

ELEMENTARY AND MULTIELEMENTARY REPRESENTATIONS OF VECTROIDS

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To the memory of Maurice Auslander

We prove that every finitely represented vectroid is determined, up to an isomorphism, by its completed bierdered set. Elementary and multielementary representations of such vectroids (which play a central role for biinvolutive posets) are described.

Introduction

Denote by k a fixed algebraically closed field and by $\text{mod } k$ the category of finite-dimensional right vector spaces over k . The symbol of a linear map is written to the right.

A *vectroid* \mathcal{V} (over the field k) is a small (the class of objects is a set) subcategory of the category $\text{mod } k$ which is a spectroid in the sense of [1], i.e., satisfies the following conditions:

- (i) For each pair of objects $X, Y \in \mathcal{V}$, the set $\mathcal{V}(X, Y)$ of morphisms is a linear subspace in $\text{mod } k(X, Y)$;
- (ii) For each $X \in \mathcal{V}$, the ring $\mathcal{V}(X, X)$ contains exactly two idempotents ($0_X \neq 1_X$);
- (iii) \mathcal{V} does not contain isomorphic objects.

The value $\sup \{ \dim X, X \in \mathcal{V} \}$ is called the *dimension* $\dim \mathcal{V}$ of \mathcal{V} .

Each vectroid \mathcal{V} defines a category (an aggregate in the sense of [1]) $\oplus \mathcal{V} \subset \text{mod } k$ whose objects are all finite direct sums $X_1 \oplus \dots \oplus X_m$ ($X_i \in \mathcal{V}, m \geq 0$). The category $\oplus \mathcal{V}$ (as well as any subcategory in $\text{mod } k$) can be regarded as a faithful module over itself [1].

A triple (U, f, X) consisting of the spaces $U \in \text{mod } k$ and $X \in \oplus \mathcal{V}$ and a linear map $f: U \rightarrow X$ is called a *representation* of \mathcal{V} ([1], 4.1; [2]). The *morphism* $(U, f, X) \rightarrow (U', f', X')$ is a pair (φ, ξ) that consists of a linear map $\varphi: U \rightarrow U'$ and a morphism $\xi: X \rightarrow X'$ of the category $\oplus \mathcal{V}$ such that $\varphi f' = f \xi$. Representations form the aggregate denoted by $\text{Rep } \mathcal{V}$. A vectroid is called *finitely represented* if $\text{Rep } \mathcal{V}$ has finitely many indecomposable nonisomorphic objects.

Within the notation of ([1], 4.1), the category $\text{Rep } \mathcal{V}$ coincides with the category $(\oplus \mathcal{V})^k$. In some cases, we shall consider the category M^k of representations of an arbitrary module M (not necessarily faithful) over an aggregate (see Appendix at the end of Introduction).

If $\dim \mathcal{V} = 1$, then \mathcal{V} is completely determined by the following partial ordering of the set $\text{Ob } \mathcal{V}$: $X \leq Y$ if $\mathcal{V}(X, Y) \neq 0$. The category $\text{Rep } \mathcal{V}$ can be naturally identified with the category of representations of this poset ([1], 4.1; [5]). The criterion of finite representability of posets was obtained in [6].

On the other hand, it was proved in ([1], 4.2, 4.3) and ([3], 9.1, 9.4) that the category $\text{mod } \Lambda$ of representations of an arbitrary finite-dimensional algebra Λ over k coincides with the category of representations of a certain vectroid \mathcal{V} in the following sense: There exists an injective indecomposable Λ -module P such that the category of all Λ -modules that do not contain P as a direct summand is epivalent to $\text{Rep } \mathcal{V}$.

Thus, the theory of representations of vectroids can be regarded as a generalization of the theory of representations of posets and the theory of representations of finite-dimensional algebras.

We define the *radical* of a vectroid \mathcal{V} as the ideal of \mathcal{V} generated by the spaces $\text{Rad}_{\mathcal{V}}(X, Y)$ of uninvertible morphisms from $\mathcal{V}(X, Y)$ for all $X, Y \in \mathcal{V}$. It is obvious that $\mathcal{V}(X, Y) = \text{Rad}_{\mathcal{V}}(X, Y)$ for $X \neq Y$ and $\mathcal{V}(X, X) = k1_X \oplus \text{Rad}_{\mathcal{V}}(X, X)$.

The set $\{(n_i^X), (f_i^{XY})\}$ that consists of the bases $(n_1^X, n_2^X, \dots, n_{\dim(X)}^X)$ of spaces $X \in \mathcal{V}$ and the bases $(f_1^{XY}, f_2^{XY}, \dots)$ of spaces $\text{Rad}_{\mathcal{V}}(X, Y)$ is called the *basis* of the vectroid \mathcal{V} . The maximal rank of the linear maps f_i^{XY} is called the *rank of the basis*. The basis $(n_1^X, n_2^X, \dots, n_{\dim(X)}^X)$ of an object $X \in \mathcal{V}$ is called *triangular* if the family $\{(\overline{n_i^X}), i = \overline{1, \dim X} \mid (\overline{n_i^X}) \neq 0\}$ is linearly independent for any $j \in \mathbb{N}$, where the bar means the transition to the factor space $X/X \text{Rad}_{\mathcal{V}}^j(X, X)$. The basis $\{(n_i^X), (f_i^{XY})\}$ of \mathcal{V} is called *triangular* if each basis $(n_i^X), X \in \mathcal{V}$, is triangular. A basis is *scalarly multiplicative* if the element $n_i^X f_i^{XY}$ is equal to $\lambda n_p^Y, \lambda \in k$, for all n_i^X, f_i^{XY} , and it follows from the relations $n_i^X f_i^{XY} = \lambda n_p^Y$ and $n_j^X f_l^{XY} = \mu n_p^Y, \lambda, \mu \in k^*$, that $i=j$. A scalarly multiplicative basis is called *multiplicative* if each element $n_i^X f_l^{XY}$ is equal to either 0 or n_p^Y ([1], 4.10). Every finitely represented vectroid has a multiplicative basis whose rank does not exceed two [4].

A vectroid \mathcal{V} is called a *chain vectroid* if, for every $X \in \mathcal{V}$, submodules of the module $X_{\mathcal{V}(XX)}$ are linearly ordered with respect to the inclusions

$$X = X_1 \supsetneq X_2 \supsetneq \dots \supsetneq X_{\dim X} \supsetneq 0.$$

In this case, all these submodules are cyclic, $X_i = m_i^X \mathcal{V}(X, X)$, and $m_1^X, \dots, m_{\dim X}^X$ is a triangular basis of $X \in \mathcal{V}$ (see Lemma 1).

It is known that if \mathcal{V} is a finitely represented vectroid, then \mathcal{V} is a chain vectroid and $\dim \mathcal{V} \leq 3$ ([1], 4.7 and 4.8).

For an arbitrary chain vectroid \mathcal{V} , we construct the poset

$$S(\mathcal{V}) = \bigcup_{X \in \mathcal{V}} \{X_1, X_2, \dots, X_{\dim X}\},$$

setting $X_i \leq Y_j$ if $m_i^X \varphi = m_j^Y$ for some $\varphi \in \mathcal{V}(X, Y)$.

The number $\text{def}(\mathcal{V}) = \sup \{\text{def}(X, Y) \mid X, Y \in \text{Ob } \mathcal{V}\}$, where

$$\text{def}(X, Y) = |\{(X_i, Y_j) \mid X_i < Y_j\}| - \dim \text{Rad}(X, Y),$$

is called the *defect* of \mathcal{V} . According to [4], we have $\text{def } \mathcal{V} \leq 1$ for all finitely represented vectroids \mathcal{V} (see Sec. 2).

If $\text{def } \mathcal{V} = 0$, then \mathcal{V} has a multiplicative basis of rank one and the category $\text{Rep } \mathcal{V}$ coincides with the category of representations of a weakly completed poset $S(\mathcal{V})$ (see Sec. 1). The criterion of finite representability of weakly completed posets and the classification of their indecomposable representations (in the case of finite representability) are given in [9] (see also [10]).

Let $\text{def } \mathcal{V} = 1$ and $\dim \mathcal{V} = 2$. In this case, representations of vectroids \mathcal{V} are identified with representations of a certain poset $S(\mathcal{V})$ with additional structure (the structure of a biinvolutive poset). In this case, \mathcal{V} is finitely represented if and only if a certain poset $\text{St}(S(\mathcal{V}))$ constructed for a biinvolutive poset S ([1], 5.8) is finitely represented. This criterion was formulated in [1] and proved in [7, 8].

Thus, the problem of finite representability remains open only for three-dimensional (chain) vectroids with defect one.

In Sec. 1, we associate every chain vectroid \mathcal{V} with a completed biordered set $S(\mathcal{V})$. In Sec. 2, we show that a finitely represented vectroid is uniquely determined by its completed biordered set. In Sec. 5, for every chain vectroid \mathcal{V} with $\text{def } \mathcal{V} \leq 1$, we construct a poset $C(\mathcal{V})$, which is a generalization of the poset $\text{St}(S)$ ([1], 5.8). In this connection, we introduce elementary and multielementary representations of vectroids.

Appendix. Modules over Aggregates

Let \mathcal{M} be a category of pairs (A, M) , where A is an aggregate and M is a pointwise finite (see [1], 3.1, 2.2, and 3.6) right module over A . A morphism from (A, M) to (A', M') is a pair (F, Φ) , where $F: A \rightarrow A'$ is a functor, $\Phi: M \rightarrow F^*M'$ is a morphism of A -modules, and F^*M' is a restriction of M' along F .

On the basis of the module $(A, M) \in \mathcal{M}$, we construct a *derivative module* $\text{Der}(A, M) = (\text{Rep } M, E_M)$. Here, $\text{Rep } M$ is an aggregate consisting of triples (V, f, X) with $V \in \text{mod } k$, $X \in A$, and $f \in \text{mod } k(V, M(X))$; morphisms from (V, f, X) to (V', f', X') are pairs (φ, ξ) , where $\varphi \in \text{mod } k(V, V')$ and $\xi \in A(X, X')$, such that $f \circ M\xi = \varphi \circ f'$ (cf. [1], 4.1, where $\text{Rep } M$ is denoted by M^k); E_M is a module over $\text{Rep } M$ obtained from the module (A, M) by the restriction along the functor $T: \text{Rep } M \rightarrow A$, $T(V, f, X) = X$.

Moreover, Der can be continued up to the functor $\text{Der}: \mathcal{M} \rightarrow \mathcal{M}$ that transforms the morphism $(F, \Phi): (A, M) \rightarrow (A', M')$ into the morphism

$$\text{Der}(F, \Phi) = (G, \Psi): (\text{Rep } M, E_M) \rightarrow (\text{Rep } M', E_{M'}),$$

where $G(V, f, X) = (V, f \circ \Phi(X), F(X))$, $\Psi(V, f, X) = \Phi(X)$ (here, $\Phi(X): M(X) \rightarrow M'(FX)$).

On the basis of an arbitrary vectroid \mathcal{V} , we naturally construct a module $M\mathcal{V} = (\oplus \mathcal{V}, M_{\mathcal{V}}) \in \mathcal{M}$, where $\oplus \mathcal{V}$ is the aggregate of all direct sums of objects from \mathcal{V} , and $M_{\mathcal{V}}$ is a module over $\oplus \mathcal{V}$, $M_{\mathcal{V}}(X) = X$ (recall that every object $X \in \oplus \mathcal{V}$ can be regarded as a vector space). The obtained module is faithful (i.e., $M_{\mathcal{V}}\xi \neq 0$ if $0 \neq \xi \in \oplus \mathcal{V}(X, X')$).

In what follows, we omit the indication of the map M , i.e., we write $\mathcal{M}(V, V')$ instead of $\mathcal{M}(MV, MV')$, $\text{Der } \mathcal{V}$ instead of $\text{Der}(M\mathcal{V})$, etc. Note that $\text{Rep } M\mathcal{V}$ coincides with $\text{Rep } \mathcal{V}$, and the vectroids \mathcal{V} and \mathcal{V}' are isomorphic if and only if the modules $M\mathcal{V}$ and $M\mathcal{V}'$ are isomorphic.

