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Integrable Anisotropic Evolution Equations on a Sphere

Anatoly G. MESHKOV * *and Maxim Ju. BALAKHNEV*

Orel State University, 95 Komsomol'skaya Str., Orel, 302026 Russia
E-mail: *meshkov@orel.ru*, *maxibal@yandex.ru*
* URL: http://www.orel.ru/meshkov

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Abstract. V.V. Sokolov's modifying symmetry approach is applied to anisotropic evolution equations of the third order on the n-dimensional sphere. The main result is a complete classification of such equations. Auto-Bäcklund transformations are also found for all equations.

Key words: evolution equation; equation on a sphere; integrability; symmetry classification; anisotropy; conserved densities; Bäcklund transformations

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1 Introduction

In this paper we are dealing with the problem of the symmetry classification of the integrable vector evolution equations of the third order. Completely integrable equations possess many remarkable properties and are often interesting for applications. As examples of such equations we may point out two well known modified Korteweg–de Vries equations

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u},\boldsymbol{u})\boldsymbol{u}_x, \qquad \boldsymbol{u}_t = \boldsymbol{u}_{xxx} + (\boldsymbol{u},\boldsymbol{u})\boldsymbol{u}_x + (\boldsymbol{u},\boldsymbol{u}_x)\boldsymbol{u},$$

where $(\cdot\,,\cdot)$ is a scalar product, $\boldsymbol{u} = (u^1,\dots,u^n)$. These equations are integrable by the inverse scattering method for any vector dimension. Another example of integrable anisotropic evolution equation on the sphere is the higher order Landau–Lifshitz equation

$$\boldsymbol{u}_t = \left(\boldsymbol{u}_{xx} + \frac{3}{2}(\boldsymbol{u}_x,\boldsymbol{u}_x)\boldsymbol{u}\right)_x + \frac{3}{2}(\boldsymbol{u},R\boldsymbol{u})\boldsymbol{u}_x, \qquad (\boldsymbol{u},\boldsymbol{u}) = 1, \tag{1}$$

where R is a diagonal constant matrix. Complete integrability of this equation was proved in [1].

To investigate integrability of such equations a modification of the symmetry approach was proposed in [2]. Different examples of integrable vector equations can also be found in [2]. Sokolov's method can be applied to any vector equation

$$\boldsymbol{u}_t = f_n\boldsymbol{u}_n + f_{n-1}\boldsymbol{u}_{n-1} + \cdots + f_0\boldsymbol{u} \tag{2}$$

with scalar coefficients f_i. Henceforth $\boldsymbol{u} = (u^1,\dots,u^{n+1})$ denotes an unknown vector, $\boldsymbol{u}_k = \partial^k\boldsymbol{u}/\partial x^k$ and f_i are some scalar functions depending on the scalar products $(\boldsymbol{u}_i,\boldsymbol{u}_j)$, $i \leqslant j$. Moreover, dependence of f_i on more than one scalar products $(\cdot,\cdot)_i$, $i = 1,2,\dots$ may be introduced. It is clear that any equation (2) with the Euclidean scalar product is invariant with respect to an arbitrary constant orthogonal transformation of the vector $\boldsymbol{u}$. Therefore the equation (2) with a unique scalar product is called isotropic. When f_i depend on two or more scalar products $(\boldsymbol{u}_i, \boldsymbol{u}_j)_k$, we call the equation (2) anisotropic. The scalar products may have a different nature. Only two properties of the scalar products are essential for us — bilinearity and continuity. Vector $\boldsymbol{u}$ may be both real and complex.

The symmetry approach [3, 4, 5, 6, 7] is based on the observation that all integrable evolution equations with one spatial variable possess local higher symmetries or, which is the same, higher commuting flows. Canonical conserved densities ρ_i, $i = 0, 1, \ldots$ make the central notion of this approach. These densities can be expressed in terms of the coefficients of the equation under consideration. The evolutionary derivative of ρ_i must be the total x-derivative of some local function θ_i:

$$D_t\rho_i(u) = D_x\theta_i(u), \qquad i = 0, 1, \ldots. \tag{3}$$

It follows from (3) that the variational derivative $\delta D_t\rho_i(u)/\delta u$ is zero. Both equations $\delta D_t\rho_i/\delta u = 0$, $i = 0, 1, \ldots$ and (3) are called the integrability conditions.

The modified symmetry approach has been recently applied to the equations on $\mathbb{S}^n$ [8] and on $\mathbb{R}^n$ [9, 10], where the third order equations in the form

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u}, \tag{4}$$

were classified under several restrictions: $(\boldsymbol{u}, \boldsymbol{u}) = 1$ in [8], $\boldsymbol{u}_t = (\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + f_0\boldsymbol{u})_x$ in [9] and $f_2 = 0$ in [10]. We have also tried to classify the equations (4) on $\mathbb{R}^n$ but the attempt has failed because of great computational difficulties. Moreover, the integrable equations that we found proved to be too cumbersome to be applied to any scientific problem. That is why the restrictions were used in the above-mentioned articles. In contrast with $\mathbb{R}^n$ the list of the anisotropic integrable equations on $\mathbb{S}^n$ presented in Section 2 is short and contains at the least one interesting equation (10).

Thus, the subject of the article may be defined as the symmetry classification of the equations (4) with the constraint $(\boldsymbol{u}, \boldsymbol{u}) \equiv \boldsymbol{u}^2 = 1$. The coefficients f_i are assumed to be depending on both isotropic variables $u_{[i,j]}$ and anisotropic variables $\tilde{u}_{[i,j]}$

$$u_{[i,j]} = (\boldsymbol{u}_i, \boldsymbol{u}_j), \qquad \tilde{u}_{[i,j]} = \langle \boldsymbol{u}_i, \boldsymbol{u}_j \rangle, \qquad i \leqslant j \tag{5}$$

with $i \leqslant j \leqslant 2$. The constraint $\boldsymbol{u}^2 = 1$ implies

$$(\boldsymbol{u}, \boldsymbol{u}_t) = 0, \qquad u_{[0,1]} = 0, \quad u_{[0,2]} = -u_{[1,1]}, \quad u_{[0,3]} = -3u_{[1,2]}, \quad \ldots. \tag{6}$$

From these constraints it follows that $f_0 = f_2u_{[1,1]} + 3u_{[1,2]}$ in the equation (4).

