.

Proposition 8. *The Fock bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{J}(V)} F_J \longrightarrow \mathcal{J}(V) = \mathcal{D}_{\mathrm{HS}}$$

is a Hermitian holomorphic Hilbert space bundle over $\mathcal{D}_{\mathrm{HS}}$*. It can be provided with a projective unitary action of the group* $\mathrm{Sp}_{\mathrm{HS}}(V)$*, covering the natural* $\mathrm{Sp}_{\mathrm{HS}}(V)$*-action on the Siegel disc* $\mathcal{D}_{\mathrm{HS}}$*.*

The proof of holomorphicity of the Fock bundle $\mathcal{F} \to \mathcal{D}_{\mathrm{HS}}$ is analogous to the proof of holomorphicity of the determinant bundle over the Hilbert–Schmidt Grassmannian, given in [13]. Note that the Fock bundle is trivial, since the Siegel disc $\mathcal{D}_{\mathrm{HS}}$ is contractible (even convex), so the statement follows from the Hilbert space version of the Oka principle (cf. [4]). An explicit trivialization of $\mathcal{F} \to \mathcal{D}_{\mathrm{HS}}$ is provided by the action (11). This action defines a projective unitary action of the group $\mathrm{Sp}_{\mathrm{HS}}(V)$ on $\mathcal{F}$, covering the $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on Siegel disc $\mathcal{D}_{\mathrm{HS}}$.

The infinitesimal version of this action yields a projective representation of the symplectic algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0. We present an explicit description of this representation, due to Segal.

Recall that symplectic algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ is the Lie algebra of symplectic Hilbert–Schmidt group $\mathrm{Sp}_{\mathrm{HS}}(V)$, which consists of linear operators A in $V^{\mathbb{C}}$, having the following block representations

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

Here, α is a bounded skew-Hermitian operator and β is a symmetric Hilbert–Schmidt operator on F_0. The complexified Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)^{\mathbb{C}}$ consists of operators of the form

$$A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix},$$

where α is a bounded operator, while β and $\bar{\gamma}$ are symmetric Hilbert–Schmidt operators on F_0.

The projective representation of complexified symplectic algebra $\mathrm{sp}_{\mathrm{HS}}(V)^{\mathbb{C}}$ is given by the formula

$$\mathrm{sp}_{\mathrm{HS}}(V)^{\mathbb{C}} \ni A = \begin{pmatrix} \alpha & \beta \\ \bar{\gamma} & -\alpha^t \end{pmatrix} \longmapsto \rho(A) = D_\alpha + \frac{1}{2}M_\beta + \frac{1}{2}M_\gamma^*. \tag{12}$$

Here, D_α is the derivation of F_0 in α-direction, defined by

$$D_\alpha f(\bar{w}) = \langle \alpha w, \partial_{\bar{w}} \rangle f(\bar{w}).$$

The operator M_β is the multiplication operator on F_0, defined by

$$M_\beta f(\bar{w}) = \langle \bar{\beta} w, \bar{w} \rangle f(\bar{w}),$$

and the operator M_γ^* is the adjoint of M_γ: $M_\gamma^* f(\bar{w}) = \langle \gamma \partial_w, \partial_{\bar{w}} \rangle f(\bar{w})$.

This is a projective representation with cocycle

$$[\rho(A_1), \rho(A_2)] - \rho([A_1, A_2]) = \frac{1}{2}\,\mathrm{tr}(\bar{\gamma}_2\beta_1 - \bar{\gamma}_1\beta_2)I, \tag{13}$$

intertwined with the Heisenberg representation r_0 of $\mathrm{heis}(V)$ in F_0.

Thus we have the following

Proposition 9 (Segal [16]). *There is a projective unitary representation*

$$\rho : \ \mathrm{sp}_{\mathrm{HS}}(V) \longrightarrow \mathrm{End}^* F_0,$$

given by formula (12) *with cocycle* (13)*. This representation intertwines with the Heisenberg representation* r_0 *of* $\mathrm{heis}(V)$ *in* F_0*.*

The Heisenberg representation r_0 in the Fock space F_0, described in Proposition 6, and symplectic representation ρ, constructed in Proposition 9, define together Dirac quantization of the system $(V, \widetilde{\mathcal{A}})$, where $\widetilde{\mathcal{A}}$ is the central extension of $\mathcal{A}$, determined by (13).

The constructed Lie-algebra representation of $\mathrm{sp}_{\mathrm{HS}}(V)$ in the Fock space F_0 may be also considered as Dirac quantization of a classical system, consisting of the phase space $\mathcal{D}_{\mathrm{HS}} = \mathrm{Sp}_{\mathrm{HS}}(V)/\mathrm{U}(W_+)$ and the algebra of observables, given by the central extension of Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$.