Remark 1. Some authors (e.g., Nazarova and Roiter [2]) studied the *category of subspaces* $U_{\mathcal{V}}$ of the aggregate $\oplus \mathcal{V}$, where \mathcal{V} is an arbitrary vectroid. Objects of this category are subspaces of the spaces $X \in \oplus \mathcal{V}$. The set of morphisms $U_{\mathcal{V}}(V, W)$ consists of all $\varphi \in \oplus \mathcal{V}(X, Y)$ such that $V\varphi \subset W$, where $V \subset X$, $W \subset Y$. Suppose that ${}^i\text{Rep } \mathcal{V} \subset \text{Rep } \mathcal{V}$ is a complete subcategory consisting of representations $(V, f, X) \in \text{Rep } \mathcal{V}$ such that f is a monomorphism. The spectroid of the aggregate $\text{Rep } \mathcal{V}$ contains a full subspectroid, which is "equal" to the spectroid of the aggregate ${}^i\text{Rep } \mathcal{V}$ and exactly one more object, namely, $(k, 0, 0)$. It is easy to see that the categories $U_{\mathcal{V}}$ and ${}^i\text{Rep } \mathcal{V}$ are equivalent.

1. Biordered Sets

Let α be a binary relation on a set Z . We define

$$A^\alpha(b) = \{y \in A \mid y\alpha b\}, \quad A^\alpha(B) = \bigcap_{b \in B} A^\alpha(b)$$

for $b \in Z$ and $A, B \subset Z$.

We write $A \alpha B$ if $a \alpha b$ for all $a \in A$ and $b \in B$.

For a partial order relation \leq on the set Z and for $a, b \in Z$, we write $a \asymp b$ if $a \leq b$ and $b \leq a$.

Denote by $\tilde{\alpha}$ the reflexive closure of a relation α , i.e., a relation on Z such that $z \tilde{\alpha} z$ for all $z \in Z$, and $z \tilde{\alpha} y$ for $y \neq z$ if and only if $z \alpha y$. A relation α is called a *pseudoequivalence* if α is antireflexive and $\tilde{\alpha}$ is an equivalence. In the present paper, all pseudoequivalences and the corresponding equivalence relations are denoted by \sim and \approx , respectively.

A set S is called a *biordered set* [8] if it is equipped with the partial order relation \leq and the relation \triangleleft that satisfy the following conditions:

- (a) if $a \triangleleft b$, then $a \leq b$;
- (b) if $a \triangleleft b \leq c$ or $a \leq b \triangleleft c$, then $a \triangleleft c$.

Note that the relation \triangleleft is transitive and antisymmetric but, in general, not reflexive (if \triangleleft is reflexive, it coincides with \leq).

Remark 2. If $\text{Cat}(S, \leq)$ is a poset (S, \leq) considered as a category (see [1], 2.1, Example 5), then $\triangleleft = \{(Y|X) \mid X \triangleleft Y\}$ is an ideal in $\text{Cat}(S, \leq)$. Conversely, every ideal in this category determines a biordered set.

A biordered set with a given equivalence relation \approx is called a *completed biordered set*. A completed biordered set $(S, \leq, \triangleleft, \approx)$ is called *locally finite* if all equivalence classes $S^{\approx}(s)$, $s \in S$, are finite.

A poset (S, \leq) with the equivalence relation \approx is called a *weakly completed poset*. If $S = (S, \leq, \triangleleft, \approx)$ is a completed biordered set, then $S^w = (S, \leq, \approx)$ is a weakly completed poset obtained by weakening the structure on S . In some cases, it is convenient to regard a weakly completed poset (S, \leq, \approx) as a completed biordered set in which the relations \leq and \triangleleft coincide.

Let S be a completed biordered set. We denote the number of elements that are equivalent to $a \in S$ by $d(a)$ and set $d(S) = \sup \{d(a) \mid a \in S\}$. We say that a pair (a, b) is an *edge* and write $a \Rightarrow b$ if $a < b$ and $a \ntriangleleft b$. By the definition of a biordered set, if $a \Rightarrow b$ and $a < x < b$, then $a \Rightarrow x \Rightarrow b$. Two edges $\alpha: a \Rightarrow b$ and $\alpha': a' \Rightarrow b'$ are called *equivalent* and denoted by $\alpha \approx \alpha'$ (or $\alpha \sim \alpha'$ for $\alpha \neq \alpha'$) if $a \approx a'$ and $b \approx b'$. The number of edges equivalent to $a \Rightarrow b$ is denoted by $\text{ed}(a, b)$. An edge $a \Rightarrow b$ is called *maximal* if $x \leq a < b \leq y$ and $x \Rightarrow y$ imply that $x = a$ and $y = b$. An edge $a \Rightarrow b$ is called *short* if there is no x such that $a < x < b$. A pair of equivalent edges $(a \Rightarrow b) \sim (a' \Rightarrow b')$ is called *short* if there is no $x \sim x'$ such that $a < x < b$ and $a' < x' < b'$ (see Example 1, (c) below).

Let \mathcal{V} be a chain vectroid. We introduce the structure of a completed biordered set on the poset $S(\mathcal{V})$ as follows: Let $X_i = m_i^X \mathcal{V}(X, X)$ and $Y_j = m_j^Y \mathcal{V}(Y, Y) \in S(\mathcal{V})$. We set $X_i \approx Y_j$ if $X = Y$, and $X_i \triangleleft Y_j$ if there exists a linear map $\varphi \in \mathcal{V}(X, Y)$ of rank one such that $m_i^X \varphi = m_j^Y$.

Remark 3. By analogy, we can define a completed biordered set $S(\mathcal{V})$ (of cyclic submodules) if \mathcal{V} is not a chain vectroid. However, it can be not locally finite. Note that a vectroid \mathcal{V} is a chain vectroid if and only if the module $X_{\mathcal{V}(X,X)}$ contains only a finite number of cyclic submodules for any $X \in \mathcal{V}$.

Let $X = \bigoplus_i k m_i^X$ and $Y = \bigoplus_j k m_j^Y$ be two objects of a vectroid \mathcal{V} . We define a linear map $e_{ij}^{XY}: X \rightarrow Y$ by setting $m_i^X e_{ij}^{XY} = m_j^Y$ and $m_l^X e_{ij}^{XY} = 0$ for $l \neq i$.

Example 1. (a) Consider the vectroid \mathcal{V} with two objects

$$X = \bigoplus_{i=1}^3 km_i^X \quad \text{and} \quad Y = \bigoplus_{i=1}^3 km_i^Y$$

and the following spaces of morphisms:

$$\mathcal{V}_1(X, X) = k1_X \oplus \bigoplus_{i<j} ke_{ij}^{XX}, \quad \mathcal{V}_1(Y, Y) = k1_Y \oplus \bigoplus_{i<j} ke_{ij}^{YY},$$

$$\mathcal{V}_1(X, Y) = k(e_{11}^{XY} + e_{22}^{XY}) \oplus k(e_{11}^{XY} + e_{33}^{XY}) \oplus \bigoplus_{i<j} ke_{ij}^{XY},$$

$$\mathcal{V}_1(Y, X) = \bigoplus_{i<j} ke_{ij}^{YX}.$$

Obviously, $\dim(\mathcal{V}_1) = 3$, $\text{def}(\mathcal{V}_1) = 1$, and

$$\begin{array}{ccc}
 X_1 \circ & \xrightarrow{\quad} & \circ Y_1 \\
 & \swarrow & \\
 S(\mathcal{V}_1) = X_2 \circ & \xrightarrow{\quad} & \circ Y_2 \\
 & \swarrow & \\
 X_3 \circ & \xrightarrow{\quad} & \circ Y_3
 \end{array}$$

($a \rightarrow b$ means that $a \triangleleft b$ and there is no $x \in S$ such that $a \triangleleft x \triangleleft b$ or $a < x < b$).

(b) Consider the vectroid \mathcal{V}_2 that differs from \mathcal{V}_1 only by the following space of morphisms:

$$\mathcal{V}_2(X, Y) = k(e_{11}^{XY} + e_{22}^{XY} + e_{33}^{XY}) \oplus \bigoplus_{i<j} ke_{ij}^{XY}.$$

Clearly, $\dim(\mathcal{V}_2) = 3$, $\text{def}(\mathcal{V}_2) = 2$, and $S(\mathcal{V}_2) = S(\mathcal{V}_1)$.

(c) We construct the vectroid \mathcal{V}_3 by completing \mathcal{V}_1 with the objects $Z = km_1^Z \oplus km_2^Z$ and $T = km_1^T \oplus km_2^T$ with the following spaces of morphisms:

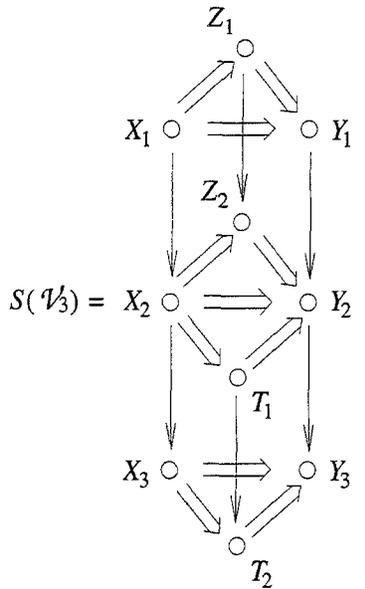
$$\mathcal{V}_3(Z, Z) = k1_Z \oplus ke_{12}^{ZZ}, \quad \mathcal{V}_3(T, T) = k1_T \oplus ke_{12}^{TT},$$

$$\mathcal{V}_3(X, Z) = k(e_{11}^{XZ} + e_{22}^{XZ}), \quad \mathcal{V}_3(Z, Y) = k(e_{11}^{ZY} + e_{22}^{ZY}),$$

$$\mathcal{V}_3(X, T) = k(e_{21}^{XT} + e_{32}^{XT}), \quad \mathcal{V}_3(T, Y) = k(e_{12}^{TY} + e_{23}^{TY}),$$

$$\mathcal{V}_3(Z, X) = \mathcal{V}_3(Y, Z) = \mathcal{V}_3(T, X) = \mathcal{V}_3(Y, T) = \mathcal{V}_3(Y, X) = 0.$$

Then



The edges $X_1 \Rightarrow Y_1$ and $X_3 \Rightarrow Y_3$ in the biordered set $S(\mathcal{V}_3)$ are long, but they form a short pair.

Example 2. Consider the vectroid \mathcal{W} with one object $X = \bigoplus_{i=1}^4 km_i^X$ and the following space of morphisms:

$$\mathcal{W}(X, X) = k1_X \oplus k(e_{12}^{XX} + e_{34}^{XX}) \oplus ke_{23}^{XX} \oplus \left(\bigoplus_{j>i+1} ke_{ij}^{XX} \right).$$

Then $\dim \mathcal{W} = 4$, $\text{def } \mathcal{W} = 2$, and

$$S(\mathcal{W}) = \begin{matrix} \circ & \xRightarrow{\quad} & \circ & \longrightarrow & \circ & \xRightarrow{\quad} & \circ \\ X_1 & & X_2 & & X_3 & & X_4 \end{matrix}$$

Let S be a locally finite completed biordered set. We define the relations \leq , \triangleleft , and \approx on the set $S \times \mathbb{N}$ in the following way:

- (i) $(s, i) \leq (t, j)$ if $s \leq t$;
- (ii) $(s, i) \triangleleft (t, j)$ if $s \triangleleft t$;
- (iii) $(s, i) \approx (t, j)$ if $s \approx t$ and $i = j$.

Note that \leq is a quasiorder ([13], II.1), \triangleleft defines an ideal in the category $\text{Cat}(S \times \mathbb{N}, \leq)$ associated with the quasiordered set $(S \times \mathbb{N}, \leq)$, and \approx is the equivalence relation on $S \times \mathbb{N}$.

A function $\varphi: S \rightarrow \mathbb{N}_0$ ($\mathbb{N}_0 = \mathbb{N} \cup 0$) is called the *dimension of φ on the set S* if $\varphi(s) = \varphi(t)$ for $s \approx t$. Denote $S_\varphi = \{(s, i) \in S \times \mathbb{N} \mid i \leq \varphi(s)\}$.

