

Local Quasitriangular Hopf Algebras

Shouchuan ZHANG ^{†‡}, Mark D. GOULD [‡] and Yao-Zhong ZHANG [‡]

[†] Department of Mathematics, Hunan University, Changsha 410082, P.R. China
E-mail: z9491@yahoo.com.cn

[‡] Department of Mathematics, University of Queensland, Brisbane 4072, Australia
E-mail: mdg@maths.uq.edu.au, yyz@maths.uq.edu.au

Received January 31, 2008, in final form April 30, 2008; Published online May 09, 2008

Original article is available at <http://www.emis.de/journals/SIGMA/2008/042/>

Abstract. We find a new class of Hopf algebras, local quasitriangular Hopf algebras, which generalize quasitriangular Hopf algebras. Using these Hopf algebras, we obtain solutions of the Yang–Baxter equation in a systematic way. The category of modules with finite cycles over a local quasitriangular Hopf algebra is a braided tensor category.

Key words: Hopf algebra; braided category

2000 Mathematics Subject Classification: 16W30; 16G10

1 Introduction

The Yang–Baxter equation first came up in the paper by Yang as factorization condition of the scattering S-matrix in the many-body problem in one dimension and in the work by Baxter on exactly solvable models in statistical mechanics. It has been playing an important role in mathematics and physics (see [2, 16]). Attempts to find solutions of the Yang–Baxter equation in a systematic way have led to the theory of quantum groups and quasitriangular Hopf algebras (see [6, 9]).

Since the category of modules with finite cycles over a local quasitriangular Hopf algebra is a braided tensor category, we may also find solutions of the Yang–Baxter equation in a systematic way.

The main results in this paper are summarized in the following statement.

Theorem 1. (i) Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\text{cf}}, C^{\{R_n\}})$, $({}_H\mathcal{M}^{\text{dcf}}, C^{\{R_n\}})$ and $({}_H\mathcal{M}^{\text{df}}, C^{\{R_n\}})$ are braided tensor categories. Furthermore, if (M, α^-) is an H -module with finite cycles and $R_{n+1} = R_n + W_n$ with $W_n \in H_{n+1} \otimes H_{(n+1)}$, then (M, α^-, δ^-) is a Yetter–Drinfeld H -module.

(ii) Assume that B is a finite dimensional Hopf algebra and M is a finite dimensional B -Hopf bimodule. Then $((T_B(M))^{\text{cop}} \bowtie_{\tau} T_{B^*}^c(M^*), \{R_n\})$ is a local quasitriangular Hopf algebra. Furthermore, if (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are arrow dual pairings with finite Hopf quiver Q , then both $((kQ^a)^{\text{cop}} \bowtie_{\tau} kQ^c, \{R_n\})$ and $((kQ^s)^{\text{cop}} \bowtie_{\tau} kQ^{sc}, \{R_n\})$ are local quasitriangular Hopf algebras.

2 Preliminaries

Throughout, we work over a fixed field k . All algebras, coalgebras, Hopf algebras, and so on, are defined over k . Books [7, 11, 15, 13] provide the necessary background for Hopf algebras and book [1] provides a nice description of the path algebra approach.

Let V and W be two vector spaces. σ_V denotes the map from V to V^{**} by defining $\langle \sigma_V(x), f \rangle = \langle f, x \rangle$ for any $f \in V^*$, $x \in V$. $C_{V,W}$ denotes the map from $V \otimes W$ to $W \otimes V$

by defining $C_{V,W}(x \otimes y) = y \otimes x$ for any $x \in V, y \in W$. Denote P by $\sum P' \otimes P''$ for $P \in V \otimes W$. If V is a finite-dimensional vector space over field k with $V^* = \text{Hom}_k(V, k)$. Define maps $b_V : k \rightarrow V \otimes V^*$ and $d_V : V^* \otimes V \rightarrow k$ by

$$b_V(1) = \sum_i v_i \otimes v_i^* \quad \text{and} \quad \sum_{i,j} d_V(v_i^* \otimes v_j) = \langle v_i^*, v_j \rangle,$$

where $\{v_i \mid i = 1, 2, \dots, n\}$ is any basis of V and $\{v_i^* \mid i = 1, 2, \dots, n\}$ is its dual basis in V^* . d_V and b_V are called evaluation and coevaluation of V , respectively. It is clear $(d_V \otimes id_U)(id_U \otimes b_V) = id_U$ and $(id_V \otimes d_V)(b_V \otimes id_V) = id_V$. ξ_V denotes the linear isomorphism from V to V^* by sending v_i to v_i^* for $i = 1, 2, \dots, n$. Note that we can define evaluation d_V when V is infinite.

We will use μ to denote the multiplication map of an algebra and use Δ to denote the comultiplication of a coalgebra. For a (left or right) module and a (left or right) comodule, denote by $\alpha^-, \alpha^+, \delta^-$ and δ^+ the left module, right module, left comodule and right comodule structure maps, respectively. The Sweedler's sigma notations for coalgebras and comodules are $\Delta(x) = \sum x_1 \otimes x_2, \delta^-(x) = \sum x_{(-1)} \otimes x_{(0)}, \delta^+(x) = \sum x_{(0)} \otimes x_{(1)}$. Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra and let $\Delta^{\text{cop}} := C_{H,H}\Delta$ and $\mu^{\text{op}} := \mu C_{H,H}$. We denote $(H, \mu, \eta, \Delta^{\text{cop}}, \epsilon)$ by H^{cop} and $(H, \mu^{\text{op}}, \eta, \Delta, \epsilon)$ by H^{op} . Sometimes, we also denote the unit element of H by 1_H .

Let A and H be two bialgebras with $\emptyset \neq X \subseteq A, \emptyset \neq Y \subseteq H$ and $P \in Y \otimes X, R \in Y \otimes Y$. Assume that τ is a linear map from $H \otimes A$ to k . We give the following notations.

(Y, R) is called almost cocommutative if the following condition satisfied:

$$(\text{ACO}) : \sum y_2 R' \otimes y_1 R'' = \sum R' y_1 \otimes R'' y_2 \quad \text{for any } y \in Y.$$

τ is called a skew pairing on $H \otimes A$ if for any $x, u \in H, y, z \in A$ the following conditions are satisfied:

$$(\text{SP1}) : \tau(x, yz) = \sum \tau(x_1, y)\tau(x_2, z);$$

$$(\text{SP2}) : \tau(xu, z) = \sum \tau(x, z_2)\tau(u, z_1);$$

$$(\text{SP3}) : \tau(x, \eta) = \epsilon_H(x);$$

$$(\text{SP4}) : \tau(\eta, y) = \epsilon_A(y).$$

P is called a copairing of $Y \otimes X$ if for any $x, u \in H, y, z \in A$ the following conditions are satisfied:

$$(\text{CP1}) : \sum P' \otimes P_1'' \otimes P_2'' = \sum P' Q' \otimes Q'' \otimes P'' \quad \text{with } P = Q;$$

$$(\text{CP1}) : \sum P_1' \otimes P_2' \otimes P'' = \sum P' \otimes Q' \otimes P'' Q'' \quad \text{with } P = Q;$$

$$(\text{CP3}) : \sum P' \otimes \epsilon_A(P'') = \eta_H;$$

$$(\text{CP4}) : \sum \epsilon_H(P') \otimes P'' = \eta_A.$$

For $R \in H \otimes H$ and two H -modules U and V , define a linear map $C_{U,V}^R$ from $U \otimes V$ to $V \otimes U$ by sending $(x \otimes y)$ to $\sum R'' y \otimes R' x$ for any $x \in U, y \in V$.

If $V = \bigoplus_{i=0}^{\infty} V_i$ is a graded vector space, let $V_{>n}$ and $V_{\leq n}$ denote $\bigoplus_{i=n+1}^{\infty} V_i$ and $\bigoplus_{i=0}^n V_i$, respectively. We usually denote $\bigoplus_{i=0}^n V_i$ by $V_{(n)}$. If $\dim V_i < \infty$ for any natural number i , then V is called a local finite graded vector space. We denote by ι_i the natural injection from V_i to V and by π_i the corresponding projection from V to V_i .

Let H be a bialgebra and a graded coalgebra with an invertible element R_n in $H_{(n)} \otimes H_{(n)}$ for any natural n . Assume $R_{n+1} = R_n + W_n$ with $W_n \in H_{(n+1)} \otimes H_{n+1} + H_{n+1} \otimes H_{(n+1)}$. $(H, \{R_n\})$ is called a local quasitriangular bialgebra if R_n is a copairing on $H_{(n)} \otimes H_{(n)}$, and

$(H_{(n)}, R_n)$ is almost cocommutative for any natural number n . In this case, $\{R_n\}$ is called a local quasitriangular structure of H . Obviously, if (H, R) is a quasitriangular bialgebra, then $(H, \{R_n\})$ is a local quasitriangular bialgebra with $R_0 = R$, $R_i = 0$, $H_0 = H$, $H_i = 0$ for $i > 0$.