In this paper we shall consider the equations (4) that are integrable for arbitrary dimension n of the sphere. In addition, we assume that the coefficients f_i do not depend on n. In virtue of the arbitrariness of n, the variables (5) will be regarded as **independent**. The functional independence of $\{u_{[i,j]}, \tilde{u}_{[i,j]}, i \leqslant j\}$ is a crucial requirement in all our considerations.

It is easy to see that the stereographic projection maps any equation (4) on $\mathbb{S}^n$ to some anisotropic equation on $\mathbb{R}^{n-1}$.

In Section 2 we present a complete list of integrable anisotropic equations of the form (4) on the sphere $\mathbb{S}^n$. And a scheme of computations is presented in Section 3.

In order to prove that all equations from the list are really integrable, we find, in Section 4, an auto-Bäcklund transformation involving a "spectral" parameter for each of the equations.

2 Classification results

In this section we formulate some classification statements concerning integrable evolution equations of third order on the n-dimensional sphere. This classification problem is much simpler than the similar problem on $\mathbb{R}^n$. Indeed, the set of the independent variables (5) on $\mathbb{S}^n$ is reduced because of the constraints (6). It is easy to see that we can express all variables of the form $u_{[0,k]}$, $k \geqslant 1$ in terms of the remaining independent scalar products. So, the complete set of dynamical variables on the sphere is

$$\{u_{[i,j]},\ 1 \leqslant i \leqslant j;\ \tilde{u}_{[i,j]},\ 0 \leqslant i \leqslant j\}. \tag{7}$$

Therefore the coefficients of the equation (4) on $\mathbb{S}^n$ *a priori* depend on nine independent variables

$$u_{[1,1]},\ u_{[1,2]},\ u_{[2,2]},\ \tilde{u}_{[0,0]},\ \tilde{u}_{[0,1]},\ \tilde{u}_{[0,2]},\ \tilde{u}_{[1,1]},\ \tilde{u}_{[1,2]},\ \tilde{u}_{[2,2]}, \tag{7$'$}$$

whereas in the case of $\mathbb{R}^n$ they are functions of twelve variables.

Let g_{ij} and $\tilde{g}_{ij}$ be the first and the second metric tensors, $u_{[k,l]} = \sum_{i,j} g_{ij} u_k^i u_l^j$, $\tilde{u}_{[k,l]} = \sum_{i,j} \tilde{g}_{ij} u_k^i u_l^j$. Then the equation (4) and the constraint $\boldsymbol{u}^2 = 1$ are obviously invariant under the transformation $\tilde{g}_{ij} \to \alpha\tilde{g}_{ij} + \lambda g_{ij}$ where α and λ are constants and $\alpha \neq 0$. It is equivalent to the following transformation of the dependent variables

$$\tilde{u}_{[k,l]} \to \alpha\tilde{u}_{[k,l]} + \lambda u_{[k,l]}. \tag{8}$$

We used this transformation to classify the integrable equations (4).

Theorem 1. *If the anisotropic equation on $\mathbb{S}^n$*

$$\boldsymbol{u}_t = \boldsymbol{u}_{xxx} + f_2\boldsymbol{u}_{xx} + f_1\boldsymbol{u}_x + (f_2 u_{[1,1]} + 3u_{[1,2]})\boldsymbol{u}, \tag{9}$$

possesses an infinite series of canonical conservation laws $(\rho_k)_t = (\theta_k)_x$, $k = 0, 1, 2\ldots$, where ρ_k and θ_k are functions of variables (7), then this equation can be reduced, with the help of the transformation (8), to one of the following equations

$$\boldsymbol{u}_t = \boldsymbol{u}_3 + \frac{3}{2}\big(u_{[1,1]} + \tilde{u}_{[0,0]}\big)\boldsymbol{u}_1 + 3u_{[1,2]}\boldsymbol{u}, \tag{10}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{11}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{u_{[1,2]}}{u_{[1,1]}}\boldsymbol{u}_2 + \frac{3}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} + \frac{u_{[1,2]}^2}{u_{[1,1]}^2} - \frac{(\tilde{u}_{[0,1]} + u_{[1,2]})^2}{(u_{[1,1]} + \tilde{u}_{[0,0]})u_{[1,1]}} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right)\boldsymbol{u}_1, \tag{12}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{2\tilde{u}_{[0,2]} + \tilde{u}_{[1,1]} + a}{2\tilde{u}_{[0,0]}} - \frac{5}{2}\frac{\tilde{u}_{[0,1]}^2}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1 + 3\left(u_{[1,2]} - \frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u}, \tag{13}$$

$$\boldsymbol{u}_t = \boldsymbol{u}_3 - 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\boldsymbol{u}_2 - 3\left(\frac{\tilde{u}_{[0,2]}}{\tilde{u}_{[0,0]}} - 2\frac{\tilde{u}_{[0,1]}^2}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1 + 3\left(u_{[1,2]} - \frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}u_{[1,1]}\right)\boldsymbol{u}, \tag{14}$$

$$\begin{aligned}\boldsymbol{u}_t = {} & \boldsymbol{u}_3 - 3\frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\big(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\big) + 3u_{[1,2]}\boldsymbol{u} \\ & + \frac{3}{2}\left(-\frac{u_{[2,2]}}{\tilde{u}_{[0,0]}} + \frac{(u_{[1,2]} + \tilde{u}_{[0,1]})^2}{\tilde{u}_{[0,0]}(\tilde{u}_{[0,0]} + u_{[1,1]})} + \frac{(\tilde{u}_{[0,0]} + u_{[1,1]})^2}{\tilde{u}_{[0,0]}} + \frac{\tilde{u}_{[0,1]}^2 - \tilde{u}_{[0,0]}\tilde{u}_{[1,1]}}{\tilde{u}_{[0,0]}^2}\right)\boldsymbol{u}_1,\end{aligned} \tag{15}$$

