TAME AND WILD SUBSPACE PROBLEMS

P. Gabriel,[1] L. A. Nazarova,[2] A. V. Roiter,[2] V. V. Sergeichuk,[2] and D. Vossieck[3]

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Assume that B is a finite-dimensional algebra over an algebraically closed field k, $\mathcal{B}_d = \operatorname{Spec} k[\mathcal{B}_d]$ is the affine algebraic scheme whose R-points are the $B \otimes_k k[\mathcal{B}_d]$-module structures on R^d, and M_d is a canonical $B \otimes_k k[\mathcal{B}_d]$-module supported by $k[\mathcal{B}_d]^d$. Further, say that an affine subscheme $\mathcal{V}$ of $\mathcal{B}_d$ is *class true* if the functor $F_{\mathcal{V}}: X \mapsto M_d \otimes_{k[\mathcal{B}]} X$ induces an injection between the sets of isomorphism classes of indecomposable finite-dimensional modules over $k[\mathcal{V}]$ and B. If $\mathcal{B}_d$ contains a class-true plane for some d, then the schemes $\mathcal{B}_e$ contain class-true subschemes of arbitrary dimensions. Otherwise, each $\mathcal{B}_d$ contains a finite number of classtrue puncture straight lines $\mathcal{L}(d, i)$ such that for each n, almost each indecomposable B-module of dimension n is isomorphic to some $F_{\mathcal{L}(d,i)}(X)$; furthermore, $F_{\mathcal{L}(d,i)}(X)$ is not isomorphic to $F_{\mathcal{L}(l,j)}(Y)$ if $(d, i) \neq (l, j)$ and $X \neq 0$. The proof uses a reduction to subspace problems, for which an inductive algorithm permits us to prove corresponding statements.

1. Notation, Terminology, Objective.

Throughout the paper, k denotes an algebraically closed field.

By $\mathcal{A}$ we denote a k-category, i.e., a category whose morphism sets $\mathcal{A}(X, Y)$ are endowed with vector space structures over k such that the composition maps are bilinear. Furthermore, we suppose that $\mathcal{A}$ is an *aggregate* (over k), i.e., that the spaced $\mathcal{A}(X, Y)$ have finite dimensions over k, that $\mathcal{A}$ has finite direct sums, and that each idempotent $e \in \mathcal{A}(X,X)$ has a kernel. As a consequence, each $X \in \mathcal{A}$ is a finite direct sum of indecomposables, and the algebra of endomorphisms of each indecomposable is local. We shall denote by $\mathfrak{S}$ a *spectroid* of $\mathcal{A}$, i.e., the full subcategory formed by chosen representatives of the isoclasses of indecomposables, and by $\mathcal{R}_{\mathcal{A}}$ and $\mathcal{R}_{\mathfrak{S}}$ the *radicals of* $\mathcal{A}$ and $\mathfrak{S}$.

Typical examples of aggregates are provided by the category $\operatorname{proj} A$ of finitely generated projective *right* modules over a finite-dimensional algebra A, or by the category $\operatorname{mod} A$ of all finite-dimensional *right* A-modules. The aggregate $\operatorname{proj} A$ has a finite spectroid; $\operatorname{mod} A$, in general, does not.

A *pointwise finite(left) module* M over $\mathcal{A}$ is, by definition, a k-linear functor from $\mathcal{A}$ to $\operatorname{mod} k$. For instance, in the examples considered above, each $N \in \operatorname{mod} A^{\mathrm{op}}$ yields a module $P \mapsto P \otimes_A N$ over $\operatorname{proj} A$ and each $L \in \operatorname{mod} A$ yields a series of modules $X \mapsto \operatorname{Ext}_A^n(L, X)$ over $\operatorname{mod} A$.

With each module M over $\mathcal{A}$ we associate a new aggregate M^k whose objects are the *M-spaces*, i.e., the triples (V, f, X) formed by a space $V \in \operatorname{mod} k$, an object $X \in \mathcal{A}$, and a linear map $f: V \to M(X)$. A morphism from (V, f, X) to (V', f', X') is determined by morphisms $\varphi: V \to V'$ and $\xi: X \to X'$ such that $f'\varphi = M(\xi)f$.

Let $\mathcal{L} = (K, J, \ldots)$ be a *bond* on M, i.e., a *finite* set of submodules. We say that $(V, f, X) \in M^k$ *avoids* $\mathcal{L}$ if $f^{-1}(L(X)) = \{0\}$ for each $L \in \mathcal{L}$. The triples which avoid $\mathcal{L}$ form a full subaggregate of M^k, which we denote by $M^k_{\mathcal{L}} = M^k_{K,J,\ldots}$.

When V and X are fixed, the triples $(V, f, X) \in M^k$ may be identified with points of the space $\operatorname{Hom}_k(V, M(X))$. The triples avoiding $\mathcal{L}$ then correspond to the points of a (Zariski-)open subset $\operatorname{Hom}_k^{\mathcal{L}}(V, M(X))$, which

[1] Zürich University, Switzerland.
[2] Institute of Mathematics, Ukrainian Academy of Sciences, Kiev.
[3] Basel University, Switzerland.

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inherits from $\operatorname{Hom}_k(V, M(X))$ the structure of an algebraic variety. Our objective is to examine the "number of parameters" occurring in an algebraic family of maps $f \in \operatorname{Hom}_k^{\mathcal{L}}(V, M(X))$ such that the triples (V, f, X) are indecomposable and pairwise nonisomorphic.

2. Formulation of the Main Theorems

2.1. With the notation introduced above, let $e = (e_0, \ldots, e_t)$ be a *coordinate system* of an affine subspace S of $\operatorname{Hom}_k(V, M(X))$, i.e., a sequence of vectors $e_i \in \operatorname{Hom}_k(V, M(X))$ such that the map

$$k^t \to \operatorname{Hom}_k(V, M(X)), \quad x \to e_0 + x_1 e_1 + \ldots + x_t e_t$$

induces a bijection $k^t \cong S$. Then e provides a functor F_e: rep $Q^t \to M^k$, where rep Q^t is the aggregate formed by the finite-dimensional representations of the quiver Q^t with 1 vertex and t arrows: F_e maps a sequence $a \in$ rep Q^t of t endomorphisms $a_i: W \to \to W$ onto the triple $(W \otimes V, f_e(a), W \otimes X)$, where $W \otimes X \in \mathcal{A}$ represents the functor $\operatorname{Hom}_k(W, \mathcal{A}(X, ?))$ $\left(\text{hence, } k^n \otimes X \cong X^n\right)$ and

$$f_e(a) = 1\!\!1_W \otimes e_0 + a_1 \otimes e_1 + \ldots + a_t \otimes e_t: W \otimes V \to W \otimes M(X) \cong M(W \otimes X).$$

The functor F_e behaves well toward affine subspaces $S' \subset S$. Let e' be a coordinate system of S', where $e_0' = e_0 + \sum_{i=1}^{t} T_{0i} e_i$ and $e_j' = \sum_{i=1}^{t} T_{ji} e_i$, $1 \leq j \leq s$. We then have $F_{e'} = F_e \circ \Phi$, where Φ: rep $Q^s \to$ rep Q^t is the functor $a' \to a$ defined by $a_i = T_{0i} 1\!\!1_W + \sum_{j=1}^{s} T_{ji} a_j'$, $1 \leq i \leq t$. In the case $S' = S$, Φ is an automorphism.

2.2. Let now R be an affine subspace of $\operatorname{Hom}_k(W, W)^t$ with coordinate system $d = (d_0, d_1, \ldots, d_s)$, where $d_j = (d_{j1}, \ldots, d_{jt})$. Then d provides a functor Φ_d: rep $Q^s \to$ rep Q^t which maps $c \in \operatorname{Hom}_k(U, U)^s$ onto $b \in \operatorname{Hom}_k(U \otimes W, U \otimes W)^t$, where $b_i = 1\!\!1_U \otimes d_{0i} + c_1 \otimes d_{1i} + \ldots + c_s \otimes d_{si}$. A simple calculation shows that $F_e \circ \Phi_d = F_f$, where f is a coordinate system of a subspace of $\operatorname{Hom}_k(W \otimes V, M(W \otimes X))$ and is defined by

$$f_0 = 1_W \otimes e_0 + d_{01} \otimes e_1 + \ldots + d_{0t} \otimes e_t$$

and

$$f_j = d_{j1} \otimes e_1 + \ldots + d_{jt} \otimes e_t, \quad 1 \leq j \leq s.$$

All compositions $\Phi_g \circ \Phi_d$ have the form Φ_h. In the case $W = k$ and $d_{ji} = T_{ji} \in k \cong \operatorname{Hom}_k(k, k)$, Φ_d coincides with the functor Φ of 2.1.

***Example* 1.** Consider the affine subspace R of $\operatorname{Hom}_k(k^{s+1}, k^{s+1})^2$ formed by the pairs of matrices

$$\left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \left[\begin{array}{ccc|cc} 0 & x_1 & 0 & 0 & 0 \\ 0 & 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & x_s \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Let d be the coordinate system of R for which x_i is the i-th coordinate of the above pair. The associated functor Φ_d: rep $Q^s \to$ rep Q^2 maps $c \in \operatorname{Hom}_k(X, X)^s$ onto the pair $b \in \operatorname{Hom}_k(X^{s+1}, X^{s+1})^2$ represented by the matrices

$$
\left[\begin{array}{ccc:cc}
0 & \mathbb{1}_x & 0 & 0 & 0 \\
0 & 0 & \mathbb{1}_x & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & \mathbb{1}_x \\
0 & 0 & 0 & 0 & 0
\end{array}\right],\quad
\left[\begin{array}{ccc:cc}
0 & c_1 & 0 & 0 & 0 \\
0 & 0 & c_2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\hdashline
0 & 0 & 0 & 0 & c_s \\
0 & 0 & 0 & 0 & 0
\end{array}\right].
$$

It follows that Φ_d factors through the full subaggregate $\operatorname{rep}_0 Q^2$ of $\operatorname{rep} Q^2$ formed by the pairs of nilpotent simultaneously trigonalizable endomorphisms. A simple calculation shows that Φ_d preserves indecomposability and *heteromorphism* ($c, c' \in \operatorname{rep} Q^s$ are isomorphic if so are the images $\Phi_d(c), \Phi_d(c')$).

***Example* 2** [1]. Consider the affine subspace U of $\operatorname{Hom}_k(k^4, k^4)^2$ formed by the pairs of matrices

$$
\begin{bmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & x_1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & x_2 \\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

If g is the coordinate system of U for which x_i is the i-th coordinate, the associated functor $\Phi_g\colon \operatorname{rep} Q^2 \to \operatorname{rep} Q^2$ factors through the full subaggregate $\operatorname{rep}_0^c Q^2$ of $\operatorname{rep}_0 Q^2$ formed by the pairs of *commuting nilpotent* matrices. The functor Φ_g preserves indecomposability and heteromorphism.

2.3. We now come back to the module M restrained by a bond $\mathcal{L}$.

Definition. *Let S be an affine subspace of dimension t of $\operatorname{Hom}_k(V, M(X))$, and e a coordinate system of S. We say that S is $\mathcal{L}$-reliable if the functor $F_e\colon \operatorname{rep} Q^t \to M^k$ factors through $M_{\mathcal{L}}^k$ and preserves indecomposability and heteromorphism.*

Lemma. *Suppose that $t = 2$, (V, e_0, X) avoids $\mathcal{L}$, and the restriction $F_e|\operatorname{rep}_0^c Q^2$ preserves indecomposability and heteromorphism. Then, for each $s \in \mathbb{N}$, there exists a $U \in \operatorname{mod} k$, a $Y \in \mathcal{A}$, and an $\mathcal{L}$-reliable subspace of $\operatorname{Hom}_k(U, M(Y))$ of dimension s.*

Proof. Let us set $W = k^{s+1}$ and choose d as in Example 1 and g as in Example 2. Then we have $F_e \circ \Phi_g \circ \Phi_d = F_f$, where f is a coordinate system of an affine subspace T of dimension s of $\operatorname{Hom}_k(V^{4(s+1)}, M(X^{4(s+1)}))$. Since $F_e|\operatorname{rep}_0^c Q^2$ and the functor $\operatorname{rep} Q^s \to \operatorname{rep}_0^c Q^2$ induced by $\Phi_g \circ \Phi_d$ preserve indecomposability and heteromorphism, so does F_f.

It suffices now to show that F_e maps $\operatorname{rep}_0 Q^2$ into $M_{\mathcal{L}}^k$. For this purpose, we call a sequence

$$0 \to (W', g', Y') \to (W, g, Y) \to (W'', g'', Y'') \to 0$$

of M^k *short exact* if the induced sequences

$$0 \to W' \to W \to W'' \to 0 \text{ and } 0 \to Y' \to Y \to Y'' \to 0$$

are exact in $\operatorname{mod} k$ and split exact in $\mathcal{A}$, respectively. Now it is clear that $F_e\colon \operatorname{rep} Q^2 \to M^k$ preserves short exact sequences and that $M_{\mathcal{L}}^k$ is closed in M^k under extensions (in the sequence above, $(W', g', Y') \in M_{\mathcal{L}}^k$ and $(W'',$

$g'', Y'') \in M_{\mathcal{L}}^k$ imply $(W, g, Y) \in M_{\mathcal{L}}^k$). It follows that $F_e^{-1}\left(M_{\mathcal{L}}^k\right)$ is closed under extensions; therefore it contains $\operatorname{rep}_0 Q^2$, which is the smallest full subaggregate of $\operatorname{rep} Q^2$, closed under extensions and containing $([0], [0]) \in F_e^{-1}\left(M_{\mathcal{L}}^k\right)$.

2.4. ***Definition.*** *The module M over $\mathcal{A}$ is called $\mathcal{L}$-wild if, for some V and X, there exists an $\mathcal{L}$-reliable affine subspace $S \subset \operatorname{Hom}_k(V, M(X))$ of dimension 2. It is called absolutely wild if it is $\mathcal{L}$-wild for all proper $\mathcal{L}$, i.e., for all $\mathcal{L}$ such that $M \notin \mathcal{L}$.*

Our objective is to examine the pairs $(M, \mathcal{L})$ such that M is not $\mathcal{L}$-wild. For this, we need the following notion. Assume that the submodules $L \in \mathcal{L}$ contain the radical $\mathcal{R}M$ of M, consider $\overline{M} = M/\mathcal{R}M$ as a module over $\overline{\mathcal{A}} = \mathcal{A}/\mathcal{R}_{\mathcal{A}}$, and denote by $\overline{\mathcal{L}}$ the set of submodules $\overline{L} = L/\mathcal{R}M$ of $\overline{M}(L \in \mathcal{L})$. We say that M is $\mathcal{L}$-*semisimple* if the obvious functor $M_{\mathcal{L}}^k \to \overline{M}_{\overline{\mathcal{L}}}^k$ is an epivalence (i.e., induces surjections on the morphism spaces, detects isomorphisms, and hits each isoclass of $\overline{M}_{\overline{\mathcal{L}}}^k$).

First main theorem. *Let M be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid. Then M is absolutely wild or $\mathcal{L}$-semisimple for some proper $\mathcal{L}$.*

2.5. For each subset $C \subset k$, we denote by $\operatorname{rep}_C Q^1$ the full subaggregate of $\operatorname{rep} Q^1$ formed by the endomorphisms with eigenvalues in C. It is clear that $\operatorname{rep}_C Q^1$ is closed in $\operatorname{rep} Q^1$ under extensions. The converse is valid: Each full subaggregate of $\operatorname{rep} Q^1$ which is closed under extensions coincides with some $\operatorname{rep}_C Q^1$.

We apply these considerations to *punched lines of M*, i.e., to subsets of some $\operatorname{Hom}_k^{\mathcal{L}}(V, M(X))$ of the form $S \setminus E$, where S is a *line* (affine subspace of dimension 1) of $\operatorname{Hom}_k(V, M(X))$ and E is a finite subset of S. If $e = (e_0, e_1)$ is a coordinate system of S, the scalars $\lambda \in k$ such that $e_0 + \lambda e_1 \in S \setminus E$ form a cofinite subset C of k. With this notation, the considerations developed above show that F_e maps $\operatorname{rep}_C Q^1$ into $M_{\mathcal{L}}^k$. Accordingly, we say that the punched line $S \setminus E \subset \operatorname{Hom}_k^{\mathcal{L}}(V, M(X))$ is $\mathcal{L}$-*reliable* if the functor $\operatorname{rep}_C Q^1 \to M_{\mathcal{L}}^k$ induced by F_e preserves indecomposability and heteromorphism.

In the second main theorem below, we say that an M-space (W, g, Y) is *produced* by the punched line $S \setminus E \subset \operatorname{Hom}_k(V, M(X))$ if it is isomorphic to some image $F_e(k^n, \lambda \mathbb{1}_n + J_n)$, where J_n is a nilpotent Jordan block, $n \geq 1$ and $\lambda \in C$. This means that there are isomorphisms $w: W \xrightarrow{\sim} V^n$ and $y: Y \xrightarrow{\sim} X^n$ such that $M(y)gw^{-1}$ is the linear map $V^n \to M(X^n)$ described by the matrix with n diagonal blocks $e_0 + \lambda e_1$:

$$\left[\begin{array}{cccc|c} e_0 + \lambda e_1 & e_1 & 0 & 0 & \\ 0 & e_0 + \lambda e_1 & e_1 & 0 & \\ 0 & 0 & e_0 + \lambda e_1 & e_1 & \\ 0 & 0 & 0 & e_0 + \lambda e_1 & \\ \hline & & & & \end{array}\right].$$

We also say that a set $\mathcal{P}$ of punched lines is *locally finite* if, for each $X \in \mathcal{A}$, $\mathcal{P}$ contains only finitely many punched lines of the form $S \setminus E \subset \operatorname{Hom}_k(V, M(Y))$, where $Y \xrightarrow{\sim} X$.

Second main theorem. *If M is not $\mathcal{L}$-wild, there is a locally finite set $\mathcal{P}$ of $\mathcal{L}$-reliable punched lines such that:*

a) For each $X \in \mathcal{A}$, the set of isoclasses of indecomposable M-spaces (V, f, X) which avoid $\mathcal{L}$ and are not produced by a punched line of $\mathcal{P}$ is finite;

b) Distinct punched lines of $\mathcal{P}$ produce nonisomorphic M-spaces.

The perspicuous description of the indecomposable M-spaces given by the second main theorem confirms us in calling M $\mathcal{L}$-*tame* (or simply *tame* in case $\mathcal{L} = \emptyset$) if it is not $\mathcal{L}$-wild.

The second main theorem also shows that M is $\mathcal{L}$-wild whenever it admits a "two-parametric family" of pairwise nonisomorphic indecomposable M-spaces avoiding $\mathcal{L}$. Thus, to prove wildness, $\mathcal{L}$-reliability is not needed even in the weak form of Lemma 2.3. We owe the following example to Th. Brüstle: Suppose that the spectroid $\mathfrak{S}$ of $\mathcal{A}$ has only one point w, that $M(w) = k^4$, and that $\mathfrak{S}(w, w)$ is the subalgebra of $k^{4\times 4}$ generated by the matrices

$$t = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad u = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

which act on k^4 by matrix multiplication. Then the M-spaces $(k^2, f_{\lambda\mu}, w)$, where $f_{\lambda\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \lambda & \mu \end{bmatrix}^T$ and λ, $\mu \in k$, are indecomposable and pairwise nonisomorphic. Hence, M is wild. But the action of the functor F: $\operatorname{rep} Q^2 \to M^k$ associated with the plane $\{f_{\lambda\mu}: \lambda, \mu \in k\}$ is already erratic on the two-dimensional representations of Q^2.

2.6. Finally, we consider a *finite-dimensional* k-algebra B and the tensor algebra $\otimes B = k \oplus B \oplus B \otimes_k B \oplus \ldots$. We identify $\operatorname{mod} B$ with a full subcategory of $\operatorname{mod} \otimes B$ by the aid of the surjective canonical homomorphism $\otimes B \to B$. Accordingly, if the right $\otimes B$-module structures on a finite-dimensional vector space V are interpreted as points of $\operatorname{Hom}_k(V \otimes_k B, V)$, the B-module structures on V are identified with the points of an algebraic subvariety $\mathcal{M}_B(V)$ of $\operatorname{Hom}_k(V \otimes_k B, V)$.

As in 2. 1, each coordinate system $e = (e_0, \ldots, e_t)$ of an affine subspace $S \subset \operatorname{Hom}_k(V \otimes_k B, V)$ gives rise to a functor F_e: $\operatorname{rep} Q^t \to \operatorname{mod} \otimes B$ which maps a sequence $a = (a_1, \ldots, a_t)$ of t endomorphisms a_i: $W \to W$ onto the space $W \otimes_k V$ equipped with the $\otimes B$-module structure

$$\mathbb{1}_W \otimes e_0 + a_1 \otimes e_1 + \ldots + a_t \otimes e_t: W \otimes V \otimes B \to W \otimes V.$$

We say that S is B-*reliable* if F_e factors through $\operatorname{mod} B$ and preserves indecomposability and heteromorphism.

In the case $t = 1$, we also consider *punched lines* $S \setminus E$, where E is a finite subset of S. Setting $C = \{\lambda \in k: e_0 + \lambda e_1 \in S \setminus E\}$ as in 2.5, we say that $S \setminus E$ is B-*reliable* if $F_e | \operatorname{rep}_C Q^1$: $\operatorname{rep}_C Q^1 \to \operatorname{mod} \otimes B$ factors through $\operatorname{mod} B$ and preserves indecomposability and heteromorphism. Under these conditions, the indecomposable B-modules isomorphic to $F_e(k^n, \lambda \mathbb{1}_n + J_n)$, where $n \geq 1$ and $\lambda \in C$, are called *produced* by $S \setminus E$.

Third main theorem. *If B is a finite-dimensional k-algebra, one and only one of the following two statements holds*:

a) B is wild, i.e., there exists a B-reliable plane;

b) There exists a family of B-reliable punched lines $S_i \setminus E_i \subset \operatorname{Hom}_k(V_i \otimes_k B, V_i)$, $i \in I$, with the following properties: For each $d \in \mathbb{N}$, the number of $i \in I$ satisfying $d = \dim V_i$ is finite, and almost all isoclasses of indecomposable B-modules of dimension d consist of modules produced by the $S_i \setminus E_i$; furthermore, if $i \neq j$, no indecomposable produced by $S_i \setminus E_i$ can be produced by $S_j \setminus E_j$.

In case *b*), the algebra B is called *tame*.

A typical example is given by the quotient $B = k[x, y]/x^3, x^2y, xy^2, y^3$ of the polynomial algebra $k[x, y]$ and by the space $V = k^{1\times 4}$ (formed by rows with four entries in k). A B-reliable plane $\{e_{a,b}: a, b \in k\}$ of

$\mathrm{Hom}_k(V \otimes_k B, V)$ is then described by the matrices

$$\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&1&0&0\\0&0&0&a\\0&0&0&1\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1&0\\0&0&0&1\\0&0&0&b\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&a\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&b\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}.$$

(The endomorphisms $v \mapsto e_{a,b}(v \otimes z)$, where z runs through the residue classes of $1, x, y, x^2, xy, y^2$, are obtained by multiplication with the given matrices; compare with 2. 2, example 2.)