A matrix M (which may have no columns or rows) is called a *representation of S of dimension φ* if its columns M_i are enumerated by the elements of the set S_φ , i.e., the bijection $n: \{1, \dots, l\} \rightarrow S_\varphi$ is given, where l is the number of columns of M . The columns M_i and M_j of the representation M are called *comparable (equivalent)* if $n(i)$ and $n(j)$ are comparable with respect to \leq (are equivalent with respect to \approx) in $S \times \mathbb{N}$.

The representation M of dimension φ is called *faithful* if $\varphi(s) \neq 0$ for every $s \in S$ and *faithful at a point* $s \in S$ if $\varphi(s) \neq 0$.

Let \mathcal{V} be a chain vectroid and let $(U, f, X) \in \text{Rep } \mathcal{V}$. Then the choice of a triangular basis of \mathcal{V} and a basis of the space U associates (U, f, X) with a representation of the completed biordered set $S(\mathcal{V})$. However, generally speaking, neither the vectroid \mathcal{V} nor the category $\text{Rep } \mathcal{V}$ is determined by $S(\mathcal{V})$. Thus, in Examples 1(a) and 1(b), we have $S(\mathcal{V}'_1) \simeq S(\mathcal{V}'_2)$ while $\mathcal{V}'_1 \neq \mathcal{V}'_2$; moreover, \mathcal{V}'_1 is finitely represented while \mathcal{V}'_2 is not finitely represented. However, if $\text{def}(\mathcal{V}) \leq 1$, then the category $\text{Rep } \mathcal{V}$ (but not \mathcal{V} itself!) is determined by the completed biordered set $S(\mathcal{V})$ in all known cases. In Sec. 2, we show that a finitely represented vectroid can be uniquely restored on the basis of $S(\mathcal{V})$.

2. Vectroids of Defect ≤ 1

Lemma 1. *Let \mathcal{V} be a chain vectroid and let $X \in \mathcal{V}$. Then*

(a) *all nonzero submodules of the module $X_{\mathcal{V}(XX)}$ are cyclic, i.e.,*

$$X = X_1 \supsetneq X_2 \supsetneq \dots \supsetneq X_{d(X)} \supsetneq 0, \quad X_i = m_i^X \mathcal{V}(X, X);$$

(b) *the generators $m_1^X, \dots, m_{d(X)}^X$ form a k -basis of the space X ;*

(c) *they can be chosen so that, for some $\varphi_X \in \text{Rad}_{\mathcal{V}}(X, X)$,*

$$m_2^X = m_1^X \varphi_X, \quad \dots, \quad m_{d(X)}^X = m_1^X \varphi_X^{d(X)-1}$$

and $\varphi_X^d = 0$.

Proof.

(a) If X_1 is not cyclic, then it can be represented as the sum of two submodules $N_1 + N_2$ such that $N_1 \not\supset N_2$ and $N_1 \not\subset N_2$.

(b) Since the field k is algebraically closed and the algebra $\mathcal{V}(X, X)$ is local, the simple module X_i/X_{i+1} ($1 \leq i < d(X)$) is isomorphic to k .

(c) We can take $m_1^X \in X_1 \setminus X_2$ and $\varphi_X \in \text{Rad}_{\mathcal{V}}(X, X) \setminus (P_1 \cup \dots \cup P_{d(X)-1})$, where $P_i = \{\psi \in \mathcal{V}(X, X) \mid X_i \psi \subset X_{i+2}\}$ are proper subspaces of the space $\text{Rad}_{\mathcal{V}}(X, X)$ and $X_{d(X)+1} = 0$.

Denote by $r(\mathcal{V})$ the least possible rank of a basis of a vectroid \mathcal{V} .

Lemma 2. *Let \mathcal{V} be a chain vectroid. Then*

(a) $\text{def}(\mathcal{V}) \geq 0$; furthermore, $\text{def}(\mathcal{V}) = 0$ if and only if $r(\mathcal{V}) \leq 1$;

(b) $\text{def}(\mathcal{V}) \geq r(\mathcal{V}) - 1$.

Proof. Let $X, Y \in \mathcal{V}$. We choose bases $m_1^X, \dots, m_{d(X)}^X$ and $m_1^Y, \dots, m_{d(Y)}^Y$ in X and Y as in Lemma 1. Since $\text{Rad}_{\mathcal{V}}(X, Y)$ is closed with respect to the multiplication by φ_X from the left and by φ_Y from the right, the space of $d(X) \times d(Y)$ matrices

$$\mathcal{R}(X, Y) = \{(\alpha_{ij}) \mid \sum \alpha_{ij} e_{ij}^{XY} \in \text{Rad}_{\mathcal{V}}(X, X)\}$$

[which defines $\text{Rad}_{\mathcal{V}}(X, Y)$] is stable under shifts rightward and upward. Therefore, it has the following staircase form (cf. [1], 4.7):

$$A = \begin{pmatrix} \begin{array}{cc} t_1 & t_2 \\ \hline & \hline \\ 0 & \hline \end{array} \end{pmatrix} \begin{matrix} s_1 \\ s_2 \end{matrix}.$$

I. Let us prove that $|\{(X_i, Y_j) \mid X_i < Y_j\}|$ is equal to the number of elements of A from $\mathcal{R}(X, Y)$ located on or above the stairs.

Indeed, the set of nodes

$$\mathcal{K}_{XY} = \{(s_1, t_1), \dots, (s_n, t_n)\}$$

$$(1 \leq s_1 < \dots < s_n \leq d(X), 1 \leq t_1 < \dots < t_n \leq d(Y))$$

consists of all minimal elements of the set

$$N_{XY} = \{(s, t) \mid \exists (\alpha_{ij}) \in \mathcal{R}(X, Y) : \alpha_{st} \neq 0\}$$

with respect to the following partial-ordering relation: $(i, j) \leq (i', j')$ if $i \geq i'$ and $j \leq j'$. It is clear that N_{XY} coincides with the set of indexing pairs of elements of the matrix A located on or above the stairs, i.e.,

$$N_{XY} = \{(i, j) \mid \exists l : 1 \leq i \leq s_l, t_l \leq j \leq d(Y)\}.$$

Let $(l, r) \in N_{XY}$. We select $\varphi = \sum \alpha_{ij} e_{ij}^{XY} \in \text{Rad}(X, Y)$ such that $\alpha_{lr} \neq 0$ and choose $f \in k[x]$ such that $\varphi f(\varphi_Y) = \sum \beta_{ij} e_{ij}^{XY}$, where $\beta_{lj} = 0$ for $j \neq r$, and $\beta_{lr} = 1$. Then $m_l^X \varphi f(\varphi_Y) = m_r^Y$, $X_l < Y_r$. Hence, there exists a bijection $N_{XY} \rightarrow \{(X_i, Y_j) \mid X_i < Y_j\}$ and $\text{def}(X, Y) = |N_{XY}| - \dim \text{Rad}(X, Y)$.

II. Let $A_\alpha = (a_{ij}^{(\alpha)})$, $1 \leq \alpha \leq t$, be a k -basis of the matrix space $\mathcal{R}(X, Y)$. Then there exists a collection of indices $J = \{(i_1, j_1), \dots, (i_t, j_t)\}$ such that the matrices $B_\alpha = (b_{ij}^{(\alpha)})$, $1 \leq \alpha \leq t$, are linearly independent. Here, $b_{ij}^{(\alpha)} = a_{ij}^{(\alpha)}$ for $(i, j) \in J$, and $b_{ij}^{(\alpha)} = 0$ for $(i, j) \notin J$. By using transformations of the form $A_\alpha \rightarrow A_\alpha + aA_\beta$ ($\alpha \neq \beta$, $a \in k$), we can pass from A_1, \dots, A_t to a new k -basis $C_\alpha = (c_{ij}^{(\alpha)})$, $1 \leq \alpha \leq t$, such that $c_{i_\beta j_\beta}^{(\alpha)} = 0$ for $\alpha \neq \beta$ and $c_{i_\alpha j_\alpha}^{(\alpha)} \neq 0$. Since every matrix C_α has the same staircase form as A , and C_α has at most $t - 1$ zero elements among those located above the stairs, we conclude that C_α has at most $|N_{XY}| - t + 1$ nonzero elements. Therefore, $\text{rank}(C_\alpha) \leq |N_{XY}| - t + 1$, $r(X, Y) \leq |N_{XY}| - \dim \mathcal{R}(X, Y) + 1$, and $\text{def}(X, Y) \geq r(X, Y) - 1$, which proves (b).

III. Let us prove (a). By virtue of (b) and step I of the proof, it is sufficient to show that $r(\mathcal{V}) = 1$ yields $\text{def}(\mathcal{V}) = 0$.

Let $A = (a_{ij}) \in \mathcal{R}(X, Y)$ and let $\text{rank}(A) = 1$. Since $\mathcal{R}(X, Y)$ is stable under shifts rightward and upward, it contains matrix units E_{ij} for all $(i, j) \geq (s, t)$, where (s, t) is the minimal element of the set $\{(i, j) \mid \alpha_{ij} \neq 0\}$ with respect to the partial-ordering relation defined in step I. Therefore, if $r(X, Y) = 1$, then $\mathcal{R}(X, Y)$ has a basis including matrix units and, by virtue of step I, $\text{def}(X, Y) = 0$. Lemma 2 is proved.

Proposition 1. *A chain vectroid whose defect does not exceed one has a scalarly multiplicative basis whose rank does not exceed two.*

Proof. We use the notation from the proof of Lemma 2.

I. Assume that $X, Y \in \mathcal{V}$. Let us show that there exist $c_1^{XY}, \dots, c_n^{XY} \in k$ (possibly, all equal to zero) such that

$$\text{Rad}_{\mathcal{V}}(X, Y) = \left\{ \sum_{(i,j) \in N_{XY}} \alpha_{ij} E_{ij}^{XY} \mid c_1^{XY} \alpha_{s_1 t_1} + \dots + c_n^{XY} \alpha_{s_n t_n} = 0 \right\}.$$

Indeed, by virtue of the condition $\text{def}(\mathcal{V}) \leq 1$, the codimension of the space $\mathcal{R}(X, Y)$ in the space of all staircase matrices of the form A does not exceed one. Therefore, there exist $c_{ij} \in k$ such that

$$\mathcal{R}(X, Y) = \left\{ \sum_{(i,j) \in N_{XY}} \alpha_{ij} E_{ij} \mid \sum_{(i,j) \in N_{XY}} c_{ij} \alpha_{ij} = 0 \right\},$$

where E_{ij} are matrix units.

Let $c_{ir} \neq 0$ for some $(i, r) \notin \mathcal{K}_{XY}$. Then $\mathcal{R}(X, Y)$ contains $B_{ij} = E_{ij} + \gamma_{ij} E_{lr}$ for all $(i, j) \in N_{XY} \setminus (i, r)$, where $\gamma_{ij} = -c_{ij} c_{lr}^{-1}$. We fix $(s, t) \in \mathcal{K}_{XY}$ such that $(s, t) < (l, r)$ and set $(l_i, r_i) = (s, t) + i[(l, r) - (s, t)]$. Let m be such that $(l_m, r_m) \in N_{XY}$ but $(l_{m+1}, r_{m+1}) \notin N_{XY}$. Let F_1 and F_2 be the matrices of the maps ϕ_X and ϕ_Y , respectively (see Lemma 1). Then $\mathcal{R}(X, Y)$ contains the matrices

$$B_{st} = E_{l_0 r_0} + \gamma_{st} E_{l_1 r_1}, \quad F_1^{s-l} B_{st} F_2^{r-t} = E_{l_1 r_1} + \gamma_{st} E_{l_2 r_2},$$

$$F_1^{2(s-l)} B_{st} F_2^{2(r-t)} = E_{l_2 r_2} + \gamma_{st} E_{l_3 r_3}, \dots, E_{l_m r_m}.$$

Hence, it contains the matrix $E_{l_1 r_1} = E_{lr}$, which contradicts the assumption that $c_{ir} \neq 0$.

Therefore, we have $c_{ij} = 0$ for all $(i, j) \in \mathcal{K}_{XY}$, which was to be proved.