The following facts are obvious: $\tau^{-1} = \tau(id_H \otimes S)$ (or $= \tau(S^{-1} \otimes id_A)$) if A is a Hopf algebra (or H is a Hopf algebra with invertible antipode) and τ is a skew pairing; $P^{-1} = (S \otimes id_A)P$ (or $= (id_H \otimes S^{-1})P$) if H is a Hopf algebra (or A is a Hopf algebra with invertible antipode) and P is a copairing.

Let A be an algebra and M be an A -bimodule. Then the tensor algebra $T_A(M)$ of M over A is a graded algebra with $T_A(M)_0 = A$, $T_A(M)_1 = M$ and $T_A(M)_n = \otimes_A^n M$ for $n > 1$. That is, $T_A(M) = A \oplus (\bigoplus_{n>0} \otimes_A^n M)$ (see [12]). Let D be another algebra. If h is an algebra map from A to D and f is an A -bimodule map from M to D , then by the universal property of $T_A(M)$ (see [12, Proposition 1.4.1]) there is a unique algebra map $T_A(h, f) : T_A(M) \rightarrow D$ such that $T_A(h, f)\iota_0 = h$ and $T_A(h, f)\iota_1 = f$. One can easily see that $T_A(h, f) = h + \sum_{n>0} \mu^{n-1} T_n(f)$, where $T_n(f)$ is the map from $\otimes_A^n M$ to $\otimes_A^n D$ given by $T_n(f)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = f(x_1) \otimes f(x_2) \otimes \cdots \otimes f(x_n)$, i.e., $T_n(f) = f \otimes_A f \otimes_A \cdots \otimes_A f$. Note that μ can be viewed as a map from $D \otimes_A D$ to D . For the details, the reader is directed to [12, Section 1.4].

Dually, let C be a coalgebra and let M be a C -bicomodule. Then the cotensor coalgebra $T_C^c(M)$ of M over C is a graded coalgebra with $T_C^c(M)_0 = C$, $T_C^c(M)_1 = M$ and $T_C^c(M)_n = \square_C^n M$ for $n > 1$. That is, $T_C^c(M) = C \oplus (\bigoplus_{n>0} \square_C^n M)$ (see [12]). Let D be another coalgebra. If h is a coalgebra map from D to C and f is a C -bicomodule map from D to M such that $f(\text{corad}(D)) = 0$, then by the universal property of $T_C^c(M)$ (see [12, Proposition 1.4.2]) there is a unique coalgebra map $T_C^c(h, f)$ from D to $T_C^c(M)$ such that $\pi_0 T_C^c(h, f) = h$ and $\pi_1 T_C^c(h, f) = f$. It is not difficult to see that $T_C^c(h, f) = h + \sum_{n>0} T_n^c(f) \Delta_{n-1}$, where $T_n^c(f)$ is the map from $\square_C^n D$ to $\square_C^n M$ induced by $T_n(f)(x_1 \otimes x_2 \otimes \cdots \otimes x_n) = f(x_1) \otimes f(x_2) \otimes \cdots \otimes f(x_n)$, i.e., $T_n^c(f) = f \otimes f \otimes \cdots \otimes f$.

Furthermore, if B is a Hopf algebra and M is a B -Hopf bimodule, then $T_B(M)$ and $T_B^c(M)$ are two graded Hopf algebra. Indeed, by [12, Section 1.4] and [12, Proposition 1.5.1], $T_B(M)$ is a graded Hopf algebra with the counit $\varepsilon = \varepsilon_B \pi_0$ and the comultiplication $\Delta = (\iota_0 \otimes \iota_0) \Delta_B + \sum_{n>0} \mu^{n-1} T_n(\Delta_M)$, where $\Delta_M = (\iota_0 \otimes \iota_1) \delta_M^- + (\iota_1 \otimes \iota_0) \delta_M^+$. Dually, $T_B^c(M)$ is a graded Hopf algebra with multiplication $\mu = \mu_B(\pi_0 \otimes \pi_0) + \sum_{n>0} T_n^c(\mu_M) \Delta_{n-1}$, where $\mu_M = \alpha_M^-(\pi_0 \otimes \pi_1) + \alpha_M^+(\pi_1 \otimes \pi_0)$.

3 Yang–Baxter equations

Assume that H is a bialgebra and a graded coalgebra with an invertible element R_n in $H_{(n)} \otimes H_{(n)}$ for any natural n . For convenience, let (LQT1), (LQT2) and (LQT3) denote (CP1), (CP2) and (ACO), respectively;

$$\text{(LQT4)} : R_{n+1} = R_n + W_n \quad \text{with} \quad W_n \in H_{(n+1)} \otimes H_{n+1} + H_{n+1} \otimes H_{(n+1)};$$

$$\text{(LQT4')} : R_{n+1} = R_n + W_n \quad \text{with} \quad W_n \in H_{n+1} \otimes H_{n+1}.$$

Then $(H, \{R_n\})$ is a local quasitriangular bialgebra if and only if (LQT1), (LQT2), (LQT3) and (LQT4) hold for any natural number n .

Let H be a graded coalgebra and a bialgebra. A left H -module M is called an H -module with finite cycles if, for any $x \in M$, there exists a natural number n_x such that $H_i x = 0$ when $i > n_x$. Let ${}_H \mathcal{M}^{\text{cf}}$ denote the category of all left H -modules with finite cycles.

Lemma 1. *Let H be a graded coalgebra and a bialgebra. If U and V are left H -modules with finite cycles, so is $U \otimes V$.*

Proof. For any $x \in U$, $y \in V$, there exist two natural numbers n_x and n_y , such that $H_{>n_x}x = 0$ and $H_{>n_y}y = 0$. Set $n_{x \otimes y} = 2n_x + 2n_y$. It is clear that $H_{>n_{x \otimes y}}(x \otimes y) = 0$. Indeed, for any $h \in H_i$ with $i > n_{x \otimes y}$, we see

$$h(x \otimes y) = \sum h_1x \otimes h_2y = 0 \quad (\text{since } H \text{ is graded coalgebra}). \quad \blacksquare$$

Lemma 2. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then for any left H -modules U and V with finite cycles, there exists an invertible linear map $C_{U,V}^{\{R_n\}} : U \otimes V \rightarrow V \otimes U$ such that*

$$C_{U,V}^{\{R_n\}}(x \otimes y) := C^{R_n}(x \otimes y) = \sum R_n''y \otimes R_n'x$$

with $n > 2n_x + 2n_y$, for $x \in U$, $y \in V$.

Proof. We first define a map f from $U \times V$ to $V \otimes U$ by sending (x, y) to $\sum R_n''y \otimes R_n'x$ with $n > 2n_x + 2n_y$ for any $x \in U$, $y \in V$. It is clear that f is well defined. Indeed, if $n > 2n_x + 2n_y$, then $C^{R_{n+1}}(x \otimes y) = C^{R_n}(x \otimes y)$ since $R_{n+1} = R_n + W_n$ with $W_n \in H_{(n+1)} \otimes H_{n+1} + H_{n+1} \otimes H_{(n+1)}$. f is a k -balanced function. Indeed, for $x, y \in U$, $z, w \in V$, $\alpha \in k$, let $n > 2n_x + 2n_y + 2n_z$. See

$$\begin{aligned} f(x + y, z) &= \sum R_n''z \otimes R_n'(x + y) = \sum R_n''z \otimes R_n'x + \sum R_n''z \otimes R_n'y \\ &= f(x, z) + f(y, z). \end{aligned}$$

Similarly, we can show that $f(x, z + w) = f(x, z) + f(x, w)$, $f(x\alpha, z) = f(x, \alpha z)$. Consequently, there exists a linear map $C_{U,V}^{\{R_n\}} : U \otimes V \rightarrow V \otimes U$ such that

$$C_{U,V}^{\{R_n\}}(x \otimes y) = C^{R_n}(x \otimes y)$$

with $n > 2n_x + 2n_y$, for $x \in U$, $y \in V$.

The inverse $(C_{U,V}^{\{R_n\}})^{-1}$ of $C_{U,V}^{\{R_n\}}$ is defined by sending $(y \otimes x)$ to $\sum (R_n^{-1})'x \otimes (R_n^{-1})''y$ with $n > 2n_x + 2n_y$ for any $x \in U$, $y \in V$. \blacksquare

Theorem 2. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\text{cf}}, C^{\{R_n\}})$ is a braided tensor category.*

Proof. Since H is a bialgebra, we have that $({}_H\mathcal{M}, \otimes, I, a, r, l)$ is a tensor category by [14, Proposition XI.3.1]. It follows from Lemma 1 that $({}_H\mathcal{M}^{\text{cf}}, \otimes, I, a, r, l)$ is a tensor subcategory of $({}_H\mathcal{M}, \otimes, I, a, r, l)$. $C^{\{R_n\}}$ is a braiding of ${}_H\mathcal{M}^{\text{cf}}$, which can be shown by the way similar to the proof of [14, Proposition VIII.3.1, Proposition XIII.1.4]. \blacksquare

An H -module M is called a graded H -module if $M = \bigoplus_{i=0}^{\infty} M_i$ is a graded vector space and $H_i M_j \subseteq M_{i+j}$ for any natural number i and j .