The restriction of this construction to the smooth part $\mathcal{S} = \mathrm{Diff}_+(S^1)/\mathrm{Möb}(S^1)$ of the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$ yields the Dirac quantization of $\mathcal{S}$. Namely, we have the following

Proposition 10. *The restriction of the Fock bundle* $\mathcal{F} \to \mathcal{D}_{\mathrm{HS}}$ *to* $\mathcal{S}$ *is a Hermitian holomorphic Hilbert space bundle*

$$\mathcal{F} := \bigcup_{J \in \mathcal{S}} F_J \longrightarrow \mathcal{S}$$

over $\mathcal{S}$*. This bundle is provided with a projective unitary action of the diffeomorphism group* $\mathrm{Diff}_+(S^1)$*, covering the natural* $\mathrm{Diff}_+(S^1)$*-action on* $\mathcal{S}$*.*

The $\mathrm{Diff}_+(S^1)$-action on the Fock bundle, mentioned in Proposition, was explicitly constructed in [7]. The infinitesimal version of this action yields a unitary projective representation of the Lie algebra $\mathrm{Vect}(S^1)$ in the Fock space F_0. We can consider this construction as Dirac quantization of the phase space $\mathcal{S}$, provided with the algebra of observables, given by the central extension of the Lie algebra $\mathrm{Vect}(S^1)$, called the *Virasoro algebra*.

IV. Quantization of $\mathcal{T}$

9 Dirac versus Connes quantization

Unfortunately, the method, used in previous Chapter for the quantization of $\mathcal{S}$, does not apply to the whole space $\mathcal{T}$. Though we still can embed $\mathcal{T}$ into the Siegel disc $\mathcal{D}$, we are not able to construct a projective action of symplectic group $\mathrm{Sp}(V)$ on Fock spaces. According to theorem of Shale, it is possible only for the Hilbert–Schmidt subgroup $\mathrm{Sp}_{\mathrm{HS}}(V)$ of $\mathrm{Sp}(V)$. So one should look for another way of quantizing the universal Teichmüller space $\mathcal{T}$. We are going to use for that the "quantized calculus" of Connes and Sullivan, presented in Chapter IV of the Connes' book [5] and [12].

Recall that in Dirac's approach we quantize a classical system $(M, \mathcal{A})$, consisting of the phase space M and the algebra of observables $\mathcal{A}$, which is a Lie algebra, consisting of smooth functions on M. The quantization of this system is given by a representation r of $\mathcal{A}$ in a Hilbert space H, sending the Poisson bracket $\{f, g\}$ of functions $f, g \in \mathcal{A}$ into the commutator $\frac{1}{i}[r(f), r(g)]$ of the corresponding operators. In Connes' approach the algebra of observables $\mathfrak{A}$ is an associative involutive algebra, provided with an exterior differential d. Its quantization is, by definition, a representation π of $\mathfrak{A}$ in a Hilbert space H, sending the differential df of a function $f \in \mathfrak{A}$ into the commutator $[S, \pi(f)]$ of the operator $\pi(f)$ with a self-adjoint symmetry operator S with $S^2 = I$. The differential here is understood in the sense of non-commutative geometry, i.e. as a linear map $d : \mathfrak{A} \to \Omega^1(\mathfrak{A})$, satisfying the Leibnitz rule (cf. [5]).

In the following table we compare Connes and Dirac approaches to quantization.

	Dirac approach	Connes approach
Classical system	$(M, \mathcal{A})$ where: M – phase space $\mathcal{A}$ – involutive Lie algebra of observables	$(M, \mathfrak{A})$ where: M – phase space $\mathfrak{A}$ – involutive associative algebra of observables with differential $d\colon \mathfrak{A} \to \Omega^1(\mathfrak{A})$
Quantization	Lie-algebra representation $r\colon \mathcal{A} \to \mathrm{End}\, H$, sending $\{f, g\} \mapsto \frac{1}{i}[r(f), r(g)]$	representation $\pi\colon \mathfrak{A} \to \mathrm{End}\, H$, sending $df \mapsto [S, \pi(f)]$, where $S = S^*$, $S^2 = I$

Reformulating the notion of Connes quantization of algebra of observables $\mathfrak{A}$, one can say that it is a representation of the algebra $\mathrm{Der}(\mathfrak{A})$ of derivations of $\mathfrak{A}$ in the Lie algebra $\mathrm{End}\, H$. Recall that a *derivation* of an algebra $\mathfrak{A}$ is a linear map: $\mathfrak{A} \to \mathfrak{A}$, satisfying the Leibnitz rule. Clearly, derivations of an algebra $\mathfrak{A}$ form a Lie algebra, since the commutator of two derivations is again a derivation.