II. A scalarly multiplicative basis of the vectroid \mathcal{V} can be obtained by supplementing the vectors m_i^X with morphisms f_l^{XY} of the following form:

- (a) e_{ij}^{XY} for $(i, j) \in N_{XY} \setminus \mathcal{K}_{XY}$ and for $(i, j) = (s_r, t_r) \in \mathcal{K}_{XY}$ such that $c_r^{XY} = 0$;
- (b) $e_{s_r(1)t_r(1)}^{XY} - (c_{r(1)}^{XY} / c_{r(i)}^{XY}) e_{s_r(i)t_r(i)}^{XY}$ for all $2 \leq i \leq q$, where $\{r(1), \dots, r(q)\} = \{r \mid c_r^{XY} \neq 0\}$ and $r(1) \leq \dots \leq r(q)$.

Lemma 3. *Let \mathcal{V} be a vectroid of defect ≤ 1 . The completed biordered set $S(\mathcal{V})$ satisfies the following conditions:*

- I. *The equivalence class of every element is linearly ordered.*
- II. *If $a_1 < a_2 < \dots < a_n$ and $a_1 \Rightarrow a_n$, then there exist $a'_1 \sim a_1, \dots, a'_n \sim a_n$ such that $a'_1 < a'_2 < \dots < a'_n$ and $a'_1 \Rightarrow a'_n$.*
- III. *If $\text{ed}(a, b) \geq 3$, then the edge $a \Rightarrow b$ is maximal.*
- IV. *$a \triangleleft a$ if and only if $d(a) = 1$.*

Proof. Let \mathcal{V} be a chain vectroid whose defect does not exceed one and let $\{(m_i^X), (f_i^{XY})\}$ be its scalarly multiplicative basis of rank two constructed in the proof of Proposition 1.

Condition I follows from the equality $m_i^X \varphi_X = m_{i+1}^X$ [$i < \dim(X)$].

It is sufficient to verify condition II for $n=2$ and $n=3$. For $n=2$, the proof is obvious. Indeed, let $X_i \Rightarrow Y_j$. Then $m_i^X f_l^{XY} = \alpha m_j^Y$ for some $f_l^{XY} = \alpha e_{ij}^{XY} + \beta e_{i'j'}^{XY}$, $i \neq i', j \neq j'$, $\alpha, \beta \in k^*$, i.e., $X_{i'} \Rightarrow X_{j'}$. Let us prove condition II for $n=3$. Assume that $X_i < Y_j < Z_r$ and $X_i \Rightarrow Z_r$. Then there exist f_l^{XY} and f_p^{YZ} such that $m_i^X f_l^{XY} = \alpha m_j^Y$ and $m_j^Y f_p^{YZ} = \beta m_r^Z$, $\alpha \neq 0 \neq \beta$. Since $X_i \Rightarrow Z_r$, we have $\text{rank}(f_l^{XY} f_p^{YZ}) \neq 1$. Consequently, $f_l^{XY} = \alpha e_{ij}^{XY} + \gamma e_{i'j'}^{XY}$, $f_p^{YZ} = \beta e_{jr}^{YZ} + \delta e_{j'r'}^{YZ}$, $\gamma \neq 0 \neq \delta$, and $X_{i'} < Y_{j'} < Z_{r'}$. If $X_{i'} \triangleleft Z_{r'}$, then $e_{i'r'}^{XZ} \in \text{Rad}(X, Z)$, and $f_l^{XY} f_p^{YZ} - \gamma \delta e_{i'r'}^{XZ}$ has rank one and maps m_i^X onto $\alpha \beta m_r^Z$, which contradicts the assumption that $X_i \Rightarrow Z_r$. Therefore, $X_{i'} \Rightarrow Z_{r'}$.

Let us prove condition III. Let $X_i \Rightarrow Y_j$, $X_{i'} \Rightarrow Y_{j'}$, and $X_{i''} \Rightarrow Y_{j''}$ be different edges. Assume, e.g., that $X_i \Rightarrow Y_j$ is not maximal from the right, i.e., there exists $Z_r > Y_j$ such that $X_i \Rightarrow Z_r$. Then $X_i \Rightarrow Y_j \Rightarrow Z_r$ and there is a basic morphism $f_l^{YZ} = \alpha e_{jr}^{YZ} + \beta e_{i'r'}^{YZ}$, $\alpha \neq 0 \neq \beta$, $j \neq r$. We may take $j' \neq r$. According to step I of the proof of Proposition 1, there exists $\psi = \gamma e_{ij}^{XY} + \delta e_{i'j'}^{XY} \in \text{Rad}_\varphi(X, Y)$, $\gamma \neq 0 \neq \delta$. The existence of the morphism $\psi f_l^{YZ} = \alpha \gamma e_{ir}^{XZ}$ contradicts $X_i \Rightarrow Z_r$.

Condition IV follows from the fact that the ring $\mathcal{V}(X, X)$ is local: If $m_i^X f = m_i^X$, then $f = 1$ and $\text{rank}(f) = \dim X$.

Lemma 4. *The following property is a consequence of conditions I–III: If $(a \Rightarrow b) \sim (a' \Rightarrow b')$, then either $a < a'$ and $b < b'$ or $a > a'$ and $b > b'$.*

Proof. Let $a = a'$. By condition II, there exists an edge $(a'' \Rightarrow b'') \sim (a \Rightarrow b)$ such that $a'' \neq a$ and $b'' \neq b$. Hence, $\text{ed}(a, b) \geq 3$ and the edges $a \Rightarrow b$ and $a \Rightarrow b'$ are maximal by virtue of condition III. However, $b = b'$, $a \Rightarrow b'$. By condition I, $b < b'$ or $b > b'$, and $a \Rightarrow b$ or $a \Rightarrow b'$ is not maximal. Therefore, $a \neq a'$ and, similarly, $b \neq b'$. If, for example, $a < a'$ and $b > b'$, then, since $a < a' < b' < b$ and $a \Rightarrow b$, we get $a \Rightarrow b'$. However, we have just proved that such an edge does not exist. Therefore, we have either $a < a'$ and $b < b'$ or $a > a'$ and $b > b'$.

Lemma 5.

- (a) *Let $\text{char}(k) \neq 2$ and let \mathcal{V} be a chain vectroid whose defect does not exceed one. If \mathcal{V} has a multiplicative basis of rank at most two, then $S(\mathcal{V})$ satisfies the following condition:*

V. If $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \dots, (\alpha_t, \alpha_{t+1})$ are long pairs of edges and t is even, then (α_1, α_{t+1}) is a short pair of edges.

(b) On the basis of a locally finite completed biordered set S satisfying conditions I–V, one can construct a chain vectroid \mathcal{V} with defect ≤ 1 that has a multiplicative basis and for which $S(\mathcal{V}) \simeq S$.

Proof. (a) Assume that \mathcal{V} is a chain vectroid whose defect does not exceed one and $S(\mathcal{V})$ contains long pairs of edges $(\alpha_1, \alpha_2), (\alpha_2, \alpha_3), \dots, (\alpha_t, \alpha_{t+1}), (\alpha_1, \alpha_{t+1})$, where $\alpha_l: X_{i_l} \Rightarrow Y_{j_l}$ ($1 \leq l \leq t+1$). Since (α_1, α_2) is a long pair, we have $X_{i_l} \Rightarrow Z_{p_l} \Rightarrow Y_{j_l}$ ($l = 1, 2$) for some Z . According to condition III, $\text{ed}(X, Z) = \text{ed}(Z, Y) = 2$. Assume that \mathcal{V} has a multiplicative basis whose rank does not exceed two. Then $e_{i_1 p_1}^{XZ} + e_{i_2 p_2}^{XZ}$ and $e_{p_1 j_1}^{ZY} + e_{p_2 j_2}^{ZY}$ are basic morphisms and $e_1 + e_2 \in \mathcal{V}(X, Y)$, where $e_l = e_{i_l j_l}^{XY}$ ($1 \leq l \leq t+1$). By analogy, $e_2 + e_3, \dots, e_t + e_{t+1}, e_1 + e_{t+1} \in \mathcal{V}(X, Y)$. For even t and $\text{char}(k) \neq 2$, we get $e_1 = e_{i_1 j_1}^{XY} \in \mathcal{V}(X, Y)$, which contradicts $X_{i_1} \Rightarrow X_{j_1}$.

(b) Assume that S is a locally finite completed biordered set satisfying conditions I–V. Let us construct a vectroid $\mathcal{V} = \text{Vect}(S)$ whose objects are the vector spaces $X = kx_1 \oplus \dots \oplus kx_{d(X)}$, where $\{x_1, \dots, x_{d(X)}\} \subset S$ are equivalence classes and $x_1 < x_2 < \dots < x_{d(X)}$. The space $\text{Rad}_{\mathcal{V}}(X, Y)$ is generated by linear maps e_{ij}^{XY} for all $x_i < y_j$ and by all maps from a certain set $L(X, Y) \subset \{e_{ij}^{XY} + e_{i'j'}^{XY} \mid (x_i \Rightarrow y_j) \sim (x_{i'} \Rightarrow y_{j'})\}$. This set contains $e_{ij}^{XY} + e_{i'j'}^{XY}$ for all long pairs $(x_i \Rightarrow y_j, x_{i'} \Rightarrow y_{j'})$ and is maximal with respect to the following property: The linear span of $L(X, Y)$ contains no maps of the form e_{ij}^{XY} . In order to uniquely choose $L(X, Y)$, we impose the following condition: If $(x_i \Rightarrow y_j, x_{i'} \Rightarrow y_{j'})$ and $(x_s \Rightarrow y_t, x_{s'} \Rightarrow y_{t'})$, $i < i', s < s'$, are two short pairs and $e_{ij}^{XY} + e_{i'j'}^{XY} \in L(X, X)$, $e_{st}^{XY} + e_{s't'}^{XY} \notin L(X, X)$, then either $i < s$ or $i = s, i' < s'$. Note that we construct a vectroid with fixed multiplicative basis.

Let us prove that \mathcal{V} is well defined, i.e., that $fg \in \text{Rad}_{\mathcal{V}}(X, Y)$ for all $f \in \text{Rad}_{\mathcal{V}}(X, Y)$ and $g \in \text{Rad}_{\mathcal{V}}(Y, Z)$. We can assume that f and g are generating maps [i.e., $f = e_{ij}^{XY}$, $x_i < y_j$, or $f \in L(X, Y)$]. Let $f = e_{ij}^{XY} + \dots$, $g = e_{j'l}^{YZ} + \dots$ (hence, $x_i < y_j < z_l$). If $x_i < z_l$, then $e_{il}^{XZ} \in \text{Rad}_{\mathcal{V}}(X, Z)$ and $fg \in \text{Rad}_{\mathcal{V}}(X, Z)$. Now assume that $x_i \Rightarrow z_l$. Then $x_i \Rightarrow y_j \Rightarrow z_l$ and, according to condition II, there exist other $x_{i'} \Rightarrow z_{l'}, x_{i'} \Rightarrow y_{j'} \Rightarrow z_{l'}$. By virtue of condition III, $\text{ed}(x_i, y_j) = \text{ed}(y_j, z_l) = 2$. Therefore, $f = e_{ij}^{XY} + e_{i'j'}^{XY}$ and $g = e_{j'l}^{YZ} + e_{j'l'}^{YZ}$. Since $(x_i \Rightarrow z_l, x_{i'} \Rightarrow z_{l'})$ is a long pair of edges, we get $fg = e_{il}^{XZ} + e_{i'l'}^{XZ} \in \text{Rad}_{\mathcal{V}}(X, Z)$ by the definition of $L(X, Z)$.

Let us investigate the form of $\text{Rad}_{\mathcal{V}}(X, Y)$. Let $\mathcal{K}_{XY} = \{(s_1, t_1), \dots, (s_n, t_n)\}$ be the set of minimal elements of the set $N_{XY} = \{(i, j) \mid x_i < y_j\}$. If $(i, j) \in N_{XY}$ and $(i, j) < (i', j')$, then $x_{i'} \leq x_i < y_j \leq y_{j'}$. Hence, $(i', j') \in N_{XY}$ and N_{XY} has the form described in step I of the proof of Lemma 2. By Lemma 4, if $x_i \Rightarrow y_j$, then $x_{i'} < y_{j'}$ and $(i, j) \in \mathcal{K}_{XY}$. Since $L(X, Y)$ is maximal, the space $\text{Rad}_{\mathcal{V}}(X, Y)$ has the form described in step I of the proof of Proposition 1, and $\text{def}(X, Y) \leq 1$. Lemma 5 is proved.