2.7. Our third main theorem raises the question of the factorization of the functor $F_e : \mathrm{rep}Q^t \to \mathrm{mod} \otimes B$ of 2.6 through $\mathrm{mod}\, B$. The answer is surprisingly simple. Let $b_0 = 1_B, b_1, \ldots, b_n$ be a basis of the vector space B and let $b_i b_j = \sum_{l=0}^{n} c_{ij}^l b_l$, $1 \le i, j \le n$, be the multiplication law. Let us further set $e_{pi}(v) = e_p(v \otimes b_i)$ for all $v \in V, p$ and $i \ge 0$ (2. 6). Then $F_e(W, a)$ lies in $\mathrm{mod}\, B$ if and only if $\sum_{p=0}^{t} a_p \otimes e_{p0} = \mathbb{1}_W \otimes \mathbb{1}_V$ and

$$\left(\sum_{q=0}^{t} a_q \otimes e_{qj} \right) \left(\sum_{p=0}^{t} a_p \otimes e_{pi} \right) = \sum_{l=0}^{n} c_{ij}^l \left(\sum_{s=0}^{t} a_s \otimes e_{sl} \right)$$

for all $i, j \ge 1$, where $a_0 = \mathbb{1}_W$. This condition is satisfied *for all* $(W, a) \in \mathrm{rep}^c Q^t$, *i.e., for all* (W, a) *with commuting endomorphisms* $a_1, \ldots, a_t$, *if and only if* $e_{00} = \mathbb{1}_V, e_{10} = \ldots = e_{t0} = 0$, *and*

$$e_{0j} e_{0i} = \sum_{l=0}^{n} c_{ij}^l e_{0l}, \quad e_{0j} e_{pi} + e_{pj} e_{0i} = \sum_{l=0}^{n} c_{ij}^l e_{pl},$$

$$e_{pj} e_{pi} = 0, \quad e_{qj} e_{pi} + e_{pj} e_{qi} = 0$$

for all $i, j \ge 1$ and all p, q such that $1 \le p < q$. These equations simply mean that *the affine subspace* S of $\mathrm{Hom}_k(V \otimes B, V)$ *is contained in the algebraic variety* $\mathcal{M}_B(V)$ (2.6). Accordingly, *if* S *is a line*, we have $\mathrm{rep}^c Q^1 = \mathrm{rep}\, Q^1$, and F_e *factors through* $\mathrm{mod}\, B$ *if and only if* $S \subset \mathcal{M}_B(V)$.

If we require that $F_e(W, a) \in \mathrm{mod} B$ for all $(W, a) \in \mathrm{rep}\, Q^t$, we must further impose the conditions $e_{qj} e_{pi} = 0$ for all $i, j \ge 1$ and all p, q such that $1 \le p < q$. Thus, $F_e : \mathrm{rep}\, Q^t \to \mathrm{mod} \otimes B$ *factors through* $\mathrm{mod}\, B$ *if and only if* $S \subset \mathcal{M}_B(V)$ *and* $F_e(k^{1\times 2}, a(p, q)) \in \mathrm{mod}\, B$ *for all* p, q *such that* $1 \le p < q$; here we set $a(p, q)_s = 0$ if $s \ne p, q$, whereas $a(p, q)_p$ and $a(p, q)_q$ are the multiplications by the matrices $\begin{bmatrix} 0&1\\0&0 \end{bmatrix}$ and $\begin{bmatrix} 0&1\\0&0 \end{bmatrix}$.

Of course, we can also interpret the equations displayed above by saying that F_e factors through $\mathrm{mod}\, B$ if and only if $F_e(W, a) \in \mathrm{mod}\, B$ holds for one single (W, a) such that the endomorphisms $\mathbb{1}_W$, a_i, and $a_i a_j$, $1 \le i, j \le t$, are linearly independent. In the case $t = 2$, for instance, we can choose $W = k^{1\times 3}$ and

$$a_1 = \begin{bmatrix} 0&1&0\\0&0&1\\0&0&0 \end{bmatrix}, \qquad a_2 = \begin{bmatrix} 0&0&0\\1&0&0\\0&1&0 \end{bmatrix}.$$

2.8. The functor $F_e : \mathrm{rep}\, Q^t \to \mathrm{mod}\, B$ admits the following more traditional interpretation. Let $C_t = k\langle x_1, \ldots, x_t \rangle$ denote the free associative algebra generated by $x_1, \ldots, x_t$. The free left C_t-module $M_t = C_t \otimes_k V$ is then equipped with a right $\otimes B$-module structure defined by the map

$$C_t \otimes V \otimes B \xrightarrow[\mathbb{1}\otimes e_0 + \dot{x}_1 \otimes e_1 + \ldots + \dot{x}_t \otimes e_t]{} C_t \otimes V,$$

where, for each $c \in C_t$, $\dot{c}$ denotes the map $C_t \to C_t$, $y \mapsto yc$. The $C_t - \otimes B$-bimodule thus obtained gives rise to a functor

$$\operatorname{rep} Q^t \to \operatorname{mod} \otimes B, \quad (W, a) \mapsto W \otimes_{C_t} M_t$$

which is isomorphic to F_e. (We define a right C_t-module structure on W by setting $wx_i = a_i(w)$, $\forall w \in W$.) The argument produced in 2.8 shows that this functor factors through $\operatorname{mod} B$ if and only if the right $\otimes B$-module structure on M_t factors through B.

Thus, our third main theorem improves results conjectured by Donovan and Freislich [2] and proved by Drozd [3] and Grawley-Boevey [4, 5] with the sophisticated technique of Roiter's boxes [6].

3. Preparative Lemmas

3.1. Lemma. *The module $M : X \mapsto X^3$ over the aggregate $\mathcal{A} = \operatorname{mod} k$ is absolutely wild.*

Proof. We must show that M is $\mathcal{L}$-wild for all proper $\mathcal{L}$. For this, we may assume that $\mathcal{L} = \{L_1, \ldots, L_r\}$ consists of maximal submodules of M, and hence, that there exist scalars λ_i, μ_i, ν_i such that

$$L_i(X) = \{v \in X^3 : \lambda_i v_1 + \mu_i v_2 + \nu_i v_3 = 0\}.$$

Transforming $\mathcal{L}$ by an automorphism of M (i.e., by an invertible 3×3 matrix) if necessary, we may assume furthermore that $\lambda_i \neq 0$ for all i. Under these assumptions, we consider the plane $S \subset \operatorname{Hom}_k(k, M(k)) \cong k^3$ formed by the columns $[1 \quad a \quad b]^{\mathrm{T}}$. If e_0, e_1, e_2 are the natural basis columns, the functor F_e: $\operatorname{rep} Q^2 \to M^k$ maps $(A, B) \in (k^{n \times n})^2$ onto the linear map $k^n \to M(k^n) = k^{3n}$ represented by the matrix $[\ \mathbb{1} \quad A^{\mathrm{T}} \quad B^{\mathrm{T}}\ ]^{\mathrm{T}}$. We infer that F_e is fully faithful. Moreover, since nilpotent simultaneously trigonalizable matrices A, B give rise to invertible matrices $\lambda_i \mathbb{1}_n + \mu_i A + \nu_i B$, F_e maps $\operatorname{rep}_0 Q^2$ into $M^k_{\mathcal{L}}$. By Lemma 2. 3, M is $\mathcal{L}$-wild.

3.2. Lemma. *The module M: $(X, Y) \mapsto X^2 \oplus Y^2$ over the aggregate $\mathcal{A} = \operatorname{mod} k \times \operatorname{mod} k$ is absolutely wild.*

Proof. The group of automorphisms of M is now identified with $\mathrm{GL}_2(k) \times \mathrm{GL}_2(k)$. This group acts on the finite sets of proper submodules. We may therefore suppose that, for each $L \in \mathcal{L}$, one of the columns $[1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}}$ and $[0 \quad 0 \quad 1 \quad 0]^{\mathrm{T}}$ does not belong to $L(k) \subset M(k) = k^2 \oplus k^2 \cong k^4$. The plane $S \subset \operatorname{Hom}_k(k, M(k))$ attached to the matrices $[1 \quad a \quad 1 \quad b]^{\mathrm{T}}$ with coordinates a, b then provides a fully faithful functor F_e: $\operatorname{rep} Q^2 \to M^k$ which maps $\operatorname{rep}_0 Q^2$ into $M^k_{\mathcal{L}}$.

3.3. For each natural number $t \geq 1$, we define a module M_t over a spectroid $\mathfrak{S}_t$ with two points x and y as follows. Denoting by $k[e, f]$ the algebra of polynomials in 2 indeterminates e and f, we set $\mathfrak{S}_t(x, x) = k\mathbb{1}_x$, $\mathfrak{S}_t(y, y) = k\,\mathbb{1}_y$, $\mathfrak{S}_t(x, y) = \bigoplus_{i=0}^{t-1} ke^{t-1-i} f^i$, $\mathfrak{S}_t(y, x) = 0$ and $M_t(x) = ke \oplus kf$, $M_t(y) = \bigoplus_{j=0}^{t} ke^{t-j} f^j$. The structural map from $\mathfrak{S}_t(x, y) \otimes M_t(x)$ to $M_t(y)$ is induced by the multiplication of polynomials.

For instance, if $t = 4$, $\mathfrak{S}_t$ is identified with the k-category of paths of the quiver $x \substack{\to \\ \to \\ \to \\ \to} y$, and the linear maps

$M_t(x) \to M_t(y)$ associated with the four arrows are represented in the natural bases by the matrices

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}.$$

Of course, we can interpret $\mathfrak{L}_t$ as the spectroid of an aggregate $\mathcal{A}_t$ whose objects are the formal direct sums $x^p \oplus y^q$, and M_t can be extended to $\mathcal{A}_t$ by setting $M_t(x^p \oplus y^q) = M_t(x)^p \oplus M_t(y)^q$.

Lemma. *The module M_t over the aggregate $\mathcal{A}_t$ is absolutely wild.*

Proof. We may suppose that $\mathcal{L}$ consists of maximal submodules $L_1, \ldots, L_r$ of M_t, where $L_j(y) = M_t(y)$ and $L_j(x) = \{ue + vf : \lambda_i u + \mu_i v = 0\}$ for some $(\lambda_i, \mu_i) \in k^2 \setminus (0,0)$. Because of the obvious equivariant action of $\mathrm{GL}_2(k)$ on $\mathfrak{L}_t$ and M_t, we may suppose that $\lambda_i \neq 0$ for all i. Under these assumptions, we consider the plane $S \subset \mathrm{Hom}_k(k^2, M_t(x^2 \oplus y))$ formed by the maps $k^2 \to M_t(x) \oplus M_t(x) \oplus M_t(y)$ represented by the matrices

$$\begin{array}{cccccccc} \left[\begin{matrix} 1 \\ 0 \end{matrix}\right. & \begin{matrix} 0 \\ 1 \end{matrix} & \begin{matrix} 0 \\ a \end{matrix} & \begin{matrix} 1 \\ b \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \,\vdots & \begin{matrix} 0 \\ 0 \end{matrix} & \left.\begin{matrix} 0 \\ 1 \end{matrix}\right]^{\mathrm{T}}. \\ e & & f & & e^t & e^{t-1}f & ef^{t-1} & f^t \end{array}$$

Choosing a and b as coordinates of these matrices, we obtain a functor F_e: $\mathrm{rep}\, Q^2 \to M_t^k$ whose restriction $F_e | \mathrm{rep}_0\, Q^2$ factors through M_{tL}^k, preserves indecomposability, and detects isomorphisms.

3.4. The examples produced in 3.3 admit the following variations. We denote by $\overline{\mathfrak{L}}_t$ the spectroid with one point x, endomorphism algebra $\overline{\mathfrak{L}}_t(x, x) = k\mathbb{1}_x \oplus ke^{t-1} \oplus ke^{t-2}f \oplus \ldots \oplus kf^{t-1}$, radical $ke^{t-1} \oplus \ldots \oplus kf^{t-1}$ and radical square zero. The formal direct sums x^p give rise to an aggregate $\overline{\mathcal{A}}_t$.

We further denote by $\overline{M}_t$ the $\overline{\mathcal{A}}_t$-module with stalk $\overline{M}_t(x) = ke \oplus kf \oplus ke^t \oplus ke^{t-1}f \oplus \ldots \oplus kf^t$ and radical $ke^t \oplus \ldots \oplus kf^t$ whose structural map $\overline{\mathfrak{L}}_t(x, x) \otimes (ke \oplus kf) \to \overline{M}_t(x)$ is induced by the multiplication of $k[e, f]$.

Lemma. *The module $\overline{M}_t$ over the aggregate $\overline{\mathcal{A}}_t$ is absolutely wild.*

Proof. Use the affine plane of $\mathrm{Hom}_k(k, \overline{M}_t(x))$ formed by the maps represented by the matrices

$$\begin{array}{cccccccc} [1 & a & 0 & 0 & \ldots & 0 & b]^{\mathrm{T}}. \\ e & f & e^t & e^{t-1}f & & ef^{t-1} & f^t \end{array}$$

Remark. Let L denote the submodule $(X, Y) \mapsto X^2$ of the module $M : (X, Y) \mapsto X^2 \oplus Y$ over $\mathrm{mod}\, k \times \mathrm{mod}\, k$. Then M is $\emptyset$-wild but not $\{L\}$-wild.

3.5. We now turn to the general case of a pointwise finite $\mathcal{A}$-module M. Our objective is to compare the representation types of M and of its factor modules M/N. For this, *we first suppose in* 3.5 *and* 3.6 *that N is a simple module located at some $s \in \mathfrak{L}$* $(\dim N(s) = 1, N(x) = 0$ if $x \in \mathfrak{L}$ and $x \neq s)$.

Let $(V, \bar{e}, X)$ be a space over $\overline{M} := M/N$ and let $e: V \to M(X)$ be a factorization of $\bar{e}: V \to \overline{M}(X)$. We call *transporter* T_e of V into $N(s)$ the set of all maps $V \to N(s)$ induced by morphisms $\mu \in \mathcal{R}_{\mathcal{A}}(X, s)$ such that $\mathrm{Im}\, M(\mu)e \subset N(s)$. We choose some basis $g_1, \ldots, g_n$ of a supplement U of T_e in $\mathrm{Hom}_k(V, N(s))$, set

$$V' := \mathrm{Hom}_k(V, N(s)) = T_e \oplus U,$$

and denote by g the induced composition

$$V \xrightarrow{[g_1 \cdots g_n]^{\mathrm{T}}} N(s)^n \xrightarrow{\text{incl.}} M(s)^n \xrightarrow{\sim} M(s^n).$$

Setting $d = [eg]^{\mathrm{T}}$, we thus obtain an M-space $(V, d, X \oplus s^n)$ which, up to isomorphism, does not depend on the basis $g_1, \ldots, g_n$ of U.

Lemma 1. *$(V, d, X \oplus s^n)$ avoids each submodule L of M such that $L \cap N = 0$.*

Proof. Clearly, $e^{-1}(L(X)) \subset K := \bigcap_{\tau \in T_e} \operatorname{Ker} \tau$. Since T_e and $g_1, \ldots, g_n$ generate $V' = \operatorname{Hom}_k(V, N(s))$, we infer that $\bigcap_i (K \cap \operatorname{Ker} g_i) = 0$, and hence, that $d = [e \quad g]^{\mathrm{T}}$ avoids L.

Lemma 2. *If $(V, \bar{e}, X) \in \overline{M}^k$ is indecomposable, then so is $(V, d, X \oplus s^n) \in M^k$.*

Proof. We may, of course, suppose that $V \neq 0$. Let us further assume that $(V, d, X \oplus s^n) \in M^k$ is decomposable. Since $(V, \bar{d}, X \oplus s^n) \in \overline{M}^k$ is the direct sum of $(V, \bar{e}, X)$ and $(0, 0, s^n)$, $(V, d, X \oplus s^n)$ admits a direct summand of the form $(0, 0, s)$ and a retraction $(0, \rho): (V, d, X \oplus s^n) \to (0, 0, s)$, where $\rho \in \mathcal{A}(X \oplus s^n, s)$. Since $(V, \bar{e}, X) \in \overline{M}^k$ has no direct summand of the form $(0, 0, s)$, $\rho \mid X$ cannot be a retraction. It follows that $\rho \mid s^n$ is a retraction, i.e., that $\rho \mid s^n = a_1\pi_1 + \ldots + a_n\pi_n + \mathscr{k}$ where π_i denote the canonical projections $s^n \to s$, the scalars a_i are not all zero, and $\mathscr{k}$ is radical. This yields

$$0 = M(\rho)d = M(\rho \mid X)e + M(\rho \mid s^n)g = M(\rho \mid X)e + \sum_{i=n}^{n} a_i g_i,$$

where $M(\rho \mid X)e \in T_e$. This provides the desired contradiction, since $g_1, \ldots, g_n$ is a basis of a supplement of T_e.

3.6. Lemma. *Consider fixed maps $e_0, e_1, e_2 \in \operatorname{Hom}_k(V, M(X))$ and variable spaces $W \in \operatorname{mod} k$ equipped with commuting endomorphisms a, b. Let further $e(a, b): W \otimes V \to W \otimes M(X) \xrightarrow{\sim} M(W \otimes X)$ denote the map $\mathbb{1}_W \otimes e_0 + a \otimes e_1 + b \otimes e_2$ and $T_{e(a,b)}$ denote the associated transporter of $W \otimes V$ into $N(s)$. Then there is a nonzero polynomial p in two indeterminates and a fixed subspace U of $V' = \operatorname{Hom}_k(V, N(s))$ such that*

$$\operatorname{Hom}_k(W \otimes V, N(s)) \xrightarrow{\sim} W^{\mathrm{T}} \otimes V' = T_{e(a,b)} \oplus W^{\mathrm{T}} \otimes U$$

whenever $p(a, b)$ is invertible.

By W^{T} we denote the dual of the vector space W.

Proof. Let us denote by u and v the compositions

$$\mathcal{R}_{\mathcal{A}}(W \otimes X, s) \xrightarrow{\text{can.}} \operatorname{Hom}_k(M(W \otimes X), M(s)) \xrightarrow{e(a,b)^*} \operatorname{Hom}_k(W \otimes V, M(s))$$

and

$$\operatorname{Hom}_k(W \otimes V, N(s)) \xrightarrow{\text{incl.}} \operatorname{Hom}_k(W \otimes V, M(s)) \xrightarrow{\text{can.}} \operatorname{Coker} u,$$

where we set $f^* = \mathrm{Hom}_k(f, M(s))$. The transporter $T_{e(a,\,b)}$ then equals $\mathrm{Ker}\, v$. On the other hand, u and v are identified with the compositions

$$W^{\mathrm{T}} \otimes \mathcal{R}_{\mathcal{A}}(X, s) \xrightarrow{\mathbb{1}\otimes \text{can.}} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(M(X), M(s)) \xrightarrow{\mathbb{1}\otimes e_0^* + a^{\mathrm{T}}\otimes e_1^* + b^{\mathrm{T}}\otimes e_2^*}$$

$$\xrightarrow{\mathbb{1}\otimes e_0^* + a^{\mathrm{T}}\otimes e_1^* + b^{\mathrm{T}}\otimes e_2^*} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, M(s))$$

and

$$W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, N(s)) \xrightarrow{\mathbb{1}\otimes \text{incl.}} W^{\mathrm{T}} \otimes \mathrm{Hom}_k(V, M(s)) \xrightarrow{\text{can.}} \mathrm{Coker}\, u.$$

Interpreting a^{T} and b^{T} as multiplication by x and y in W^{T} equipped with a module structure over $\Lambda = k[x, y]$, we obtain a description of u and v as tensor products $W^{\mathrm{T}} \otimes_{\Lambda} u_0$ and $W^{\mathrm{T}} \otimes_{\Lambda} v_0$, where u_0 and v_0 are Λ-linear compositions

$$\Lambda \otimes \mathcal{R}_{\mathcal{A}}(X, s) \xrightarrow{\mathbb{1}\otimes \text{can.}} \Lambda \otimes \mathrm{Hom}_k(M(X), M(s)) \xrightarrow{\mathbb{1}\otimes e_0^* + x\otimes e_1^* + y\otimes e_2^*}$$

$$\xrightarrow{\mathbb{1}\otimes e_0^* + x\otimes e_1^* + y\otimes e_2^*} \Lambda \otimes \mathrm{Hom}_k(V, M(s))$$

and

$$\Lambda \otimes \mathrm{Hom}_k(V, N(s)) \xrightarrow{\mathbb{1}\otimes \text{incl.}} \Lambda \otimes \mathrm{Hom}_k(V, M(s)) \xrightarrow{\text{can.}} \mathrm{Coker}\, u_0.$$

Now, there is a nonzero polynomial $q \in k[x, y]$ such that the kernels, images, and cokernels of $\Lambda[q^{-1}] \otimes_{\Lambda} u_0$ and $\Lambda[q^{-1}] \otimes_{\Lambda} v_0$ are free. This implies that

$$T_{e(a,\,b)} = \mathrm{Ker}\, v \xrightarrow{\sim} W^{\mathrm{T}} \otimes_{\Lambda[q^{-1}]} \mathrm{Ker}(\Lambda[q^{-1}] \otimes_{\Lambda} v_0) \xrightarrow{\sim} W^{\mathrm{T}} \otimes_{\Lambda} \mathrm{Ker}\, v_0,$$

whenever $q(a, b)$ is invertible.

To conclude, we choose arbitrary scalars $\xi, \eta \in k$ satisfying $q(\xi, \eta) = 0$ and an arbitrary supplement U of $T_{e(\xi,\eta)}$ in $\mathrm{Hom}_k(V, N(s))$. The canonical map

$$w_0 \colon \mathrm{Ker}\, v_0 \oplus \Lambda \otimes U \longrightarrow \Lambda \otimes \mathrm{Hom}_k(V, N(s))$$

then becomes bijective if we "specialize" x, y to ξ, η. Hence, there is a nonzero polynomial r such that $\Lambda[r^{-1}] \otimes_{\Lambda} w_0$ is bijective. So we may finally set $p = qr$.

3.7. We now return to the case of an *arbitrary submodule* N of M and denote by $\overline{\mathcal{L}} = \{L/N : L \in \mathcal{L}$ and $L \supset N\}$ the bond on $\overline{M} = M/N$ induced by a bond $\mathcal{L}$ on M.

Proposition. *M is $\mathcal{L}$-wild if M/N is $\overline{\mathcal{L}}$-wild.*

Proof. For each $L \in \mathcal{L}$ not containing N, let $s_L \in \mathcal{S}$ be such that $L(s_L)$ does not contain $N(s_L)$. Assume further that $\bar{e} = (\bar{e}_0, \bar{e}_1, \bar{e}_2)$ is a coordinate system of an $\overline{\mathcal{L}}$-reliable plane in $\mathrm{Hom}_k(V, \overline{M}(X))$ and $e = (e_0, e_1, e_2)$ is a system of factorizations of the $\bar{e}_i$ through $M(X)$. Restricting $\mathcal{S}$ to the finite full subspectroid formed by the support of X and all points s_L, and proceeding by induction on the length of N, we are reduced to the case where

N is simple and located at some s. Let then $p \in k[x,y]$ and $U \subset \mathrm{Hom}_k(V, N(s))$ be chosen according to Lemma 3.6. Assume finally that $g_1, \ldots, g_n$ denotes a basis of U, $g: V \to N(s)^n \subset M(s^n)$ the induced map and $\mathrm{rep}_p^c Q^2$ denotes the full subcategory of $\mathrm{rep}\, Q^2$ formed by the (W, a, b) such that a, b commute and $p(a, b)$ is invertible. Setting

$$d_0 = \begin{bmatrix} e_0 & g \end{bmatrix}^{\mathrm{T}} \in \mathrm{Hom}_k(V, M(X \oplus s^n))$$

and $d_1 = \begin{bmatrix} e_1 & 0 \end{bmatrix}^{\mathrm{T}}, d_2 = \begin{bmatrix} e_2 & 0 \end{bmatrix}^{\mathrm{T}}$, we prove that *the restriction*

$$F_d \mid \mathrm{rep}_p^c Q^2 : \mathrm{rep}_p^c Q^2 \longrightarrow M^k$$

preserves indecomposability and heteromorphism and factors through $M_{\mathcal{L}}^k$. Our proposition will then follow from Lemma 2.3 applied to a coordinate system $(d_0 + \xi d_1 + \eta d_2, d_1, d_2)$, where $(\xi, \eta) \in k^2$ satisfies $p(\xi, \eta) \neq 0$.