Lemma 3. *Assume that H is a local finite graded coalgebra and bialgebra. If $M = \bigoplus_{i=0}^{\infty} M_i$ is a graded H -module, then the following conditions are equivalent:*

- (i) M is an H -module with finite cycles;
- (ii) Hx is finite dimensional for any $x \in M$;
- (iii) Hx is finite dimensional for any homogeneous element x in M .

Proof. (i) \Rightarrow (ii). For any $x \in M$, there exists a natural number n_x such that $H_i x = 0$ with $i > n_x$. Since $H/(0 : x)_H \cong Hx$, where $(0 : x)_H := \{h \in H \mid h \cdot x = 0\}$, we have that Hx is finite dimensional.

(ii) \Rightarrow (iii). It is clear.

(iii) \Rightarrow (i). We first show that, for any homogeneous element $x \in M_i$, there exists a natural number n_x such that $H_j x = 0$ with $j > n_x$. In fact, if the above does not hold, then there exists $h_j \in H_{n_j}$ such that $h_j x \neq 0$ with $n_1 < n_2 < \dots$. Considering $h_j x \in M_{n_j+i}$ we have that $\{h_j x \mid j = 1, 2, \dots\}$ is linear independent in Hx , which contradicts to that Hx is finite dimensional.

For any $x \in M$, then $x = \sum_{i=1}^l x_i$ and x_i is a homogeneous element for $i = 1, 2, \dots, l$. There exists a natural number n_{x_i} such that $H_j x_i = 0$ with $j > n_{x_i}$. Set $n_x = \sum_{s=1}^l n_{x_s}$. Then $H_j x = 0$ with $j > n_x$. Consequently, M is an H -module with finite cycles. \blacksquare

Note that if $M = \bigoplus_{i=0}^{\infty} M_i$ is a graded H -module, then both (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) hold in Lemma 3.

Let ${}_H\mathcal{M}^{\text{gf}}$ and ${}_H\mathcal{M}^{\text{gcf}}$ denote the category of all finite dimensional graded left H -modules and the category of all graded left H -modules with finite cycles. Obviously, they are two tensor subcategories of ${}_H\mathcal{M}^{\text{cf}}$. Therefore we have

Theorem 3. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra. Then $({}_H\mathcal{M}^{\text{gf}}, C^{\{R_n\}})$ and $({}_H\mathcal{M}^{\text{gcf}}, C^{\{R_n\}})$ are two braided tensor categories.*

Therefore, if M is a finite dimensional graded H -module (or H -module with finite cycles) over local quasitriangular Hopf algebra $(H, \{R_n\})$, then $C_{M,M}^{\{R_n\}}$ is a solution of Yang–Baxter equation on M .

It is easy to prove the following.

Theorem 4. *Assume that $(H, \{R_n\})$ is a local quasitriangular Hopf algebra and $R_{n+1} = R_n + W_n$ with $W_n \in H_{n+1} \otimes H_{(n+1)}$. If (M, α^-) is an H -modules with finite cycles then (M, α^-, δ^-) is a Yetter–Drinfeld H -module, where $\delta^-(x) = \sum R_n'' \otimes R_n' x$ for any $x \in M$ and $n \geq n_x$.*

4 Relation between tensor algebras and co-tensor coalgebras

Lemma 4 (See [3, 7]). *Let A, B and C be finite dimensional coalgebras, $(M, \delta_M^-, \delta_M^+)$ and $(N, \delta_N^-, \delta_N^+)$ be respectively a finite dimensional A - B -bicomodule and a finite dimensional B - C -bicomodule. Then*

- (i) $(M^*, \delta_{M^*}^-, \delta_{M^*}^+)$ is a finite dimensional A^* - B^* -bimodule;
- (ii) $(M \square_{B^*} N, \delta_{M \square_{B^*} N}^-, \delta_{M \square_{B^*} N}^+)$ is an A - C -bicomodule with structure maps $\delta_{M \square_{B^*} N}^- = \delta_M^- \otimes id_N$ and $\delta_{M \square_{B^*} N}^+ = id_M \otimes \delta_N^+$;
- (iii) $M^* \otimes_{B^*} N^* \cong (M \square_{B^*} N)^*$ (as A^* - C^* -bimodules).

Lemma 5 (See [3, 7]). *Let A, B and C be finite dimensional algebras, $(M, \alpha_M^-, \alpha_M^+)$ and $(N, \alpha_N^-, \alpha_N^+)$ be respectively a finite dimensional A - B -bimodule and a finite dimensional B - C -bimodule. Then*

- (i) $(M^*, \alpha_{M^*}^-, \alpha_{M^*}^+)$ is a finite dimensional A^* - B^* -bicomodule;
- (ii) $(M \otimes_B N, \alpha_{M \otimes_B N}^-, \alpha_{M \otimes_B N}^+)$ is an A - C -bimodule with structure maps $\alpha_{M \otimes_B N}^- = \alpha_M^- \otimes id_N$ and $\alpha_{M \otimes_B N}^+ = id_M \otimes \alpha_N^+$;
- (iii) $M^* \square_{B^*} N^* \cong (M \otimes_B N)^*$ (as A^* - C^* -bicomodules).

Proof. (i) and (ii) are easy.

(iii) Consider

$$M^* \square_{B^*} N^* \cong (M^* \square_{B^*} N^*)^{**} \cong (M \otimes_B N)^* \quad (\text{by Lemma 4}).$$

Of course, we can also prove it in the dual way of the proof of Lemma 4, by sending $f \otimes_k g$ to $f \otimes_{B^*} g$ for any $f \in M^*, g \in N^*$ with $f \otimes g \in M^* \square_{B^*} N^*$. \blacksquare

Theorem 5. *If A is a finite dimensional algebra and M is a finite dimensional A -bimodule, then $T_A(M)$ is isomorphic to subalgebra $\sum_{n=0}^{\infty} (\square_{A^*}^n M^*)^*$ of $(T_{A^*}^c(M^*))^0$ under map $\sigma_{T_A(M)}$ and $\sigma_{T_A(M)} = \sigma_A + \sum_{n>0} \mu^{n-1} T_n(\sigma_M)$ with $\mu^{n-1} T_n(\sigma_M) = \sigma_{\otimes_A^n M}$.*

Proof. We view $\bigoplus_{n=0}^{\infty} (\square_{A^*}^n M^*)^*$ as inner direct sum of vector spaces. It is clear that σ_A is algebra homomorphism from A to $A^{**} \subseteq (T_{A^*}^c(M^*))^*$ and σ_M is a A -bimodule homomorphism from M to $M^{**} \subseteq (T_{A^*}^c(M^*))^*$. Thus it follows from [12, Proposition 1.4.1] that $\phi = \sigma_A + \sum_{n>0} \mu^{n-1} T_n(\sigma_M)$ is an algebra homomorphism from $T_A(M)$ to $(T_{A^*}^c(M^*))^*$.

It follows from Lemma 4 (iii) that $\mu^{n-1} T_n(\sigma_M) = \sigma_{\otimes_A^n M}$. Indeed, we use induction on $n > 0$. Obviously, the conclusion holds when $n = 1$. Let $n > 1$, $N = \otimes_A^{n-1} M$, $L = (\square_{A^*}^{n-1} M^*)^*$ and $\zeta = \mu^{n-2} T_{n-1}(\sigma_M)$. Obviously, $\mu^{n-1} T_n(\sigma_M) = \mu(\zeta \otimes \sigma_M)$. By inductive assumption, $\zeta = \sigma_N$ is an A -bimodule isomorphism from N to L . See

$$\begin{aligned} \otimes_A^n M &= N \otimes_A M \stackrel{\nu_1}{\cong} L \otimes_A M^{**} \quad (\text{by inductive assumption}) \\ &\stackrel{\nu_2}{\cong} (\square_{A^*}^{n-1} M^*)^* \otimes_{A^{**}} M^{**} \\ &\stackrel{\nu_3}{\cong} ((\square_{A^*}^{n-1} M^*) \square_{A^*} M^*)^* \quad (\text{by Lemma 4 (iii)}) \\ &= (\square_{A^*}^n M^*)^*, \end{aligned}$$

where $\nu_1 = \sigma_N \otimes_A \sigma_M$, $\nu_2(f^{**} \otimes_A g^{**}) = f^{**} \otimes_{A^{**}} g^{**}$ and $\nu_3(f^{**} \otimes_{A^{**}} g^{**}) = f^{**} \otimes_k g^{**}$ for any $f^{**} \in (\square_{A^*}^{n-1} M^*)^*$, $g^{**} \in M^{**}$. Now we have to show $\nu_3 \nu_2 \nu_1 = \sigma_{\otimes_A^n M} = \mu^{n-1} T_n(\sigma_M)$. For any $f^* \in \square_{A^*}^{n-1} M^*$, $g^* \in M^*$, $x \in \otimes_A^{n-1} M$, $y \in M$, on the one hand

$$\langle \sigma_{\otimes_A^n M}(x \otimes_A y), f^* \otimes_k g^* \rangle = \langle f^*, x \rangle \langle g^*, y \rangle.$$