The category $\text{Rep}(\mathcal{V})$ of representations of the vectroid $\mathcal{V} = \text{Vect}(S)$ constructed in the proof of assertion (b) of Lemma 5 is called the *category* $\text{Rep}(S)$ of representations of a locally finite completed biordered set S satisfying conditions I–V.

Remark 4. If \mathcal{V} is a chain vectroid such that $\text{r}(\mathcal{V}) = 2$ and $\text{def}(\mathcal{V}) > 1$, we can define an equivalence relation \simeq_{ed} on the set of edges of $S(\mathcal{V})$. For this purpose, we set $(X_i \Rightarrow Y_j) \simeq_{\text{ed}} (X_{i'} \Rightarrow Y_{j'})$ if and only if there exists a morphism $\xi \in \mathcal{V}(X, Y)$ such that

$$X_i \xi \subset Y_j, \quad (X_i \text{Rad } \mathcal{V}(X, X)) \xi \subset Y_j \text{Rad } \mathcal{V}(Y, Y),$$

$$X_{i'} \xi \subset Y_{j'}, \quad (X_{i'} \text{Rad } \mathcal{V}(X, X)) \xi \subset Y_{j'} \text{Rad } \mathcal{V}(Y, Y),$$

$$X_i \xi \not\subset Y_j \text{Rad } \mathcal{V}(Y, Y), \quad X_{i'} \xi \not\subset Y_{j'} \text{Rad } \mathcal{V}(Y, Y),$$

and there is no $\zeta \in \mathcal{V}(X, Y)$ such that $m_i^X \zeta = m_j^Y$, $m_{i'}^X \zeta = 0$. On the basis of the completed biordered set S with given equivalence relation \approx_{ed} on edges, we can similarly define a vectroid $\text{Vect}(S, \approx_{\text{ed}})$ such that $S(\text{Vect}(S, \approx_{\text{ed}})) \approx (S, \approx_{\text{ed}})$. Note that if $\text{def } \mathcal{V} \leq 1$, then the equivalence relation \approx_{ed} coincides with the equivalence relation for edges introduced in Sec. 1.

Lemma 6. *If a vectroid \mathcal{V} is finitely represented, then $\text{def } \mathcal{V} \leq 1$ and the completed biordered set $S(\mathcal{V})$ satisfies conditions I–V and the following one:*

VI. $d(a) \leq 3$ for all $a \in S(\mathcal{V})$.

Proof. Condition VI is satisfied because \mathcal{V} is a chain vectroid and $\dim \mathcal{V} \leq 3$. Let us prove that $\text{def } \mathcal{V} \leq 1$. According to Lemma 1 in [4], if $\dim X = 2$, then

(i) $\mathcal{V}(X, X) = k 1_X \oplus k e_{12}^{XX}$,

and if $\dim X = 3$, then

(ii) $\mathcal{V}(X, X) = k 1_X \oplus k e_{12}^{XX} \oplus k e_{23}^{XX} \oplus k e_{13}^{XX}$

or

(iii) $\mathcal{V}(X, X) = k 1_X \oplus k (e_{12}^{XX} + \lambda e_{23}^{XX}) \oplus k e_{13}^{XX}$, $\lambda \in k^*$.

It is clear that $\text{def}(X, X) = 0$ in cases (i) and (ii) and $\text{def}(X, X) = 1$ in case (iii).

Let $X, Y \in \mathcal{V}$ and let $X \neq Y$. According to Lemma 5 in [4] (this also follows from the proof of Lemma 2), the space $\mathcal{V}(X, Y)$ admits a basis consisting of linear maps of the form e_{ij}^{XY} and $e_{ij}^{XY} + \alpha e_{rl}^{XY}$ ($\alpha \neq 0$, $i \neq r$, and $j \neq l$). Moreover, it contains at most two morphisms of the second form, and if there are two such morphisms, then they have the form $e_{11}^{XY} + \lambda e_{22}^{XY}$, $e_{11}^{XY} + \mu e_{33}^{XY}$, $\lambda, \mu \in k^*$. Therefore, $\text{def}(X, Y) = 1$ and $\text{def } \mathcal{V} \leq 1$. By virtue of Lemma 2, $S(\mathcal{V})$ satisfies conditions I–IV. Condition V is satisfied because the equivalence class containing more than two edges has the following form: $\{x_i \Rightarrow y_j : i = 1, 2, 3\}$. According to Proposition 1 in [4], it contains a short pair of edges.

Proposition 2. *The map $\mathcal{V} \mapsto S(\mathcal{V})$ establishes a bijection between the isoclasses of finitely represented vectroids and the isoclasses of finitely represented locally finite completed biordered sets satisfying conditions I–VI.*

Proof. Let M_1 be the class of all finitely represented vectroids and let M_2 be the class of all finitely represented locally finite completed biordered sets satisfying conditions I–VI. By virtue of Lemma 6, if $\mathcal{V} \in M_1$, then $S(\mathcal{V}) \in M_2$. In the proof of Lemma 5, we have constructed, for every $S \in M_2$, a vectroid $\text{Vect}(S) \in M_1$ such that $S(\text{Vect}(S)) \approx S$. It remains to prove that $\text{Vect}(S(\mathcal{V})) \approx \mathcal{V}$ for all $\mathcal{V} \in M_1$.

Let $\mathcal{V} \in M_1$. Every space $\text{Rad}_{\mathcal{V}}(X, Y)$ has the form described in step I of the proof of Proposition 1, where $(s_1, t_1), \dots, (s_n, t_n)$ are minimal elements of the set $N_{XY} = \{(i, j) \mid X_i < Y_j, X_i, Y_j \in S(\mathcal{V})\}$. By analogy with the proof of Lemma 5, $S(\mathcal{V})$ defines the set of linear maps $L(X, Y)$. According to Propositions 1 and 2 in [4], the bases $(m_1^X, \dots, m_{d(X)}^X)$ of the spaces $X \in \mathcal{V}$ can be chosen so that $L(X, Y) \subset \text{Rad}_{\mathcal{V}}(X, Y)$, and m_i^X together with e_{ij}^{XY} ($X_i < Y_j$) and morphisms from $L(X, Y)$ forms a multiplicative basis of the vectroid \mathcal{V} . Thus, we get $\text{Vect}(S(\mathcal{V})) \simeq \mathcal{V}$.

Remark 5. Two vectroids \mathcal{V} and \mathcal{V}' are called *locally isomorphic* if there exists a bijection $f: \text{Ob } \mathcal{V} \rightarrow \text{Ob } \mathcal{V}'$ and, for every pair of objects $X, Y \in \mathcal{V}$ (including $X = Y$), there exist nondegenerate linear maps $\varphi: X \rightarrow f(X)$ and $\psi: Y \rightarrow f(Y)$ such that $\mathcal{V}(X, Y)\psi = \varphi \mathcal{V}'(f(X), f(Y))$. Let us prove the following assertion: *If finitely represented vectroids \mathcal{V} and \mathcal{V}' are locally isomorphic, then they are isomorphic.*

It is sufficient to show that the equality $\mathcal{V}(X, Y)\psi = \varphi \mathcal{V}'(f(X), f(Y))$ implies that $\mathcal{K}_{XY} = \mathcal{K}_{f(X)f(Y)}$ and $\{r \mid c_r^{XY} \neq 0\} = \{r \mid c_r^{f(X)f(Y)} \neq 0\}$ for all $X, Y \in \mathcal{V}$ (because, in this case, $S(\mathcal{V}) \simeq S(\mathcal{V}')$ and we can use Proposition 2).

For $\dim X = 2$, the statement is obvious. Let $\dim X = 3$ and let $X = Y$. It follows from the proof of Lemma 6 that $\mathcal{V}(X, X)$ has the form $k1_X \oplus k(e_{12}^{XX} + e_{23}^{XX}) \oplus ke_{13}^{XX}$ or $k1_X \oplus ke_{12}^{XX} \oplus ke_{23}^{XX} \oplus ke_{13}^{XX}$. Since the map $\alpha \mapsto \varphi^{-1}\alpha\psi$ defines an isomorphism of the spaces $\mathcal{V}(X, X)$ and $\mathcal{V}'(f(X), f(X))$, we can conclude that $\mathcal{V}(X, X)$ and $\mathcal{V}'(f(X), f(X))$ have the same dimension and, hence, the same form.

Let $X \neq Y$. We can write the following equality in the matrix form: $V\Psi = \Phi V'$, where $V = \mathcal{R}_{\mathcal{V}}(X, Y)$, $V' = \mathcal{R}_{\mathcal{V}'}(f(X), f(Y))$, and Ψ and Φ are the matrices of the maps ψ and φ , respectively. Since $V = SVR$ and $V' = S'V'R'$ for arbitrary upper triangular matrices S, R, S' , and R' with unit diagonal, we can replace the matrices Ψ and Φ by $R\Psi R'^{-1}$ and $S^{-1}\Phi S$ with exactly one nonzero element in each column and each row (i.e., by permutation matrices). However, it is impossible to pass from any staircase form of $\mathcal{R}_{\mathcal{V}}(X, Y)$ to another one by permutations. Remark 5 is proved.

3. S-Graphs

Let S denote a completed biordered set. A collection $(\mathcal{B}, \Gamma, -, \sim, \varphi_{\mathcal{B}})$ is called an *S-graph* if the following conditions are satisfied:

- (i) \mathcal{B} is a finite set (of *vertices* of the *S-graph*);
- (ii) $\Gamma \subset \mathcal{B}$ is a subset (of *nondegenerate* vertices);
- (iii) \sim is a pseudoequivalence relation on \mathcal{B} ;
- (iv) $-$ is a symmetric binary relation on Γ ;
- (v) $\varphi_{\mathcal{B}}: \mathcal{B} \rightarrow S$,

and, moreover,

- (vi) if $x - y$ belongs to Γ , then $\varphi_{\mathcal{B}}(x) \approx \varphi_{\mathcal{B}}(y)$; $|\Gamma^-(x)| \leq 1$ for any $X \in \Gamma$;
- (vii) if $x_1 \sim x_2$ (in \mathcal{B}), then $\varphi_{\mathcal{B}}(x_1) \sim \varphi_{\mathcal{B}}(x_2)$ (in S); $\varphi_{\mathcal{B}}(\mathcal{B}^{\sim}(x)) = S^{\sim}(\varphi_{\mathcal{B}}(x))$ for any $x \in \mathcal{B}$.

The set of vertices of the S -graph $(\mathcal{B}, \Gamma, -, \sim, \varphi_{\mathcal{B}})$ can be decomposed into nonintersecting subsets $\mathcal{B}^{\approx}(x)$, $x \in \mathcal{B}$. We call them *nodes* of the S -graph. The node $\mathcal{B}^{\approx}(x)$ is called the *node of the vertex* x or the *node that belongs to the equivalence class* $S^{\approx}(\varphi_{\mathcal{B}}(x))$.

The nodes of the S -graph \mathcal{B} form a graph $K(\mathcal{B})$ (nonoriented and, possibly, possessing loops and multiple edges), in which the edges between two nodes $X, Y \in K(\mathcal{B})$ are the pairs $\{x, y\}$, where $x \in X, y \in Y$, and $x - y$.

The S -graph \mathcal{B} is called *connected* if the graph $K(\mathcal{B})$ is connected. The S -graph \mathcal{B} is called *nondegenerate* if $\Gamma \cap \mathcal{B}^{\approx}(x) \neq \emptyset$ for all $x \in \mathcal{B}$.

Morphisms of S -graphs are defined in a natural way. In particular, one can speak about isomorphic S -graphs and S -subgraphs. Denote by $\hat{C}(S)$ the set of isoclasses of connected S -graphs.

Remark 6. The definition of an S -graph immediately implies that $\varphi_{\mathcal{B}}|_{\mathcal{B}^{\approx}(x)}$ is a bijection of $\mathcal{B}^{\approx}(x)$ onto $S^{\approx}(\varphi_{\mathcal{B}}(x))$ for every $x \in \mathcal{B}$. A connected S -graph is always nondegenerate except the case where $\Gamma = \emptyset$ and $|\mathcal{B}/\approx| = 1$.