The composition

$$\mathrm{rep}\, Q^2 \xrightarrow{F_d} M^k \xrightarrow{\text{can.}} \overline{M}^k$$

maps (W, a, b) into $F_{\bar{e}}(W, a, b) \oplus (0, 0, W \otimes s^n)$. Since $F_{\bar{e}}$ preserves heteromorphism, so do F_d and $F_d \mid \mathrm{rep}_p^c Q^2$.

In order to prove the remaining two statements, we consider some $(W, a, b) \in \mathrm{rep}_p^c Q^2$ and set

$$\bar{e}(a,b) = \mathbb{1} \otimes \bar{e}_0 + a \otimes \bar{e}_1 + b \otimes \bar{e}_2 \colon W \otimes V \longrightarrow W \otimes \overline{M}(X) \xrightarrow{\sim} \overline{M}(W \otimes X),$$

$$e(a,b) = \mathbb{1} \otimes e_0 + a \otimes e_1 + b \otimes e_2 \colon W \otimes V \longrightarrow W \otimes M(X) \xrightarrow{\sim} M(W \otimes X).$$

On account of Lemma 3.6, $W^{\mathrm{T}} \otimes U$ is a supplement of the transporter $T_{e(a,b)}$ of $W \otimes V$ into $N(s)$. The M-space $(W \otimes V, \begin{bmatrix} e(a,b) & \varphi \end{bmatrix}^{\mathrm{T}}, W \otimes X \oplus W \otimes s^n)$ provided by a basis $\varphi_1, \ldots, \varphi_m$ of W^{T} and the associated map

$$\varphi\colon W \otimes V \longrightarrow N(s)^{m \times n}, \quad w \otimes v \mapsto [\varphi_i(w) g_j(v)]$$

avoids $\mathcal{L}$ by Lemma 1 of 3.5. By Lemma 2, it is indecomposable if so is (W, a, b). It is isomorphic to $F_d(W, a, b)$, as shown in the next diagram

$$\begin{tikzcd} W \otimes V \arrow[r, "\mathbb{1} \otimes g"] \arrow[dr, "\varphi"'] & W \otimes N(s)^m \arrow[d, "\iota"] \\ & N(s)^{m \times n} \end{tikzcd}, \quad \iota(w \otimes z) = [\varphi_i(w) z_j].$$

4. Proof of the First Main Theorem

4.1. Lemma. *Suppose that $\mathcal{J}$ is an ideal of an aggregate $\mathcal{A}$ with spectroid $\mathcal{S}$, M is a pointwise finite left module over $\mathcal{A}$, N is the annihilator of $\mathcal{J}$ in M, and $\tilde{M}$ is the module $M / \mathcal{J}M$ over $\tilde{\mathcal{A}} = \mathcal{A}/\mathcal{J}$. We further assume that the induced maps $\mathcal{J}(x, y) \to \mathrm{Hom}_k((M/N)(x), (\mathcal{J}M)(y))$ are surjective for all $x, y \in \mathcal{S}$. Then*:

a) *either $\mathcal{J}^2 M = 0$, the induced functor $P: M_N^k \to \tilde{M}_{N/\mathcal{J}M}^k$ is quasisurjective, and the indecomposables* *annihilated by* P *are isomorphic to some* $(0,0,s)$, *where* $s \in \mathfrak{K}$, $M(s)=0$, *and* $\mathbb{1}_s \in \mathfrak{I}$;

b) *or* $\mathfrak{I}$ *contains the identity* $\mathbb{1}_t$ *of one point* $t \in \mathfrak{K}$ *such that* $\dim M(t)=1$, *the induced functor* Q: $M_N^k \to \tilde{M}^k$ *is quasisurjective, and the indecomposables annihilated by* Q *are isomorphic to* $(0,0,t)$ *or to some* $(0,0,s)$, *where* $s \in \mathfrak{K}$, $M(s)=0$, *and* $\mathbb{1}_s \in \mathfrak{I}$.

The proof of the first main theorem uses Statement a) only. Statement b) will be used in Section 9.

Proof. We first show that Q induces surjections of the morphism spaces. Let (V,f,X) and (V',f',X') be two objects of M_N^k, and $\varphi \in \mathrm{Hom}_k(V,V')$, $\xi \in \mathcal{A}(X,X')$ two morphisms which induce a morphism $(\varphi,\bar{\xi})$: $(V,\tilde{f},X) \to (V',\tilde{f}',X')$ of $\tilde{M}^k_{N/\mathfrak{I}M}$. By definition, we then have $M(x)f - f'\varphi = ig$ for some $g \in \mathrm{Hom}_k(V,(\mathfrak{I}M)(X'))$, where $i:(\mathfrak{I}M)(X') \to M(X')$ denotes the inclusion. Since (V,f,X) avoids N, the obvious maps

$$(X,X') \longrightarrow \mathrm{Hom}_k((M/N)(X),(\mathfrak{I}M)(X')) \longrightarrow \mathrm{Hom}_k(V,(\mathfrak{I}M)(X'))$$

are both surjective and g is the image of some $\eta \in \mathfrak{I}(X,X')$. This means that $ig = M(\eta)f$ and implies $M(\xi-\eta)f = f'\varphi$. We infer that $(\varphi,\bar{\xi})$: $(V,\tilde{f},X) \to (V',\tilde{f}',X')$ is the image of $(\varphi,\xi-\eta)$: $(V,f,X) \to (V',f',X')$.

Now, in the case $\mathfrak{I}^2M=0$, Q maps M_N^k into $\tilde{M}^k_{N/\mathfrak{I}M}$, and P is surjective on the objects. This implies a).

In the case $\mathfrak{I}^2M \neq 0$, $\mathfrak{K}$ admits a point t such that $(\mathfrak{I}M)(t)$ is not contained in $N(t)$. The image of $\mathrm{Hom}_k((M/N)(t),(\mathfrak{I}M)(t))$ in $\mathrm{End}_k M(t)$ then contains an idempotent of rank 1. A preimage of this idempotent in $\mathfrak{I}(t,t)$ must be invertible in $\mathfrak{K}(t,t)$, because $\mathfrak{K}(t,t)$ is local. We infer that $\mathbb{1}_t \in \mathfrak{I}$ and that $\dim M(t)=1$. The last statement of b) now follows from the fact that $\mathfrak{K}$ contains no point $r \neq t$ such that $\mathbb{1}_r \in \mathfrak{I}$ and $M(r) \neq 0$. Otherwise, there would be morphisms $\sigma \in \mathfrak{I}(t,r)$ and $\rho \in \mathfrak{I}(t,r)$ such that $M(\rho\sigma) = \mathbb{1}_{M(t)}$, and the simple $\mathfrak{K}(t,t)$-module $M(t)$ would not be annihilated by the radical. So it remains to prove that Q hits each isoclass of $\tilde{M}^k$. Indeed, for each $\tilde{M}$-space $(V,\tilde{f},X)$, we can choose a factorization $f: V \to M(X)$ of $\tilde{f}$ and an isomorphism $g: V \xrightarrow{\sim} M(t)^d$, where $d=\dim V$; then $(V,[f\ d]^T, X \oplus t^n)$ avoids N, and its image in $\tilde{M}^k$ is isomorphic to $(V,\tilde{f},X)$.

4.2. *Remarks.* a) The assumptions of our lemma remain valid if we factor the annihilator of M out of $\mathcal{A}$. Hence, we might restrict ourselves to the case where M is faithful. In this case, the maps

$$\mathfrak{I}(x,y) \to \mathrm{Hom}_k((M/N)(x),(\mathfrak{I}M)(y))$$

are bijective. In subcase b), it follows that $\mathfrak{I}(x,y)$ is identified with $\mathfrak{K}(t,y) \otimes_k \mathfrak{K}(x,t)$. In both subcases, $\mathfrak{I}$ can be completely "described" in terms of the vector spaces $I(x)=(\mathfrak{I}M)(x) \subset N(x) \subset M(x)$ (where $x \neq t$ in case b). Accordingly, formal examples are constructed with ease.

b) Our concrete examples are the following. We start with a morphism $\mu \in \mathfrak{K}(s,t)$ such that $M(\mu)$: $M(s) \to M(t)$ has rank 1. Setting $S = \mathrm{Im}\, M(\mu)$, we denote by C_S the submodule of $\mathcal{A}(?,t)$ which consists of the morphisms $\xi; X \to t$ of $\mathcal{A}$ mapping $M(X)$ into S. Then we claim that *the assumptions of our lemma are satisfied by the ideal* $\mathfrak{I}$ *generated by any submodule* C *of* C_S *which contains* μ. Indeed, for all $x,y \in \mathfrak{K}$, the composition of $\mathfrak{K}$ maps $\mathfrak{K}(t,y) \otimes_k C(x)$ onto $\mathfrak{I}(x,y)$, and $N(x)$ is the annihilator of $C(x)$ in $M(x)$. Hence, the obvious map $(M/N)(x) \to \mathrm{Hom}_k(C(x),S)$ is injective, and the transposed map $C(x) \to \mathrm{Hom}_k((M/N)(x),S)$ is surjective. Taking into account that $(\mathfrak{I}M)(y)$ is the image of $\mathfrak{K}(t,y) \otimes_k S$, we infer that the double-headed arrows of the diagram

$$\begin{tikzcd}
\mathcal{S}(t,y)\otimes C(x) \arrow[r, two heads] \arrow[d] & \mathcal{S}(t,y)\otimes \mathrm{Hom}_k\left(\left(\frac{M}{N}\right)(x), S\right) \arrow[r, "\sim"] & \mathrm{Hom}_k\left(\left(\frac{M}{N}\right)(x), \mathcal{S}(t,y)\otimes C(x)\right) \arrow[d] \\
\mathcal{I}(x,y) \arrow[rr] & & \mathrm{Hom}_k\left((M/N)(x), (\mathcal{I}M)(y)\right)
\end{tikzcd}$$

are surjective. Hence, so is the lower arrow.

4.3. Let us now consider pairs $(\mathcal{A}, M)$ formed by an aggregate $\mathcal{A}$ and a pointwise finite $\mathcal{A}$-module M. We say that two such pairs $(\mathcal{A}, M)$ and $(\mathcal{A}', M')$ are *equivalent* if there exist a k-linear equivalence $E:\mathcal{A} \longrightarrow \mathcal{A}'$ and an isomorphism $M \xrightarrow{\sim} M'E$. And we say that the $\mathcal{A}$-module M is *climacteric* if the pair $(\mathcal{A}/\mathcal{N}_M, M)$, where $\mathcal{N}_M$ denotes the annihilator of M in $\mathcal{A}$, is equivalent to one of the absolutely wild pairs examined in 3.1, 3.2, 3.3, and 3.4.

Lemma. *Let M be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid $\mathcal{S}$. If M is not semisimple and has no climacteric quotient, $\mathcal{S}$ admits a morphism $\mu \in \mathcal{R}_{\mathcal{S}}(x, y)$ such that $M(\mu): M(x) \to M(y)$ has rank* 1 *and* $M(\lambda\mu) = 0 = M(\mu\nu)$ *for all $\lambda \in \mathcal{R}_{\mathcal{S}}(y, z)$, $\nu \in \mathcal{R}_{\mathcal{S}}(z, x)$, and $z \in \mathcal{S}$.*

Proof. a) *Reduction to the case of height* 2: Let us assume that M has height $h > 2$, and that the proposition is true for modules of height 2. We then denote by S_iM the annihilator of $\mathcal{R}^i_{\mathcal{A}}$ in M. Thus $\overline{M} = M/S_{h-2}M$ has height 2. If it admits a climacteric quotient, then so does M. Otherwise, there is a $\rho \in \mathcal{R}_{\mathcal{A}}(x, y)$ such that $\overline{M}(\rho)$ has rank 1 and vanishes on $(\mathcal{R}\overline{M})(x)$. Since $\rho\overline{M}(x) \neq 0$, we have $\sigma\rho M(x) \neq 0$ for some $\sigma \in \mathcal{R}^{h-2}_{\mathcal{A}}(y, z)$. On the other hand, $\sigma\rho \in \mathcal{R}^{h-1}_{\mathcal{A}}(x, z)$ annihilates $(\mathcal{R}M)(x)$, and $M(\sigma\rho)$ admits a factorization

$$M(x) \xrightarrow{\ \rho_*\ } M(y)/(S_{h-2}M)(y) \xrightarrow{\ \sigma_*\ } M(z).$$

where ρ_* is induced by ρ and σ_* by σ. We infer that $M(\sigma\rho)$ has rank 1.

b) Finally, we suppose that M *has height* 2. Factoring out the annihilator of M in $\mathcal{A}$ if necessary, we may suppose that the module M is faithful. We then consider four cases.

If M/S_1M has an isotypic component of dimension 1 supported, say, by $x \in \mathcal{S}$, then each nonzero radical morphism $\mu: x \to y$ of $\mathcal{S}$ suits.

If M/S_1M has an isotypic component of dimension ≥ 3, then M has a climacteric quotient of type 3.1.

If M/S_1M has at least 2 isotypic components of dimension 2, then M has a climacteric quotient of type 3.2.

If M/S_1M is isotypic of dimension 2 and supported by $x \in \mathcal{S}$, then we choose any $y \in \mathcal{S}$ such that $\mathcal{R}_{\mathcal{A}}(x, y) \neq 0$ and consider two subclasses. If $M(\mu)$ has rank 1 for some $\mu \in \mathcal{R}_{\mathcal{A}}(x, y)$, then μ suits. If $M(\rho)$ has rank 2 for all nonzero $\rho \in \mathcal{R}_{\mathcal{A}}(x, y)$, we denote by M' the sum of the isotypic components of S_1M not supported by y. Then $N = M/M'$ has a quotient of type 3.3 or 3.4 accordingly as $x \neq y$ or $x = y$:

To prove this, we choose two vectors $e, f \in N(x)$ whose classes modulo S_1N form a basis of $(N/S_1N)(x)$. The module structure of N then provides two maps $\varepsilon, \varphi: \mathcal{R}_{\mathcal{A}}(x, y) \overset{\rightarrow}{\rightarrow} (S_1N)(y)$ defined by $\varepsilon(\rho) = \rho e$ and $\varphi(\rho) = \rho f$. Since $M(\rho)$ has rank 2 for each $\rho \neq 0$, $a\varepsilon + b\varphi$ is injective for all $(a, b) \in k^2\setminus(0, 0)$. By Kronecker's classification of pairs of linear maps, we can therefore choose bases $n = (n_0, \ldots, n_t)$ of $(S_1N)(y)$ and $r = (r_i)_{i \in I}$ of $\mathcal{R}_{\mathcal{A}}(x, y)$, where $I \subset \{0, 1, \ldots, t-1\}$, such that $r_ie = \varepsilon(r_i) = n_i$ and $r_if = \varphi(r_i) = n_{i+1}$ for all $i \in I$. A typical example is

$$\begin{tikzcd}
& r_0 \arrow[dl, "\varepsilon"'] \arrow[dr, "\varphi"] & & & r_2 \arrow[dl, "\varepsilon"'] \arrow[dr, "\varphi"] & & r_3 \arrow[dl, "\varepsilon"'] \arrow[dr, "\varphi"] & & \\
n_0 & & n_1 & n_2 & & n_3 & & n_4 & n_5
\end{tikzcd},$$

where $t = 5$ and $I = \{0, 2, 3\}$.

Now we choose natural numbers $a < b$ such that $\{x \in \mathbb{N}: a \le x < b\} \subset I$ and $a - 1 \notin I, b \notin I$ (for instance, $a = 2, b = 4$ in the case of our diagram). Factoring out the basis vectors n_i for $i < a$ and for $b < i$, we obtain a quotient N' of N such that $(N'/S_1N')(x) \tilde{\to} ke \oplus kf$ and $(S_1N')(y) \tilde{\to} \bigoplus_{a \le i \le b} kn_i$. If $\mathcal{N}$ denotes the annihilator of N', the pair $(\mathcal{A}/\mathcal{N}, N')$ is equivalent to one of the pairs $(\mathcal{A}_{b-a}, M_{b-a})$ or $(\overline{\mathcal{A}}_{b-a}, \overline{M}_{b-a})$ examined in 3.3 and 3.4.

4.4. ***Proof of the first main theorem*** **(2.4).** We proceed by induction on the length of M. If M is not semisimple and has no climacteric quotient, we choose a morphism $\mu \in \mathcal{R}_{\mathfrak{s}}(x, y)$ according to Lemma 4.3 and denote by $\mathcal{I}$ the ideal of $\mathcal{A}$ generated by μ. Then the annihilator N of $\mathcal{I}$ in M is a maximal submodule of M, and M/N is supported by x. By 4.2 b), the assumptions of 4.1 a) are satisfied. If $\tilde{M} = M/\mathcal{I}M$ is considered as a module over $\tilde{\mathcal{A}} = \mathcal{A}/\mathcal{I}$, the canonical functor $M_N^k \to \tilde{M}^k_{N/\mathcal{I}M}$ is an epivalence. By the induction hypothesis, $\tilde{M}$ admits a bond $\mathcal{K}$ formed by submodules $L_i/\mathcal{I}M \supset \mathcal{R}M/\mathcal{I}M$, $1 \le i \le r$, such that $\tilde{M}^k_{\mathcal{K}} \to \overline{M}^k_{\overline{\mathcal{K}}}$ is an epivalence with the notation of 2.4 ($\overline{M} = \tilde{M}/\mathcal{R}\tilde{M} = M/\mathcal{R}M \ldots$). If we set $\mathcal{L} = \{L_1, \ldots \ldots, L_r, N\}$ and $\tilde{\mathcal{L}} = \mathcal{K} \cup \{N\}$, Lemma 4.1 implies that the composition $M^k_{\mathcal{L}} \to \tilde{M}^k_{\tilde{\mathcal{L}}} \to \overline{M}^k_{\overline{\mathcal{L}}}$ is an epivalence.

5. Pencils.

As in Sec. 4, $\mathcal{A}$ here denotes an aggregate with finite spectroid $\mathfrak{s}$. If M is a pointwise finite module on $\mathcal{A}$, we denote by $\dot{M} := \{x \in \mathfrak{s}: (\mathcal{R}M)(x) \ne M(x)\}$ the *generation indicator* of M. For each $p \in \dot{M}$, we write M_p for the submodule of M such that $M_p(p) = (\mathcal{R}M)(p)$ and $M_p(x) = M(x)$ if $x \in \mathfrak{s} \setminus p$.

5.1. ***Definition.*** *A pencil over $\mathcal{A}$ is a pointwise finite $\mathcal{A}$-module P restrained by a proper bond $\mathcal{K}$ such that:*

a) *P is not $\mathcal{K}$-wild;*

b) *there is no proper bond $\mathcal{B}$ on P for which $P^k_{\mathcal{B}}$ has a finite spectroid.*

Condition b) obviously implies that P admits infinitely many maximal submodules or, equivalently, that $\dim P/P_d \ge 2$ for some $d \in \dot{P}$. Proposition 4.3 implies that such a d is unique and satisfies $\dim P/P_d = 2$. We therefore call $d_p := d$ the *double point* of P; any other point $s \in \dot{P}$ satisfies $\dim P/P_s = 1$ and will be called *ordinary*.

Proposition. *Let $(P, \mathcal{K})$ be a pencil with double point d, and $(u_s)_{s \in \dot{P} \setminus d}$ a family of elements $u_s \in P(s) \setminus (\mathcal{R}P)(s)$. Let us further suppose that $\mathcal{K}$ is not empty and that P is $\mathcal{K}$-semisimple. Then*

$$u + \sum_{s \in \dot{P} \setminus d} u_s \in P\Big(d \oplus \bigoplus_s s\Big)$$

generates a maximal submodule of P for each $u \in P(d) \setminus \bigcup_{K \in \mathcal{K}} K(d)$.

We recall that, according to our terminology, each $K \in \mathcal{K}$ contains $\mathcal{R}P$ (2.4).

Proof. If Q is the module generated by $u + \sum_s u_s =: v$, it suffices to show that $Q \supset \mathcal{R}P$ if $u \in P(d) \setminus \bigcup_K K(d)$. For this purpose, we set $\sum = d \oplus \bigoplus_s s$ and consider any $r \in (\mathcal{R}P)(x)$, $x \in \mathfrak{s}$. The P-spaces

$\left(k, [v'\ 0]^{\mathrm{T}}, \sum \oplus x\right)$ and $\left(k, [v'\ r']^{\mathrm{T}}, \sum \oplus x\right)$, where $v'(\lambda) = \lambda v$ and $r'(\lambda) = \lambda r$, then avoid $\mathcal{K}$ and give rise to the same $\overline{P}$-space. They are, therefore, connected by a morphism $\left(\mathbb{1}, \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right)$ which is congruent to the identity modulo $\mathcal{R}_{\mathcal{A}}$ (2.4). This means that $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\begin{bmatrix} v \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ r \end{bmatrix}$ and implies that $\gamma v = r$ with $\gamma \in \mathcal{R}_{\mathcal{A}}(\Sigma, x)$.

5.2. Proposition 5. 1 only concerned the module structure of a pencil. We now examine its bond.

Proposition. *For each ordinary point $s \in \dot{P}$ of a pencil $(P, \mathcal{K}), P_s$ belongs to $\mathcal{K}$.*

Proof. Suppose that $P_s \notin \mathcal{K}$ and set $N = P_d \cap P_s, \overline{P} = P/N$, and $\overline{\mathcal{K}} = \{K/N : N \subset K \in \mathcal{K}\}$, where $d = d_P$. Then $\overline{P}$ is a semisimple pencil supported by d and s. The functor $F : \operatorname{rep} Q^2 \to \overline{P}^k$ associates (2.1) with the two-parametric affine family of

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & x \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & y \end{bmatrix}^{\mathrm{T}}$$

preserves indecomposability and heteromorphism. The $\overline{P}$-spaces represented by the displayed matrices avoid all proper submodules of $\overline{P}$ except $\overline{P}_s = P_s/N \notin \overline{\mathcal{K}}$. We infer that $\overline{P}$ is $\overline{\mathcal{K}}$-wild, and P is $\mathcal{K}$-wild (3.7).

5.3. From now on and throughout Section 5, M *denotes a pointwise finite $\mathcal{A}$-module restrained by a bond $\mathcal{L}$ for which M is not $\mathcal{L}$-wild.* All submodules P of M are implicitly supposed to be restrained by the trace $\mathcal{L} \cap P := \{L \cap P : L \in \mathcal{L}\}$ of $\mathcal{L}$. Our objective is to investigate the *pencils* of M, i.e., the submodules P of M such that $(P, \mathcal{L} \cap P)$ is a pencil. Our first result is easily derived from 5.2.