If all observables are smooth real-valued functions on M, the two approaches are equivalent to each other. Indeed, the differential df of a smooth function f is symplectically dual to the Hamiltonian vector field X_f and this establishes a relation between the associative algebra $\mathfrak{A}$ of functions f on M and the Lie algebra $\mathcal{A}$ of Hamiltonian vector fields on M. (This Lie algebra is isomorphic for a simply connected M to a Lie algebra of Hamiltonians, associated with $\mathcal{A}$.) A symmetry operator S is determined by a polarization $H = H_+ \oplus H_-$ of the quantization space H. Evidently, $S = iJ$, where J is the complex structure operator, defining the polarization $H = H_+ \oplus H_-$. (By this reason we do not make distinction between symmetry and complex structure operators.)

In the case when the algebra of observables $\mathcal{A}$ contains non-smooth functions, its Dirac quantization is not defined in the classical sense. In Connes approach the differential *df* of a non-smooth observable $f \in \mathfrak{A}$ is also not defined classically, but its quantum counterpart $d^q f$, given by

$$d^q f := [S, \pi(f)],$$

may still be defined, as it is demonstrated by the following example, borrowed from [5].

Suppose that $\mathfrak{A}$ is the algebra $L^\infty(S^1, \mathbb{C})$ of bounded functions on the circle S^1. Any function $f \in \mathfrak{A}$ defines a bounded multiplication operator in the Hilbert space $H = L^2(S^1, \mathbb{C})$:

$$M_f: \ v \in H \longmapsto fv \in H.$$

The operator S is given by the *Hilbert transform* $S : L^2(S^1, \mathbb{C}) \to L^2(S^1, \mathbb{C})$:

$$(Sf)(e^{i\varphi}) = \frac{1}{2\pi}\, V.P. \int_0^{2\pi} K(\varphi, \psi) f(e^{i\psi}) d\psi,$$

where the integral is taken in the principal value sense and $K(\varphi, \psi)$ is the *Hilbert kernel*

$$K(\varphi, \psi) = 1 + i \cot \frac{\varphi - \psi}{2}. \tag{14}$$

The differential *df* of a general observable $f \in \mathfrak{A}$ is not defined in the classical sense, but its quantum analogue

$$d^q f := [S, M_f]$$

is correctly defined as an operator in H for functions $f \in V$. Namely, we have the following

Proposition 11 (Nag–Sullivan [12]). *A function $f \in V$ if and only if the corresponding quantum differential $d^q f$ is a Hilbert–Schmidt operator on H (and on V). Moreover, the Hilbert–Schmidt norm of $d^q f$ coincides with the V-norm of f.*

Indeed, the commutator $d^q f := [S, M_f]$ is an integral operator on H with the kernel, given by $K(\varphi, \psi)(f(\varphi) - f(\psi))$. This operator is Hilbert–Schmidt if and only if its kernel is square integrable on $S^1 \times S^1$, i.e.

$$\int_0^{2\pi} \int_0^{2\pi} \frac{|f(\varphi) - f(\psi)|^2}{\sin^2 \frac{1}{2}(\varphi - \psi)}\, d\varphi\, d\psi < \infty.$$

This inequality is equivalent to the condition $f \in V$ (cf. [12]).

The quantum differential $d^q f = [S, M_f]$ of a function $f \in V$ is an integral operator on V, given by

$$d^q f(h)(e^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} k(\varphi, \psi) h(e^{i\psi}) d\psi$$

with the kernel, given by

$$k(\varphi, \psi) := K(\varphi, \psi)(f(\varphi) - f(\psi)),$$

where $K(\varphi, \psi)$ is defined by (14).

Note that the quasiclassical limit of this operator, defined by taking the value of the kernel on the diagonal (i.e. by taking the limit for $s \to t$), coincides (up to a constant) with the multiplication operator $h \mapsto f'h$, so the quantization means in this case essentially the replacement of the derivative by its finite-difference analogue. This finite-difference analogue is an integral operator, given by

$$\delta f(v)(e^{i\varphi}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(\varphi)-f(\psi)}{\varphi-\psi} v(e^{i\psi})d\psi. \tag{15}$$

The correspondence between functions $f \in \mathfrak{A}$ and operators M_f on H has the following remarkable properties (cf. [14]):

1. The differential $d^q f$ is a finite rank operator if and only if f is a rational function.
2. The differential $d^q f$ is a compact operator if and only if the function f belongs to the class $\mathrm{VMO}(S^1)$.
3. The differential $d^q f$ is a bounded operator if and only if the function f belongs to the class $\mathrm{BMO}(S^1)$.