$$\begin{aligned}\boldsymbol{u}_t = {} & \boldsymbol{u}_3 + 3\left(\frac{\tilde{u}_{[0,1]}\tilde{u}_{[0,2]}}{\xi} - \frac{\tilde{u}_{[1,2]}\tilde{u}_{[0,0]}}{\xi} + \frac{\tilde{u}_{[0,1]}}{\tilde{u}_{[0,0]}}\right)\big(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}\big) + 3u_{[1,2]}\boldsymbol{u} \\ & + \frac{3}{2\xi^2\tilde{u}_{[0,0]}^2}\Big(\tilde{u}_{[0,0]}^3\tilde{u}_{[2,2]}\xi - \xi(\xi + \tilde{u}_{[0,2]}\tilde{u}_{[0,0]})^2 + (\tilde{u}_{[0,0]}^2\tilde{u}_{[1,2]} - 2\xi\tilde{u}_{[0,1]}\end{aligned}$$

$$- \tilde u_{[0,0]}\tilde u_{[0,1]}\tilde u_{[0,2]}\big)^2\Big)\boldsymbol{u}_1 - a\frac{\tilde u_{[0,0]}^2 u_{[1,1]} + \tilde u_{[0,1]}^2}{\tilde u_{[0,0]}\xi}\boldsymbol{u}_1, \qquad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2, \tag{16}$$

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}\tilde u_{[0,2]}}{\xi} - \frac{\tilde u_{[1,2]}\tilde u_{[0,0]}}{\xi} + \frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}}\right)(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}) + 3u_{[1,2]}\boldsymbol{u}\\ &\quad + \frac{3}{\xi}\left(\tilde u_{[0,0]}\tilde u_{[2,2]} - 2\tilde u_{[0,1]}\tilde u_{[1,2]} - \frac{(\tilde u_{[0,2]}\tilde u_{[0,0]} - 2\tilde u_{[0,1]}^2)(\xi + \tilde u_{[0,2]}\tilde u_{[0,0]})}{\tilde u_{[0,0]}^2}\right)\boldsymbol{u}_1,\\ &\quad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2,\end{aligned} \tag{17}$$

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 - 3\frac{a\tilde u_{[0,1]}}{\eta}\boldsymbol{u}_2 + 3\frac{u_{[1,2]}\eta - a\tilde u_{[0,1]}u_{[1,1]}}{\eta}\boldsymbol{u} + \frac{3}{2}\left(\frac{\tilde u_{[2,2]}}{\eta} + \frac{a\xi - (\tilde u_{[0,2]} + \eta)^2}{\eta\tilde u_{[0,0]}}\right)\boldsymbol{u}_1\\ &\quad + \frac{3}{2}\left(\frac{\big(\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}(2a\tilde u_{[0,0]} + b + \tilde u_{[0,2]})\big)^2}{\eta\xi\tilde u_{[0,0]}} - b\frac{a\tilde u_{[0,1]}^2 + \eta\tilde u_{[0,0]}u_{[1,1]}}{\eta^2\tilde u_{[0,0]}}\right)\boldsymbol{u}_1,\\ &\quad \eta = a\tilde u_{[0,0]} + b, \qquad \xi = \tilde u_{[0,0]}(\eta - \tilde u_{[1,1]}) + \tilde u_{[0,1]}^2,\end{aligned} \tag{18}$$

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}} + \frac{\tilde u_{[0,1]}\tilde u_{[0,2]}}{\xi} - \frac{\tilde u_{[0,0]}\tilde u_{[1,2]}}{\xi}\right)(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}) + 3u_{[1,2]}\boldsymbol{u}\\ &\quad + \frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[2,2]}}{\xi} + b\tilde u_{[0,0]}\frac{u_{[1,1]}\eta + a\tilde u_{[0,1]}^2}{\eta\xi} - \frac{(\tilde u_{[0,0]}\tilde u_{[0,2]} + \xi)^2}{\tilde u_{[0,0]}^2\xi}\right)\boldsymbol{u}_1\\ &\quad + \frac{3}{2}\frac{\big(\tilde u_{[0,0]}^2(a\xi\tilde u_{[0,1]} - \eta\tilde u_{[1,2]}) + \eta\tilde u_{[0,1]}(\tilde u_{[0,0]}\tilde u_{[0,2]} + \xi)\big)^2}{\eta\tilde u_{[0,0]}^2(\xi + \eta)\xi^2}\boldsymbol{u}_1,\\ &\quad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2, \qquad \eta = (a\tilde u_{[0,0]} + b)\tilde u_{[0,0]},\end{aligned} \tag{19}$$

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 + \frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}\tilde u_{[0,2]}}{\mu(\mu + \tilde u_{[0,0]})} - 2\frac{\tilde u_{[0,1]}}{\mu}\right)(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}) + 3u_{[1,2]}\boldsymbol{u}\\ &\quad + \frac{3/2}{\tilde u_{[0,0]}(\mu + \tilde u_{[0,0]})}\Big[\mu^{-2}\big(\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}\tilde u_{[0,2]}\big)^2 + \tilde u_{[0,0]}\tilde u_{[2,2]} - \tilde u_{[0,2]}^2\\ &\quad - 2\mu^{-2}\tilde u_{[0,1]}\big(\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}\tilde u_{[0,2]}\big)(\mu + 2\tilde u_{[0,0]})\Big]\boldsymbol{u}_1 + \big(6\mu^{-2}\tilde u_{[0,1]}^2 - 3\tilde u_{[0,0]}^{-1}\tilde u_{[0,2]}\big)\boldsymbol{u}_1,\\ &\quad \mu^2 = \tilde u_{[0,1]}^2 + \tilde u_{[0,0]}^2 - \tilde u_{[0,0]}\tilde u_{[1,1]}.\end{aligned} \tag{20}$$

Remark 1. The equations (10)–(12) were given in [8]; the equation (10) coincides with (1).