Corollary. *If P is a pencil of M with double point d, P/P_d is the socle of M/P_d. As a consequence, $P/\mathcal{R}P$ is the socle of $M/\mathcal{R}P$.*

Proof. Replacing $M \supset P$ by $M/P_d \supset P/P_d$ and applying 3.7, we are reduced to thecase where P is semisimple and $\dot{P} = \{d\}$. Then let Q denote the socle of M. Since Q is not $\mathcal{L} \cap Q$-wild, Q is a pencil of M which satisfies $d_Q = d$. In the case $Q \neq P, \dot{Q}$ has a simple point t outside $\dot{P}$ and $\mathcal{L}$ contains an L such that $L \cap Q = Q_t \supset P$: a contradiction to the assumption that $\mathcal{L} \cap P$ a proper bond on P.

5.4. Our next result rests on the classical *submodule algorithm* [7]. Starting from a submodule P of M we consider a new aggregate $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ and modules $\hat{R}$ on $\hat{\mathcal{A}}$ associated with submodules R of M and defined by

$$\hat{R}(W, g, X) = (g(W) + R(X))/g(W) \subset M(X)/g(W) = \hat{M}(W, g, X).$$

By $\hat{\mathcal{L}}$, we denote the bond on $\hat{M}$ formed by $\hat{P}$ and the submodules $\hat{L}, L \in \mathcal{L}$. Thus, we obtain a functor

$$E : M^k_{\mathcal{L}} \to \hat{M}^k_{\hat{\mathcal{L}}},\ (V, f, X) \mapsto (V/V', f'', (V', f', X)),$$

where V' equals $f^{-1}(P(X))$ and $f' : V' \to P(X)$, $f'' : V/V' \to M(X)/f(V')$ are induced by f. This functor is an epivalence, and even an equivalence if $\mathcal{L} \neq \emptyset$.

Proposition. *If P is a pencil of M, $P(X) = M(X)$ holds for all $x \in \dot{P}$. Accordingly, M contains only finitely many pencils.*

Proof. Restricting M, P, and all $L \in \mathcal{L}$ to $\dot{P}$, we may suppose that $\dot{P} = \mathcal{L}$. Arguing by contradiction and replacing M by a submodule if necessary, we may further suppose that M/P is simple, i.e., that $\dim M(x) = 1 + \dim P(x)$ for some $x \in \dot{P}$ and $M(y) = P(y)$ for all $y \in \dot{P}/x$. Setting $N = P_d \cap P_x$ and replacing M by M/N, we are reduced to the case where P is semisimple and where $\dot{P}$ consists of two points $d \neq x$ or of one point $d = x$.

a) *Case* $d \neq x$. For each submodule R of M, we then denote by R' the restriction of $\hat{R}$ to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ whose spectroid consists of the indecomposables $(0, 0, x) \in \hat{\mathcal{A}}$ and $p = (k^3, \bar{p}, d^4 \oplus x^3)$, where

$$\bar{p} = \begin{array}{c} \left[\begin{array}{cccc:cccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]^{T} \\ \begin{array}{cc} \qquad\qquad d \qquad\qquad & \qquad x \qquad \end{array} \end{array}.$$

The module M' admits a submodule Q such that $Q(0, 0, x) = P'(0, 0, x) = P(x)$ and $Q(p) = M'(p) = M(d^4 \oplus x^3)/\operatorname{Im}\bar{p} \supset P'(p)$. To prove this, it suffices to show that each morphism $(0, \mu): (k^3, \bar{p}, d^4 \oplus x^3) \to (0, 0, x)$ maps $M(d^4 \oplus x^3)$ into $P(x)$. For this, it is enough to show that $\mu: d^4 \oplus x^3 \to x$ is radical. This is due to the fact that a section σ of μ would provide a section $(0, \sigma)$ of $(0, \mu)$.

The restriction $\mathcal{L}' = \{L' : L \in \mathcal{L}\} \cup \{P'\}$ of $\hat{\mathcal{L}}$ to M' induces a proper bond $\mathcal{L}' \cap Q$ on Q, because $P' = P' \cap Q \neq Q$ and $L' \cap P' \neq P'$ for each $L \in \mathcal{L}$. Therefore, it suffices to show that $\dim Q(p)/(\mathcal{R}Q)(p) \geq 3$ (3.1). This follows from $\dim (M/P)(d^4 \oplus x^3) = \dim (M/P)(x^3) = 3$ and from $(\mathcal{R}Q)(p) \subset P(d^4 \oplus x^3)/\operatorname{Im}\bar{p}$. The inclusion is due to the fact that each morphism $(0, 0, x) \to (k^3, \bar{p}, d^4 \oplus x^3)$ of $\mathcal{A}'$ maps $Q(0, 0, x) = P(x)$ into $P(d^4 \oplus x^3)$, and that each radical endomorphism of p is induced by a radical endomorphism of $d^4 \oplus x^3$ which annihilates $(M/P)(d^4 \oplus x^3)$.

b) *Case* $d = x$. Then the argument is simpler. We focus on the sole indecomposable $q = (k^2, \bar{q}, d^3)$ of $\hat{\mathcal{A}}$, where $\bar{q} = \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]^{T}$. Each element of $\hat{\mathcal{L}}$ induces a proper subspace of $\hat{M}(q) = M(d^3)/\operatorname{Im}\bar{q}$, and each radical endomorphism of q maps $\hat{M}(q)$ into $\hat{P}(q)$. Replacing $\hat{M}$ by its restriction M' to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}}$ defined by q, we infer that $\dim M'(q)/(\mathcal{R}M)(q) \geq \dim M(d^3)/P(d^3) = 3$, and we conclude with 3.1.

5.5. Proposition. *Let* K *be maximal in* $\mathcal{L}$ *and not contained in the pencil* P *of* M. *Then* $\sum_{x \in \dot{P}} \dim M(x)/K(x) = 1$.

Proof. Suppose that the statement is wrong. Then we can find submodules $R_1 \subset Q_1$ of $M \mid \dot{P}$ which contain $K \mid \dot{P}$ and are of colength 2 and 1. We denote by Q_0 the maximal submodule of P such that $Q_0 \mid \dot{P} = Q_1$, by R the maximal submodule of $Q = Q_0 + K$ such that $R \mid \dot{P} = R_1$. (Of course, R contains K.)

We set $d = d_P$ and $\Sigma = \bigoplus_s s$, where $s \in \dot{P} \setminus d$. Up to isomorphism there is a unique indecomposable P-space of the form $p = (k^3, \bar{p}, d^4 \oplus \Sigma^3)$, which avoids all maximal submodules of P. Applying the submodule algorithm to $P \subset M$, we denote by M' and $\mathcal{L}'$ the restrictions of $\hat{M}$ and $\hat{\mathcal{L}}$ to the full subaggregate $\mathcal{A}'$ of $\hat{\mathcal{A}} = P^k_{\mathcal{L} \cap P}$ whose spectroid consists of p and of the $(0, 0, y)$, where $y \in \dot{R}$. The desired contradiction will follow from the fact that M' is $\mathcal{L}'$-wild.

To prove this, we consider the submodule N of M' such that $N(p) = Q(d^4 \oplus \Sigma^3) \bmod \operatorname{Im}\bar{p}$ and $N(0, 0, y) = R(y)$ if $y \in \dot{R}$. Such a submodule exists because each morphism $(0, \mu): (k^3, \bar{p}, d^4 \oplus \Sigma^3) \to (0, 0, y)$ maps $Q(d^4 \oplus$

Σ^3) into $R(y)$. Otherwise, μ would induce an isomorphism of a summand y' of $d^4 \oplus \Sigma^3$ onto y, and $(0, \mu)$ would admit a section.

Let X' denote the submodule of M' induced by a submodule X of M. Then N is not contained in K', because p avoids each proper submodule of P; hence, $R(d^4 \oplus \Sigma^3)$ and $Q(d^4 \oplus \Sigma^3)$ are identified with their images in $M(d^4 \oplus \Sigma^3)/\operatorname{Im}\bar{p}$, and we have $K'(p) \subset R(d^4 \oplus \Sigma^3) \neq Q(d^4 \oplus \Sigma^3) \xrightarrow{\sim} N(p)$. On the other hand, each $L \in (\mathcal{L} \setminus \mathcal{K}) \cup \{P\}$ intersects R properly; it follows that $L'(0,0,y) = L(y) \neq R(y) = N(0,0,y)$ for some $y \in \dot{R}$ and that $\mathcal{L}'$ is a proper bond on N. Hence, it suffices to prove that $\dim(N/\mathcal{R}N)(p) \geq 3$, which implies that N is absolutely wild and M' is $\mathcal{L}'$-wild.

The announced inequality is due to the fact that each radical endomorphism of p is induced by a radical endomorphism of $d^4 \oplus \Sigma^3$ and maps $N(p) \xrightarrow{\sim} Q(d^4 \oplus \Sigma^3)$ into $R(d^4 \oplus \Sigma^3)$. We conclude that $(\mathcal{R}N)(p) \subset R(d^4 \oplus \Sigma^3)$ and that

$$\dim(N/\mathcal{R}N)(p) \geq \dim(Q/R)(d^4 \oplus \Sigma^3) = 4 \text{ or } 3.$$

5.6. If $\tilde{\mathcal{L}}$ denotes the set of all maximal elements of $\mathcal{L}$, it is clear that $M^k_{\mathcal{L}} = M^k_{\tilde{\mathcal{L}}}$. Therefore we may always restrict ourselves to the case where $\mathcal{L}$ is *irredundant*, i.e., where $\mathcal{L} = \tilde{\mathcal{L}}$.

Corollary. *Suppose that* $\mathcal{L}$ *is an irredundant bond on* M *and that* $s \in \dot{P}$ *is an ordinary point of a pencil* P *of* M. *The conditions* $L \in \mathcal{L}$ *and* $L(s) \neq M(s)$ *then imply* $L \cap P = P_s$.

5.7. Corollary. *Let* K *be a submodule of* M *which is neither contained in the pencil* P *of* M *nor in any* $L \in \mathcal{L}$. *Then* $\sum_{x \in \dot{P}} \dim M(x)/K(x) \leq 1$.

Proof. The corollary follows from Proposition 5.5 applied to a new bond $\mathcal{L} \cup \{K\}$.

5.8. Corollary. *Suppose that the* $\mathcal{L}$*-pencils* P *and* Q *of* M *are not comparable. Then* $d_P \notin \dot{Q}$ *and* $d_Q \notin \dot{P}$.

Proof. Suppose that $d_Q \notin \dot{P}$ and that $u \in Q(d_Q) \neq M(d_Q)$ lies outside $L(d_Q)$ whenever $L \in \mathcal{L}$ satisfies $L(d_Q) \neq M(d_Q)$. Let further K denote a maximal submodule of Q such that $u \in K(d_Q) \neq M(d_Q)$. Then K is not contained in P and $\mathcal{L} \cap K$ is a proper bond on K. On the other hand, we have $K(d_Q) \neq M(d_Q)$ and $K(s) = Q(s) \neq$ $\neq M(s)$ for some $s \in \dot{P}$, hence

$$\sum_{x \in \dot{P}} \dim M(x)/K(x) \geq 2,$$

in contradiction to 5.7.

5.9. Corollary. *If the* $\mathcal{L}$*-pencils* P *and* Q *of* M *are not comparable, then* $(\mathcal{R}P)(s) = (\mathcal{R}Q)(s)$ *for all* $s \in \dot{P} \cap \dot{Q}$.

Proof. Indeed, s is ordinary by 5.8. If L is maximal in $\mathcal{L}$ and such that $L \cap P = P_s$ (5.2), we have $L \cap Q = Q_s$ by 5.6; hence, $(\mathcal{R}P)(s) = L(s) = (\mathcal{R}Q)(s)$.

5.10. For each submodule N of M, we set $\check{N} = \{x \in \mathfrak{S} : N(x) = M(x)\}$. Thus we have $\dot{P} \subset \check{P}$ if P is a pencil of M.

Corollary. *If P, Q, and R are 3 pairwise incomparable pencils of M, the equality $\dot{P} \setminus \check{R} = \dot{Q} \setminus \check{R}$ implies $\dot{R} \setminus \check{P} = \dot{R} \setminus \check{Q}$.*

Proof. Let $s \in \dot{P} \cap \dot{Q}$ be such that $R(s) \neq M(s)$, and L a maximal element of $\mathcal{L}$ such that $L \cap P = P_s$ and $L \cap Q = Q_s$ (5.6). If $t \in \dot{R}$ is such that $M(t) = R(t) \neq L(t)$, we have $P(t) = P_s(t) \subset L(t)$ and $Q(t) = Q_s(t) \subset L(t)$; hence, $\dot{R} \setminus \check{P} = \{t\} = \dot{R} \setminus \check{Q}$.

6. Proof of the Second Main Theorem (Reduction).

Our objective is to propose a general "construction" of locally finite sets $\mathcal{D} = \mathcal{D}(M, \mathcal{L})$ of $\mathcal{L}$-reliable punched lines which satisfy the conditions a) and b) of the second main theorem. Our sets $\mathcal{D}$ are the unions of subsets $\mathcal{D}_n = \mathcal{D}_n(M, \mathcal{L})$ formed by punched lines $D \setminus E \subset \operatorname{Hom}_k(V, M(X))$ whose points have *space dimension* $\dim V = n$. We construct the slices $\mathcal{D}_n(M, \mathcal{L})$ by induction on n and *simultaneously* for "all" nonwild pairs $(M, \mathcal{L})$. The construction is rather precise and rather involved, as nature seems to be.

In order to classify the indecomposable M-spaces, we can examine the finite full subspectroids $\mathfrak{S}'$ of $\mathfrak{S}$ separately and focus on the M-spaces with "support" $\mathfrak{S}'$. We are thus reduced to the case examined in the present section where *the spectroid $\mathfrak{S}$ of $\mathcal{A}$ is assumed to be finite. From* 6.2 *until the end of the section, we assume that M is not $\mathcal{L}$-wild.*

6.1. Since our construction proceeds by induction on the space dimension, we first examine indecomposable M-spaces with space dimension 1. For this purpose, no restriction is needed on the representation type of $(M, \mathcal{L})$.

Proposition. *The map $(V, f, X) \mapsto \mathcal{A}f(V)$, which assigns to (V, f, X) the submodule of M generated by $f(V)$, induces a bijection between the set of isoclasses of indecomposables in $M_{\mathcal{L}}^k$ with space dimension 1 and the set of submodules N of M for which $\mathcal{L} \cap N$ is a proper bond.*

Proof. The inverse bijection is obtained as follows. For each N, we choose a projective cover n: $\mathcal{A}(X, ?) \to N$ and set $n' = n(X)(\mathbb{1}_X) \in N(X)$. To N we then assign the isoclass of $(k, ?n', X) \in M_{\mathcal{L}}^k$.

6.2. Let us now return to the case where M is not $\mathcal{L}$-wild. Each pencil P of $(M, \mathcal{L})$ with double point d gives rise to a one-parametric family of maximal submodules Q of P such that $P_d \subset Q \subset P$. The other maximal submodules of P have the form P_s, where s is an ordinary point of $\dot{P}$; their number is finite, and the induced bond $\mathcal{L} \cap P_s$ is not proper (5.2).

Proposition. *Besides maximal submodules of pencils, M contains only finitely many submodules N for which $\mathcal{L} \cap N$ is a proper bond.*

Proof. We proceed by induction on the number of pencils of $(M, \mathcal{L})$, which is finite by 5.4. If M contains no pencil, we denote by $\mathcal{N}$ the set of all $N \subset M$ such that $\mathcal{L} \cap N$ is proper. Each element of $\mathcal{N}$ has finitely many (direct) predecessors. Since $\mathcal{N}$ has finite height and (at most) one maximal element, $\mathcal{N}$ is finite.

If M contains pencils, we consider a minimal pencil P (with double point d) and maximal submodules $Q_1, \dots, Q_s$ $(s \geq 1)$ of P containing P_d and such that each $u \in P(d) \setminus \bigcup_{i=1}^{s} Q_i(d)$ satisfies the statement of Proposition 5.1. Then each nonmaximal submodule of P is contained in some Q_i or some P_s with $s \in \dot{P} \setminus d$. And each nonmaximal submodule $N \subset P$ for which $\mathcal{L} \cap N$ is proper is contained in some Q_i. Together with $Q_1, \dots, Q_s$, these N form a poset $\mathcal{N}$ which has finite height and a finite number of maximal elements. Since each element of $\mathcal{N}$ has a finite number of (direct) predecessors, $\mathcal{N}$ is finite.

On the other hand, since $(M, \mathcal{L} \cup \{P\})$ admits fewer pencils than $(M, \mathcal{L})$, we know by induction that there are only finitely many submodules N' which are not contained in P, which are not maximal in a pencil of $(M, \mathcal{L})$ and for which $\mathcal{L}' \cap N'$ is proper.

6.3. ***The construction of*** $\mathcal{D}_1$. For each pencil P of M, we pick vectors $u_s \in P(s)\setminus(\mathcal{R}P)(s)$, $s \in \dot{P}\setminus d_P$, and a basis (u, v) of a supplement of $(\mathcal{R}P)(d_P)$ in $P(d_P)$. Thus, we obtain a straight line

$$D_P = \{u + \lambda v + \sum_s u_s : \lambda \in k\}$$

of $M(d_P \oplus \underset{s}{\oplus} s) \xrightarrow{\sim} \mathrm{Hom}_k(k, M(d_P \oplus \underset{s}{\oplus} s))$ whose associated functor F: rep $Q^1 \to M^k$ preserves indecomposability and heteromorphism. Erasing from D_P the points lying in the various subspaces $L(d_P \oplus \underset{s}{\oplus} s)$, $L \in \mathcal{L}$, we get an $\mathcal{L}$-reliable punched line, which seems to be a good applicant for a position in $\mathcal{D}_1$. Unfortunately, if the lines D_P are to be retained, the present state of our technology urges us to overpunch them, as will be explained below.

First we consider the *minimal pencils* of M, which we stack up in a finite set $\mathcal{P}$ equipped with an arbitrary *linear order*. If $\mathcal{P} \neq \emptyset$, we construct an ideal $\mathcal{J}$ of $\mathcal{A}$ and a bond $\mathcal{K}$ on M which satisfy the statements of Lemma 6.4 below. Finally, for each $P \in \mathcal{P}$, we construct a proper bond $\mathcal{K}'_P$ on P, formed by maximal submodules N such that P is $\mathcal{K}'_P$-semisimple and $v \in N(d_P)$ for some N. The submodules N give birth to a bond

$$\mathcal{L}'_P = (\mathcal{L} \cap P) \cup (\mathcal{K} \cap P) \cup \mathcal{K}'_P \cup \{X : P > X \in \mathcal{P}\}$$

on P and to a finite subset

$$E_P = \bigcup_{L \in \mathcal{L}'_P} D_P \cap L(d_P \oplus \underset{s}{\oplus} s)$$

of the straight line D_P. The associated punched lines $D_P \setminus E_P$ are the first selected constituents of $\mathcal{D}_1$.

The restraint imposed by $\mathcal{K}$ will permit us to prove Lemma 6.4 below. As a result of the insertion of $\mathcal{K}'_P$ into $\mathcal{L}'_P$, *all maximal elements of* $\mathcal{L}'_P$ *and all proper submodules* K *of* P *for which* $\mathcal{L}'_P \cap K$ *is proper are maximal in* P (5.1). Accordingly, each $u + \lambda v + \sum_s u_s \in D_P \setminus P_P$ generates a maximal submodule of P.

In order to puncture the lines D_P *when* P *is not minimal*, we now set $\mathcal{P}_1 := \mathcal{P}$ and $\mathcal{K}_1 := \mathcal{K}$. We denote by $\mathcal{P}_2$ the set of minimal pencils of $(M, \mathcal{L} \cup \mathcal{P}_1)$ or, equivalently, of $(M, \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1)$, by $\mathcal{P}_3$ the set of minimal pencils of $(M, \mathcal{L} \cup \mathcal{P}_1 \cup \mathcal{P}_2)$.... Replacing $\mathcal{L}$ by $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$, we construct a bond $\mathcal{K}_2$ which satisfies the statements of Lemma 6.4 for $(M, \mathcal{L}_1)$. Adapting the technique above to the new data, we obtain a proper bond $\mathcal{L}'_P$ on each $P \in \mathcal{P}_2$ and the associated finite subset $E_P \subset D_P$. Then replacing $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$ by $\mathcal{L}_2 = \mathcal{L}_1 \cup \mathcal{K}_2 \cup \mathcal{P}_2$, we construct bonds $\mathcal{K}_3$ on M and $\mathcal{L}'_P$ on each $P \in \mathcal{P}_3$, thus obtaining finite sets $E_P \subset D_P$ for all $P \in \mathcal{P}_3$.... If $\mathcal{P}_h$ is the last nonempty set of pencils constructed in this way, we finally set

$$\mathcal{D}_1(M, \mathcal{L}) = \{D_P \setminus E_P : P \in \mathcal{P}_i,\ 1 \leq i \leq h\}.$$

If M contains no pencil, $\mathcal{D}_1(M, \mathcal{L})$ is empty.

6.4. Lemma. *Suppose that* M *is a pointwise finite module over an aggregate* $\mathcal{A}$ *with finite spectroid* $\mathfrak{S}$, $\mathcal{L}$ *is a bond on* M *such that* M *is not* $\mathcal{L}$*-wild,* $\mathcal{P}$ *is a nonempty set of pairwise incomparable pencils* (5.1) *of* M, *and* $R = \bigcap_{P \in \mathcal{P}} \mathcal{R}P$ *is the intersection of their radicals. Then there is an ideal* $\mathcal{J} \subset \mathcal{R}_{\mathcal{A}}$ *and a bond* $\mathcal{K}$

on M such that:

a) $\mathcal{J}M \subset R \subset B \cap P \neq P$ *and* $(\mathcal{J}M)(x) = (\mathcal{R}P)(x)$ *for all* $B \in \mathcal{K}$, *all* $P \in \mathcal{P}$, *and all* $x \in \dot{P}$;

b) *If* $M / \mathcal{J}M$ *is considered as a module over* $\mathcal{A}/\mathcal{J}$ *and* $\mathcal{K}/\mathcal{J}M$ *denotes the set of all* $B / \mathcal{J}M$, $B \in \mathcal{K}$, *then the canonical functor* $M^k_{\mathcal{K}} \to (M / \mathcal{J}M)^k_{\mathcal{K}/\mathcal{J}M}$ *is an epivalence.*

The proof of the lemma is given in 7.1 below.

6.5. *The construction of* $\mathcal{D}_r$, $r \geq 2$. The construction is based on a *sequence of submodules* of M which we must present beforehand. First supposing $\mathcal{P}_1 \neq \emptyset$, we consider the submodules X such that: a) $\mathcal{L} \cap X$ is a proper bond on X; b) X is contained in a module belonging to $\mathcal{K} = \mathcal{K}_1$ or to some $\mathcal{K}'_P$, where $P \in \mathcal{P} = \mathcal{P}_1$ (6.3). These submodules form a *finite* set (6.2), which we denote by $O_0 = O_0(M, \mathcal{L})$ and equip with some linear order $\leq$ such that $X \subset Y$ implies $X \leq Y$. By construction, O_0 contains all the nonmaximal submodules N of P, $P \in \mathcal{P}_1$, for which $\mathcal{L} \cap N$ is a proper bond.