Remark 7. We depict nondegenerate vertices of the S -graph by dots with the values of the map $\varphi_{\mathcal{B}}$ written above them. Two dots corresponding to vertices $x, y \in \Gamma$ are joined by a wavy line if $x \sim y$ and by a straight line if $x - y$. It follows from Remark 6 that a nondegenerate S -graph can be uniquely reconstructed by the quadruple $(\Gamma, -, \sim_{\Gamma}, \varphi_{\mathcal{B}}|_{\Gamma})$, where \sim_{Γ} is the restriction of \sim to Γ .

Remark 8. The definition of an S -graph does not take into account the relation \triangleleft on S , i.e., it is defined by S^w . In Sec. 5, we define the set $C(S)$ of (connected) S -graphs with marked vertex and introduce an order relation depending on \triangleleft on this set.

The sequence of vertices of an S -graph $(x'_0, x_1, x'_1, \dots, x_{n-1}, x'_{n-1}, x_n)$, $n \in \mathbb{N}$, such that $\mathcal{B}^{\approx}(x'_0) = X$, $\mathcal{B}^{\approx}(x_n) = Y$, $x_i \sim x'_i$ if $i = \overline{1, n-1}$, and $x'_i - x_{i+1}$ if $i = \overline{0, n-1}$, is called a *path in the S -graph \mathcal{B} from the node X to the node Y* . Note that $(\mathcal{B}^{\approx}(x'_0), \mathcal{B}^{\approx}(x_1), \mathcal{B}^{\approx}(x_2), \dots, \mathcal{B}^{\approx}(x_n))$ is a path in $K(\mathcal{B})$ from X to Y . Conversely, if $(X = X_0, X_1, \dots, X_{n-1}, X_n = Y)$ is a path in $K(\mathcal{B})$ from X to Y , then, in the S -graph \mathcal{B} , there exists a path $(x'_0, x_1, x'_1, \dots, x_{n-1}, x'_{n-1}, x_n)$ from X to Y such that $\mathcal{B}^{\approx}(x_i) = X_i$, $i = \overline{1, n}$. 'By virtue of Remark 6, a path in the S -graph \mathcal{B} consists of only nondegenerate vertices.

An S -graph \mathcal{B} is called *acyclic* if the graph $K(\mathcal{B})$ is a forest (i.e., contains no cycles). A completed bi-ordered set S is called *acyclic* if every S -graph is acyclic. It obviously follows from the definitions introduced that if \mathcal{B} is an acyclic S -graph, then

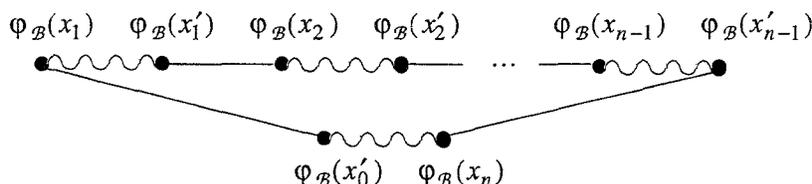
- (i) \mathcal{B} is a connected graph if and only if $K(\mathcal{B})$ is a tree,
- (ii) if \mathcal{B} is a connected graph, then the path in \mathcal{B} from X to Y exists and is unique for any nodes $X, Y \in K(\mathcal{B})$.

Lemma 7. Let S be acyclic, let $X, Y, Z, T \in K(\mathcal{B})$ be nodes of the connected S -graph \mathcal{B} such that $\varphi_{\mathcal{B}}(X) \approx \varphi_{\mathcal{B}}(Y) \approx \varphi_{\mathcal{B}}(Z) \approx \varphi_{\mathcal{B}}(T)$ in S , let (x'_0, \dots, x_n) be a path in \mathcal{B} from X to Y , and let (z'_0, \dots, z_m) be a path in \mathcal{B} from Z to T . Then the following relations hold:

- (a) $\varphi_{\mathcal{B}}(x'_0) = \varphi_{\mathcal{B}}(x_n)$,

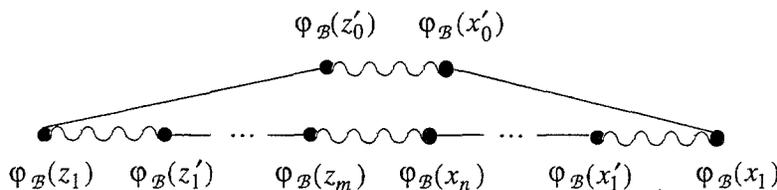
(b) $\varphi_{\mathcal{B}}(z'_0) = \varphi_{\mathcal{B}}(x'_0)$.

Proof. (a) Let $\varphi_{\mathcal{B}}(x'_0) \neq \varphi_{\mathcal{B}}(x_n)$. Then



is an S -graph, which contradicts the assumption that S is acyclic.

(b) By virtue of (a), we have $\varphi_{\mathcal{B}}(z'_0) = \varphi_{\mathcal{B}}(z_m)$. Therefore, it follows from the relation $\varphi_{\mathcal{B}}(z'_0) \neq \varphi_{\mathcal{B}}(x'_0)$ that $\varphi_{\mathcal{B}}(z_m) \neq \varphi_{\mathcal{B}}(x_n)$ and



is an S -graph, which contradicts the assumption that S is acyclic.

Lemma 8. *If S is acyclic, the S -graph \mathcal{B} is connected, and $K(\mathcal{B})$ contains at least three vertices, then there exists a node $X \in K(\mathcal{B})$ such that the following relations hold:*

(a) $\varphi_{\mathcal{B}}(X) \neq \varphi_{\mathcal{B}}(Y)$ for an arbitrary node $Y \in K(\mathcal{B})$,

(b) X contains at most one nondegenerate vertex.

Proof. (a) Assume that $\mathcal{A} = (A^1, \dots, A^m)$ is the set of all classes from S/\approx such that there exists a node $Y \in K(\mathcal{B})$ with $\varphi_{\mathcal{B}}(Y) \subset A^i$ for a proper $i = \overline{1, m}$. Let us transform \mathcal{A} into an oriented graph. For this purpose, we assume that the arrow $A^i \rightarrow A^j$ exists if there are nodes X_i, Y_i , and Z_j in \mathcal{B} such that $\varphi_{\mathcal{B}}(X_i) \subset A^i$, $\varphi_{\mathcal{B}}(Y_i) \subset A^i$, $\varphi_{\mathcal{B}}(Z_j) \subset A^j$, and the path from X_i to Y_i [in $K(\mathcal{B})$] passes through Z_j .

Let us prove that \mathcal{A} contains no oriented cycles. Indeed, assume that, after a proper enumeration, it contains the cycle

$$\begin{array}{ccc} A^2 & \rightarrow & A^3 \rightarrow \dots \rightarrow A^{n-1} \\ \uparrow & & \downarrow, \quad n \geq 1. \\ A^1 & \leftarrow & A^n \end{array}$$

This means that, for every $i = \overline{1, n}$, there exists a path $(s_i | \mathbf{a}_i | t_i, p_i | \mathbf{b}_i | r_i)$ in the S -graph \mathcal{B} with $\varphi_{\mathcal{B}}(s_i)$,

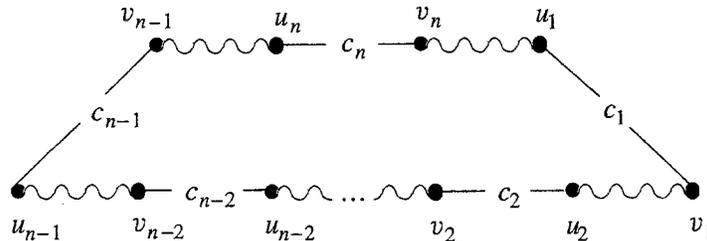
$\varphi_{\mathcal{B}}(r_i) \in A^i$, $\varphi_{\mathcal{B}}(t_i)$, $\varphi_{\mathcal{B}}(p_i) \in A^{i'}$, and $t_i \sim p_i$ in \mathcal{B} ; here, \mathbf{a}_i and \mathbf{b}_i are paths in \mathcal{B} , the symbol $|$ denotes linking of paths, and

$$i' = \begin{cases} i+1, & i = \overline{1, n-1}; \\ 1, & i = n. \end{cases}$$

By virtue of Lemma 7, $\varphi_{\mathcal{B}}(s_i) = \varphi_{\mathcal{B}}(r_i)$. By the definition of a path, $\varphi_{\mathcal{B}}(t_i) \sim \varphi_{\mathcal{B}}(p_i)$. Let

$$(u_i | \mathbf{c}_i | v_i) = \begin{cases} (s_i | \mathbf{a}_i | t_i) & \text{if } \varphi_{\mathcal{B}}(t_i) \neq \varphi_{\mathcal{B}}(s_{i'}), \\ (r_i | \bar{\mathbf{b}}_i | p_i) & \text{if } \varphi_{\mathcal{B}}(t_i) = \varphi_{\mathcal{B}}(s_{i'}). \end{cases}$$

Here, $\bar{\mathbf{b}}$ is the path \mathbf{b} passed in the opposite direction. Then



is an S -graph, which contradicts the assumption that S is acyclic.

Thus, either \mathcal{A} is a disconnected union of points or \mathcal{A} contains at least one sink (i.e., a point with no outgoing arrows) with an arrow really entering this point. In the first case, the number of nondegenerate vertices of a node is not less than the number of edges of $K(\mathcal{B})$ originating from this node, and two nodes from this equivalence class in S cannot be joined by an edge [in $K(\mathcal{B})$]. Therefore, as the required node X , one can take any node such that at least two edges originate from it.

In the second case, we assume that $B \in \mathcal{A}$ is a sink, $A \in \mathcal{A}$, and $A \rightarrow B$. Then, in $K(\mathcal{B})$, there exists a path of the form $Y_1 - \dots - X - \dots - Y_2$, where Y_1 and Y_2 belong to A , X belongs to B , and X contains at least two vertices. If $X' \in K(\mathcal{B})$ and $X' \neq X$ is a node that also belongs to B , then it cannot be joined with X by a straight line in $K(\mathcal{B})$ and there is a node between X' and X that does not belong to B . Hence, B is not a sink. Thus, X is the required node.

Proposition 3. *A completed biordered set S is finite and acyclic if and only if $|\hat{C}(S)| < \infty$.*

Proof. The necessity is obvious. To prove the proposition it suffices to show that, for a finite acyclic biordered set S , we have $\sup \{|\Gamma| \mid (\mathcal{B}, \Gamma, -, \sim, \varphi_{\mathcal{B}}) \in \hat{C}(S)\} < \infty$. For this purpose, we define functions $\bar{\varphi}_{\mathcal{B}}: \mathcal{B} \rightarrow S/\approx$ and $\bar{\varphi}_{\mathcal{B}} = \text{can} \circ \varphi_{\mathcal{B}}$ and note the following: By virtue of Lemma 7, for any path (x'_0, x_1, \dots, x_n) in the acyclic S -graph \mathcal{B} and for any $i = \overline{1, n}$, there exists at most one $j = \overline{1, n}$ such that $\bar{\varphi}_{\mathcal{B}}(x_i) = \bar{\varphi}_{\mathcal{B}}(x_j)$. Therefore, the length of a path that joins two arbitrary vertices of $K(\mathcal{B})$ does not exceed $2|S/\approx|$. Since $K(\mathcal{B})$ is a tree, $|\Gamma/\approx|$ does not exceed the number of vertices of a complete tree of height $2|S/\approx|$ on the set S/\approx . This immediately implies that the value $|\Gamma|$ is bounded.

4. Elementary Representations

For a chain vectroid \mathcal{V} , we set $S = S(\mathcal{V})$ and $\hat{C}(\mathcal{V}) = \hat{C}(S(\mathcal{V}))$. A vectroid \mathcal{V} is called *quasifinite* if $\hat{C}(\mathcal{V})$ is a finite set and $\text{def } \mathcal{V} \leq 1$. Below, we show that finitely represented vectroids are quasifinite.

We fix triangular bases (m_i^X) of objects X of the vectroid \mathcal{V} and identify $s \in S(\mathcal{V})$ with the corresponding vector.