This list may be supplemented by further function-theoretic properties of elements of $\mathfrak{A}$, having curious operator-theoretic characterizations (cf. [5]).

10 Quantization of the universal Teichmüller space

We apply these ideas to the universal Teichmüller space $\mathcal{T}$. In Section 5.1 we have defined a natural action of quasisymmetric homeomorphisms on V. As we have remarked, this action does not admit the differentiation, so classically there is no Lie algebra, associated with $\mathrm{QS}(S^1)$ or, in other words, there is no classical algebra of observables, associated to $\mathcal{T}$. (The situation is similar to that in the example above.) We would like to define a quantum algebra of observables, associated to $\mathcal{T}$.

First of all, extend the $\mathrm{QS}(S^1)$-action on V to symmetry operators by setting

$$S^h := h \circ S \circ h^{-1} \tag{16}$$

for $h \in \mathrm{QS}(S^1)$. This action agrees with a natural action of $\mathrm{QS}(S^1)$ on the universal Teichmüller space $\mathcal{T} = \mathrm{QS}(S^1)/\mathrm{Möb}(S^1)$, considered as a space of compatible complex structures on V. The quantized infinitesimal version of the action (16) is given by the integral operator $d^q h : V \to V$, equal to $d^q h = [S, \delta h]$ with δh given by (15).

Let us recall the steps of the Dirac quantization of Sobolev space V:

1) we start from $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on V;
2) extend it to $\mathrm{Sp}_{\mathrm{HS}}(V)$-action on complex structure operators J;
3) this action generates a projective unitary action of $\mathrm{Sp}_{\mathrm{HS}}(V)$ on Fock spaces $F(V, J)$;
4) the infinitesimal version of this action yields a projective unitary representation of the Lie algebra $\mathrm{sp}_{\mathrm{HS}}(V)$ in Fock space F_0, described in Section 8.2.

In the case of $\mathcal{T}$ we have:

1) $\mathrm{QS}(S^1)$-action on V;
2) this action extends to $\mathrm{QS}(S^1)$-action on symmetry operators S, given by $h \mapsto S^h$.

However, compared to Dirac quantization of V, the next step in the quantization scheme is absent. Because of the Shale theorem, we cannot extend the $\mathrm{QS}(S^1)$-action on symmetry operators S to Fock spaces $F(V, S)$. Also we cannot differentiate the $\mathrm{QS}(S^1)$-action on V. But we have a quantized infinitesimal version of $h : S \mapsto S^h$, given by quantum differential $d^q h = [S, \delta h]$.

We extend this operator to F_0 by defining it first on the basis elements (9) of the Fock space F_0 with the help of Leibnitz rule, and then by linearity to all finite elements of F_0. The completion of the obtained operator yields an operator $d^q h$ on F_0. These extended operators $d^q h$ with $h \in \mathrm{QS}(S^1)$ generate a *quantum derivation algebra* $\mathrm{Der}^q(\mathrm{QS})$, associated to $\mathcal{T}$. This algebra should be considered as a quantum Lie algebra of observables, associated to $\mathcal{T}$. So, instead of steps (3), (4) in the Dirac quantization of V, we construct directly a quantum Lie algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, corresponding to the non-existing classical Lie algebra of observables on $\mathcal{T}$.

Moreover, we can use the quantum Lie algebra $\mathrm{Der}^q(\mathrm{QS})$ as a substitution of a classical Lie algebra of $\mathrm{QS}(S^1)$.

Conclusion. The Connes quantization of the universal Teichmüller space $\mathcal{T}$ consists of two stages:

1. The first stage ("first quantization") is a construction of quantized infinitesimal version of $\mathrm{QS}(S^1)$-action on V, given by quantum differentials $d^q h = [S, \delta h]$ with $h \in \mathrm{QS}(S^1)$.
2. The second step ("second quantization") is an extension of quantum differentials $d^q h$ to the Fock space F_0. The extended operators $d^q h$ with $h \in \mathrm{QS}(S^1)$ generate the quantum algebra of observables $\mathrm{Der}^q(\mathrm{QS})$, associated to $\mathcal{T}$.

We note also that the correspondence principle for the constructed Connes quantization of $\mathcal{T}$ means that this quantization reduces to the Dirac quantization while restricted to $\mathcal{S}$.

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