Replacing $\mathcal{L}$ by $\mathcal{L}_1 = \mathcal{L} \cup \mathcal{K}_1 \cup \mathcal{P}_1$, then by $\mathcal{L}_2 = \mathcal{L}_1 \cup \mathcal{K}_2 \cup \mathcal{P}_2, \ldots, \mathcal{L}_h = \mathcal{L}_{h-1} \cup \mathcal{K}_h \cup \mathcal{P}_h$, we may repeat the construction of O_0 and obtain further linearly ordered sets $O_1 = O_0(M, \mathcal{L}_1)$, $O_2 = O_0(M, \mathcal{L}_2), \ldots, O_{h-1} = O_0(M, \mathcal{L}_{h-1})$. To these sets we add a set O_h, formed by the submodules N of M for which $\mathcal{L}_h \cap N$ is proper, and also equipped with a linear order $\leq$ such that $X \subset Y$ implies $X \leq Y$. Together with the linear orders imposed onto $\mathcal{P}_1$, $\mathcal{P}_2, \ldots, \mathcal{P}_h$, we finally obtain a *finite linearly ordered set* $\mathcal{Q}$ which has M as maximum and is formed by the disjoint intervals

$$O_0 < \mathcal{P}_1 < O_1 < \mathcal{P}_2 < O_2 < \ldots < \mathcal{P}_n < O_h.$$

If M contains no pencil, O_0 denotes the set of all submodules X of M for which $\mathcal{L} \cap X$ is proper. We then set $\mathcal{Q} = O_0$.

Our construction of $\mathcal{D}_r(M, \mathcal{L})$ now results from an application of our *main algorithm* to each submodule $N \in \mathcal{Q}$ and to the associated bond $\mathcal{B}N = \mathcal{L} \cup \{X \in \mathcal{Q}; X < N\}$ on M. For this sake, we introduce the aggregate $\mathcal{A}^N = N^k_{\mathcal{B}N \cap N}$, its spectroid $\mathcal{S}^N$, the module M^N on $\mathcal{A}^N$ defined by $M^N(W, g, X) = M(X)/g(W)$, and a bond $\hat{\mathcal{B}}N$ on M^N, which consists of the submodules of M^N induced by N and the modules $X \in \mathcal{B}N$. The resulting epivalence $M^k_{\mathcal{B}N} \to M^{Nk}_{\hat{\mathcal{B}}N}$ will allow us to lift various slices of the desired $\mathcal{D}_r(M, \mathcal{L})$ from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{B}N)$. We distinguish two cases:

1) Case $N \in O_i$. Then $\mathcal{B}N \cap N$ contains all maximal submodules of N. The spectroid $\mathcal{S}^N$ is finite and contains one point $(k, g, \bigoplus_{s \in \dot{N}}, s)$ with space dimension 1. The remaining points of $\mathcal{S}^N$ have the form $(0. 0. t)$, $t \in \mathcal{S}$.

Obviously, M^N is not $\hat{\mathcal{B}}N$-wild, because two-parametric families of indecomposables could be lifted from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{L})$. Proceeding by induction on r, we may therefore suppose that the sets $\mathcal{D}_s(M^N, \hat{\mathcal{B}}N)$ are at our disposal for all $s < r$. Here we are concerned with $\hat{\mathcal{B}}N$-reliable punched lines formed by M^N-spaces (U, h, Z) whose bases $Z = (W, g, X) \in N^k_{\mathcal{B}N \cap N}$ have a space dimension $\dim W =: t \geq 1$. These lines form a subset $\mathcal{D}^t_s(M^N, \hat{\mathcal{B}}N)$ of $\mathcal{D}_s(M^N, \hat{\mathcal{B}}N)$. Lifting the lines of $\mathcal{D}^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{B}N)$, we finally obtain a set $\tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ of $\mathcal{L}$-reliable punched lines and the requested contribution of N to $\mathcal{D}_r(M, \mathcal{L})$:

$$\bigcup_{t=1}^{r-1} \tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N).$$

2) Case $N \in \mathcal{P}_i$. We then proceed as in case 1, the difference being that $\mathcal{S}^N$ is infinite. According to Lemma 6.6 below, $\mathcal{S}^N$ contains a *finite* full subspectroid $\mathcal{S}^N_r$ which supports the bases $Z = (W, g, X)$ of all

indecomposables $(U, h, Z) \in M^{Nk}_{\hat{\mathcal{B}}N}$ such that $1 \le \dim U$ and $\dim W < r$. (More precisely, $\mathfrak{S}^N_r$ is formed by the points $(0, 0, t)$, $t \in \mathfrak{S}$, and by at most $5(r-1)$ points of the form (W', g', X') with $1 \le \dim W' < r$.) Since $\mathfrak{S}^N_r$ is finite, our induction provides us with finite sets $\mathcal{D}_s(M^N \mid \mathfrak{S}^N_r, \hat{\mathcal{B}}N \mid \mathfrak{S}^N_r)$ for $s < r$. As in case 1, these sets are partitioned into subsets $\mathcal{D}^t_s(M^N \mid \mathfrak{S}^N_r, \hat{\mathcal{B}}N \mid \mathfrak{S}^N_r)$. Lifted from $(M^N, \hat{\mathcal{B}}N)$ to $(M, \mathcal{L})$, these subsets give rise to finite sets of $\mathcal{B}N$-reliable punched lines denoted by $\tilde{\mathcal{D}}^t_s(M^N, \hat{\mathcal{B}}N)$.

Putting together the various pieces obtained above, we finally set

$$\mathcal{D}_r(M, \mathcal{L}) = \bigcup_{N \in Q} \bigcup_{t=1}^{r-1} \tilde{\mathcal{D}}^t_{r-t}(M^N, \hat{\mathcal{B}}N). \tag{*}$$

The fact that $\mathcal{D}(M, \mathcal{L}) = \bigcup_{r \ge 1} \mathcal{D}_r(M, \mathcal{L})$ satisfies the statements of the second main theorem is easy and will be checked in 6.7.

6.6. Let us provisionally consider an arbitrary pointwise finite module M' over an aggregate $\mathcal{A}'$ and a bond $\mathcal{L}'$ on M'. We then say that an indecomposable $s \in \mathcal{A}'$ is $(M', \mathcal{L}')$-relevant if s is a direct summand of the base X of some indecomposable $(V, f, X) \in M'^k_{\mathcal{L}'}$.

Lemma. *With the notation of* 6.5, *let* N *be a pencil of* M *and* $r \ge 2$. *Then there are at most* $5(r-1)$ *isoclasses of indecomposable* N*-spaces* (W, g, X) *which avoid* $\mathcal{B}N \cap N$, *satisfy* $1 \le \dim W < r$, *and are* $(M^N, \hat{\mathcal{B}}N)$*-relevant.*

6.7. ***Checking the statements of the second main theorem.*** The statements result almost immediately from the construction.

Since $\mathfrak{S}$ is assumed to be finite, the finiteness of the cardinality of $\mathcal{D}_r(M, \mathcal{L})$ follows from 6.5 (*).

In order to prove statement a), we denote by $\nu_r(M, \mathcal{L})$ the number of isoclasses of indecomposable M-spaces $(V, f, X) \in M^k_{\mathcal{L}}$ which have space dimension r and are not produced by punched lines of $\mathcal{D}(M, \mathcal{L})$. We shall prove that $\nu_r(M, \mathcal{L})$ is finite by induction on r. Clearly, $\nu_0(M, \mathcal{L})$ is equal to the number of points of $\mathfrak{S}$. So let us assume that $r = 1$. By 6.1, the isoclasses of the indecomposables $(k, f, X) \in M^k_{\mathcal{L}}$ with space dimension 1 correspond bijectively to the submodules $X = \mathcal{A}f(k)$ for which $\mathcal{L} \cap X$ is proper. In the case $\mathcal{A}f(k) \notin Q$, (k, f, X) is produced by $\mathcal{D}(M, \mathcal{L})$ and $\mathcal{A}f(k)$ is a maximal submodule of a pencil. *We infer that* $\nu_1(M, \mathcal{L}) = |Q|$.

In the case $r \ge 2$, let $(V, f, X) \in M^k_{\mathcal{L}}$ be an indecomposable with space dimension r which is not produced by $\mathcal{D}(M, \mathcal{L})$, and let N be the smallest element of Q such that $t = \dim f^{-1}(N(X)) \ge 1$. *If* N *is not a pencil,* our induction hypothesis and the finiteness of $\mathfrak{S}^N$ imply that $M^{Nk}_{\hat{\mathcal{B}}N}$ has a finite number, say, $\nu^t_{r-t}(M^N, \hat{\mathcal{B}}N)$, of isoclasses of indecomposables (U, h, Z) not produced by $\mathcal{D}(M^N, \hat{\mathcal{B}}N)$ and such that $\dim U = r - t$ and that Z has space dimension $t \ge 1$. The contribution of N to $\nu_r(M, \mathcal{L})$ is therefore equal to $\sum_{t=1}^r \nu^t_{r-t}(M^N, \hat{\mathcal{B}}N)$. (We recall that $\nu^r_0(M^N, \hat{\mathcal{B}}N) = 0$ in the considered case $r \ge 2$.)

If N *is a pencil,* the numbers $\nu^t_{r-t}(M^N, \hat{\mathcal{B}}N) \in \mathbb{N} \cup \{\infty\}$ can still be defined. Now $\nu^r_0(M^N, \hat{\mathcal{B}}N) = 1$. In the case $1 \le t < r$, the finiteness of $\nu^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ follows from the fact that the bases Z of the indecomposables (U, h, Z) considered above are supported by a *finite* subspectroid $\mathfrak{S}^N_r$ of $\mathfrak{S}^N$ (6.5, case 2, and 6.6). It follows that N still has a finite contribution $\sum_{t=1}^r \nu^t_{r-t}(M^N, \hat{\mathcal{B}}N)$ and that

$$\nu_r(M, \mathcal{L}) = \sum_{N \in Q} \sum_{t=1}^{r} \nu^t_{r-t}(M^N, \hat{\mathcal{B}}N).$$

Finally, in order to check statement b), we prove by induction on r that indecomposable M-spaces $(V, f, X) \in M_{\mathcal{L}}^k$ and $(V', f', X') \in M_{\mathcal{L}}^k$ cannot be isomorphic if they are produced by different punched lines D and D' of $\mathcal{D}_{\leq r}(M, \mathcal{L})$: $= \bigcup_{s \leq r} \mathcal{D}_s(M, \mathcal{L})$. This is clear by construction if $D \in \mathcal{D}_1(M, \mathcal{L})$ or $D' \in \mathcal{D}_1(M, \mathcal{L})$. Otherwise, r is ≥ 2. Then we consider the smallest elements N and N' of Q which are not avoided by (V, f, X) and (V', f', X'), respectively. Our claim is clear if $N \neq N'$. In the case $N \neq N'$, D and D' are obtained by lifting punched lines defined on finite spectroids $\mathfrak{S}^N$ or $\mathfrak{S}_r^N$. These punched lines consist of M^N-spaces with space dimension less than r. They produce the M^N-spaces associated with (V, f, X) and (V', f', X'). Since these M^N-spaces are not isomorphic by induction hypothesis, (V, f, X) and (V', f', X') are not isomorphic either.

7. Simultaneous Eradication of Incomparable Pencils

7.1. Theorem. *Let M be a pointwise finite module over an aggregate $\mathcal{A}$ with finite spectroid $\mathfrak{S}$, $\mathcal{L}$ a bond on M such that M is not $\mathcal{L}$-wild, $\mathcal{P}$ a nonempty set of pairwise incomparable pencils of M, and $R = \bigcap_{P \in \mathcal{P}} \mathcal{R}P$ the intersection of their radicals. We suppose that $R(q) \neq 0$, where $q \in \mathfrak{S}$ satisfies $R(q) = M(q)$ or belongs to the generation indicator $\dot{P} = \{x \in \mathfrak{S}: P(x) \neq (\mathcal{R}P)(x)\}$ of some $P \in \mathcal{P}$. Then R contains a simple submodule S such that the transporter* Transp(M, S), *i.e., the ideal of $\mathcal{A}$ formed by the radical morphisms $\mu: X \to Y$ satisfying $\mu M(X) \subset S(Y)$, annihilates no $P \in \mathcal{P}$.*

Before presenting the proof of the theorem, we show that it implies Lemma 6.4 given above.

In the notation of 6.4, we proceed by induction on $d = \sum_x \dim R(x)$, where $x \in \bigcup_{P \in \mathcal{P}} \dot{P}$. In the case $d = 0$, we set $\mathcal{J} = \{0\}$ and $\mathcal{K} = \emptyset$. In the case $d > 0$, we apply our theorem, setting $\mathcal{I} = \text{Transp}\,(M, S)$ and $B = N + R$, where N is the annihilator of $\mathcal{I}$ in M. Considering $\overline{M} = M / S = M / \mathcal{I}M$ as a module over $\overline{\mathcal{A}} = \mathcal{A} / \mathcal{I}$, we then obtain an epivalence $M_B^k \to \overline{M}_{B/S}^k$ (4.2.b). Applying the induction hypothesis to $\overline{M}$ and $\overline{\mathcal{P}} = \{P / S: P \in \mathcal{P}\}$, we get an ideal $\overline{\mathcal{J}}$ of $\overline{\mathcal{A}}$ and a bond $\overline{\mathcal{K}}$ on $\overline{M}$ which satisfy the statements of the lemma mutatis mutandis. For $\mathcal{J}$, it then suffices to choose the inverse image of $\overline{\mathcal{J}}$ in $\mathcal{A}$ for $\mathcal{K}$, the set formed by B and by the inverse images of the submodules in $\overline{\mathcal{K}}$.

7.2. ***Beginning of the proof of Theorem* 7.1.** The proof occupies the whole Section 7. We are really interested in the case $q \in \dot{P}$; the alternative $R(q) = M(q)$ only serves our inductive argument.

If $\mathcal{P}$ has cardinality $|\mathcal{P}| = 1$, we apply Lemma 4.3 to P and use the fact that $P(x) = M(x)$ for all $x \in \dot{P}$ (5.4). Hence, we may suppose that $|\mathcal{P}| \geq 2$ and proceed by induction on $|\mathcal{P}|$. We set $\dot{\mathcal{P}} = \bigcup_{P \in \mathcal{P}} \dot{P}$ and call a point $s \in \dot{P}$ *double* if $s = d_P$ for some $P \in \mathcal{P}$; otherwise, s is called *ordinary*.

Lemma. *For each $p \in \mathcal{P}$ and each $x \in \dot{P}$, we have $R(x) = (\mathcal{R}P)(x)$. Accordingly, $R(x)$ has codimension 1 in $M(x)$ if x is ordinary and codimension 2 if $x = d_P$.*

Proof. Consider any $Q \in \mathcal{P} \setminus P$. If $x \in \dot{Q}$, x is ordinary (5.8), and we have $(\mathcal{R}Q)(x) = (\mathcal{R}P)(x)$ by 5.9. If $x \notin \dot{Q}$, we have $(\mathcal{R}Q)(x) = Q(x)$; on the other hand, the restriction $Q \,|\, \dot{P}$ is a maximal submodule of $P \,|\, \dot{P}$ (5.7); it follows that $Q \,|\, \dot{P} \supset \mathcal{R}(P \,|\, \dot{P}) = \mathcal{R}(P) \,|\, \dot{P}$, hence $Q(x) \supset (\mathcal{R}P)(x)$. Accordingly, $(\mathcal{R}Q)(x)$ contains $(\mathcal{R}P)(x)$ in all cases.

7.3. ***First reduction.*** Let $\mathcal{T}$ denote the full subspectroid of $\mathfrak{S}$ formed by $\dot{\mathcal{P}}$ and by the points $x \in \mathfrak{S}$ such that $R(x) = M(x)$. Let further $n \in \mathbb{N}$ be such that $\mathcal{R}_{\mathfrak{S}}^{n+1}$ annihilates all $R(x)$, $x \in \mathcal{T}$, whereas $\mathcal{R}_{\mathfrak{S}}^n(t, s)R(t) \neq 0$ for some $t \in \mathcal{T}$ and some $s \in \mathfrak{S}$. Denoting by R' the annihilator of $\mathcal{R}_{\mathfrak{S}}^n$ in R, we replace M by M / R', $\mathcal{L}$ by $\mathcal{L} / R' = \{L / R': R' \subset L \in \mathcal{L}\}$, and $\mathcal{P}$ by $P / R' = \{\mathcal{P} / R': P \in \mathcal{P}\}$.

We claim that our theorem is true if it holds for M / R', $\mathcal{L} / R'$, and $\mathcal{P} / R'$. Indeed, let N / R' be a simple submodule of R / R' such that the transporter $\mathcal{J}$ of M / R' into N / R' annihilates no P / R', $P \in \mathcal{P}$. If N / R' is located at $x \in \mathfrak{S}$, there is a morphism $\mu \in \mathcal{R}^n_{\mathfrak{S}}(x, y)$ and a simple submodule S of M such that $S(y) = \mu N(x) \neq 0$. Our claim then follows from the observation that the ideal $\mathcal{I}$ such that $\mathcal{I}(z, y) = \mu \mathcal{J}(z, x)$ and $\mathcal{J}(z, t) = 0$ in the case $t \neq y$ is contained in Transp (M, S) and annihilates no $P \in \mathcal{P}$.

Thus, we are reduced to the case where $\mathcal{R}_{\mathfrak{S}}$ annihilates all $R(t)$, $t \in \mathcal{T}$, and $R(q) \neq 0$ for some $q \in \mathcal{T}$. Restricting M to the full subspectroid of $\mathfrak{S}$ formed by $\dot{\mathcal{P}}$ and q, we are further reduced to the case where R is semisimple. Factoring out the submodule R' of R such that $R'(q) = 0$ and $R'(t) = R(t)$ if $t \neq q$, *we are finally reduced to the following situation, to which we restrict ourselves in the sequel: R is a semisimple module vanishing outside some point $q \in \mathfrak{S}$; the set of points of $\mathfrak{S}$ is $\dot{\mathcal{P}} \cup \{q\}$; and, finally, $M(q) = (RM)(q) = R(q)$ if $q \notin \dot{\mathcal{P}}$.*

7.4. ***Second reduction and dichotomy of the proof.*** Suppose that there is an ordinary point $s \in \dot{\mathcal{P}}$ such that $P(s) = M(s)$ for all $P \in \mathcal{P}$ and $\mathcal{R}_{\mathfrak{S}}(s, q)M(s) \neq 0$. Then we have

$$\mathcal{R}_{\mathfrak{S}}(s, q)M(s) \subset \bigcap\nolimits_{P \in \mathcal{P}} (\mathcal{R}P)(q) = R(q),$$

and each $\mu \in \mathcal{R}_{\mathfrak{S}}(s, q)$ satisfying $\mu M(s) \neq 0$ determines a simple submodule S of R such that $S(q) = \mu M(s)$ (7.2). Since Transp (M, S) contains μ, it annihilates no $P \in \mathcal{P}$.

Thus, we are reduced to the case considered in the sequel, where $\mathcal{R}_{\mathfrak{S}}(s, q)M(s) = 0$ for each ordinary $s \in \dot{\mathcal{P}}$ such that $P(s) = M(s)$, $\forall\, P \in \mathcal{P}$.

From now on, we fix a pencil $F \in \mathcal{P}$ subjected to the sole condition that $q \in \dot{F}$ if $q \in \dot{\mathcal{P}}$. Since we have $M \neq F$ and $M(t) = F(t)$ for all $t \in \dot{F}$ (5.4), the generation indicator $\dot{M}$ of M is not contained in $\dot{F}$. Thus $\dot{M} \setminus \dot{F}$ contains a double or an ordinary point. The two cases are examined separately in 7.5 and 7.6 below.

7.5. ***First half:*** *Suppose that $\dot{M} \setminus \dot{F}$ contains the double point $d = d_Y$ of some $Y \in \mathcal{P}$.*

Let us then examine any $X \in \mathcal{P}$ different from Y. Since $d \notin \dot{X}$ (5.8), we have $X(d) = (\mathcal{R}X)(d) \subset (\mathcal{R}M)(d) \neq M(d) = Y(d)$. Since the restriction $X \cap Y | \dot{Y}$ is a maximal submodule of $Y \mid \dot{Y}$ (5.7), $X(d) = (\mathcal{R}M)(d)$ is a hyperplane of $M(d)$ containing $(\mathcal{R}Y)(d) = R(d)$. Thus, we can choose vectors $u \in M(d) \setminus X(d)$ and $v \in X(d) \setminus R(d)$ such that $M(d) = ku \oplus kv \oplus R(d)$ and $R(q) \subset (\mathcal{R}Y)(q) = \mathcal{R}_{\mathfrak{S}}(d, q)u + \sum_s \mathcal{R}_{\mathfrak{S}}(s, q)Y(s)$, where s runs through the ordinary points of $\dot{Y}$ (5.1).

If $X_1 \in \mathcal{P}$ differs from Y and X, we have $X_1(s) = M(s) = X(s)$ for all ordinary $s \in \dot{Y}$. Using 7.4, we infer that $\mathcal{R}_{\mathfrak{S}}(s, q)Y(s) = 0$ and $(\mathcal{R}Y)(q) = \mathcal{R}_{\mathfrak{S}}(d, q)u$. On the other hand, we have $\mathcal{R}_{\mathfrak{S}}(d, q)v \subset R(q)$ because v belongs to $Y(d) = M(d)$ and to all $X_1(d) = (\mathcal{R}M)(d) = X(d)$.

Now set $E = \{\mu \in \mathcal{R}_{\mathfrak{S}}(d, q) : \mu u \in R(q)\}$. Since $\mathcal{R}_{\mathfrak{S}}(d, q)u = (\mathcal{R}Y)(q)$ contains $R(q)$, the multiplication by u provides a surjection $?u \colon E \to R(q)$. This implies that the representation $?u, ?v \colon E \rightrightarrows R(q)$ of the double arrow is a direct sum of tubular and preinjective indecomposables. We distinguish two cases:

a) Case $?v \neq 0$. Our representation then admits an indecomposable summand which is isomorphic neither to 1, $0 \colon k \rightrightarrows k$ nor to $0, 0, k \rightrightarrows 0$. Such a summand contains vectors $\mu, \nu \in E$ satisfying $0 \neq \mu u = \nu v =: r$ and $\mu v \in kr$. Accordingly, if $S \subset R$ is the simple module such that $S(q) = kr$, μ belongs to Transp (M, S), and Transp (M, S) does not annihilate Y. On the other hand, each $X \in \mathcal{P} \setminus Y$ satisfies some relation $v \in \varphi w + R(d)$, where $w \in X(s)$, $s \in \dot{X}$, and $\varphi \in \mathcal{R}_{\mathfrak{S}}(s, d)$. From $\nu\varphi M(s) \subset \nu X(d) = k\nu v$ and $\nu\varphi w = \nu v = r$ we infer that Transp (M, S) contains $\nu\varphi$ and does not annihilate X.

b) Case $?v = 0$. Then we apply our induction hypothesis to $\mathcal{P} \setminus Y$. Since q satisfies $R(q) = M(q)$ or $q \in \dot{F}$, where $F \in \mathcal{P} \setminus Y$, we infer that R contains a simple submodule S located at q and such that Transp (M, S) annihilates no $X \in \mathcal{P} \setminus Y$. On the other hand, since $S(q) \subset R(q) \subset \mathcal{R}_{\mathfrak{S}}(d, q)u$, there exists a $\varphi \in \mathcal{R}_{\mathfrak{S}}(d, q)$ such that $\varphi v = 0 \neq \varphi u \in S(q)$; thus, Transp (M, S) also contains φ and does not vanish on Y.

7.6. ***Second half:*** *Suppose that* $\dot{M} \setminus \dot{F}$ *contains an ordinary point* y.