Remark 2. Each of the equations (10)–(20) can contain the term $c\boldsymbol{u}_1$ in its right-hand side. We removed these terms by the Galilean transformation as trivial. The constants a and b are arbitrary. One can set $a = 0$ in (13) and in (16). In (18) we may choose $a = 0$ or $b = 0$ but $\{a, b\} \neq 0$. If we set in (18) $a = 0$, then it takes the following form

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 + \frac{3}{2}\left(\frac{\tilde u_{[2,2]}}{b} - \frac{(\tilde u_{[0,2]} + b)^2}{b\tilde u_{[0,0]}} + \frac{\big(\tilde u_{[0,0]}\tilde u_{[1,2]} - \tilde u_{[0,1]}(\tilde u_{[0,2]} + b)\big)^2}{b\xi\tilde u_{[0,0]}} - u_{[1,1]}\right)\boldsymbol{u}_1\\ &\quad + 3u_{[1,2]}\boldsymbol{u},\end{aligned} \tag{18$'$}$$

where $\xi = \tilde u_{[0,0]}(b - \tilde u_{[1,1]}) + \tilde u_{[0,1]}^2$. If we set in (19) $a = 0$ and then $b = 0$, then the equation is reduced to the following form

$$\begin{aligned}\boldsymbol{u}_t &= \boldsymbol{u}_3 + 3\left(\frac{\tilde u_{[0,1]}}{\tilde u_{[0,0]}} + \frac{\tilde u_{[0,1]}\tilde u_{[0,2]} - \tilde u_{[0,0]}\tilde u_{[1,2]}}{\xi}\right)(\boldsymbol{u}_2 + u_{[1,1]}\boldsymbol{u}) + 3u_{[1,2]}\boldsymbol{u}\\ &\quad + \frac{3}{2}\left(\frac{\tilde u_{[0,0]}\tilde u_{[2,2]}}{\xi} - \frac{(\xi + \tilde u_{[0,0]}\tilde u_{[0,2]})^2}{\xi\tilde u_{[0,0]}^2}\right)\boldsymbol{u}_1, \qquad \xi = \tilde u_{[0,0]}\tilde u_{[1,1]} - \tilde u_{[0,1]}^2.\end{aligned} \tag{19$'$}$$

Remark 3. In the classifying process we did not consider the isotropic equations because they were found earlier in [8]. Nevertheless, all equations (10)–(20) admit the reduction $\tilde{u}_{[i,j]} = ku_{[i,j]}$ that will be referred as the isotropic reduction. Each integrable isotropic equation can be obained from the list (10)–(20). The diagram of the isotropic reductions takes the following form:

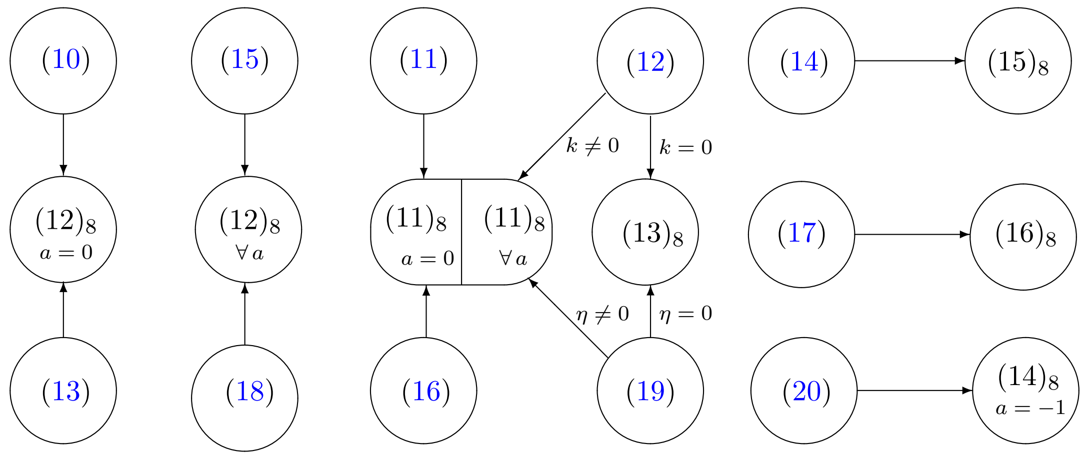

Diagram of the isotropic reductions $\tilde{u}_{[i,j]} = ku_{[i,j]}$. Here $(n)_8$ means the equation (n) from [8].

We stress that the equation (19) with $\eta \neq 0$ is reduced to (11) from [8]. But its reduction under $\eta = 0$ — (19′), on the contrary, is reduced to the vector Schwartz–KdV equation (13) from [8]. Hence the properties of the equations (19) and (19′) are essentially different.

Remark 4. While proving the main theorem we found that all equations (10)–(20) have non-trivial local conserved densities of the orders 2, 3, 4 and 5. All these densities can be obtained from the formula (21). For example, the equation (11) have the following canonical conserved densities:

$$\rho_0 = \frac{u_{[1,2]}}{u_{[1,1]}}, \qquad \rho_1 = -\frac{1}{2}\left(\frac{u_{[2,2]}}{u_{[1,1]}} - \frac{u^2_{[1,2]}}{u^2_{[1,1]}} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right) - D_x\frac{u_{[2,2]}}{u_{[1,1]}},$$
$$\rho_2 = D_x\left(\frac{u_{[1,3]}}{u_{[1,1]}} + \frac{3}{2}\frac{u_{[2,2]}}{u_{[1,1]}} - 2\frac{u^2_{[1,2]}}{u^2_{[1,1]}} + \frac{\tilde{u}_{[1,1]}}{u_{[1,1]}}\right), \quad \ldots.$$

3 A sketchy proof of the main theorem

The equation (4) can be rewritten in the form

$$L\boldsymbol{u} = 0, \qquad L = -D_t + D_x^3 + f_2D_x^2 + f_1D_x + f_0.$$

This operator L is used for obtaining the canonical conserved densities by a technique proposed in [11]. Motivation and explanation of the technique, for vector equations, have been presented earlier in [8]. For more details, see also [12].