On the other hand,

$$\begin{aligned} \langle \nu_3 \nu_2 \nu_1(x \otimes_A y), f^* \otimes_k g^* \rangle &= \langle \nu_3 \nu_2(\sigma_N(x) \otimes_A \sigma_M(y)), f^* \otimes_k g^* \rangle \\ &= \langle \sigma_M(x) \otimes_k \sigma_N(y), f^* \otimes_k g^* \rangle = \langle f^*, x \rangle \langle g^*, y \rangle. \end{aligned}$$

Thus $\nu_3 \nu_2 \nu_1 = \sigma_{\otimes_A^n M}$. See

$$\begin{aligned} \langle \mu(\zeta \otimes_A \sigma_M)(x \otimes_A y), f^* \otimes_k g^* \rangle &= \langle \zeta(x) \otimes_A \sigma_M(y), \Delta(f^* \otimes_k g^*) \rangle \\ &= \langle \zeta(x), f^* \rangle \langle \sigma_M(y), g^* \rangle = \langle f^*, x \rangle \langle g^*, y \rangle. \end{aligned}$$

Thus $\sigma_{\otimes_A^n M} = \mu^{n-1} T_n(\sigma_M)$.

Finally, for any $x \in T_A(M)$ with $x = x^{(1)} + x^{(2)} + \cdots + x^{(n)}$ and $x^{(i)} \in \otimes_A^i M$,

$$\phi(x) = \sum_{i=1}^n \phi(x^{(i)}) = \sum_{i=1}^n \sigma_{\otimes_A^i M}(x^{(i)}) = \sigma_{T_A(M)}(x). \quad \blacksquare$$

5 (Co-)tensor Hopf algebras

Lemma 6. *Assume that B is a finite dimensional Hopf algebra and M is a finite dimensional B -Hopf bimodule. Let $A := T_B(M)^{\text{cop}}$, $H := T_{B^*}^c(M^*)$. Then*

(i) $\phi := \sigma_B + \sum_{i>0} \mu^{i-1} T_i(\sigma_M)$ is a Hopf algebra isomorphism from $T_B(M)$ to the Hopf subalgebra $\sum_{i=0}^{\infty} (\square_{B^*}^i M^*)^*$ of $(T_{B^*}^c(M^*))^0$;

(ii) Let $\phi_n := \phi|_{A_{(n)}}$ for any natural number $n \geq 0$. Then there exists $\psi_n : (H_{(n)})^* \rightarrow A_{(n)}$ such that $\phi_n \psi_n = \text{id}_{(H_{(n)})^*}$ and $\psi_n \phi_n = \text{id}_{A_{(n)}}$, and $\psi_{n+1}(x) = \psi_n(x)$ for any $x \in (H_{(n)})^*$. Furthermore, ϕ_n and ψ_n preserve the (co)multiplication operations of $T_B(M)$ and $(T_{B^*}^c(M^*))^0$, respectively.

Proof. (i) We first show that $(\square_{B^*}^n M^*)^* \subseteq (T_{B^*}^c(M^*))^0$. For any $f \in (\square_{B^*}^n M^*)^*$, $\sum_{i=n+1}^{\infty} \square_{B^*}^i M^* \subseteq \ker f$ and $\sum_{i=n+1}^{\infty} \square_{B^*}^i M^*$ is a finite codimensional ideal of $T_{B^*}^c(M^*)$. Consequently, $f \in (T_{B^*}^c(M^*))^0$.

Next we show that $\phi := \sigma_B + \sum_{n>0} \mu^{n-1} T_n(\sigma_M)$ (see the proof of Theorem 5) is a coalgebra homomorphism from $T_B(M)$ to $\sum_{i=0}^{\infty} (\square_{B^*}^i M^*)^*$. For any $x \in \otimes_B^n M$, $f, g \in T_{B^*}^c(M^*)$, on the one hand

$$\begin{aligned} \langle \phi(x), f * g \rangle &= \langle f * g, x \rangle \quad (\text{by Theorem 5}) \\ &= \sum_x \langle f, x_1 \rangle \langle g, x_2 \rangle = \sum_x \langle \phi(x_1), f \rangle \langle \phi(x_2), g \rangle. \end{aligned}$$

On the other hand

$$\langle \phi(x), f * g \rangle = \sum \langle (\phi(x))_1, f \rangle \langle (\phi(x))_2, g \rangle,$$

since $\phi(x) \in (T_{B^*}^c(M^*))^0$. Considering $T_{B^*}^c(M^*) = \bigoplus_{n \geq 0} \square_{B^*}^n M^* \cong \bigoplus_{n \geq 0} (\otimes_B^n M)^*$ as vector spaces, we have that $T_{B^*}^c(M^*)$ is dense in $(T_B(M))^*$. Consequently, $\sum \phi(x_1) \otimes \phi(x_2) = \sum (\phi(x))_1 \otimes (\phi(x))_2$, i.e. ϕ is a coalgebra homomorphism.

(ii) It follows from Theorem 5. ■

Recall the double cross product $A_\alpha \bowtie_\beta H$, defined in [17, p. 36] and [14, Definition IX.2.2]. Assume that H and A are two bialgebras; (A, α) is a left H -module coalgebra and (H, β) is a right A -module coalgebra. We define the multiplication m_D , unit η_D , comultiplication Δ_D and counit ϵ_D on $A \otimes H$ as follows:

$$\begin{aligned} \mu_D((a \otimes h) \otimes (b \otimes g)) &= \sum a\alpha(h_1, b_1) \otimes \beta(h_2, b_2)g, \\ \Delta_D(a \otimes h) &= \sum (a_1 \otimes h_1 \otimes a_2 \otimes h_2), \end{aligned}$$

$\epsilon_D = \epsilon_A \otimes \epsilon_H$, $\eta_D = \eta_A \otimes \eta_H$ for any $a, b \in A$, $h, g \in H$. We denote $(A \otimes H, \mu_D, \eta_D, \Delta_D, \epsilon_D)$ by $A_\alpha \bowtie_\beta H$, which is called the double cross product of A and H .

Lemma 7 (See [8]). *Let H and A be two bialgebras. Assume that τ is an invertible skew pairing on $H \otimes A$. If we define $\alpha(h, a) = \sum \tau(h_1, a_1) a_2 \tau^{-1}(h_2, a_3)$ and $\beta(h, a) = \sum \tau(h_1, a_1) h_2 \tau^{-1}(h_3, a_2)$ then the double cross product $A_\alpha \bowtie_\beta H$ of A and H is a bialgebra. Furthermore, if A and H are two Hopf algebras, then so is $A_\alpha \bowtie_\beta H$.*

Proof. We can check that (A, α) is an H -module coalgebra and (H, β) is an A -module coalgebra step by step. We can also check that (M1)–(M4) in [17, pp. 36–37] hold step by step. Consequently, it follows from [17, Corollary 1.8, Theorem 1.5] or [14, Theorem IX.2.3] that $A_\alpha \bowtie_\beta H$ is a Hopf algebra. ■

In this case, $A_\alpha \bowtie_\beta H$ can be written as $A \bowtie_\tau H$.

Lemma 8. *Let H and A be two Hopf algebra. Assume that there exists a Hopf algebra monomorphism $\phi : A^{\text{cop}} \rightarrow H^0$. Set $\tau = d_H(\phi \otimes id_H)C_{H,A}$. Then $A \bowtie_\tau H$ is Hopf algebra.*

Proof. Using [10, Proposition 2.4] or the definition of the evaluation and coevaluation on tensor product, we can obtain that τ is a skew pairing on $H \otimes A$. Considering Lemma 7, we complete the proof. ■

Lemma 9. (i) *If $H = \bigoplus_{n=0}^{\infty} H_n$ is a graded bialgebra and H_0 has an invertible antipode, then H has an invertible antipode.*

(ii) *Assume that B is a finite dimensional Hopf algebra and M is a B -Hopf bimodule. Then both $T_B(M)$ and $T_B^c(M)$ have invertible antipodes.*

Proof. (i) It is clear that H^{op} is a graded bialgebra with $(H^{\text{op}})_0 = (H_0)^{(\text{op})}$. Thus H^{op} has an antipode by [12, Proposition 1.5.1]. However, the antipode of H^{op} is the inverse of antipode of H .