Our premise implies the existence of pencils $X, Y \in \mathcal{P}$ such that $y \notin \dot{X}$ and $y \in \dot{Y}$; hence, $X(y) = (\mathcal{R}X)(y) \subset (\mathcal{R}M)(y) \neq M(y) = Y(y)$. By 5.7 there is a unique point $x_X = x \in \dot{X}$ such that $Y(x) \neq M(x) = X(x)$; by 5.10 x_X depends only on X and y, but not on Y.

Let us now examine the points $z \in \dot{Y} \setminus y$ *such that* $\mathcal{R}_{\mathfrak{s}}(z, q)M(z) \neq 0$. By 5.7, z satisfies $X(z) = M(z) = Y(z)$; by 7.4 z is the double point d_Y of Y or satisfies $Y_1(z) \neq Y(z)$ for some $Y_1 \in \mathcal{P}$, whose indicator $\dot{Y}_1$ runs through y (5.7). In both cases, $z \notin \dot{X}$. This follows from 5.8 if $z = d_Y$, from $Y_1(z) \neq M(z)$, $Y_1(x) \neq M(x)$, and 5.7 if not. *We conclude that*

$$M(z) = (\mathcal{R}X)(z) = \sum_{t \in \dot{X}} \mathcal{R}_{\mathfrak{s}}(t, z)X(t) = \mathcal{R}_{\mathfrak{s}}(x, z)X(x) = \mathcal{R}_{\mathfrak{s}}(x, z)n \tag{$*$}$$

for all $n \in X(x) \setminus Y(x)$. The last equalities result from the fact that each $t \in \dot{X} \setminus x$ satisfies $X(t) = Y(t)$ (5.7); hence we have $\mathcal{R}_{\mathfrak{s}}(t, z)X(t) \subset \mathcal{R}(Y)(z) = R(z)$ (7.2) and $\mathcal{R}_{\mathfrak{s}}(x, z)Y(x) \subset R(z)$; but $y \notin \dot{F}$ implies $z \notin \dot{F}$ (as we have seen above in the case of X); hence $z \neq q$ and $R(z) = 0$.

When Y varies, the points $z \in \dot{Y}$ considered above give rise to a subset of $\dot{\mathcal{P}}$, which we denote by Z. *The contribution*

$$R^Z = \sum_{z \in Z} \mathcal{R}_{\mathfrak{s}}(z, q)M(z)$$

of Z *to* $M(q)$ *is contained in* $R(q)$. Indeed, this is clear if $R(q) = M(q)$ and follows from

$$R^Z = \sum_{z \in Z} \mathcal{R}_{\mathfrak{s}}(z, q)\mathcal{R}_{\mathfrak{s}}(x_F, z)F(x_F) \subset (\mathcal{R}F)(q) = R(q)$$

if $q \in \dot{F}$ (Lemma 7.3). On the other hand, we have $R(q) \subset R^Z + \mathcal{R}_{\mathfrak{s}}(y, q)M(y)$ because each Y satisfies

$$R(q) \subset (\mathcal{R}Y)(q) = \sum_{z \in \dot{Y}} \mathcal{R}_{\mathfrak{s}}(s, q)M(s) = \mathcal{R}_{\mathfrak{s}}(y, q)M(y) + \sum_{z \in Z \cap \dot{Y}} \mathcal{R}_{\mathfrak{s}}(z, q)M(z).$$

Thus, we are led to distinguish the following three cases:

a) Case $R^Z + \mathcal{R}_{\mathfrak{s}}(y, q)M(y) \neq 0$. The nonzero intersection then contains some

$$r = \sum_{z \in Z} \varphi_z m_z = \varphi_y m_y \neq 0,$$

where $\varphi_s \in \mathcal{R}_{\mathfrak{s}}(s, q)$ and $m_s \in M(s)$. If $S \subset R$ denotes the simple module such that $S(q) = kr$, φ_Y clearly belongs to Transp (M, S). On the other hand, for each $X \in \mathcal{P}$ satisfying $y \notin \dot{X}$ and each $z \in Z \cap \dot{Y}$, m_z can be written as $m_z = \psi_z n$ with $\psi_z \in \mathcal{R}_{\mathfrak{s}}(x_X, z)$, where $n \in M(x_X) \setminus \bigcup_Y Y(x_X)$ (see ($*$) above). We infer that $r = \varphi_x n$, where $\varphi_x = \sum_{z \in Z} \varphi_z \psi_z$ vanishes on $Y(x_X)$ together with ψ_z, hence has rank 1 and belongs to Transp (M, S).

b) Case $R^Z = 0$, i.e., $Z = \emptyset$. In this case, we have

$$R(q) \subset (\mathcal{R}Y)(q) = \mathcal{R}_{\mathfrak{s}}(y, q)M(y)$$

for all $Y \in \mathcal{P}$ such that $y \in \dot{Y}$. Removing these Y from $\mathcal{P}$, we obtain a set $\mathcal{P}'$ of smaller cardinality which contains F and satisfies the assumptions of Theorem 7.1 because $R(q) \neq M(q)$ implies $q \in \dot{F}$. The induction hypothesis then guarantees the existence of a simple submodule S of R such that Transp (M, S) annihilates no $X \in \mathcal{P}'$, and no $Y \in \mathcal{P}\setminus\mathcal{P}'$ because of $0 \neq S(q) \subset R(q) \subset \mathcal{R}_{\mathfrak{S}}(y, q)M(y)$, $\mathcal{R}_{\mathfrak{S}}(y, q)R(y) = 0$, and $\dim M(y)/R(y) = 1$.

c) Case $R^Z \neq 0$ and $R^Z \cap \mathcal{R}_{\mathfrak{S}}(y, q)R(y) = 0$. Then we set $\mathcal{P}' = \{Y \in \mathcal{P}: y \in \dot{Y}\}$, and accordingly, $\dot{\mathcal{P}}' = \bigcup_{Y \in \mathcal{P}'} \dot{Y}$. We denote by $\mathfrak{S}'$ the full subspectroid of $\mathfrak{S}$ supported by $\{q\} \cup \dot{\mathcal{P}}'$, by $\mathcal{A}'$ the corresponding full subaggregate of $\mathcal{A}$. We finally set $Y' = Y \mid \mathcal{A}'$ for each $Y \in \mathcal{P}'$, $M' = \sum_{Y \in \mathcal{P}'} Y'$ and $R' = \bigcap_{Y \in \mathcal{P}'} \mathcal{R}Y'$. Thus we have $R'(s) = 0$ if $s \in \dot{\mathcal{P}}' \setminus q$ and

$$R'(q) = R^Z \oplus \mathcal{R}_{\mathfrak{S}}(y, q)M(y) = (\mathcal{R}M')(q);$$

in particular, $R'(q) = M'(q)$ holds if $(\mathcal{R}M')(q) = M'(q)$, hence if $q \notin \dot{\mathcal{P}}'$. It follows that M' and $\mathcal{P}' \mid \mathcal{A}' = \{Y': Y \in \mathcal{P}'\}$ satisfy the assumptions of Theorem 7.1. (But we may, of course, have $q \notin \dot{\mathcal{P}}'$ even if $q \in \mathcal{P}'$. Here is precisely the point where the alternative $R(q) = M(q)$ of Theorem 7.1 enters the inductive argument.)

The assumptions of 7.1 pass from M' and $\mathcal{P}' \mid \mathcal{A}'$ to $M'' = M'/N$ and $\mathcal{P}' = \{Y'/N: Y \in \mathcal{P}'\}$, where N denotes the submodule of R' such that $N(q) = \mathcal{R}_{\mathfrak{S}}(y, q)M(y)$; we then have

$$R'' := \bigcap_{T \in \mathcal{P}} \mathcal{R}T = R'/N.$$

Applying our induction hypothesis to M'' and $\mathcal{P}''$, we find a simple submodule S'' of R'' such that Transp (M'', S'') annihilates no $T = Y'/N$. Since $R^Z \xrightarrow{\sim} R''(q)$, S'' can be "lifted" to a simple submodule S' of R' such that $S'(q) \subset R^Z$. Extending S' by 0 to $\mathcal{A}$, we finally obtain the required $S \subset R$. Indeed, the construction implies that each $Y \in \mathcal{P}'$ contains a point $z \in Z \cap \dot{Y}$ such that $M(z)$ is not annihilated by Transp (M, S). Since z satisfies $M(z) = \mathcal{R}_{\mathfrak{S}}(x_X, z)M(x_X)$ for each $X \in \mathcal{P}\setminus\mathcal{P}'$, Transp (M, S) does not annihilate X either.

8. The Case of a Semisimple Pencil.

Our main objective in this section is to prove Lemma 6.6 above.

Sticking to our previous notation and assumptions, we further suppose throughout Sections 8.1, 8.2, and 8.4–8.10 that M is a *faithful* module over $\mathcal{A}$ and P a *semisimple* $\mathcal{L}$-pencil. This implies that P *is the socle of* M (5.3) and that the points $x \in \mathfrak{S}$ satisfy either $0 \neq P(x) = M(x)$ or $P(x) = 0 \neq M(x)$ (5.4). In the case $0 \neq P(x)$, we keep the basis chosen in 6.3, setting $M(x) = ku_x$ if x is an ordinary point of $\dot{P}$ and $M(d) = ku \oplus kv$ if $d = d_P$ is the double point. Finally, we set $\mathcal{K} = \{L \in \mathcal{L}: L(d) = M(d)\}$.

To help intuition, we may and shall choose $\mathcal{A}$ as the aggregate of all finite-dimensional projective modules over some finite-dimensional algebra. Accordingly, if $\mathcal{A}_p$ denotes the full subaggregate of $\mathcal{A}$ formed by the objects isomorphic to p^n, where $p \in \dot{P}$ is fixed and n ranges over $\mathbb{N}$, the inclusion $\mathcal{A}_p \to \mathcal{A}$ admits a *canonical* right adjoint which maps $X \in \mathcal{A}$ onto the largest submodule X_p belonging to $\mathcal{A}_p$; moreover, if p is an ordinary point of $\dot{P}$ and $Y \in \mathcal{A}_p$, each vector subspace of $M(Y)$ is identified with $M(Z)$ for some submodule $Z \in \mathcal{A}_p$ of Y.

8.1. We first apply our *main algorithm* to the submodule P of M and to the bond $\mathcal{K}$ defined above. As usual, we set $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$, $\hat{L}(W, h, Z) = (L(Z) + h(W))/h(W)$ for all submodules $L \subset M$ and all $(W, h, Z) \in \hat{\mathcal{A}}$, and $\hat{\mathcal{K}} = \{\hat{L}: L \in \mathcal{K}\} \cup \{\hat{P}\}$. The canonical epivalence $M^k_{\mathcal{K}} \to \overline{M}^k_{\hat{\mathcal{K}}}$ (5.4) then reduces the investigation of $M^k_{\mathcal{K}}$ to $\overline{M}^k_{\hat{\mathcal{K}}}$, and we are lead to examine $\hat{\mathcal{A}}$.

The relevant part of $\mathcal{K} \cap P$ consists of the maximal submodules P_s, where $s \in \dot{P} \setminus d$ (5.2). In order to choose a spectroid of $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$, we consider a pair of adjoint functors

$$(P \mid \mathcal{A}_d)^k \underset{S}{\overset{R}{\rightleftarrows}} P^k.$$

The right adjoint R is defined by $R(V, g, Y) = (V, g_d, Y_d)$, where g_d is the d-component of $g: V \to P(Y) = \bigoplus_{p \in \dot{P}} P(Y_p)$. The left adjoint is such that $S(W, h, Z) = (W, \bar{h}, Z \oplus W \otimes \Sigma)$, where $\Sigma = \oplus\, s \in \mathcal{A}$ is the sum of all $s \in \dot{P} \setminus d$ and $\bar{h}$ maps $x \in W$ onto

$$(h(x), (x \otimes u_s)) \in P(Z) \oplus (\bigoplus_s W \otimes P(s)).$$

This left adjoint factors through $P^k_{\mathcal{K} \cap P}$ and is fully faithful and exact (for the short exact sequences considered in 2.3). Accordingly, the indecomposables $\Lambda_n, T^\lambda_n, V_n$ of $(P \mid \mathcal{A}_d)^k$ are associated with pairwise nonisomorphic indecomposables of $P^k_{\mathcal{K} \cap P}$ of the following form:

$$S\Lambda_n = (k^{n-1}, a_n, d^n \oplus \Sigma^{n-1}), \quad a_{nd} = [\mathbb{1}_{n-1} 0 \mid 0\, \mathbb{1}_{n-1}]^{\mathrm{T}},$$

$$ST^\lambda_n = (k^n, t^\lambda_n, d^n \oplus \Sigma^n), \quad t^\lambda_{nd} = [\mathbb{1}_n \mid \lambda \mathbb{1}_n + J_n]^{\mathrm{T}},$$

$$ST^\infty_n = (k^n, t^\infty_n, d^n \oplus \Sigma^n), \quad t^\infty_{nd} = [J_n \mid \mathbb{1}_n]^{\mathrm{T}},$$

$$SV_n = (k^n, z_n, d^{n-1} \oplus \Sigma^n), \quad z_{nd} = \left[\begin{array}{c} \mathbb{1}_{n-1}\ 0 \\ \hline 0\ \mathbb{1}_{n-1} \end{array}\right].$$

The scalar λ ranges over k, n is ≥ 1, J_n is a nilpotent Jordan block, $a_{nd}: k^{n-1} \to P(d^n)$ is the component of a_n relative to $d, \ldots$.

As a spectroid $\hat{\mathfrak{A}}$ of $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$ we choose the indecomposables $S\Lambda_n, ST^\lambda_n, SV_n$ $(n \geq 1, \lambda \in k \cup \infty)$ and the P-spaces $(0, 0, x), x \in \mathfrak{A} \setminus d$.

Proposition. *There are at most four "scalars" $\lambda \in k \cup \infty$ such that ST^λ_n is $(\hat{M}, \hat{\mathcal{K}})$-relevant* (6.7) *for some $n \geq 1$.*

Sections 8.4 – 8.9 are devoted to the proof of the proposition. First, we shall show that the proposition implies Lemma 6.6 above.

8.2. Proposition 8.1 deals with a lopped bond $\mathcal{K}$ on M, not with the given $\mathcal{L}$. So it remains for us to adapt the arguments of 8.1 to $\mathcal{L}$. First, we must replace $\hat{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$ by a full subaggregate $\tilde{\mathcal{A}} = P^k_{\mathcal{K} \cap P}$. The corresponding spectroid $\tilde{\mathfrak{A}}$ is obtained from $\hat{\mathfrak{A}}$ by deletion of some SV_n and some ST^λ_n. For each submodule L of M, the $\hat{\mathcal{A}}$-module $\hat{L}$ is then replaced by its restriction $\tilde{L} = \hat{L} \mid \tilde{\mathcal{A}}$, and $\tilde{M}$ is restrained by $\tilde{\mathcal{L}} = \{\tilde{L}: L \in \mathcal{L}\} \cup \{\tilde{P}\}$. The resulting aggregate $\tilde{M}^k_{\tilde{\mathcal{L}}}$ is identified with a full subaggregate of $\hat{M}^k_{\hat{\mathcal{K}}}$. Thus we finally obtain the following corollary of Proposition 8.1.

Proposition. *With the preceding notation, there are at most four scalars $\lambda \in k \cup \infty$ such that ST^λ_n is $(\tilde{M}, \tilde{\mathcal{L}})$-relevant for some $n \geq 1$.*

8.3. ***Proof of Lemma* 6.6.** The lemma follows directly from Proposition 8.2 when M is faithful and $N = P$ semisimple. Our objective here is to reduce the general case to the particular one. If $N \in \mathcal{P}_e$ with $e \geq 2$, we first replace $\mathcal{L}$ by $\mathcal{L}_{e-1}$ (6.3) and are thus reduced to the case of a minimal pencil $N \in \mathcal{P}_1$. We may also replace $\mathcal{L}$ by $\mathcal{L} \cup \mathcal{K} \cup \bigcup_{P \in \mathcal{P}_1} \mathcal{K}'_P$, hence, suppose that $O_0 = \emptyset$ (6.5). Our further reduction consists of three steps.

First Step. Here we factor out the ideal $\mathcal{J}$ of 6.4, replacing $\mathcal{A}$ by $\overline{\mathcal{A}} = \mathcal{A} / \mathcal{J}, M$ by $\overline{M} = M / \mathcal{J}M$, and N by $\overline{N} = N / \mathcal{J}M$. The bond $\mathcal{B}N$ is replaced by the set of all $X / \mathcal{J}M$ such that $\mathcal{J}M \subset X \in \mathcal{B}N$. This set equals $\mathcal{B}\overline{N}$ if $\mathcal{L}$ is replaced by the corresponding bond on $\overline{M}$. Applying the main algorithm to the submodules N and $\overline{N}$ of M and $\overline{M}$, we obtain the diagram

$$\begin{array}{ccc} M^k_{\mathcal{B}N} & \xrightarrow{F} & \overline{M}^k_{\mathcal{B}\overline{N}} \\ \downarrow & & \downarrow \\ M^{Nk}_{\hat{\mathcal{B}}N} & \xrightarrow{G} & \overline{M}^{\overline{N}k}_{\hat{\mathcal{B}}\overline{N}} \end{array}$$

Since some $Y \in \mathcal{B}N$ give no contribution to $\mathcal{B}\overline{N}$, it is possible that F is not an epivalence. But it is the restriction of an epivalence to a full subcategory. Hence it is surjective on the morphism spaces and detects isomorphisms. Since the vertical arrows of the diagram are equivalences, G preserves indecomposability and heteromorphism. We infer that $\mathfrak{S}^N$ (6.5) has fewer "relevant points" than $\mathfrak{S}^{\overline{N}}$, and the required statements can be lifted from $\overline{M}$ to M.

Second Step. We suppose that $(\mathcal{R}N)(x) = 0$ for all $x \in \dot{N}$. Under this condition, we now set $\overline{M} = M / \mathcal{R}N$, $\overline{N} = N / \mathcal{R}N$, and equip $\overline{M}$ with the bond formed by all $L / \mathcal{R}N$, where $\mathcal{R}N \subset L \in \mathcal{B}N$. Applying the main algorithm to $N \subset M$ and $\overline{N} \subset \overline{M}$, we obtain modules M^N and $\overline{M}^{\overline{N}}$ over some aggregates with spectroids $\mathfrak{S}^N$ and $\mathfrak{S}^{\overline{N}}$. The induced functor $\mathfrak{S}^N \to \mathfrak{S}^{\overline{N}}$ is an isomorphism because, for each $Z = (W, g, X) \in \mathfrak{S}^N$ with space dimension $\dim W \geq 1$, X is supported by $\dot{N}$ which is disjoint from the support of $\mathcal{R}N$. Accordingly, if $\mathcal{R}(N)^N$ denotes the submodule of M^N associated with $\mathcal{R}N$, we have $(\mathcal{R}N)^N(Z) = 0$, and we may identify $\mathfrak{S}^N$ with $\mathfrak{S}^{\overline{N}}$ and $M^N / (\mathcal{R}N)^N$ with $\overline{M}^{\overline{N}}$. The equality $(\mathcal{R}N)^N(Z) = 0$ implies that, for any M^N-space (U, h, Z'), the canonical map

$$M^{Nk}((U, h, Z'), (0, 0, Z)) \to \overline{M}^{\overline{N}k}((U, h, Z'), (0, 0, Z))$$

is bijective. Therefore, Z is relevant with respect to $(M^N, \hat{\mathcal{B}}N)$ if it is so with respect to $(\overline{M}^{\overline{N}}, \hat{\mathcal{B}}\overline{N})$. Thus we are reduced from M to $\overline{M}$.

Third Step. Here we may suppose that $\mathcal{R}N = 0$. But formally we still have to reduce our statement to the case where M is faithful. For this sake, we denote by $\overline{\mathcal{A}}$ the residue category of $\mathcal{A}$ modulo the annihilator of M. If $\overline{M}$ and $\overline{N}$ are the $\overline{\mathcal{A}}$-modules associated with M and N, the canonical functor $M^{Nk}_{\hat{\mathcal{B}}N} \to \overline{M}^{\overline{N}k}_{\hat{\mathcal{B}}\overline{N}}$ is quasisurjective. Therefore, the isoclasses of "relevant" points of $\mathfrak{S}^N$ correspond bijectively to those of $\mathfrak{S}^{\overline{N}}$.

8.4. We now return to Proposition 8.1. Before entering its proof, we examine the notion of *relevance*. Let us provisionally consider an arbitrary pointwise finite module M' over an aggregate $\mathcal{A}'$ and a bond $\mathcal{L}'$ on M'. Equipped with the short exact sequences defined in 2.3, $M'^k_{\mathcal{L}'}$ is an exact category. Accordingly, an M'-space $(V, f, X) \in M'^k_{\mathcal{L}'}$ is called *$(M', \mathcal{L}')$-injective* if, for each short exact sequence

$$0 \longrightarrow (W', g', Y') \xrightarrow[(i,j)]{} (W, g, Y) \xrightarrow[(p,q)]{} (W'', g'', Y'') \longrightarrow 0$$

formed by M'-spaces avoiding $\mathcal{L}'$, each morphism from (W', g', Y') to (V, f, X) factors through (i, j). It is equivalent to say that, for each $(W, g, Y) \in M'^{k}_{\mathcal{L}'}$, each linear map $m: W \to M(X)/f(V)$ is a composition of the form

$$W \xrightarrow{g} M(Y) \xrightarrow{M(\eta)} M(X) \xrightarrow{\text{can.}} M(X)/f(V).$$

The indecomposable $(M', \emptyset)$-injectives are easy to describe; they have the form $(k, 0, 0)$ or $(M', (s), \mathbb{1}, s)$. The general case $\mathcal{L}' \neq \emptyset$ seems to be more intricate. In the following lemma we examine indecomposables $s \in \mathcal{A}'$ such that $(0, 0, s)$ is $(M', \mathcal{L}')$-injective; then we simply say that s is *$(M', \mathcal{L}')$-injective.*

Lemma. *An indecomposable $s \in \mathcal{A}'$ is $(M', \mathcal{L}')$-irrelevant if and only if s is $(M', \mathcal{L}')$-injective and satisfies $L'(s) = M'(s)$ for each maximal element L' of $\mathcal{L}'$.*

Proof. a) The condition is *sufficient*: If $(V, [fg]^{\mathrm{T}}, Y \oplus s)$ avoids $\mathcal{L}'$, the equalities $L'(s) = M'(s)$ considered above imply that $(V, f, X) \in M^{k}_{\mathcal{L}'}$. Hence, we have a short exact sequence

$$0 \longrightarrow (0, 0, s) \longrightarrow (V, [fg]^{\mathrm{T}}, Y \oplus s) \longrightarrow (V, f, Y) \longrightarrow 0$$

of $M'^{k}_{\mathcal{L}'}$, which splits because s is $(M', \mathcal{L}')$-injective.

b) The condition is *necessary*. In order to show that s is $(M', \mathcal{L}')$-injective, it suffices to prove that the exact sequence

$$0 \longrightarrow (0, 0, s) \xrightarrow[(0,\,[0\,\mathbb{1}]^{\mathrm{T}})]{} (V, [fg]^{\mathrm{T}}, Y \oplus s) \xrightarrow[(\mathbb{1},\,[\mathbb{1}\,0]^{\mathrm{T}})]{} (V, f, Y) \longrightarrow 0$$

splits if (V, f, Y) is indecomposable. But this is clear if $(V, f, Y) \xrightarrow{\sim} (0, 0, s)$. If not, Y has no direct summand isomorphic to s. Decomposing the middle term into indecomposables, we obtain an isomorphism

$$(V, [fg]^{\mathrm{T}}, Y \oplus s) \xrightarrow[i]{\sim} (V, h, Y) \oplus (0, 0, s)$$

whose components are, say $(e, [a\,b])$ and $(0, [c\,d])$. The composition of i with $(0, [0\,\mathbb{1}]^{\mathrm{T}})$ is a section with components $(0, b)$ and $(0, d)$. Since b cannot be a section, d is an isomorphism, and our short exact sequence splits.