Let ρ and θ be the generating functions for the canonical densities and fluxes correspondingly:

$$\rho = k^{-1} + \sum_{i=0}^{\infty}\rho_i k^i, \qquad \theta = k^{-3} + \sum_{i=0}^{\infty}\theta_i k^i, \qquad D_t\rho = D_x\theta.$$

Then, from the equation $L\exp(D_x^{-1}\rho)=0$, the recursion formula follows

$$\begin{aligned}\rho_{n+2}&=\frac{1}{3}\left[\theta_n-f_0\delta_{n,0}-2f_2\rho_{n+1}-f_2D_x\rho_n-f_1\rho_n\right]\\&\quad-\frac{1}{3}\left[f_2\sum_{s=0}^{n}\rho_s\rho_{n-s}+\sum_{0\leqslant s+k\leqslant n}\rho_s\rho_k\rho_{n-s-k}+3\sum_{s=0}^{n+1}\rho_s\rho_{n-s+1}\right]\\&\quad-D_x\left[\rho_{n+1}+\frac{1}{2}\sum_{s=0}^{n}\rho_s\rho_{n-s}+\frac{1}{3}D_x\rho_n\right],\qquad n\geqslant 0,\end{aligned}\tag{21}$$

where $\delta_{i,j}$ is the Kronecker delta and ρ_0, ρ_1 are

$$\rho_0=-\frac{1}{3}f_2,\qquad \rho_1=\frac{1}{9}f_2^2-\frac{1}{3}f_1+\frac{1}{3}D_xf_2.\tag{22}$$

The corresponding functions θ_i can be found from (3). The fact that the left-hand sides of (3) are total x-derivatives imposes rigid restrictions (see below) on the coefficients f_i of (4).

Expressions for the next functions ρ_i involve, besides f_k, the functions θ_j with $j\leqslant i-2$. For example

$$\rho_2=-\frac{1}{3}f_0+\frac{1}{3}\theta_0-\frac{2}{81}f_2^3+\frac{1}{9}f_1f_2-D_x\left(\frac{1}{9}f_2^2+\frac{2}{9}D_xf_2-\frac{1}{3}f_1\right)$$

and so on.

It is shown in [8] that all even canonical densities ρ_{2n} are trivial and we have the following strengthened conditions of the integrability

$$D_t\rho_{2n+1}(u)=D_x\theta_{2n+1}(u),\qquad \rho_{2n}(u)=D_x\sigma_{2n}(u),\qquad n=0,1,\ldots\tag{23}$$

instead of (3).

To show how to use the conditions (23), we consider the equations (9) on $\mathbb{S}^n$. Obviously, we have to replace f_0 by $f_2u_{[1,1]}+3u_{[1,2]}$ in the formulas (21).

Lemma 1. *Suppose the equation (9) on $\mathbb{S}^n$ admits the canonical conserved density ρ_0 then the equation has the following form*

$$\boldsymbol{u}_t=\boldsymbol{u}_3-\frac{3}{2}\boldsymbol{u}_2D_x\ln(g_0)+f_1\boldsymbol{u}_1+\left(3u_{[1,2]}-\frac{3}{2}u_{[1,1]}D_x\ln(g_0)\right)\boldsymbol{u},\tag{24}$$

where g_0 depends on $\tilde u_{[0,0]}$, $\tilde u_{[0,1]}$, $\tilde u_{[1,1]}$ and $u_{[1,1]}$, and f_1 is a function of the variables (7′).

Proof. From (22) and (23) we have $f_2=D_x\sigma_0$ for some function σ_0. Since f_2 does not depend on the third order variables, we see that σ_0 may only depend on the variables $\tilde u_{[0,0]}$, $\tilde u_{[0,1]}$, $\tilde u_{[1,1]}$ and $u_{[1,1]}$. Setting for convenience $\sigma_0=-3/2\ln(g_0)$ we obtain (24). ■

Lemma 2. *Suppose the equation (24) on $\mathbb{S}^n$ admits the canonical conserved densities ρ_1 and ρ_2, then*

$$\begin{aligned}f_1&=\frac{c_1u_{[2,2]}+c_2\tilde u_{[2,2]}}{g_0}+f_3u_{[1,2]}^2+f_4\tilde u_{[0,2]}^2+f_5\tilde u_{[1,2]}^2+f_6u_{[1,2]}\tilde u_{[0,2]}\\&\quad+f_7u_{[1,2]}\tilde u_{[1,2]}+f_8\tilde u_{[1,2]}\tilde u_{[0,2]}+f_9u_{[1,2]}+f_{10}\tilde u_{[0,2]}+f_{11}\tilde u_{[1,2]}+f_{12},\end{aligned}\tag{25}$$

where f_i are some functions of the variables $\tilde u_{[0,0]}$, $\tilde u_{[0,1]}$, $\tilde u_{[1,1]}$ and $u_{[1,1]}$.

Proof. To specify the form of the coefficient f_1, we consider the condition $D_t\rho_1 = D_x\theta_1$, where ρ_1 is given by (22).