A representation M of a locally finite completed biorordered set S is called *elementary* if the following conditions are satisfied:

- (i) the elements of the matrix M are equal to either 0 or 1;
- (ii) every row of M contains at most two unit entries; each of its columns contains at most one unit entry;
- (iii) if there are two nonzero elements in a row of M , they belong to incomparable columns.

In particular, by setting $S = S(\mathcal{V})$, we arrive at the notion of *elementary representation of the vectroid \mathcal{V}* [in the basis (m_i^X)].

For every matrix of elementary representation nondegenerate with respect to rows, we construct an $S(\mathcal{V})$ -graph as follows:

- (i) vertices of the graph are considered as columns of the matrix M of the given representation;
- (ii) nonzero columns of the matrix M are regarded as nondegenerate vertices;
- (iii) above each vertex, we write the element of $S(\mathcal{V})$ related to the corresponding column;
- (iv) we join two vertices by a straight line if these columns have a unit entry in a common row;
- (v) we join two vertices by a wavy line if these columns are equivalent.

Conversely, an arbitrary S -graph $\mathcal{B} \in \hat{C}(\mathcal{V})$ is associated with a nondegenerate elementary representation $R(\mathcal{B}) \in \text{Rep } \mathcal{V}$. For a node $X \in K(\mathcal{B})$, we denote by V_X a uniquely defined object of the vectroid \mathcal{V} such that $\varphi_{\mathcal{B}}(X) \subset V_X$. Let

$$\mathcal{X}_{\mathcal{B}} = \bigoplus_{X \in K(\mathcal{B})} V_X,$$

let $U_{\mathcal{B}}$ be a subspace of $\mathcal{X}_{\mathcal{B}}$ spanned either on the vectors $\varphi_{\mathcal{B}}(x)$ if $x \in \Gamma$ and $\Gamma^-(x) = \emptyset$ or on the vectors $\varphi_{\mathcal{B}}(x) + \varphi_{\mathcal{B}}(y) \in V_{\mathcal{B}^-(x)} \oplus V_{\mathcal{B}^-(y)}$ if $x - y$ in Γ , and let $i_{\mathcal{B}}: U_{\mathcal{B}} \rightarrow \mathcal{X}_{\mathcal{B}}$ be the imbedding of subspaces. Thus, we obtain the representation $R(\mathcal{B}) = (U_{\mathcal{B}}, i_{\mathcal{B}}, \mathcal{X}_{\mathcal{B}})$, which has the matrix required in the definition of elementary representations in the basis $S(\mathcal{V})$ of the vectroid \mathcal{V} and in the basis $U_{\mathcal{B}}$ formed of the generating vectors. Furthermore, $R(\mathcal{B}) \in {}^i \text{Rep } \mathcal{V}$ because $i_{\mathcal{B}}$ is an injection. In what follows, we identify a vertex x of the S -graph \mathcal{B} and the vector $\varphi_{\mathcal{B}}(x) \in V_{\mathcal{B}^-(x)}$. Note that $\{x \mid x \in \mathcal{B}\}$ is a basis of the space $\mathcal{X}_{\mathcal{B}}$.

Note that R establishes a bijection between the set of classes of isomorphisms of $S(\mathcal{V})$ -graphs and the set of matrices of elementary representations nondegenerate with respect to the rows of the vectroid \mathcal{V} in the given basis $S(\mathcal{V})$; here, the matrices are considered up to permutations of rows and columns.

In view of the structure of the bijection R , we can assume that every node $\mathcal{B}^\approx(x)$ of the S -graph \mathcal{B} consists of elements of the set $S \times \mathbb{N}$ of the form (s, i) , where $s \in S^\approx(\varphi_{\mathcal{B}}(x))$ and i is the number of the corresponding columns in bands determined by elements s of a certain matrix of the representation $R(\mathcal{B})$.

Example 3. Generally speaking, an elementary representation in one basis can be equivalent to a nonelementary representation in another basis. Let \mathcal{W} be the vectroid defined in Example 2 and let $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ be the matrix of a representation of \mathcal{W} . In this case, the representation is elementary in the basis $m_1^X, m_2^X, m_3^X, m_4^X$. However, in the basis $\bar{m}_1^X = m_1^X - m_2^X, \bar{m}_2^X = m_2^X, \bar{m}_3^X = m_3^X, \bar{m}_4^X = m_4^X$, it is determined by the matrix $\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$, which is not equivalent to an elementary one.

Let \mathcal{V}'_1 and \mathcal{V}'_2 be two chain vectroids, $S_1 = S(\mathcal{V}'_1), S_2 = S(\mathcal{V}'_2)$, and let f be an injective map $S_1 \rightarrow S_2$ (which does not take into account structures on S_1 and S_2). Assume that f is induced by a morphism $(F, \Phi) \in \mathcal{M}(\mathcal{V}'_1, \mathcal{V}'_2)$.

For a nondegenerate S_1 -graph $(\mathcal{B}, \Gamma, -, \sim, \varphi_{\mathcal{B}})$, we define a nondegenerate S_2 -graph $(\mathcal{D}, \Delta, -', \sim', \varphi_{\mathcal{D}})$ as follows: $\Delta = \Gamma$ and, for $x, y \in \Delta$,

- (i) $x -' y$ if $x - y$ (in Γ) and $f \circ \varphi_{\mathcal{B}}(x) \approx f \circ \varphi_{\mathcal{B}}(y)$ (in S_2);
- (ii) $x \sim' y$ if $x \sim y$ (in Γ) and $f \circ \varphi_{\mathcal{B}}(x) \sim f \circ \varphi_{\mathcal{B}}(y)$ (in S_2);
- (iii) $\varphi_{\mathcal{D}}|_{\Delta} = f \circ \varphi_{\mathcal{B}}|_{\Gamma}$.

Then we reconstruct \mathcal{D} by using Remark 6. Obviously, $f_*(\mathcal{B})$ is a nondegenerate S_2 -graph.

Let f be such that, for $s, t \in S_1$, it follows from $s \approx t$ that $f(s) \approx f(t)$. Then, clearly, $\text{Der}(F, \Phi)(R(\mathcal{B})) = R(f_*(\mathcal{B}))$.

Example 4. Let \mathcal{B} be a nondegenerate S -graph.

- (a) Let \mathcal{V} be a vectroid, $S = S(\mathcal{V})$, let $S^w = (S, \leq, \approx)$ be a weakly completed poset (Sec. 1), and let $w^s: S \rightarrow S^w$ be an identity map of supporting sets. By using S^w , we construct a vectroid \mathcal{V}^w (whose rank does not exceed one) and obtain a morphism $W \in \mathcal{M}(\mathcal{V}, \mathcal{V}^w)$ from w^s . Then $R(w^s_*(\mathcal{B})) = \text{Der}(W)(R(\mathcal{B}))$.
- (b) For a weakly completed poset (S, \leq, \approx) and $Q \subset S$, we construct $S^Q = (S, \leq, \approx')$; let $d^Q: S \rightarrow S^Q$ be an identity map and assume that $x \sim' y$ if and only if $x \sim y$ and $x, y \notin Q$. The operation of passing from S to S^Q with the help of d^Q (or a morphism of the corresponding vectroids $d^Q: \mathcal{V} \rightarrow \mathcal{V}^Q$) is called a *wave break* for Q . The following equality is true:

$$R(d^Q_*(\mathcal{B})) = \text{Der}(D^Q)(R(\mathcal{B})).$$

Below, we obtain criteria for quasifinite vectroids to be indecomposable and for elementary representations to be equivalent. First, note that if an S -graph \mathcal{B} is disconnected, then the representation $R(\mathcal{B})$ is decomposable. Indeed, let $\mathcal{B} = \mathcal{B}_1 \amalg \mathcal{B}_2$. In this case, $R(\mathcal{B}) = R(\mathcal{B}_1) \oplus R(\mathcal{B}_2)$, which directly follows from the construction.

Proposition 4. *Suppose that a vectroid \mathcal{V} is acyclic, \mathcal{B} and \mathcal{D} are S -graphs, and the S -graph \mathcal{B} is connected. Then the following statements are true:*

- (a) *If $R(\mathcal{B}) \simeq R(\mathcal{D})$, then \mathcal{B} is isomorphic to \mathcal{D} .*
- (b) *$R(\mathcal{B})$ is indecomposable.*

Proof. First, note that Example 4(a) allows us to regard S as a weakly completed poset. We prove the proposition by induction on $\mathbf{d}(\mathcal{B})$ for all weakly completed posets S and S -graphs \mathcal{B} simultaneously. Denote by $\mathbf{d}(\mathcal{B})$ the number of waves in Γ , i.e., $\mathbf{d}(\mathcal{B}) = |\{\{x, y\} \subseteq \Gamma \mid x \sim y\}|$. In this case, if there is a node in \mathcal{B} that contains at least two nondegenerate vertices and one of them belongs to an element of \mathcal{Q} , then $\mathbf{d}(d_*^{\mathcal{Q}}(\mathcal{B})) < \mathbf{d}(\mathcal{B})$. If $\Gamma = \emptyset$, the proposition is obvious (see Remark 6). Therefore, we assume that $\Gamma \neq \emptyset$.

We take $\mathbf{d}(\mathcal{B}) = 0$ as a basis of induction. By virtue of connectedness, \mathcal{B} has the form

$$\mathcal{B}_1 = \underset{s}{\bullet} \quad \text{or} \quad \mathcal{B}_2 = \underset{s}{\bullet} \text{---} \underset{t}{\bullet},$$

where $s, t \in S$. In this case, the proposition can easily be proved by using the fact that S is acyclic and the exchange theorem ([1], 3.3, b).

Induction step. By virtue of Lemma 8, there are two possibilities for the connected S -graph \mathcal{B} , namely, either it contains no vertices joined by wavy lines, i.e., $\mathbf{d}(\mathcal{B}) = 0$, or there is a node containing at least two vertices and there is no other node in \mathcal{B} that belongs to the same equivalence class in S . By choosing an arbitrary vertex x from this node and setting $\mathcal{Q} = \{\varphi_{\mathcal{B}}(x)\}$, we get $\mathcal{B}' = d_*^{\mathcal{Q}}(\mathcal{B})$ and $\mathcal{D}' = d_*^{\mathcal{Q}}(\mathcal{D})$, where $\mathcal{B}' = \mathcal{B}_1 \amalg \mathcal{B}_2$, \mathcal{B}_1 and \mathcal{B}_2 are connected $S(\mathcal{V}^{\mathcal{Q}})$ -graphs, and $R(\mathcal{B}') \simeq R(\mathcal{D}')$ [see Example 4(b)].

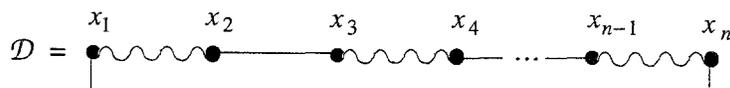
By the induction hypothesis and the uniqueness of the decomposition in direct sum in $\text{Rep } \mathcal{V}^{\mathcal{Q}}$, we establish that $\mathcal{D}' = \mathcal{B}' = \mathcal{B}_1 \amalg \mathcal{B}_2$, $R(\mathcal{D}') = R(\mathcal{B}_1) \oplus R(\mathcal{B}_2)$, and $R(\mathcal{B}_1)$ and $R(\mathcal{B}_2)$ are indecomposable. Therefore, either $\mathcal{D} = \mathcal{B}$ or $\mathcal{D} = \mathcal{B}_1 \amalg \mathcal{B}_2$. In the latter case, $\mathcal{X}_{\mathcal{D}}$ contains two direct summands of the form $V_{\mathcal{B}^{\varphi(x)}}$ and $\mathcal{X}_{\mathcal{B}}$ contains one such summand, which is impossible. If $R(\mathcal{B})$ is decomposable, i.e., $R(\mathcal{B}) = (U_1, i_1, I_1) \oplus (U_2, i_2, I_2)$, then $\text{Der}(\mathcal{D}^{\mathcal{Q}})(U_1, i_1, I_1) \simeq R(\mathcal{B}_1)$ and $\text{Der}(\mathcal{D}^{\mathcal{Q}})(U_2, i_2, I_2) \simeq R(\mathcal{B}_2)$. We again get two direct summands of the form $V_{\mathcal{B}^{\varphi(x)}}$ in $I_1 \oplus I_2$, but $\mathcal{X}_{\mathcal{B}}$ has one such summand. Thus, $R(\mathcal{B})$ is indecomposable. Proposition 4 is proved.