Let us now turn to a maximal $L' \in \mathcal{L}'$. In the case $L'(s) \neq M'(s)$, we consider the submodule N' of M' which is generated by L' and $M'(s)$. Since the generation indicator of N' contains s, the indecomposable M'-space associated with N' in 6.1 has the form $(k, f, Y \oplus s)$ and avoids $\mathcal{L}'$. This contradicts our assumptions that s is $(M', \mathcal{L}')$-irrelevant.

8.5. We now return to the *assumptions of Proposition* 8.1 and start with the proof. By 5.6, each $L \in \mathcal{K}$ satisfies $L \cap P = P_s$ for some ordinary point $s \in \dot{P}$. It easily follows that $\hat{K}(ST_n^{\lambda}) = \hat{P}(ST_n^{\lambda}) = \hat{M}(ST_n^{\lambda})$ holds for each $\hat{K} \in \hat{\mathcal{K}}$. Hence, ST_n^{λ} is $(\hat{M}, \hat{\mathcal{K}})$-relevant if and only if it is not $(\hat{M}, \hat{\mathcal{K}})$-injective.

Thus, our objective is to show that $\operatorname{Ext}(X, (0, 0, ST_n^{\lambda})) = 0$ *for all* $X \in \hat{M}^{k}_{\hat{\mathcal{K}}}$, *provided* λ *avoids some finite set* e. The extension groups $\operatorname{Ext}(X, (W, h, Z))$ considered here can be computed within the surrounding category $\hat{M}^{k}$ with the help of an injective resolution of (W, h, Z) in $\hat{M}^{k}$ of the following form:

$$0 \longrightarrow (W, h, Z) \longrightarrow (\operatorname{Ker} h, 0, 0) \oplus (\hat{M}(Z), \mathbb{1}, Z) \longrightarrow (\operatorname{Coker} h, 0, 0) \longrightarrow 0.$$

The resolution shows that Ext is right exact on the short exact sequences of $\hat{M}^{k}$ considered here (2.3).

We display the spectroid $\hat{\mathfrak{S}}$ of $\hat{\mathcal{A}}$ (8.1) in such a way that all morphisms from the right to the left vanish ($s \in \hat{\mathfrak{S}} \setminus d$, $\lambda \in k \cup \infty$):

$$(0, 0, s), S\Lambda_1, S\Lambda_2, S\Lambda_3, \ldots, ST_n^\lambda, \ldots, SV_3, SV_2, SV_1.$$

In particular, $\operatorname{Hom}(SF, (0, 0, s)) = 0$ for all $s \in \hat{\mathfrak{S}} \setminus d$ and all $F \in (P \mid \mathcal{A}_d)^k$. It follows that each $A \in \hat{\mathcal{A}}$ gives rise to a canonical split sequence

$$0 \longrightarrow A_P \xrightarrow[\iota]{} A \xrightarrow[\pi]{} A / A_P \longrightarrow 0,$$

where A_P is isomorphic to some SF, and A/A_P to some $\bigoplus_{i \in I} (0, 0, s_i)$ with $s_i \in \hat{\mathfrak{S}} \setminus d$. Accordingly, each $(U, f, A) \in \hat{M}^k$ gives rise to an *exact sequence*

$$0 \longrightarrow (0, 0, A_P) \xrightarrow[(0,\iota)]{} (U, f, A) \xrightarrow[(\mathbb{1},\pi)]{} (U,\ \operatorname{can} \circ f, A / A_P) \longrightarrow 0 \qquad (*)$$

of $\hat{M}^k$. In the case $(U, f, A) \in \hat{M}^k_{\hat{\mathcal{K}}}$, the end terms $(0, 0, A_P)$ and $(U,\ \operatorname{can} \circ f, A/A_P)$ also belong to $\hat{M}^k_{\hat{\mathcal{K}}}$ because $\hat{L}(SF) = \hat{M}(SF)$, $\forall\ L \in \mathcal{K}$, $\forall\ F \in (P \mid \mathcal{A}_d)^k$. We shall denote by $\hat{M}_1^k$ and $\hat{M}_2^k$ the *full subaggregates* of $\hat{M}^k_{\hat{\mathcal{K}}}$ formed by the (U, f, A) such that $A_P = A$ and $A_P = 0$, respectively.

Now, since we have $\operatorname{Ext}((0, 0, A_P), (0, 0, ST_n^\lambda)) = 0$ by the definition of the exact sequences of $\hat{M}^k$, we infer that the map

$$\operatorname{Ext}((U, f, A), (0, 0, ST_n^\lambda)) \leftarrow \operatorname{Ext}((U,\ \operatorname{can} \circ f, A/A_P), (0, 0, ST_n^\lambda))$$

is surjective, and we are reduced to proving the following lemma.

Lemma. *If M is not $\mathcal{L}$-wild, there exists a subset $e \subset k \cup \infty$ of cardinality ≤ 4 such that* $\operatorname{Ext}(X, (0, 0, ST_n^\lambda)) = 0$ *for all $X \in \hat{M}_2^k$,* all $n \geq 1$, *and all $\lambda \in (k \cup \infty) \setminus e$.*

8.6. Lemma 8.5 concerns the aggregate $\hat{M}^k_{\hat{\mathcal{K}}}$. Our next step brings us back to $M^k_{\mathcal{K}}$ via the rum functor

$$\Phi : \hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}},\ (U, f, (W, h, Z)) \mapsto (V, g, Z) \oplus (\operatorname{Ker} h, 0, 0),$$

where $V \subset M(Z)$ is the inverse image of $f(U) \subset M(Z)/h(W)$ and g the inclusion. This functor induces a bijection between the sets of isoclasses of $\hat{M}^k_{\hat{\mathcal{K}}}$ and $M^k_{\mathcal{K}}$. It is a quasiinverse of the classical equivalence $M^k_{\mathcal{K}} \to \hat{M}^k_{\hat{\mathcal{K}}}$ if $\mathcal{K} \neq \varnothing$, i.e., if $\dot{P} \setminus d \neq \varnothing$. In general, the main virtue of Φ is to be exact, whereas $\hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}}$ is not because $M^k_{\mathcal{K}}$ has "more" exact sequences than $\hat{M}^k_{\hat{\mathcal{K}}}$. In fact, for all $A_1, A_2 \in \hat{M}^k_{\hat{\mathcal{K}}}$, Φ induces an *injection*

$$\operatorname{Ext}(A_2, A_1) \to \operatorname{Ext}(\Phi A_2, \Phi A_1),$$

whose image consists of all classes of short exact sequences

$$0 \to \Phi A_1 = (V_1, g_1, Z_1) \to (V_3, g_3, Z_3) \to \Phi A_2 = (V_2, g_2, Z_2) \to 0$$

of $M^k_{\mathcal{K}}$ such that the induced sequence

$$0 \to (g_1^{-1}(PZ_1), g_1', Z_1) \to (g_3^{-1}(PZ_3), g_3', Z_3) \to (g_2^{-1}(PZ_2), g_2', Z_2) \to 0$$

is split exact in $\hat{\mathcal{A}} = P^k_{\mathcal{K}\cap P}$. Such exact sequences of $M^k_{\mathcal{K}}$ will be called *P-exact.*

In particular, if (U, f, A) ranges over $\hat{M}^k_{\hat{\mathcal{K}}}$, the images of the sequences (*) under Φ are short exact sequences of $M^k_{\mathcal{K}}$. Up to isomorphism, they can be described directly as follows. Let us consider the two pairs of adjoint functors

$$(P \mid \mathcal{A}_d)^k \underset{R}{\overset{S}{\rightleftarrows}} P^k_{\mathcal{K}\cap P} \underset{R'}{\overset{S'}{\rightleftarrows}} M^k_{\mathcal{K}},$$

where R, S are defined as in 8.1, S' is the functor $(W, h, Z) \mapsto (W, h, Z)$ induced by the inclusion $P \to M$, and R' is the *trace functor* $(V, g, Y) \mapsto (g^{-1}(PY), g', Y)$ already considered above. With each $(V, g, Y) \in M^k_{\mathcal{K}}$, the adjoint pair $(RR', S'S)$ associates a *canonical short exact sequence*

$$0 \to (g^{-1}(PY), \bar{g}_d, Y') \xrightarrow{(\nu,\iota)} (V, g, Y) \xrightarrow{(\varphi,\pi)} (V/g^{-1}(PY), g'', Y/Y') \to 0, \qquad (**)$$

of $M^k_{\mathcal{K}}$, where $Y' = Y_d \oplus g^{-1}(PY) \otimes \Sigma$. These sequences are related to the short exact sequences (*) of 8.5 via the rum Φ. If we denote by M_1^k and M_2^k *the full subaggregates* of $M^k_{\mathcal{K}}$ formed by the pairs (V, g, Y), which induce isomorphisms (ν, ι) and (φ, π) respectively, then $S'S$ induces an equivalence $(P \mid \mathcal{A}_d)^k \xrightarrow{\sim} M_1^k$, whereas M_2^k is equivalent to $M'_{\mathcal{K}'\cap P'}$ where M', $\mathcal{K}'$, P' denote the restrictions of M, $\mathcal{K}$, P to $\mathfrak{L}\backslash d$. The functor $\Phi : \hat{M}^k_{\hat{\mathcal{K}}} \to M^k_{\mathcal{K}}$ maps $\hat{M}_1^k$ into M_1^k and induces an equivalence $\hat{M}_2^k \xrightarrow{\sim} M_2^k$. Moreover, in the case $A_1 \in \hat{M}_1^k$ and $A_2 \in \hat{M}_2^k$, all short exact sequences

$$0 \to \Phi A_1 \to E \to \Phi A_2 \to 0$$

of $M^k_{\mathcal{K}}$ are obviously P-exact. Hence, Φ induces a bijection

$$\operatorname{Ext}(A_2, A_1) \xrightarrow{\sim} \operatorname{Ext}(\Phi A_2, \Phi A_1),$$

and Lemma 8.5 is reduced to the following lemma, where we set $E^{\mathfrak{L}} = S'SE$ for all $E \in (P \mid \mathcal{A}_d)^k$.

Lemma. *If M is not $\mathcal{L}$-wild, there exists a subset $e \subset k \cup \infty$ of cardinality ≤ 4 such that* $\operatorname{Ext}(H, T_n^{\lambda\mathfrak{L}}) = 0$ *for all $H \in M_2^k$, all $n \geq 1$, and all $\lambda \in (k \cup \infty) \backslash e$.*

8.7. In order to prove Lemma 8.6, we start with an arbitrary $H \in M^k$ and some $F = E^{\mathfrak{L}} \in M_1^k$, where $E \in (P \mid \mathcal{A}_d)^k$. For the exact structure defined in 2.3, M^k admits almost split sequences [8, 9]. If τH denotes the cotranslate of H, we know that

$$\operatorname{Ext}(H, F) \xrightarrow{\sim} \underline{\operatorname{Hom}}(F, \tau H)^{\mathrm{T}},$$

where W^{T} denotes the dual of a vector space W and $\underline{\operatorname{Hom}}(F, \tau H)$ the residue space of $\operatorname{Hom}(F, \tau H)$ obtained by annihilation of the morphisms factoring through injectives of M^k. Now, since F admits an injective resolution whose indecomposable injective summands have the form $(k, 0, 0)$ or $(M(p), \mathbb{1}, p)$, $p \in \dot{P}$, it suffices to annihilate the morphisms factoring through these injectives. But τH has no nonzero injective direct summand. It easily follows that all morphisms from $(k, 0, 0)$ or $(M(p), \mathbb{1}, p)$ to τH vanish and that

$$\mathrm{Ext}(H,F) \xrightarrow{\sim} \mathrm{Hom}(E^{\mathfrak{S}}, \tau H)^{\mathrm{T}} \xrightarrow{\sim} \mathrm{Hom}(E,(\tau H)_d)^{\mathrm{T}}$$

if we set $K_d = RR'K \in (P \mid \mathcal{A}_d)^k$ for all $K \in M^k$.

Now, *in the case* $H \in M_2^k$, the following lemma states that $(\tau H)_d$ is a direct sum of indecomposables Λ_n and T_n^λ, where λ belongs to some subset $e \subset k \cup \infty$ of cardinality ≤ 4. If follows that $\mathrm{Hom}\,(E,(\tau H)_d) = 0$ if $E = V_n$ or $E = T_n^\mu$ with $\mu \subset k \cup \infty \setminus e$. So it remains for us to prove the following lemma.

Lemma. *Let* $e \subset k \cup \infty$ *be the set of all* $\lambda \subset k \cup \infty$ *such that, for some* $n \geq 1$ *and some* $H \in M_2^k$, T_n^λ *is isomorphic to a direct summand of* $(\tau H)_d$. *Then the cardinality of* e *is* ≤ 4. *Furthermore, if* $H \in M_2^k$, $(\tau H)_d$ *has no direct summand isomorphic to* V_n, $n \geq 1$.

8.8. Lemma 8.7 will finally result from the virtues of some restriction $\overline{M}$ of the module $\hat{M}$ examined in 8.1. Let $\overline{\mathfrak{S}}$ denote the finite full subspectroid of $\tilde{\mathfrak{S}}$ formed by $S\Lambda_3$ and all $(0,0,s)$, $s \in \mathfrak{S} \setminus d_P$. Let $\overline{\mathcal{A}}$ be the full subaggregate of $\hat{\mathcal{A}}$ formed by the points of $\overline{\mathfrak{S}}$, all isomorphic indecomposable, and their finite direct sums. The restrictions $\overline{M} = \hat{M} \mid \overline{\mathcal{A}}$ and $\overline{\mathcal{K}} = \{K \mid \overline{\mathcal{A}} : K \in \hat{\mathcal{K}}\}$ then satisfy the following lemma.

Lemma. $\overline{M}$ *is not* $\overline{\mathcal{K}}$*-wild.*

Proof. We know that the module $\tilde{M}$ of 8.2 is not $\tilde{\mathcal{L}}$-wild. It has a submodule N which vanishes at $S\Lambda_3$, $S\Lambda_2$, $S\Lambda_1$, and all $(0,0,s)$ with $s \in \mathfrak{S} \setminus d_P$, and which takes the same values as $\tilde{M}$ at all other points of $\tilde{\mathfrak{S}}$. By 3.7, $\tilde{M}/N$ is not $(\tilde{\mathcal{L}}/N)$-wild if we set $\tilde{\mathcal{L}}/N = \{K/N : N \subset K \in \tilde{\mathcal{L}}\}$. The condition $N \subset K$ eliminates all K of the form $K = \tilde{L}$ with $L(d_P) \neq M(d_P)$. Hence, only $\mathcal{K}$ contributes to $\tilde{\mathcal{L}}/N$, and $\overline{M}, \overline{\mathcal{K}}$ are identified with the restrictions of $\tilde{M}/N$, $\tilde{\mathcal{L}}/N$ to $\overline{\mathcal{A}}$.

8.9. ***Proof of Lemma* 8.7.** a) Obviously, $\overline{M}_{\overline{\mathcal{K}}}^k$ can be identified with the full subcategory of $\hat{M}_{\hat{\mathcal{K}}}^k$ formed by the $\hat{M}$-spaces (U,f,A) such that A_P (8.5) is a direct sum of copies of $S\Lambda_3$. Setting $X = (U, \mathrm{can} \circ f, A/A_P) \in \hat{M}_2^k$ and denoting by

$$\varepsilon \in \mathrm{Ext}(X,(0,0,A_P)) \xrightarrow{\sim} \mathrm{Hom}_k(\mathrm{Hom}(A_P, S\Lambda_3), \mathrm{Ext}(X,(0,0,S\Lambda_3))$$

the extension associated with an $\overline{M}$-space $(U,f,A) \in \overline{M}_{\overline{\mathcal{K}}}^k$ and with the sequence

$$0 \longrightarrow (0,0,A_P) \xrightarrow[(0,\iota)]{} (U,f,A) \xrightarrow[(\mathbb{1},\pi)]{} X = (U, \mathrm{can} \circ f, A/A_P) \longrightarrow 0$$

in 8.5, we obtain an epivalence

$$\Psi : \overline{M}_{\overline{\mathcal{K}}}^{k\mathrm{op}} \longrightarrow \hat{E}^k, \quad (U,f,A) \mapsto (\mathrm{Hom}(A, S\Lambda_3), \varepsilon, X),$$

where $\hat{E}$ is the module on $\hat{M}_2^{k\mathrm{op}}$ such that $\hat{E}(X) = \mathrm{Ext}(X,(0,0,S\Lambda_3))$. This epivalence can be composed with an equivalence $\hat{E}^k \xrightarrow{\sim} E^k$ which results from the equivalence $\hat{M}_{\hat{\mathcal{K}}}^k \xrightarrow{\sim} M_{\mathcal{K}}^k$ and from the invariance

$$\mathrm{Ext}(A_2, A_1) \xrightarrow{\sim} \mathrm{Ext}(\Phi A_2, \Phi A_1),\ A_1 \in \hat{M}_1^k, A_2 \in \hat{M}_2^k$$

examined in 8.6. By E we here denote the module

$$H \mapsto \operatorname{Ext}(H, \Lambda_3^{\$}) \xrightarrow{\sim} \operatorname{Hom}(\Lambda_3, (\tau H)_d)^{\mathrm{T}},$$

which is defined on the aggregate $M_2^{k\,\mathrm{op}}$ (8.6).

b) In the epivalence $\overline{M}_{\overline{\mathcal{K}}}^{k\,\mathrm{op}} \to E^k$ derived above, the point is that E is free of any bond. Before exploiting this point, we must transfer "tameness" from $\overline{M}$ to E.

Lemma. *E is not wild.*

Proof. It suffices to prove that $\hat{E}$ is tame. If not, there is a plane coordinate system

$$e_0, e_1, e_2 \in \operatorname{Ext}((U, g, B), (0, 0, W^T \otimes S\Lambda_3)) \xrightarrow{\sim} \operatorname{Hom}_k(W, \hat{E}(U, g, B))$$

such that the induced functor $\operatorname{rep}Q^2 \to \hat{E}^k$ preserves indecomposability and heteromorphism. The extensions e_i are the classes of short exact sequences, which we may write as follows:

$$0 \longrightarrow (0, 0, W^T \otimes S\Lambda_3) \xrightarrow[(0,\iota)]{} \left(U, \begin{bmatrix} h_i \\ g \end{bmatrix}, W^T \otimes S\Lambda_3 \oplus B\right) \xrightarrow[(\mathbb{1},\pi)]{} (U, g, B) \longrightarrow 0$$

where ι and π are the canonical immersion and projection. Setting $f_0 = [h_0\, g]^T$ and $f_i = [h_i\;\, 0]^T$ for $i = 1, 2$, we obtain a plane coordinate system

$$f_0, f_1, f_2 \in \operatorname{Hom}_k(U, \overline{M}(W^T \otimes S\Lambda_3 \oplus B)).$$

The induced functor $F_f\colon \operatorname{rep}Q^2 \to \overline{M}^k$ factors through $\overline{M}_{\overline{\mathcal{K}}}^k$ by construction. We claim that the composition

$$\operatorname{rep}Q^2 \xrightarrow[D]{} (\operatorname{rep}Q^2)^{\mathrm{op}} \xrightarrow[F_f]{} \overline{M}_{\overline{\mathcal{K}}}^{k\,\mathrm{op}} \xrightarrow[\Psi]{} \hat{E}^k,$$

where D is induced by the duality of vector spaces, is isomorphic to F_e. This implies that F_f preserves indecomposability and heteromorphisms, a contradiction to Lemma 8.8.

Our claim follows from the observation that the map

$$\operatorname{Hom}_k(U, \overline{M}(C)) \longrightarrow \operatorname{Ext}((U, g, B), (0, 0, C)),\ h \mapsto \overline{h},$$

where $\overline{h}$ denotes the class of the short exact sequence

$$0 \longrightarrow (0, 0, C) \xrightarrow[(0,\iota)]{} \left(U, \begin{bmatrix} h_i \\ g \end{bmatrix}, C \oplus B\right) \xrightarrow[(\mathbb{1},\pi)]{} (U, g, B) \longrightarrow 0, \qquad (***)$$

is k-linear for all $C = W^T \otimes S\Lambda_3$. To ascertain this point, we compute the extension group using the injective resolution

$$0 \longrightarrow (0, 0, C) \xrightarrow[(0,\mathbb{1})]{} (\overline{M}(C), \mathbb{1}, C) \xrightarrow[(\mathbb{1},0)]{} (\overline{M}(C), 0, 0) \longrightarrow 0$$

of $(0, 0, C)$ in $\overline{M}^k$. The induced linear map

$$\operatorname{Hom}((U, g, B), (\overline{M}(C), 0, 0)) \longrightarrow \operatorname{Ext}((U, g, B), (0, 0, C))$$

maps $(h, 0)$ onto the induced pull-back of the chosen resolution. This pull-back is isomorphic to (***).

c) Let us now suppose that Lemma 8.7 is false, and *let* $H \in M_2^k$ *be such that* $(\tau H)_d$ *has a direct summand of the form* V_n. Then we may further assume that H is indecomposable and denote by $\mathcal{H}$ the full subaggregate of M_2^k formed by the objects isomorphic to H^r, $r \in \mathbb{N}$. If m is the smallest number satisfying $\mathrm{Hom}(V_m, (\tau H)_d) \neq 0$, then $\mathrm{Hom}(V_m, (\tau H)_d) \otimes V_m$ is identified with a nonzero direct summand of $(\tau H)_d$, and

$$X \mapsto \mathrm{Hom}(V_m, (\tau H)_d) \otimes \mathrm{Hom}(\Lambda_3, V_m)$$

with a submodule of

$$E^{\mathrm{T}} \mid \mathcal{H}: X \mapsto \mathrm{Hom}(\Lambda_3, (\tau H)_d) \xrightarrow{\sim} \mathrm{Ext}(X, \Lambda_3^{\natural})^{\mathrm{T}}.$$

Accordingly, each simple submodule S of $X \mapsto \mathrm{Hom}(V_m, (\tau H)_d)$ provides a semisimple submodule $S \otimes \mathrm{Hom}(\Lambda_3, V_m)$ of $E^{\mathrm{T}} \mid \mathcal{H}$ such that

$$\dim S(H) \otimes \mathrm{Hom}(\Lambda_3, V_m) = \dim\mathrm{Hom}(\Lambda_3, V_m) = m + 2 \geq 3.$$

We infer that $E \mid \mathcal{H}^{\mathrm{op}}$ has a semisimple residue module whose dimension at H is ≥ 3; and hence, that E is wild in contradiction to the lemma of part b).

d) *Let us finally suppose that* $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ *are distinct scalars and* H *is an object of* M_2^k *such that, for each* i, $(\tau H)_d$ *has a direct summand of the form* $T_{n_i}^{\lambda_i}$. We then denote by $\mathcal{H}$ the full subaggregate of M_2^k formed by the objects isomorphic to direct summands of H^r, $r \in \mathbb{N}$. The restriction $E^{\mathrm{T}} \mid \mathcal{H}$ contains a direct sum of five nonzero submodules of the form

$$X \mapsto \mathrm{Hom}(T_1^{\lambda_i}, (\tau H)_d) \otimes \mathrm{Hom}(\Lambda_3, T_1^{\lambda_i}).$$

Accordingly, if S_i is a simple submodule of $X \mapsto \mathrm{Hom}(T_1^{\lambda_i}, (\tau H)_d)$, $E^{\mathrm{T}} \mid \mathcal{H}$ has a semisimple submodule of the form

$$\bigoplus_{i=1}^{5} S_i \otimes \mathrm{Hom}(\Lambda_3, T_1^{\lambda_i}),$$

and $E \mid \mathcal{H}^{\mathrm{op}}$ has a semisimple residue module of length 5. We infer that E is wild in contradiction to the lemma of part b).