To simplify the equation $D_t\rho_1 = D_x\theta_1$ we use the equivalence relation that will be denoted as $\sim$. We say that F_1 and F_2 are equivalent ($F_1 \sim F_2$) if $F_1 - F_2 = D_xF_3$ for some function F_3. Thus we may write $D_t\rho_1 \sim 0$, and this equivalence remains true after adding any x-derivative: $D_t\rho_1 + D_xF \sim 0$, $\forall\, F$. We call the transformation $D_t\rho_1 \to D_t\rho_1 + D_xF$ the equivalence transformation. Using the equivalence transformation we can reduce the order of $D_t\rho_1$ step by step. For example, $f(\tilde{u}_{[0,0]})\tilde{u}_{[0,1]} \sim f(\tilde{u}_{[0,0]})\tilde{u}_{[0,1]} - \frac{1}{2}D_x \int f(\tilde{u}_{[0,0]})d\tilde{u}_{[0,0]} = 0$.

Reducing the order of $D_t\rho_1$ by the equivalence transformation we found that $D_t\rho_1$ is equivalent to a third degree polynomial of the third order variables $u_{[i,3]}$, $i = 1,2,3$ and $\tilde{u}_{[i,3]}$, $i = 0,1,2,3$. As this polynomial must be equivalent to zero, and it is obvious that any total derivative D_xF is linear with respect to highest order variables, then the second and third degree terms must vanish. By equating the third degree terms to zero one can find that all third order derivatives of f_1 with respect to second order variables vanish, and, moreover,

$$\frac{\partial^2 f_1 g_0}{\partial u_{[2,2]}\partial u_{[i,j]}} = \frac{\partial^2 f_1 g_0}{\partial u_{[2,2]}\partial \tilde{u}_{[i,j]}} = \frac{\partial^2 f_1 g_0}{\partial \tilde{u}_{[2,2]}\partial u_{[i,j]}} = \frac{\partial^2 f_1 g_0}{\partial \tilde{u}_{[2,2]}\partial \tilde{u}_{[i,j]}} = 0, \qquad \forall\, [i,j].$$

Integrating all these equations we obtain (25). ■

Remark 5. We do not use the equations $\delta D_t\rho_i(u)/\delta u = 0$ because such computations are only possible with the help of a supercomputer. Unfortunately, our IBM PC with 1 GB RAM does not permit us to do it.

Lemma 3. *The following three cases are only possible in* (25)*:*

$$(\mathbf{A})\ c_1 = c_2 = 0; \qquad (\mathbf{B})\ c_1 = 1, \quad c_2 = 0; \qquad (\mathbf{C})\ c_1 = 0, \quad c_2 = 1.$$

Proof. If $c_1 \neq 0$, $c_2 = 0$ we can change $g_0 \to c_1 g_0$ and it is equivalent to $c_1 = 1$. If $c_2 \neq 0$, the transformation (8) with $\lambda = -c_1/c_2$, $\alpha = 1/c_2$ gives $c_1 = 0$ and $c_2 = 1$. ■

Then we specified the functions $g_0, f_3, \ldots, f_{12}$ in (24), (25) in the cases A, B, and C using the next integrability conditions (23). These computations are very cumbersome and we can not present it in a short article. The result of the computations is the Theorem 1.

4 Bäcklund transformations

To prove integrability of all equations from the list (10)–(20) we present in this section first order auto-Bäcklund transformations for all equations. Such transformations involving an arbitrary parameter allow us to build up both multi-soliton and finite-gap solutions even if the Lax representation is not known (see [7]). That is why the existence of an auto-Bäcklund transformation with additional "spectral" parameter λ is a convincing evidence of integrability.

For a scalar evolution equation, a first order auto-Bäcklund transformation is a relation between two solutions u and v of the same equation and their derivatives u_x and v_x. Writing this constraint as $u_x = \phi(u, v, v_x)$, we can express all derivatives of u in terms of $u, v, v_x, \ldots, v_i, \ldots$. These variables are regarded as independent.

In the vector case, the independent variables are vectors

$$\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{v}_1,\ \boldsymbol{v}_2,\ \ldots,\ \boldsymbol{v}_i,\ \ldots, \tag{26}$$

and all their scalar products

$$\tilde{u}_{[0,0]} = \langle \boldsymbol{u}, \boldsymbol{u}\rangle, \qquad v_{[i,j]} = (\boldsymbol{v}_i, \boldsymbol{v}_j), \qquad \tilde{v}_{[i,j]} = \langle \boldsymbol{v}_i, \boldsymbol{v}_j\rangle,$$

$$w_i = (\boldsymbol{u}, \boldsymbol{v}_i), \qquad \tilde{w}_i = \langle \boldsymbol{u}, \boldsymbol{v}_i \rangle, \qquad i, j \geqslant 0. \tag{27}$$

Following [8], we consider in this paper special vector auto-Bäcklund transformations of the form

$$\boldsymbol{u}_1 = h\boldsymbol{v}_1 + f\boldsymbol{u} + g\boldsymbol{v}, \tag{28}$$

where f, g and h are scalar functions of the variables (27) with $i, j \leqslant 1$. Since $\boldsymbol{v}$ belongs to the sphere, $(\boldsymbol{v}, \boldsymbol{v}) = 1$, we assume, without loss of generality, that the arguments of f, g and h are

$$\tilde{u}_{[0,0]},\ \tilde{v}_{[0,0]},\ w_0,\ \tilde{w}_0,\ v_{[1,1]},\ \tilde{v}_{[0,1]},\ \tilde{v}_{[1,1]},\ w_1,\ \tilde{w}_1. \tag{29}$$

Since $(\boldsymbol{u}, \boldsymbol{u}) = 1$, and $(\boldsymbol{u}, \boldsymbol{u}_1) = 0$, it follows from (28) that

$$f = -w_0 g - w_1 h.$$

To find an auto-Bäcklund transformation for the equation (2), we differentiate (28) with respect to t in virtue of (2) and express all vector and scalar variables in terms of the independent variables (26) and (27). By definition of the Bäcklund transformation, the expression thus obtained must be identically equal to zero. Splitting this expression with respect to the vector variables (26) and the scalar variables (27) different from (29) we derive an overdetermined system of non-linear PDEs for the functions f and g. If the system has a solution depending on an essential parameter λ, this solution gives us the auto-Bäcklund transformation we are looking for.