(ii) It follows from (i). ■

Lemma 10. *Let $A = \bigoplus_{n=0}^{\infty} A_n$ and $H = \bigoplus_{n=0}^{\infty} H_n$ be two graded Hopf algebras with invertible antipodes. Let τ be a skew pairing on $(H \otimes A)$ and P_n be a copairing of $H_{(n)} \otimes A_{(n)}$ for any natural number n . Set $D = A \bowtie_{\tau} H$ and $[P_n] = 1_A \otimes P_n \otimes 1_H$. Then $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$ if and only if*

$$\begin{aligned} \text{(ACO1)} : & \sum P' y_1 \otimes P'' \otimes y_2 = \sum y_4 P' \otimes P''_2 \otimes y_2 \tau(y_1, P''_1) \tau^{-1}(y_3, P''_3) \quad \text{for any } y \in H_{(n)}; \\ \text{(ACO2)} : & \sum x_2 \otimes P' \otimes x_1 P'' = \sum x_2 \otimes P'_2 \otimes P'' x_4 \tau(P'_1, x_1) \tau^{-1}(P'_3, x_3) \quad \text{for any } x \in A_{(n)}. \end{aligned}$$

Proof. It is clear that $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$ if and only if the following holds:

$$\begin{aligned} & \sum x_2 \otimes y_4 P' \otimes x_1 P''_2 \otimes y_2 \tau(y_1, P''_1) \tau^{-1}(y_3, P''_3) \\ & = \sum x_2 \otimes P'_2 y_1 \otimes P'' x_4 \otimes y_2 \tau(P'_1, x_1) \tau^{-1}(P'_3, x_3) \end{aligned} \quad (1)$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$.

Assume that both (ACO1) and (ACO2) hold. See that

$$\text{the left hand of (1)} \stackrel{\text{by (ACO1)}}{=} \sum x_2 \otimes P' y_1 \otimes x_1 P'' \otimes y_2 \stackrel{\text{by (ACO2)}}{=} \text{the right hand of (1)}$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$. That is, (1) holds.

Conversely, assume that (1) holds. Thus we have that

$$\begin{aligned} & \sum x_2 \otimes P' \otimes x_1 P''_2 \otimes \epsilon_H(1_H) \tau(1_H, P''_1) \tau^{-1}(1_H, P''_3) \\ & = \sum x_2 \otimes P'_2 \otimes P'' x_4 \otimes \epsilon_H(1_H) \tau(P'_1, x_1) \tau^{-1}(P'_3, x_3) \end{aligned}$$

and

$$\begin{aligned} & \sum \epsilon_A(1_A) \otimes y_4 P' \otimes P''_2 \otimes y_2 \tau(y_1, P''_1) \tau^{-1}(y_3, P''_3) \\ & = \sum \epsilon_A(1_A) \otimes P'_2 y_1 \otimes P'' \otimes y_2 \tau(P'_1, 1_A) \tau^{-1}(P'_3, 1_A) \end{aligned}$$

for any $x \in A_{(n)}$, $y \in H_{(n)}$. Consequently, (ACO1) and (ACO2) hold. ■

Lemma 11. *Let $A = \bigoplus_{n=0}^{\infty} A_n$ and $H = \bigoplus_{n=0}^{\infty} H_n$ be two graded Hopf algebras with invertible antipodes. Let τ be a skew pairing on $(H \otimes A)$ and P_n be a copairing of $(H_{(n)} \otimes A_{(n)})$ with $P_{n+1} = P_n + W_n$ and $W_n \in H_{(n+1)} \otimes A_{n+1} + H_{n+1} \otimes A_{(n+1)}$ for any natural number n . Set $D = A \bowtie_{\tau} H$. If $\tau(P'_n, x) P''_n = x$ and $\tau(y, P''_n) P'_n = y$ for any $x \in A_{(n)}$, $y \in H_{(n)}$, then $(D, \{[P_n]\})$ is a local quasitriangular Hopf algebra.*

Proof. It follows from Lemma 7 that $D = \bigoplus_{n=0}^{\infty} D_n$ is a Hopf algebra. Let $D_n = \sum_{i+j=n} A_i \otimes H_j$. It is clear that $D = \bigoplus_{n=0}^{\infty} D_n$ is a graded coalgebra. We only need to show that $(D, \{[P_n]\})$ is almost cocommutative on $D_{(n)}$. Now fix n . For convenience, we denote P_n by P and Q in the following formulae. For any $x \in A_i$, $y \in H_j$ with $i + j \leq n$,

$$\text{the right hand of (ACO1)} \stackrel{\text{by (CP1)}}{=} \sum y_4 Q' P' \otimes Q''_1 \otimes y_2 \tau(y_1, P'') \tau^{-1}(y_3, Q''_2)$$

$$\begin{aligned}
& \text{by assumption} \sum y_4 Q' y_1 \otimes Q''_1 \otimes y_2 \tau^{-1}(y_3, Q''_2) \stackrel{\text{by (CP1)}}{=} \sum y_4 Q' P' y_1 \otimes P'' \otimes y_2 \tau^{-1}(y_3, Q'') \\
& = \sum y_4 Q' P' y_1 \otimes P'' \otimes y_2 \tau(S^{-1}(y_3), Q'') \stackrel{\text{by assumption}}{=} \sum y_4 S^{-1}(y_3) P' y_1 \otimes P'' \otimes y_2 \\
& = \sum P' y_1 \otimes P'' \otimes y_2 = \text{the left hand of (ACO1)}.
\end{aligned}$$

Similarly, we can show that (ACO2) holds on $A_{(n)}$. \blacksquare

Theorem 6. *Assume that B is a finite dimensional Hopf algebra and M is a finite dimensional B -Hopf bimodule. Let $A := T_B(M)^{\text{cop}}$, $H := T_{B^*}^c(M^*)$ and $D = A \bowtie_{\tau} H$ with $\tau := d_H(\phi \otimes id)C_{H,A}$. Then $((T_B(M))^{\text{cop}} \bowtie_{\tau} T_{B^*}^c(M^*), \{R_n\})$ is a local quasitriangular Hopf algebra. Here $P_n = (id \otimes \psi_n)b_{H(n)}$, $R_n = [P_n] = 1_B \otimes (id \otimes \psi_n)b_{H(n)} \otimes 1_{B^*}$, ϕ and ψ_n are defined in Lemma 6.*

Proof. By Lemma 9 (ii), A and H have invertible antipodes. Assume that $e_1^{(i)}, e_2^{(i)}, \dots, e_{n_i}^{(i)}$ is a basis of H_i and $e_1^{(i)*}, e_2^{(i)*}, \dots, e_{n_i}^{(i)*}$ is an its dual basis in $(H_i)^*$. Then $\{e_j^{(i)} \mid i = 0, 1, 2, \dots, n; j = 1, 2, \dots, n_i\}$ is a basis of $H_{(n)}$ and $\{e_j^{(i)*} \mid i = 0, 1, 2, \dots, n; j = 1, 2, \dots, n_i\}$ is its dual basis in $(H_{(n)})^*$. Thus $b_{H(n)} = \sum_{i=0}^n \sum_{j=1}^{n_i} e_j^{(i)} \otimes e_j^{(i)*}$. See that

$$\begin{aligned}
P_{n+1} &= \sum_{i=0}^n \sum_{j=1}^{n_i} e_j^{(i)} \otimes \psi_{n+1}(e_j^{(i)*}) + \sum_{j=1}^{n_{n+1}} e_j^{(n+1)} \otimes \psi_{n+1}(e_j^{(n+1)*}) \\
&= P_n + \sum_{j=1}^{n_{n+1}} e_j^{(n+1)} \otimes \psi_{n+1}(e_j^{(n+1)*}).
\end{aligned}$$

Obviously, $\sum_{j=1}^{n_{n+1}} e_j^{(n+1)} \otimes \psi_{n+1}(e_j^{(n+1)*}) \in H_{n+1} \otimes A_{n+1}$. It is clear that P_n is a copairing on $H_{(n)} \otimes A_{(n)}$ and τ is a skew pairing on $H \otimes A$ with $\sum \tau(P'_n, x)P''_n = x$ and $\sum \tau(y, P''_n)P'_n = y$ for any $x \in A_{(n)}$, $y \in H_{(n)}$. We complete the proof by Lemma 11. \blacksquare

Note that $R_{n+1} = R_n + W_n$ with $W_n \in D_{n+1} \otimes D_{n+1}$ in the above theorem.

6 Quiver Hopf algebras

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 and Q_1 are the sets of vertices and arrows, respectively; s and t are two maps from Q_1 to Q_0 . For any arrow $a \in Q_1$, $s(a)$ and $t(a)$ are called its start vertex and end vertex, respectively, and a is called an arrow from $s(a)$ to $t(a)$. For any $n \geq 0$, an n -path or a path of length n in the quiver Q is an ordered sequence of arrows $p = a_n a_{n-1} \cdots a_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n-1$. Note that a 0-path is exactly a vertex and a 1-path is exactly an arrow. In this case, we define $s(p) = s(a_1)$, the start vertex of p , and $t(p) = t(a_n)$, the end vertex of p . For a 0-path x , we have $s(x) = t(x) = x$. Let Q_n be the set of n -paths, $Q_{(n)}$ be the set of i -paths with $i \leq n$ and Q_{∞} be the set of all paths in Q . Let ${}^y Q_n^x$ denote the set of all n -paths from x to y , $x, y \in Q_0$. That is, ${}^y Q_n^x = \{p \in Q_n \mid s(p) = x, t(p) = y\}$. A quiver Q is *finite* if Q_0 and Q_1 are finite sets.