9. From Subspaces to Modules.

In the present section, we apply our second main theorem (2.5) to a finite-dimensional k-algebra B. For this sake, we consider a proper quotient $\overline{\mathcal{T}}$ of a spectroid $\mathcal{T}$ of B and reduce $\mathrm{mod}\, B \xrightarrow{\sim} \mathrm{mod}\, \mathcal{T}$ to a "subspace category" M_N^k, where M and N are suitable left modules over $\mathrm{mod}\, \overline{\mathcal{T}}$.

9.0. Since we prefer working with finite spectroids rather than with finite-dimensional algebras, we first adapt the language introduced in 2.6 to the case of a *finite spectroid* $\mathcal{T}$.

First, we introduce the k-category $\otimes\mathcal{T}$ whose objects are the points of $\mathcal{T}$ and whose morphism spaces are defined by

$$(\otimes\mathcal{T})(r, s) = \bigoplus_{x} \mathcal{T}(x_{n-1}, s) \otimes_k \ldots \otimes_k \mathcal{T}(x_1, x_2) \otimes_k \mathcal{T}(r, x_1),$$

where x ranges over the sequences of points of $\mathcal{T}$ of length $n \geq 0$. (In the case $n = 0$, the displayed tensor product coincides with $\mathcal{T}(r, s)$.) The composition of $\otimes\mathcal{T}$ is induced by tensor multiplication.

Let $\operatorname{mod}\otimes\mathcal{T}$ and $\operatorname{mod}\mathcal{T}$ denote the categories of all finite-dimensional right modules over $\otimes\mathcal{T}$ and $\mathcal{T}$, i.e., of all *contravariant* k-linear functors from $\otimes\mathcal{T}$ and $\mathcal{T}$ to $\operatorname{mod} k$. An object of $\operatorname{mod}\otimes\mathcal{T}$ is given by a family $U = (U(s))_{s\in\mathcal{T}}$ of "*stalks*" $U(s) \in \operatorname{mod} k$ and by a family of linear maps lying in

$$H_U := \prod_{r,s\in\mathcal{T}} \operatorname{Hom}_k(U(r)\otimes \mathcal{T}_k(r,s), U(s)).$$

We shall identify $\operatorname{mod}\mathcal{T}$ with a full subcategory of $\operatorname{mod}\otimes\mathcal{T}$ with the aid of the canonical functor $\otimes\mathcal{T}\to\mathcal{T}$.

Each coordinate system $e = (e_0,\dots, e_t)$ of an affine subspace $S \subset H_U$ gives rise to a functor $F_e : \operatorname{rep}Q^t \to \operatorname{mod}\otimes\mathcal{T}$ which maps a sequence $a = (a_1,\dots, a_t)$ of t endomorphisms $a_i : W \to W$ onto the family $W\otimes U = (W\otimes_k U(s))_{s\in\mathcal{T}}$ equipped with the linear maps

$$\mathbb{1}_W\otimes e_0(r,s) + a_1\otimes e_1(r,s) + \dots + a_t\otimes e_t(r,s) : W\otimes U(r)\otimes\mathcal{T}(r,s) \to W\otimes U(s).$$

The space S is called *$\mathcal{T}$-reliable* if F_e factors through $\operatorname{mod}\mathcal{T}$ and preserves indecomposability and heteromorphism. And $\mathcal{T}$ is called *wild* if it admits a $\mathcal{T}$-reliable plane. If not, $\mathcal{T}$ is *tame*.

Lemma. *Let B be a finite-dimensional algebra with spectroid $\mathcal{T}$. Then B is wild if so is $\mathcal{T}$.*

Proof. We may suppose that the points of $\mathcal{T}$ are projective B-modules $\varepsilon_1 B,\dots,\dots,\varepsilon_m B$, where the ε_1 denote primitive idempotents. Choosing an isomorphism $B \xrightarrow{\sim} \bigoplus_{i=1}^{m}(\varepsilon_i B)^{n_i}$ of $\operatorname{mod} B$, we then identify the algebra B with the matrix algebra $\bigoplus_{i,j}(\varepsilon_i B\,\varepsilon_j)^{n_i\times n_j}$.

Now let $U = (U_i)_{1\le i\le m}$ be a family of stalks and $e_0, e_1, e_2 \in \prod_{i,j}\operatorname{Hom}_k(U_i\otimes\varepsilon_i B\,\varepsilon_j, U_j)$ be a coordinate system of a $\mathcal{T}$-reliable plane. If V denotes the direct sum of the spaces $U_i^{1\times n_i}$ formed by the rows with n_i entries in U_i, we obtain a coordinate system $f_0, f_1, f_2 \in \operatorname{Hom}_k(V\otimes B, V)$ of a B-reliable plane by setting

$$f_p(v\otimes b) = \Bigl(\sum_{i=1}^{m} v^i e_p(i,j;b^{ij})\Bigr)_{1\le j\le m} \in \bigoplus_{j=1}^{m} U_j^{1\times n_j} = V$$

for all $v = (v^i) \in \bigoplus_i U^{1\times n_i} = V$ and all $b = (b^{ij}) = \bigoplus_{i,j}(\varepsilon_i B\,\varepsilon_j)^{n_i\times n_j} = B$. Here

$$e_p(i,j;b^{ij}) \in \operatorname{Hom}_k(U_i, U_j)^{n_i\times n_j}$$

denotes a matrix whose entries are defined by

$$e_p(i,j;b^{ij})_{rs}(x) = e_p(x\otimes b^{ij}_{rs}).$$

In the case $t = 1$, we also consider *punched lines* $S\setminus E$, where E is a finite subset of S. Setting $C = \{\lambda\in k : e_0 + \lambda e_1 \in S\setminus E)$ as in 2.5 and 2.6, we say that $S\setminus E$ is $\mathcal{T}$-reliable if $F_e : \operatorname{rep}_C Q^1 \to \operatorname{mod}\otimes\mathcal{T}$ factors through $\operatorname{mod}\mathcal{T}$

and preserves indecomposability and heteromorphism. As in the case of reliable planes considered above, $\mathcal{T}$-reliable punched lines give rise to B-reliable punched lines whenever B is a finite-dimensional algebra with spectroid $\mathcal{T}$. Thus, in order to prove our third main theorem, it suffices to construct suitable $\mathcal{T}$-reliable punched lines whenever $\mathcal{T}$ is tame and to carry them over to B. As a corollary, we obtain the converse of the lemma above (B is tame if so is $\mathcal{T}$), which of course could also be proved directly.

9.1. Assume that $\mathcal{T}$ is an arbitrary *finite spectroid* over k, $\sigma \in \mathcal{R}_{\mathcal{T}}(s, t)$ is a nonzero radical morphism of $\mathcal{T}$ such that $\mathcal{R}_{\mathcal{T}}(t, x)\sigma = 0 = \sigma\mathcal{R}_{\mathcal{T}}(x, s)$ for all $x \in \mathcal{T}$, and $\overline{\mathcal{T}} = \mathcal{T}/\sigma$. For each $X \in \operatorname{mod}\mathcal{T}$, we denote by $\underline{X}$ the largest submodule of X annihilated by σ. Concretely, $\underline{X}$ satisfies $\underline{X}(x) = X(x)$ for all $x \in \mathcal{T}\setminus t$, whereas $\underline{X}(t)$ is the kernel of $X(\sigma) : X(t) \to X(s)$. Accordingly, $X/\underline{X}$ is semisimple and located at t. The obvious exact sequence

$$0 \longrightarrow \underline{X} \longrightarrow X \longrightarrow X/\underline{X} \longrightarrow 0,$$

therefore, provides a linear map

$$\varepsilon_X \in \operatorname{Hom}_k(\operatorname{Hom}_{\mathcal{T}}(t^-, X/\underline{X}), \operatorname{Ext}^1_{\mathcal{T}}(t^-, \underline{X})) \xleftarrow{\sim} \operatorname{Ext}^1_{\mathcal{T}}(X/\underline{X}, \underline{X}),$$

where $t^- \in \operatorname{mod}\mathcal{T}$ is the simple module located at t. Finally, we obtain an epivalence

$$G : \operatorname{mod} \mathcal{T} \longrightarrow M_N^k, \quad X \longrightarrow (\operatorname{Hom}_{\mathcal{T}}(t^-, X/\underline{X}), \varepsilon_X, \underline{X}),$$

where M and N are the left modules over $\mathcal{A} = \operatorname{mod}\overline{\mathcal{T}}$ such that $N(Z) = \operatorname{Ext}^1_{\overline{\mathcal{T}}}(t^-, Z) \subset M(Z) = \operatorname{Ext}^1_{\mathcal{T}}(t^-, Z)$ ([9], 4.2).

Our proof of the third main theorem uses the epivalence $\operatorname{mod} \mathcal{T} \to M_N^k$, the second main theorem, and the following statement. There, $\operatorname{ind}\overline{\mathcal{T}}$ denotes the chosen spectroid $\mathcal{S}$ of $\mathcal{A} = \operatorname{mod}\overline{\mathcal{T}}$.

Proposition. *With the notation above, suppose that M is not N-wild. Then, for each $d \in \mathbb{N}$, $\operatorname{ind}\overline{\mathcal{T}}$ contains only finitely many (M, N)-relevant modules of length d* (6.6).

The proposition will be proved in 9.6.

9.2. Proposition. *$\mathcal{T}$ is wild if M is N-wild.*

Proof. Let $e = (e_0, e_1, e_2)$ be a coordinate system of an N-reliable plane in some $\operatorname{Hom}_k(V, M(X)) \xleftarrow{\sim} \operatorname{Ext}^1_{\mathcal{T}}(V \otimes_k t^-, X)$ ($V \in \operatorname{mod}k, X \in \mathcal{A}$). To produce a $\mathcal{T}$-reliable plane, we start from the tensor product

$$0 \longrightarrow V \otimes_k \underline{p} \longrightarrow V \otimes_k p \longrightarrow V \otimes_k t^- \longrightarrow 0 \tag{*}$$

of V with the obvious sequence (9.1) associated with $p = \mathcal{T}(?, t)$.

The induced connecting homomorphism $\operatorname{Hom}_{\mathcal{T}}(V \otimes_k \underline{p}, X) \to \operatorname{Ext}^1_{\mathcal{T}}(V \otimes_k t^-, X)$ is subjective and maps $f : V \otimes_k \underline{p} \to X$ onto the class of the push-out of (*) along f. Choosing the preimages h_i of the given e_i, we construct the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & W \underset{k}{\otimes} V \underset{k}{\otimes} \underline{p} & \to & W \underset{k}{\otimes} V \underset{k}{\otimes} p & \to & W \underset{k}{\otimes} V \underset{k}{\otimes} t^{-} & \to & 0 \\ & & {\scriptstyle \mathbb{1}_W \otimes h_0 + a_W \otimes h_1 + b_W \otimes h_2} \downarrow & & c \downarrow & & \| & & \\ 0 & \to & W \otimes_k X & \xrightarrow{d} & Y_{a,b} & \to & W \underset{k}{\otimes} V \underset{k}{\otimes} t^{-} & \to & 0 \end{array}, \qquad (**)$$

where a_W and b_W map $w \in W$ onto wa and wb.

For $Y_{a,b}$, we choose the following concrete construction. Let $Y = (Y(q))$ be a family of stalks such that $Y(t) = X(t) \oplus V$ and $Y(r) = X(r)$ if $r \neq t$. We set $Y_{a,b}(q) = W \otimes_k Y(q)$ for all $q \in \mathcal{T}$. Thus, the stalks of $W \otimes X$ are subspaces of the stalks $Y_{a,b}(q)$; on these subspaces, the structure maps

$$f_{a,b}(r, q) : Y_{a,b}(q) \otimes \mathcal{T}(r, q) \longrightarrow Y_{a,b}(r)$$

coincide with those of $W \otimes X$. Accordingly, d is an inclusion, and it remains for us to describe c and the restriction

$$Y_{a,b}(t) \otimes \mathcal{R}_{\mathcal{T}}(r, t) \longrightarrow Y_{a,b}(r)$$

of $f_{a,b}(r, t)$. The morphism c is determined by the commutativity of the left square of (**) and by the equations $c(w \otimes v \otimes \mathbb{1}_p) = w \otimes v$. These equations imply

$$f_{a,b}(r, t)(w \otimes v) = w \otimes h_0(v \otimes \mu) + wa \otimes h_1(v \otimes \mu) + wb \otimes h_2(v \otimes \mu)$$

for all $\mu \in \mathcal{R}_{\mathcal{T}}(n, t)$. Thus, we have

$$f_{a,b}(r, q) = \mathbb{1}_W \otimes f_0(r, q) + a \otimes f_1(r, q) + b \otimes f_2(r, q),$$

where $f_1(r, q)$, $f_2(r, q)$ vanish on $X(q) \otimes \mathcal{T}(r, q)$, whereas $f_0(r, q)$ coincides there with the structure map of X. In other words, we have $Y_{a,b} = F_f(W, a, b)$ where $f = (f_0, f_1, f_2) \in H_Y^3$ (9.0).

Furthermore, the construction of $Y_{a,b}$ as a push-out shows that the composition

$$\operatorname{rep} Q^2 \xrightarrow[F_f]{} \operatorname{mod} \mathcal{T} \xrightarrow[G]{} M_N^k$$

of F_f with the epivalence G of 9.1 coincides with F_e. Since F_e preserves indecomposability and heteromorphism, so does F_f.

9.3. ***Proof of the third main theorem.*** Supposing that $\mathcal{T}$ is not wild, we shall construct a family of $\mathcal{T}$-reliable punched lines which (mutatis mutandis) satisfy statement b) of 2.6 (see 9.0 above).

Using induction on the dimension $\sum_{a, b \in \mathcal{T}} \dim \mathcal{T}(a, b)$ of $\mathcal{T}$, we may suppose that such a family is already available for $\overline{\mathcal{T}} = \mathcal{T}/\sigma$. Hence, we restrict our attention to the "new" indecomposables, which are not annihilated by σ, i.e., are transformed by $\operatorname{mod}\mathcal{T} \to M_N^k$ into M-spaces with nonzero first components. By 9.2, M is not N-wild. By 9.1, the full subaggregate $\mathcal{A}_d$ of $\mathcal{A}$ "generated" by the indecomposables X of dimension $\leq d$, which are (M, N)-relevant, has a finite spectroid for each $d \geq 1$. Denoting by M_d and N_d the restrictions of M and N to $\mathcal{A}_d$, there exists a locally finite set $\mathcal{D}^d$ of N_d-reliable punched lines which, for each $X \in \mathcal{A}_d$, produce almost all indecomposables of $(M_d)^k_{N_d}$ of the form (V, f, X) up to isomorphism. Of course, we may and shall assume that $\mathcal{D}^1 \subset \mathcal{D}^2 \subset \dots$.

Now assume that $S \setminus E$ is an element of $\mathcal{D} = \bigcup_{d \geq 1} \mathcal{D}^d$, $e = (e_0, e_1)$ is a coordinate system of S, and $C = \{\lambda \in k \mid e_0 + \lambda e_1 \in S \setminus E\}$. As in the proof of 9.2, we can construct a $\mathcal{T}$-reliable punched line with coordinate system $f = (f_0, f_1) \in H^2_Y$ such that the composition $\operatorname{rep} Q^1 \xrightarrow{F_f} \operatorname{mod} \mathcal{T} \xrightarrow{G} M^k_N$ is isomorphic to $\operatorname{rep}_C Q^1 \xrightarrow{F_e} M^k_N$. It is easy to check that the punched lines arising in this way from $\mathcal{D}$ "parametrize" the new indecomposables over $\mathcal{T}$ as wanted.

9.4. We now turn to the proof of Proposition 9.1. Our first objective is to shake off the bond $N = \operatorname{Ext}^1_{\overline{\mathcal{T}}}(t^-, ?)$ on $M = \operatorname{Ext}^1_{\mathcal{T}}(t^-, ?)$. For this sake, we resort to the injective $\mathcal{T}$-module $i = \mathcal{T}(s, ?)^{\mathrm{T}}$. The largest submodule $\underline{i}$ of i annihilated by σ is identified with $\overline{\mathcal{T}}(s, ?)^{\mathrm{T}}$, and $i/\underline{i}$ can be identified with t^- via

$$i(t) = \mathcal{T}(s, t)^{\mathrm{T}} \to k, \quad f \mapsto f(\sigma).$$

It easily follows that $0 = N(\underline{i}) \subset M(\underline{i}) = k\varepsilon_i$, where ε_i denotes the extension associated with the exact sequence $0 \to \underline{i} \to i \to t^- \to 0$. As a consequence, the submodule of M generated by $\varepsilon_i \in M(\underline{i})$ coincides with $\mathfrak{I}M$, where $\mathfrak{I}$ is the ideal of $\mathcal{A} = \operatorname{mod} \overline{\mathcal{T}}$ generated by $\mathbb{1}_{\underline{i}}$. In the following proposition, $\overline{M} := M / \mathfrak{I}M$ is considered as a module over the aggregate $\overline{\mathcal{A}} = \mathcal{A}/\mathfrak{I}$, whose spectroid $\overline{\mathfrak{S}}$ is obtained by deleting the point $\underline{i}$ from the quotient $\mathfrak{S}/\mathbb{1}_{\underline{i}}$ of the spectroid $\mathfrak{S} = \operatorname{ind} \overline{\mathcal{T}}$ of $\mathcal{A} = \operatorname{mod} \overline{\mathcal{T}}$.

Proposition. *The canonical functor* $M^k_N \to \overline{M}^k$ *is quasisurjective. Up to isomorphism, it annihilates just one indecomposable* $(0, 0, \underline{i}) \in M^k_N$.

We postpone the proof to 9.7.

9.5. Proposition. *With the notation of* 9.4, *suppose that* $\overline{M}$ *is not wild. Then, for each* $d \in \mathbb{N}$, $\overline{M}$ *vanishes on almost all modules in* $\overline{\mathfrak{S}}$ *of length* d.

It seems advisable here to recall that the points of $\overline{\mathfrak{S}}$ are *genuine* modules over $\overline{\mathcal{T}}$, even though the morphisms of $\overline{\mathfrak{S}}$ are classes of morphisms of $\operatorname{mod} \overline{\mathcal{T}}$.

Proof. Let us denote by $\overline{\mathfrak{S}}_d$ the full subspectroid of $\overline{\mathfrak{S}}$ formed by the modules of dimension d, by $\overline{M}_d$ the restriction of $\overline{M}$ to $\overline{\mathfrak{S}}_d$. By the lemma of Harada and Sai ([9], 3.2, Example 2), the radical $\mathcal{R}_d$ of $\overline{\mathfrak{S}}_d$ is nilpotent. If $\overline{M}_d(x) \neq 0$ for infinitely many $x \in \overline{\mathfrak{S}}_d$, we infer that $(\mathcal{R}^n_d \overline{M}_d / \mathcal{R}^{n+1}_d \overline{M}_d)(x) \neq 0$ for some $n \in \mathbb{N}$ and (at least!) five points $x \in \overline{\mathfrak{S}}_d$. This means that $\overline{M}_d$ has a subquotient which is the sum of five nonisomorphic simple modules. Hence, the subquotient is wild, and so are $\overline{M}_d$ and $\overline{M}$.

9.6. ***Proof of proposition*** **9.1.** a) We first show that M *is* N*-wild if* $\overline{M}$ *is wild.* Indeed, let ${}_{\mathcal{A}}\overline{M}$ denote the quotient $M / \mathfrak{I}M$ considered as a module over $\mathcal{A}$. If $\overline{M}$ is wild, it is clear that ${}_{\mathcal{A}}\overline{M}$ is wild. Since ${}_{\mathcal{A}}\overline{M}$ is a quotient of M and N does not contain $\mathfrak{I}M$, Proposition 3.7 implies that M is N-wild.

b) Suppose now that M is not N-wild. Then $\overline{M}$ is not wild. Hence, for each $d \in \mathbb{N}$, $\overline{\mathfrak{S}}$ has a finite number $n(d)$ of points x of dimension d such that $\overline{M}(x) \neq 0$. Of course, all these $x \in \mathfrak{S} \setminus \underline{i}$ are (M, N)-relevant. On the other hand, if $y \in \mathfrak{S} \setminus \underline{i}$ is (M, N)-relevant, M^k_N admits an indecomposable $(V, f, y \otimes Y)$ such that $V \neq 0$. Since this triple is also indecomposable as an object of $\overline{M}^k$ (9.4), we have $\overline{M}(y) \neq 0$. We infer that, besides $\underline{i}$, $\mathfrak{s}$ has $n(d)$ points of dimension d which are (M, N)-relevant.

9.7. *It remains for us to prove Proposition* 9.4, which follows from 4.2 b), 4.1, and the following lemma.

Lemma. *The annihilator of* $\mathfrak{I}$ *in* $M = \mathrm{Ext}^1_{\mathcal{T}}(t^-, ?)$ *is* $N = \mathrm{Ext}^1_{\overline{\mathcal{T}}}(t^-, ?)$.

Proof. For each $Z \in \mathcal{A}$, the annihilator of $\mathfrak{I}$ in $M(Z)$ consists of the classes of short exact sequences $0 \longrightarrow Z \xrightarrow[\iota]{} Y \xrightarrow[\pi]{} t^- \longrightarrow 0$ of $\mathrm{mod}\,\mathcal{T}$ whose push-out splits for each $\mu \in \mathrm{Hom}_{\mathcal{T}}(Z, \underline{i})$. If the class belongs to $\mathrm{mod}\overline{\mathcal{T}}$, Y is a $\overline{\mathcal{T}}$-module and the push-out splits because $\underline{i}$ is injective in $\mathrm{mod}\overline{\mathcal{T}}$. Hence, N is contained in the annihilator.

Conversely, suppose that the class of (ι, π) is annihilated by $\mathfrak{I}$. Since each $\mu \in \mathrm{Hom}_{\mathcal{T}}(Z, \underline{i})$ factors through Y, the first row of

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}_{\mathcal{T}}(t^-, \underline{i}) & \to & \mathrm{Hom}_{\mathcal{T}}(Y, \underline{i}) & \to & \mathrm{Hom}_{\mathcal{T}}(Z, \underline{i}) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathrm{Hom}_{\mathcal{T}}(t^-, i) & \to & \mathrm{Hom}_{\mathcal{T}}(Y, i) & \to & \mathrm{Hom}_{\mathcal{T}}(Z, i) & \to & 0
\end{array}$$

is exact. Since the first and the second vertical arrows are invertible, so is the second. Since i is, up to isomorphism, the only indecomposable injective $\mathcal{T}$-module outside $\mathrm{mod}\,\overline{\mathcal{T}}$, we infer that $Y \in \mathrm{mod}\,\overline{\mathcal{T}}$.

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