We present below the result of our computations.

In the case of the equation (10) the auto-Bäcklund transformation reads as follows

$$\boldsymbol{u}_x + \boldsymbol{v}_x = 2\frac{(\boldsymbol{u}, \boldsymbol{v}_x)(\boldsymbol{u} + \boldsymbol{v}) + f(\boldsymbol{v} - (\boldsymbol{u}, \boldsymbol{v})\boldsymbol{u})}{(\boldsymbol{u} + \boldsymbol{v})^2},$$

where $f^2 = \langle \boldsymbol{u} + \boldsymbol{v}, \boldsymbol{u} + \boldsymbol{v} \rangle + \lambda(\boldsymbol{u} + \boldsymbol{v})^2$.

Next two equations, (11) and (12), are integrable on $\mathbb{R}^n$ not only on $\mathbb{S}^n$. In fact, f_1 and f_2 do not depend on $u_{[0,i]}$. Hence

$$D_t u_{[i,j]} = \big(D_x^i(\boldsymbol{u}_3 + f_2\boldsymbol{u}_2 + f_1\boldsymbol{u}_1), \boldsymbol{u}_j\big) + \big(\boldsymbol{u}_i, D_x^j(\boldsymbol{u}_3 + f_2\boldsymbol{u}_2 + f_1\boldsymbol{u}_1)\big)$$

do not depend on $u_{[0,i]}$ for any $i, j > 0$. This implies that all ρ_n and θ_n for (11) and (12) do not depend on $u_{[0,i]}$ (see (22) and (21)). This means that the conditions $D_t\rho_n = D_x\theta_n$ are valid both on $\mathbb{R}^n$ and on $\mathbb{S}^n$.

The equations (11) and (12) on $\mathbb{R}^n$ have the auto-Bäcklund transformations of the form

$$\boldsymbol{u}_x = F\left(\frac{w_1 - v_{[0,1]}}{\varphi}(\boldsymbol{u} - \boldsymbol{v}) - \boldsymbol{v}_x\right),$$

where $\varphi = \frac{1}{2}(\boldsymbol{u} - \boldsymbol{v})^2$. The function F reads as

$$F = \sqrt{\frac{\lambda\varphi - \tilde{\varphi}}{v_{[1,1]}}} - 1, \qquad \tilde{\varphi} = \tilde{u}_{[0,0]} + \tilde{v}_{[0,0]} - 2\tilde{w}_0$$

for (11) and as

$$F = \frac{\lambda\varphi + \tilde{w}_0}{\tilde{v}_{[0,0]}}\left(\sqrt{1 + \tilde{v}_{[0,0]} v_{[1,1]}^{-1}}\sqrt{1 - \frac{\tilde{u}_{[0,0]}\tilde{v}_{[0,0]}}{(\lambda\varphi + \tilde{w}_0)^2}} - 1\right)$$

for (12). On $\mathbb{S}^n$ we have $v_{[0,1]} = 0$ and $\varphi = 1 - w_0$.

The equations (13) and (14) have the auto-Bäcklund transformations of the form

$$\boldsymbol{u}_x = f\left[\boldsymbol{v}_x - w_1\boldsymbol{u} + \left(\frac{f\tilde{v}_{[0,1]} - \tilde{w}_1}{f\tilde{v}_{[0,0]} - \tilde{w}_0} + g\right)(w_0\boldsymbol{u} - \boldsymbol{v})\right], \qquad f^2 = \frac{\tilde{u}_{[0,0]}}{\tilde{v}_{[0,0]}},$$

where $g = \lambda$ for (14) and g satisfies, for (13), the following equation

$$g^2 = a\tilde{v}_{[0,0]}\frac{f^2 - 2fw_0 + 1}{(f\tilde{v}_{[0,0]} - \tilde{w}_0)^2} + \frac{\lambda f\tilde{v}_{[0,0]}}{f\tilde{v}_{[0,0]} - \tilde{w}_0}.$$

The auto-Bäcklund transformation for the equation (15) is defined by the following equation

$$\boldsymbol{u}_x = f\left(\boldsymbol{v}_x - \boldsymbol{u}w_1 + \frac{(\lambda w_1 + h)(\boldsymbol{v} - \boldsymbol{u}w_0)}{f\tilde{v}_{[0,0]} - \tilde{w}_0}\right), \qquad f^2 = \frac{\tilde{u}_{[0,0]}}{\tilde{v}_{[0,0]}},$$

where

$$h^2 = (\tilde{v}_{[0,0]} + v_{[1,1]})\left(\lambda^2 - (\lambda w_0 - \tilde{w}_0 + f\tilde{v}_{[0,0]})^2\right).$$

The auto-Bäcklund transformations for the equations (16), (17) and (18) take the form

$$\boldsymbol{u}_x = F(\boldsymbol{v}_x + g\boldsymbol{v} - (w_1 + gw_0)\boldsymbol{u}),$$

where

$$F = h + f, \qquad f^2 = \frac{\tilde{u}_{[0,0]}}{\tilde{v}_{[0,0]}}, \qquad g = -\frac{f\tilde{v}_{[0,1]} - \tilde{w}_1}{f\tilde{v}_{[0,0]} - \tilde{w}_0},$$
$$h^2 = \frac{2}{3}\tilde{v}_{[0,0]}\frac{a(f^2 + 1 - 2fw_0) - \lambda(\tilde{u}_{[0,0]} - \tilde{w}_0 f)}{\tilde{v}_{[0,0]}\tilde{v}_{[1,1]} - \tilde{v}^2_{[0,1]}}$$

for (16);

$$F = f - \lambda\frac{\tilde{v}_{[0,1]}\tilde{w}_0 - \tilde{w}_1\tilde{v}_{[0,0]}}{\tilde{v}_{[0,0]}\tilde{v}_{[1,1]} - \tilde{v}^2_{[0,1]}}, \qquad f^2 = \frac{\tilde{u}_{[0,0]}}{\tilde{v}_{[0,0]}}, \qquad g = -\frac{f\tilde{v}_{[0,1]} - \tilde{w}_1}{f\tilde{v}_{[0,0]} - \tilde{w}_0}$$