Let G be a group. Let $\mathcal{K}(G)$ denote the set of conjugate classes in G . $r = \sum_{C \in \mathcal{K}(G)} r_C C$ is called a *ramification* (or *ramification data*) of G , if r_C is the cardinal number of a set for any $C \in \mathcal{K}(G)$. We always assume that the cardinal number of the set $I_C(r)$ is r_C . Let $\mathcal{K}_r(G) := \{C \in \mathcal{K}(G) \mid r_C \neq 0\} = \{C \in \mathcal{K}(G) \mid I_C(r) \neq \emptyset\}$.

Let G be a group. A quiver Q is called a *quiver of G* if $Q_0 = G$ (i.e., $Q = (G, Q_1, s, t)$). If, in addition, there exists a ramification r of G such that the cardinal number of ${}^y Q_1^x$ is equal to r_C for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$, then Q is called a *Hopf quiver with respect to the ramification data r* . In this case, there is a bijection from $I_C(r)$ to ${}^y Q_1^x$. Denote by (Q, G, r) the

Hopf quiver of G with respect to r . e denotes the unit element of G . $\{p_g \mid g \in G\}$ denotes the dual basis of $\{g \mid g \in G\}$ of finite group algebra kG .

Let $Q = (G, Q_1, s, t)$ be a quiver of a group G . Then kQ_1 becomes a kG -bicomodule under the natural comodule structures:

$$\delta^-(a) = t(a) \otimes a, \quad \delta^+(a) = a \otimes s(a), \quad a \in Q_1, \quad (2)$$

called an *arrow comodule*, written as kQ_1^c . In this case, the path coalgebra kQ^c is exactly isomorphic to the cotensor coalgebra $T_{kG}^c(kQ_1^c)$ over kG in a natural way (see [3] and [4]). We will regard $kQ^c = T_{kG}^c(kQ_1^c)$ in the following. Moreover, kQ_1 becomes a $(kG)^*$ -bimodule with the module structures defined by

$$p \cdot a := \langle p, t(a) \rangle a, \quad a \cdot p := \langle p, s(a) \rangle a, \quad p \in (kG)^*, \quad a \in Q_1, \quad (3)$$

written as kQ_1^a , called an *arrow module*. Therefore, we have a tensor algebra $T_{(kG)^*}(kQ_1^a)$. Note that the tensor algebra $T_{(kG)^*}(kQ_1^a)$ of kQ_1^a over $(kG)^*$ is exactly isomorphic to the path algebra kQ^a . We will regard $kQ^a = T_{(kG)^*}(kQ_1^a)$ in the following.

Lemma 12 (See [4], Theorem 3.3, and [5], Theorem 3.1). *Let Q be a quiver over group G . Then the following statements are equivalent:*

- (i) Q is a Hopf quiver.
 - (ii) Arrow comodule kQ_1^c admits a kG -Hopf bimodule structure.
- If Q is finite, then the above statements are also equivalent to the following:
- (iii) Arrow module kQ_1^a admits a $(kG)^*$ -Hopf bimodule structure.

Assume that Q is a Hopf quiver. It follows from Lemma 12 that there exist a left kG -module structure α^- and a right kG -module structure α^+ on arrow comodule $(kQ_1^c, \delta^-, \delta^+)$ such that $(kQ_1^c, \alpha^-, \alpha^+, \delta^-, \delta^+)$ becomes a kG -Hopf bimodule, called a co-arrow Hopf bimodule. We obtain two graded Hopf algebras $T_{kG}(kQ_1^c)$ and $T_{kG}^c(kQ_1^c)$, called semi-path Hopf algebra and co-path Hopf algebra, written as kQ^s and kQ^c , respectively.

Assume that Q is a finite Hopf quiver. Dually, it follows from Lemma 12 that there exist a left $(kG)^*$ -comodule structure δ^- and a right $(kG)^*$ -comodule structure δ^+ on arrow module $(kQ_1^a, \alpha^-, \alpha^+)$ such that $(kQ_1^a, \alpha^-, \alpha^+, \delta^-, \delta^+)$ becomes a $(kG)^*$ -Hopf bimodule, called an arrow Hopf bimodule. We obtain two graded Hopf algebras $T_{(kG)^*}(kQ_1^a)$ and $T_{(kG)^*}^c(kQ_1^a)$, called path Hopf algebra and semi-co-path Hopf algebra, written as kQ^a and kQ^{sc} , respectively.

From now on, we assume that Q is a finite Hopf quiver on finite group G . Let $\xi_{kQ_1^a}$ denote the linear map from kQ_1^a to $(kQ_1^a)^*$ by sending a to a^* for any $a \in Q_1$ and $\xi_{kQ_1^c}$ denote the linear map from kQ_1^c to $(kQ_1^c)^*$ by sending a to a^* for any $a \in Q_1$. It is easy to check the following.

Lemma 13. (i) *If $(M, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a finite dimensional B -Hopf bimodule and B is a finite dimensional Hopf algebra, then $(M^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$ is a B^* -Hopf bimodule.*

(ii) *If $(kQ_1^c, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a co-arrow Hopf bimodule, then there exist unique left $(kG)^*$ -comodule operation $\delta_{kQ_1^a}^-$ and right $(kG)^*$ -comodule operation $\delta_{kQ_1^a}^+$ such that $(kQ_1^a, \alpha_{kQ_1^a}^-, \alpha_{kQ_1^a}^+, \delta_{kQ_1^a}^-, \delta_{kQ_1^a}^+)$ becomes a $(kG)^*$ -Hopf bimodule and $\xi_{kQ_1^a}$ becomes a $(kG)^*$ -Hopf bimodule isomorphism from $(kQ_1^a, \alpha_{kQ_1^a}^-, \alpha_{kQ_1^a}^+, \delta_{kQ_1^a}^-, \delta_{kQ_1^a}^+)$ to $((kQ_1^c)^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$.*

(iii) *If $(kQ_1^a, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is an arrow Hopf bimodule, then there exist unique left kG -module operation $\alpha_{kQ_1^c}^-$ and right kG -module operation $\alpha_{kQ_1^c}^+$ such that $(kQ_1^c, \alpha_{kQ_1^c}^-, \alpha_{kQ_1^c}^+, \delta_{kQ_1^c}^-, \delta_{kQ_1^c}^+)$ become a kG -Hopf bimodule and $\xi_{kQ_1^c}$ becomes a kG -Hopf bimodule isomorphism from $(kQ_1^c, \alpha_{kQ_1^c}^-, \alpha_{kQ_1^c}^+, \delta_{kQ_1^c}^-, \delta_{kQ_1^c}^+)$ to $((kQ_1^a)^*, \delta^{-*}, \delta^{+*}, \alpha^{-*}, \alpha^{+*})$.*

(iv) *$\xi_{kQ_1^a}$ is a $(kG)^*$ -Hopf bimodule isomorphism from $(kQ_1^a, \alpha_{kQ_1^a}^-, \alpha_{kQ_1^a}^+, \delta_{kQ_1^a}^-, \delta_{kQ_1^a}^+)$ to $((kQ_1^c)^*, \delta_{kQ_1^c}^{-*}, \delta_{kQ_1^c}^{+*}, \alpha_{kQ_1^c}^{-*}, \alpha_{kQ_1^c}^{+*})$ if and only if $\xi_{kQ_1^c}$ becomes a kG -Hopf bimodule isomorphism from $(kQ_1^c, \alpha_{kQ_1^c}^-, \alpha_{kQ_1^c}^+, \delta_{kQ_1^c}^-, \delta_{kQ_1^c}^+)$ to $((kQ_1^a)^*, \delta_{kQ_1^a}^{-*}, \delta_{kQ_1^a}^{+*}, \alpha_{kQ_1^a}^{-*}, \alpha_{kQ_1^a}^{+*})$.*

Let B be a Hopf algebra and ${}^B_B\mathcal{M}_B$ denote the category of B -Hopf bimodules. Let $G\mathcal{H}opf$ denote the category of graded Hopf algebras. Define $T_B(\psi) := T_B(\iota_0, \iota_1\psi)$ and $T_B^c(\psi) := T_B^c(\pi_0, \psi\pi_1)$ for any B -Hopf bimodule homomorphism ψ .

Lemma 14. *Let B be a Hopf algebra. Then T_B and T_B^c are two functors from ${}^B_B\mathcal{M}_B$ to $G\mathcal{H}opf$.*

Proof. (i) If ψ is a B -Hopf bimodule homomorphism from M to M' , then $T_B(\iota_0, \iota_1\psi)$ is a graded Hopf algebra homomorphism from $T_B(M)$ to $T_B(M')$. Indeed, let $\Phi := T_B(\iota_0, \iota_1\psi)$. Then both $\Delta_{T_{B'}(M')} \Phi$ and $(\Phi \otimes \Phi) \Delta_{T_B(M)}$ are graded algebra maps from $T_B(M)$ to $T_B(M') \otimes T_B(M')$. Now we show that $\Delta_{T_B(M')} \Phi = (\Phi \otimes \Phi) \Delta_{T_B(M)}$. Considering that Φ is an algebra homomorphism, we only have to show that $\Delta_{T_B(M')} \Phi \iota_0 = (\Phi \otimes \Phi) \Delta_{T_B(M)} \iota_0$ and $\Delta_{T_B(M')} \Phi \iota_1 = (\Phi \otimes \Phi) \Delta_{T_B(M)} \iota_1$. Obviously, the first equation holds. For the second equation, see

$$\begin{aligned} \Delta_{T_{B'}(M')} \Phi \iota_1 &= \Delta_{T_{B'}(M')} \iota_1 \psi = (\iota_0 \otimes \iota_1) \delta_{M'}^- \psi + (\iota_1 \otimes \iota_0) \delta_{M'}^+ \psi \\ &= (\iota_0 \otimes \iota_1) (id \otimes \psi) \delta_M^- + (\iota_1 \otimes \iota_0) (\psi \otimes id) \delta_M^+ \\ &= (\iota_0 \otimes \iota_1 \psi) \delta_M^- + (\iota_1 \psi \otimes \iota_0) \delta_M^+ \\ &= (\Phi \otimes \Phi) (\iota_0 \otimes \iota_1) \delta_M^- + (\Phi \otimes \Phi) (\iota_1 \otimes \iota_0) \delta_M^+ \\ &= (\Phi \otimes \Phi) [(\iota_0 \otimes \iota_1) \delta_M^- + (\iota_1 \otimes \iota_0) \delta_M^+] \\ &= (\Phi \otimes \Phi) \Delta_{T_B(M)} \iota_1. \end{aligned}$$

Consequently, $T_B(\iota_0, \iota_1\psi)$ is a graded Hopf algebra homomorphisms.

(ii) If ψ is a B -Hopf bimodule homomorphism from M to M' , then $T_B^c(\pi_0, \psi\pi_1)$ is a graded Hopf algebra homomorphism from $T_B^c(M)$ to $T_B^c(M')$. Indeed, let $\Psi := T_B^c(\pi_0, \psi\pi_1)$. Then both $\Psi \mu_{T_B^c(M)}$ and $\mu_{T_B^c(M')} (\Psi \otimes \Psi)$ are graded coalgebra maps from $T_B^c(M) \otimes T_B^c(M)$ to $T_B^c(M')$. Since $T_B^c(M) \otimes T_B^c(M)$ is a graded coalgebra, $\text{corad}(T_B^c(M) \otimes T_B^c(M)) \subseteq (T_B^c(M) \otimes T_B^c(M))_0 = \iota_0(B) \otimes \iota_0(B)$. It follows that $(\pi_1 \Psi \mu_{T_B^c(M)}) (\text{corad}(T_B^c(M) \otimes T_B^c(M))) = 0$. Thus by the universal property of $T_B^c(M')$, in order to prove $\Psi \mu_{T_B^c(M)} = \mu_{T_B^c(M')} (\Psi \otimes \Psi)$, we only need to show $\pi_n \Psi \mu_{T_B^c(M)} = \pi_n \mu_{T_B^c(M')} (\Psi \otimes \Psi)$ for $n = 0, 1$. However, this follows from a straightforward computation dual to part (i). Furthermore, one can see $\Psi(1) = 1$. Hence Ψ is an algebra map, and so a Hopf algebra map.

(iii) It is straightforward to check $T_B(\psi)T_B(\psi') = T_B(\psi\psi')$ and $T_B^c(\psi)T_B^c(\psi') = T_B^c(\psi\psi')$ for B -Hopf bimodule homomorphisms $\psi : M' \rightarrow M''$ and $\psi' : M \rightarrow M'$.

(iv) $T_B(id_M) = id_{T_B(M)}$ and $T_B^c(id_M) = id_{T_B^c(M)}$. ■

Lemma 15. *If ψ is a Hopf algebra isomorphism from B to B' and $(M, \alpha^-, \alpha^+, \delta^-, \delta^+)$ is a B -Hopf bimodule, then $(M, \alpha^-(\psi^{-1} \otimes id_M), \alpha^+(id_M \otimes \psi^{-1}), (\psi \otimes id_M) \delta^-, (id_M \otimes \psi) \delta^+)$ is a B' -Hopf bimodule. Furthermore $T_B(\iota_0\psi, \iota_1)$ and $T_B^c(\psi\pi_0, \pi_1)$ are graded Hopf algebra isomorphisms from $T_B(M)$ to $T_{B'}(M)$ and from $T_B^c(M)$ to $T_{B'}^c(M)$, respectively.*

By Lemma 14 and Lemma 13 (ii) and (iii), $T_{(kG)^*}(\iota_0, \iota_1 \xi_{kQ_1^a})$ and $T_{kG}^c(\pi_0, \xi_{kQ_1^c} \pi_1)$ are graded Hopf algebra isomorphisms from $T_{(kG)^*}(kQ_1^a)$ to $T_{(kG)^*}((kQ_1^c)^*)$ and from $T_{kG}^c(kQ_1^c)$ to $T_{kG}^c((kQ_1^a)^*)$, respectively. $T_{(kG)^*}^c(\pi_0, \xi_{kQ_1^a} \pi_1)$ and $T_{kG}(\iota_0, \iota_1 \xi_{kQ_1^c})$ are graded Hopf algebra isomorphisms from $T_{(kG)^*}^c(kQ_1^a)$ to $T_{(kG)^*}^c((kQ_1^c)^*)$ and from $T_{kG}(kQ_1^c)$ to $T_{kG}((kQ_1^a)^*)$, respectively. Furthermore, (kQ_1^a, kQ_1^c) , (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are said to be arrow dual pairings.

Theorem 7. *Assume that (Q, G, r) is a finite Hopf quiver on finite group G . If (kQ^a, kQ^c) and (kQ^s, kQ^{sc}) are said to be arrow dual pairings, then*

(i) $((kQ^a)^{\text{cop}} \bowtie_{\tau} kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra. Here

$$R_n = \sum_{g \in G} p_e \otimes g \otimes p_g \otimes e + \sum_{q \in Q(n), q \notin G} p_e \otimes q \otimes q \otimes e$$

and $\tau(a, b) = \delta_{a,b}$, for any two paths a and b in Q , where $\delta_{a,b}$ is the Kronecker symbol.

(ii) There exist τ and $\{R_n\}$ such that $((kQ^s)^{\text{cop}} \bowtie_{\tau} kQ^{sc}, \{R_n\})$ becomes a local quasitriangular Hopf algebra.

Proof. (i) Let $B := (kG)^*$ and $M := kQ_1^a$. Thus $T_B(M) = kQ^a$. Since (kQ_1^a, kQ_1^c) is an arrow dual pairing, $\xi_{kQ_1^c}$ is a kG -Hopf bimodule isomorphism from kQ_1^c to $(kQ_1^a)^*$ by Lemma 13. See that

$$\begin{aligned} T_{B^*}^c(M^*) &= T_{(kG)^{**}}^c((kQ_1^a)^*) \\ &\cong_{\nu_1} T_{kG}^c((kQ_1^a)^*) \quad (\text{by Lemma 15}) \\ &\cong_{\nu_2} T_{kG}^c(kQ_1^c) \quad (\text{by Lemma 13 and Lemma 14}) \\ &= kQ^c, \end{aligned}$$

where $\nu_1 = T_{kG}^c(\sigma_{kG}^{-1}\pi_0, \pi_1)$, $\nu_2 = T_{(kG)^{**}}^c(\pi_0, (\xi_{kQ_1^c})^{-1}\pi_1)$.