for (17);

$$g = \frac{1}{\tilde{v}_{[0,0]}}\Bigg[\Big(\big((1 + h\tilde{w}_0)^2 - \tilde{u}_{[0,0]}\tilde{v}_{[0,0]}h^2\big)\big(\tilde{v}^2_{[0,1]} + \tilde{v}_{[0,0]}(a\tilde{v}_{[0,0]} + b - \tilde{v}_{[1,1]})\big)\Big)^{1/2}$$
$$- \tilde{v}_{[0,1]} - h(\tilde{v}_{[0,1]}\tilde{w}_0 - \tilde{w}_1\tilde{v}_{[0,0]})\Bigg],$$
$$F = f, \qquad f^2 = \frac{a\tilde{u}_{[0,0]} + b}{a\tilde{v}_{[0,0]} + b}, \qquad h = \frac{\lambda}{(a\tilde{v}_{[0,0]} + b)f - a\tilde{w}_0 - bw_0}$$

for (18).

The equation (19) has the following auto-Bäcklund transformation

$$\boldsymbol{u}_x = F\left(\boldsymbol{v}_x - w_1\boldsymbol{u} + \frac{\tilde{w}_1 + f\tilde{v}_{[0,1]}}{\tilde{w}_0 + f\tilde{v}_{[0,0]}}(w_0\boldsymbol{u} - \boldsymbol{v})\right), \tag{30}$$

where

$$F = \phi^{-1}\left(g + \sqrt{g^2 - \phi\psi}\coth\chi\right), \qquad g = \lambda(\tilde{w}_0 + f\tilde{v}_{[0,0]}) - af\tilde{v}_{[0,0]} + bw_0,$$

$$\sinh^2\chi = \frac{\tilde v_{[0,0]}\tilde v_{[1,1]} - \tilde v_{[0,1]}^2}{\tilde v_{[0,0]}\phi}, \qquad f^2 = \frac{\tilde u_{[0,0]}}{\tilde v_{[0,0]}}, \qquad \phi = a\tilde v_{[0,0]} + b, \qquad \psi = a\tilde u_{[0,0]} + b.$$

If we set here $a = 0$, $\lambda \to b\lambda$ and then $b = 0$, we obtain the following expression for F:

$$F = q + \sqrt{q^2 - 1}, \qquad q = \lambda(\tilde w_0 + f\tilde v_{[0,0]}) + w_0.$$

But it is a trivial solution because (30) is "auto-Bäcklund transformation" for (19′) when $F = F(\tilde u_{[0,0]}, \tilde v_{[0,0]}, w_0, \tilde w_0)$ is an arbitrary solution of a quasilinear system of the three first order equations. (V. Sokolov and A. Meshkov found for the vector isotropic Schwartz–KdV equation that an "auto-Bäcklund transformation" containing an arbitrary function is equivalent to a point transformation. Unfortunately, formula (49) in [8] also gives a false auto-Bäcklund transformation.)

The true auto-Bäcklund transformation for the anisotropic Schwartz–KdV equation (19′) has the following form

$$\boldsymbol{u}_x = \frac{\lambda\tilde v_{[0,0]}}{\zeta}\left((\tilde w_0 + f\tilde v_{[0,0]})(\boldsymbol{v}_x - w_1\boldsymbol{u}) + (\tilde w_1 + f\tilde v_{[0,1]})(w_0\boldsymbol{u} - \boldsymbol{v})\right),$$

where

$$f^2 = \frac{\tilde u_{[0,0]}}{\tilde v_{[0,0]}}, \quad \zeta = \tilde v_{[0,0]}\tilde v_{[1,1]} - \tilde v_{[0,1]}^2.$$

This transformation is reduced in the isotropic limit to the true auto-Bäcklund transformation for isotropic Schwartz-KdV equation that was pesented in [9].

Finally, the auto-Bäcklund transformation for the equation (20) reads as follows

$$\boldsymbol{u}_x = F\big(\boldsymbol{v}_x - w_1\boldsymbol{u} + g(\boldsymbol{v} - w_0\boldsymbol{u})\big),$$

where

$$F = \frac{1}{\tilde v_{[0,0]}}\left(\lambda\frac{\tilde v_{[0,0]}\tilde w_1 - \tilde v_{[0,1]}\tilde w_0}{\nu + \tilde v_{[0,0]}} - h\right), \qquad h^2 = \tilde u_{[0,0]}\tilde v_{[0,0]} + \lambda^2(\tilde w_0^2 - \tilde u_{[0,0]}\tilde v_{[0,0]}),$$

$$g = (\lambda^2 - 1)\frac{\tilde v_{[0,0]}\tilde w_1 - \tilde v_{[0,1]}\tilde w_0}{\tilde v_{[0,0]}(h + \tilde v_{[0,0]})} - \frac{\tilde v_{[0,1]} + \lambda(\nu + \tilde v_{[0,0]})}{\tilde v_{[0,0]}}, \qquad \nu^2 = \tilde v_{[0,1]}^2 + \tilde v_{[0,0]}^2 - \tilde v_{[0,0]}\tilde v_{[1,1]}.$$

5 Concluding remarks

Each of the presented Bäcklund transformations is reduced to the true Bäcklund transformation under the isotropic reduction $\tilde u_{[i,j]} = ku_{[i,j]}$ in accordance with the diagram of the the isotropic reductions. It is convincing evidence that all Bäcklund transformations are real.

We do not know any applied problems that lead to one of the equations from our list. If such problem emerge in future, it will be interesting to find the Lax representation for the corresponding equation.

The presented Bäcklund transformations can be used, first, for obtaining the soliton-like solutions (see [13]) and, secondly, for constructing the superposition formulas and new discrete integrable systems.

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