Let $H = T_{B^*}^c(M^*)$ and $A = T_B(M)^{\text{cop}}$. By Theorem 6, $((kQ^a)^{\text{cop}} \bowtie_{\tau} kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra. Here $\tau = d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)$; $R_n = (id_A \otimes \nu_2\nu_1 \otimes id_A \otimes \nu_2\nu_1)(1_B \otimes (id \otimes \psi_n)b_{H(n)} \otimes 1_{B^*})$, ϕ and ψ_n are defined in Lemma 6. We have to show that they are the same as in this theorem. That is,

$$d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)(a \otimes b) = \delta_{a,b} \quad (4)$$

$$\begin{aligned} (1_B \otimes b_{H(n)} \otimes 1_{B^*}) &= (id_A \otimes (\nu_2\nu_1)^{-1} \otimes \phi_n \otimes (\nu_2\nu_1)^{-1}) \\ &\quad \times \left(\sum_{g \in G} p_e \otimes g \otimes p_g \otimes e + \sum_{q \in Q_n, q \notin G} p_e \otimes q \otimes q \otimes e \right). \end{aligned} \quad (5)$$

If $b = b_n b_{n-1} \cdots b_1$ is a n -path in kQ^c with $b_i \in Q_1$ for $i = 1, 2, \dots, n$, then

$$(\nu_2\nu_1)^{-1}(b) = b_n^* \otimes b_{n-1}^* \otimes \cdots \otimes b_1^*. \quad (6)$$

If $b \in G$, then $(\nu_2\nu_1)^{-1}(b) = \sigma_{kG}(b)$. Consequently, for any $a, b \in Q_{\infty}$ with $b \in kQ^c$ and $a \in kQ^a$, we have

$$\begin{aligned} d_H(\phi \otimes id)C_{H,A}((\nu_2\nu_1)^{-1} \otimes id_A)(a \otimes b) &= d_H(\phi(a) \otimes (\nu_2\nu_1)^{-1}(b)) \\ &= \begin{cases} \langle \phi(a), \sigma_{kG}(b) \rangle = \langle \sigma_{kG}(b), a \rangle = \delta_{a,b}, & \text{when } b \in Q_0, \\ \langle (\nu_2\nu_1)^{-1}(b), a \rangle \stackrel{\text{by (6)}}{=} \delta_{a,b}, & \text{when } b \notin Q_0. \end{cases} \end{aligned}$$

Thus (4) holds. Note that Q_n is a basis of not only $(kQ^a)_n$ but also $(kQ^c)_n$ for $n > 0$. By (6), $\{\phi(q) \mid q \in Q_n\}$ is the dual basis of $\{(\nu_2\nu_1)^{-1}(q) \mid q \in Q_n\}$ for any $n > 0$. Consequently, (5) holds.

(ii) Let $B := kG$ and $M := kQ_1^c$. Thus $T_B(M) = kQ^s$. Since (kQ_1^a, kQ_1^c) is an arrow dual pairing, $\xi_{kQ_1^a}$ is a $(kG)^*$ -Hopf bimodule isomorphism from kQ_1^c to $(kQ_1^a)^*$ by Lemma 13. Thus

$$\begin{aligned} T_{B^*}^c(M^*) &= T_{(kG)^*}^c((kQ_1^c)^*) \\ &\cong_{\nu_3} T_{(kG)^*}^c(kQ_1^a) \quad (\text{by Lemma 13 and Lemma 14}) \\ &= kQ^{sc}, \end{aligned}$$

where $\nu_3 = T_{(kG)^*}^c(\pi_0, (\xi_{kQ_1^a})^{-1}\pi_1)$. By Theorem 6, the double cross product of $T_B(M)^{\text{cop}}$ and $T_{B^*}(M^*)$ is a local quasitriangular Hopf algebra. Consequently, so is the double cross product of $(kQ^s)^{\text{cop}}$ and kQ^{sc} . \blacksquare

Note (LQT4') holds in the above theorem.

Example 1. Let $G = \mathbf{Z}_2 = (g) = \{e, g\}$ be the group of order 2 with $\text{char } k \neq 2$, X and Y be respectively the set of arrows from g^0 to g^0 and the set of arrows from g to g , and $|X| = |Y| = 3$. The quiver Q is a Hopf quiver with respect to ramification $r = r_{\{e\}}\{e\}$ with $r_{\{e\}} = 3$. Let $\chi_e^{(i)} \in \mathbf{Z}_2$ and $a_{y,x}^{(i)}$ denote the arrow from x to y for $i = 1, 2, 3$. Define $\delta^-(a_{x,x}^{(i)}) = x \otimes a_{x,x}^{(i)}$, $\delta^+(a_{x,x}^{(i)}) = a_{x,x}^{(i)} \otimes x$, $g \cdot a_{x,x}^{(i)} = a_{gx,gx}^{(i)}$, $a_{e,e}^{(i)} \cdot g = \chi_e^{(i)}(g)a_{xg,xg}^{(i)}$ for $x \in G$, $i = 1, 2, 3$. By [4], kQ_1 is a kG -Hopf bialgebra. Therefore, it follows from Theorem 7 that $((kQ^a)^{\text{cop}} \bowtie_{\tau} kQ^c, \{R_n\})$ is a local quasitriangular Hopf algebra and for every finite dimensional $(kQ^a)^{\text{cop}} \bowtie_{\tau} kQ^c$ -module M , $C^{\{R_n\}}$ is a solution of Yang–Baxter equations on M .

By the way, we obtain the relation between path algebras and path coalgebras by Theorem 5.

Corollary 1. *Let Q be a finite quiver over finite group G . Then Path algebra kQ^a is algebra isomorphic to subalgebra $\sum_{n=0}^{\infty} (\square_{kG}^n kQ_1^c)^*$ of $(kQ^c)^*$.*

Proof. Let $A = (kG)^*$ and $M = kQ_1^c$. It is clear that $\xi_{kQ_1^c}$ is a kG -bicomodule isomorphism from kQ_1^c to $(kQ_1^a)^*$. See that

$$T_{B^*}^c(M^*) = T_{(kG)^{**}}^c((kQ_1^a)^*) \stackrel{\nu_1}{\cong} T_{kG}^c((kQ_1^a)^*) \stackrel{\nu_2}{\cong} T_{kG}^c(kQ_1^c) = kQ^c,$$

where $\nu_1 = T_{kG}^c(\sigma_{kG}^{-1}\pi_0, \pi_1)$, $\nu_2 = T_{kG}^c(\pi_0, (\xi_{kQ_1^c})^{-1}\pi_1)$. Obviously, ν_1 and ν_2 are coalgebra isomorphism. Consequently, it follows from Theorem 5 that kQ^a is algebra isomorphism to subalgebra $\sum_{n=0}^{\infty} (\square_{kG}^n kQ_1^c)^*$ of $(kQ^c)^*$. ■

Acknowledgement

The authors were financially supported by the Australian Research Council. S.Z. thanks Department of Mathematics, University of Queensland for hospitality.

References

- [1] Auslander M., Reiten I., Smalø S.O., Representation theory of Artin algebras, Cambridge University Press, 1995.
- [2] Belavin A.A., Drinfel'd V.G., solutions of the classical Yang–Baxter equations for simple Lie algebras, *Functional Anal. Appl.* **16** (1982), no. 3, 159–180.
- [3] Chin W., Montgomery S., Basic coalgebras, in Modular Interfaces (1995, Riverside, CA), *AMS/IP Stud. Adv. Math.*, Vol. 4, Amer. Math. Soc., Providence, RI, 1997, 41–47.
- [4] Cibils C., Rosso M., Hopf quivers, *J. Algebra* **254** (2002), 241–251, [math.QA/0009106](https://arxiv.org/abs/math.QA/0009106).
- [5] Cibils C., Rosso M., Algèbres des chemins quantiques, *Adv. Math.* **125** (1997), 171–199.
- [6] Drinfel'd V.G., Quantum groups, in Proceedings International Congress of Mathematicians (August 3–11, 1986, Berkeley, CA), Amer. Math. Soc., Providence, RI, 1987, 798–820.
- [7] Dascalescu S., Nastasescu C., Raianu S., Hopf algebras: an introduction, Marcel Dekker Inc., 2001.
- [8] Doi Y., Takeuchi M., Multiplication alteration by two-cocycles – the quantum version, *Comm. Algebra* **22** (1994), 5715–5732.
- [9] Jantzen J.C., Lectures on quantum groups, *Graduate Studies in Mathematics*, Vol. 6, Amer. Math. Soc., Providence, RI, 1996.
- [10] Majid S., Algebras and Hopf algebras in braided categories, in Advances in Hopf Algebras (1992, Chicago, IL), *Lecture Notes in Pure and Appl. Math.*, Vol. 158, Marcel Dekker, New York, 1994, 55–105.
- [11] Montgomery S., Hopf algebras and their actions on rings, *CBMS Regional Conference Series in Mathematics*, Vol. 82, Amer. Math. Soc., Providence, RI, 1993.

- [12] Nichols W., Bialgebras of type one, *Comm. Algebra* **6** (1978), 1521–1552.
- [13] Lusztig G., Introduction to quantum groups, *Progress in Mathematics*, Vol. 110, Birkhäuser, Boston, MA, 1993.
- [14] Kassel C., Quantum groups, *Graduate Texts in Mathematics*, Vol. 155, Springer-Verlag, New York, 1995.
- [15] Sweedler M.E., Hopf algebras, Benjamin, New York, 1969.
- [16] Yang C.N., Ge M.L. (Editors), Braid group, knot theory and statistical mechanics, World Scientific, Singapore, 1989.
- [17] Zhang S., Chen H.-X., The double bicrossproducts in braided tensor categories, *Comm. Algebra* **29** (2001), 31–66.