Corollary 1. *Let a vectroid \mathcal{V} be acyclic and let \mathcal{B} and \mathcal{D} be two arbitrary $S(\mathcal{V})$ -graphs. Then the relation $\mathcal{B} \simeq \mathcal{D}$ follows from $R(\mathcal{B}) \simeq R(\mathcal{D})$.*

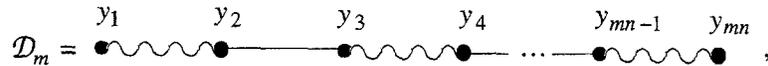
This statement is a consequence of Proposition 4 (a) and Remark 6.

Example 5. Let \mathcal{V} be a chain vectroid and let $S(\mathcal{V})$ be not acyclic. Let us prove that \mathcal{V} has infinitely many nonisomorphic indecomposable elementary representations.

Assume that the nondegenerate $S = S(\mathcal{V})$ -graph



is a cycle (note that $n > 3$). We assume that the cycle \mathcal{D} is minimal. We define the S -graph



$\varphi_{\mathcal{D}_m}(y_i) = \varphi_{\mathcal{D}}(x_{\bar{i}})$, where $1 \leq \bar{i} \leq n$ and $\bar{i} \bmod n = i \bmod n$. Then, for any $m \in \mathbb{N}$, the representation $R(\mathcal{D}_m) \in \text{Rep } \mathcal{V}$ is indecomposable. In this case, obviously, $R(\mathcal{D}_m) \neq R(\mathcal{D}_{m'})$ if $m \neq m'$. Example 4 (a) allows us to assume that S is a weakly completed poset.

For a vertex y of the S -graph \mathcal{D}_m , we denote by y' the nondegenerate vertex such that $y' \sim y$. For $y \notin \{y_1, y_{mn}\}$, we denote by \bar{y} the vertex such that $\bar{y} - y$.

Let M be the matrix of the representation $R(\mathcal{D}_m)$, $\dim U_{\mathcal{D}_m} = l$, $\dim X_{\mathcal{D}_m} = k$. Let us introduce a partial ordering \preceq on the set of nondegenerate vertices Δ_m of the S -graph \mathcal{D}_m . Let y_i and y_j belong to Δ_m . In the case $\varphi_{\mathcal{D}_m}(y_i) \neq \varphi_{\mathcal{D}_m}(y_j)$, we set $y_i \prec y_j$ if and only if $\varphi_{\mathcal{D}_m}(y_i) < \varphi_{\mathcal{D}_m}(y_j)$. If $\varphi_{\mathcal{D}_m}(y_i) = \varphi_{\mathcal{D}_m}(y_j)$, then, in order to define \prec , we construct the following two sequences of integer pairs (possibly, of zero length):

$$(a_1, b_1), \dots, (a_\alpha, b_\alpha), (a_{\alpha+1}, b_{\alpha+1}), \dots,$$

$$(c_1, d_1), \dots, (c_\alpha, d_\alpha), (c_{\alpha+1}, d_{\alpha+1}), \dots$$

If $j \in \{1, mn\}$, we set $y_i \prec y_j$ and do not start the construction; if $i \in \{1, mn\}$, we set $y_i \not\prec y_j$ and also do not start the construction. If $i, j \notin \{1, mn\}$, then $(y_{a_i}, y_{b_i}) = (y_i, y_j)$ and $(y_{c_i}, y_{d_i}) = (\bar{y}_i, \bar{y}_j)$.

We continue the construction by induction. If $(a_\alpha, b_\alpha) = (c_\alpha, d_\alpha) = (0, 0)$, then the construction is completed and we set $y_i \prec y_j$. If $(a_\alpha, b_\alpha) = (0, 0)$, then $(a_{\alpha+1}, b_{\alpha+1}) = (0, 0)$. If $(c_\alpha, d_\alpha) = (0, 0)$, then $(c_{\alpha+1}, d_{\alpha+1}) = (0, 0)$. If y'_{a_α} or y'_{c_α} belongs to $\{y_1, y_{mn}\}$, then the construction is completed and we set $y_i \not\prec y_j$. If $b_\alpha \in \{1, mn\}$, then $(a_{\alpha+1}, b_{\alpha+1}) = (0, 0)$. If $d_\alpha \in \{1, mn\}$, then $(c_{\alpha+1}, d_{\alpha+1}) = (0, 0)$. Otherwise, if $(a_\alpha, b_\alpha) \neq (0, 0)$ and $y'_{a_\alpha}, y'_{b_\alpha} \neq y_1, y_{mn}$ then $(y_{a_{\alpha+1}}, y_{b_{\alpha+1}}) = ((y'_{a_\alpha})^-, (y'_{b_\alpha})^-)$. If $(c_\alpha, d_\alpha) \neq (0, 0)$ and $y'_{c_\alpha}, y'_{d_\alpha} \neq y_1, y_{mn}$ then

$$(y_{c_{\alpha+1}}, y_{d_{\alpha+1}}) = ((y'_{c_\alpha})^-, (y'_{d_\alpha})^-).$$

By construction, the relation $y_i \preceq y_j \preceq y_i$ implies that $y_i = y_j$.

Let us introduce a partial ordering \sqsubseteq on rows of M . For this purpose, note that the set of rows of M is bijective to the set $K(\mathcal{D}_m) = \{\{y_1\}, \{y_2, y_3\}, \dots, \{y_{m\#2}, y_{m\#1}\}, \{y_{mn}\}\}$. We set $A \sqsubseteq B$, $A, B \in K(\mathcal{D}_m)$, if, for every $z \in B$, one can find $\bar{z} \in A$ such that $\bar{z} \preceq z$ (cf. [5, p. 13]).

Let $(\varphi, \xi) \in \text{End}_{\text{Rep } \mathcal{V}}(R(\mathcal{D}_m))$ be an endomorphism and let $(F = (f_{ij}), G = (g_{ij}))$ be its matrix notation. In particular, $FM = MG$. Then, by analogy with the proof of Lemma 10, we can show that $f_{ij} \neq 0$ for some (φ, ξ) if and only if $i \sqsubseteq j$, $i, j = \overline{1, l}$, and $g_{ij} \neq 0$ for some (φ, ξ) if and only if $i \preceq j$, $i, j = \overline{1, k}$. Hence, we conclude that \sqsubseteq and \preceq are transitive and, therefore, they are orders.

Let us introduce linear orders on the bases $U_{\mathcal{D}_m}$ and $X_{\mathcal{D}_m}$ so that $i \sqsubseteq j$ yields $i \leq j$ (for $U_{\mathcal{D}_m}$) and $i \preceq j$ yields $i \leq j$ (for $X_{\mathcal{D}_m}$). This can be done because every order can be extended to a linear one (see [14], VII.8, Theorem 8). For such an ordering of bases, the matrices F and G are upper triangular.

Assuming that $R(\mathcal{D}_m)$ is decomposable, we can find an idempotent (φ, ξ) . Then, for arbitrary $\lambda \in k^*$, the pair $(\varphi + \lambda \text{id}_{U_{\mathcal{D}_m}}, \xi + \lambda \text{id}_{X_{\mathcal{D}_m}})$ is an endomorphism of $R(\mathcal{D}_m)$. An arbitrary nonzero element of the matrix $(F +$

$\lambda E_l)M$ has the form $(\lambda + f_{ii})m_{it}$ or $\sum_{j \neq i} a_{ij}m_{jt}$. Similarly, every nonzero element of the matrix $M(G + \lambda E_k)$ has the form $m_{it}(\lambda + g_{tt})$ or $\sum_{j \neq i} m_{ij}g_{jt}$. Since $(\lambda E_l + F)M = M(\lambda E_k + G)$ for all $\lambda \in k^*$, the nonzero element of this matrix is equal to $(\lambda + f_{ii})m_{it} = m_{it}(\lambda + g_{tt})$ or $\sum_{j \neq i} a_{ij}m_{jt} = \sum_{j \neq i} m_{ij}g_{jt}$. Therefore, $(\bar{\varphi}, \bar{\xi}) \in \text{End}(R(\mathcal{D}_m))$ (here, $\bar{\varphi}$ is the diagonal of the matrix φ).

But then we can conclude that the S -graph \mathcal{D}_m is disconnected, which is not true. Therefore, the representation $R(\mathcal{D}_m)$ is indecomposable.

Proposition 5. *Every chain vectroid \mathcal{V} has finitely many classes of isomorphisms of indecomposable elementary representations if and only if $|\hat{C}(\mathcal{V})| < \infty$.*

This statement obviously follows from Propositions 3 and 4 and Example 5.

Corollary 2. *A finitely represented vectroid \mathcal{V} is quasifinite.*

The validity of this statement follows from Proposition 5.

5. Order on $C(S)$

A pair (\mathcal{B}, x) , where \mathcal{B} is a connected S -graph and $x \in \mathcal{B} \setminus \Gamma$, is called an S -graph with marked vertex. Denote the set of classes of isomorphisms of S -graphs with marked vertex by $C(S)$. We assume that the completed biorordered set S is acyclic.

Let $(\mathcal{B}, \Gamma, -, \sim, \varphi_{\mathcal{B}})$ be a connected S -graph and let $x \in \mathcal{B}$ be its vertex. We define (connected) S -graphs $l_x \mathcal{B}$ and $r_x \mathcal{B}$ as follows:

- (a) Let $x \in \Gamma$ and let $\Gamma^-(x) = \{y\}$. We remove the straight line that joins x and y from Γ , i.e., we introduce a relation $-'$ on Γ as follows: For $z_1, z_2 \in \Gamma$, we have $z_1 - z'_2$ if and only if $z_1 - z_2$ and $\{z_1, z_2\} \neq \{x, y\}$. Then, since S is acyclic, $(\mathcal{B}, \Gamma \setminus \{x, y\}, -', \sim, \varphi_{\mathcal{B}})$ is a disconnected S -graph with two connected components. One of these, denoted by $l_x \mathcal{B}$, contains the vertex x , and the other one, denoted by $r_x \mathcal{B}$, contains y .
- (b) Let $x \in \Gamma$ and let $\Gamma^-(x) = \emptyset$. In this case, $l_x \mathcal{B} = (\mathcal{B}, \Gamma \setminus \{x\}, -', \sim, \varphi_{\mathcal{B}})$, where $-'$ is a restriction to $\Gamma \setminus \{x\}$ and $r_x \mathcal{B} = (\mathcal{B}^{\neq}(x), \emptyset, \sim|_{\mathcal{B}^{\neq}(x)}, \varphi_{\mathcal{B}}|_{\mathcal{B}^{\neq}(x)})$.
- (c) Let $x \notin \Gamma$. In this case, $l_x \mathcal{B} = \mathcal{B}$ and $r_x \mathcal{B} = \emptyset$.

For $(\mathcal{B}, x) \in C(S)$ and $s \in S^-(\varphi_{\mathcal{B}}(x))$, we now introduce derivatives $\partial_s^1(\mathcal{B}, x)$ and $\partial_s^2(\mathcal{B}, x)$. Let $y \in \mathcal{B}^-(x)$ and let $\varphi_{\mathcal{B}}(y) = s$. We set

$$\begin{aligned} \partial_s^1(\mathcal{B}, x) &= (l_y \mathcal{B}, y), \\ \partial_s^2(\mathcal{B}, x) &= \begin{cases} (r_y \mathcal{B}, z) & \text{if } y \in \Gamma \text{ and } y - z; \\ 0 & \text{if } y \notin \Gamma; \\ 1 & \text{if } y \in \Gamma, \Gamma^-(y) = \emptyset. \end{cases} \end{aligned}$$

We denote $\partial_{x'}(\mathcal{B}, x) = \partial_{\varphi_{\mathcal{B}(x')}}(\mathcal{B}, x)$ for a vertex $x' \in \mathcal{B}^-(x)$. For an S -graph (\mathcal{D}, y) with $\varphi_{\mathcal{D}(y)} \approx \varphi_{\mathcal{B}(x)}$ and for a vertex $y' \in \mathcal{D}^-(y)$, $\varphi_{\mathcal{D}(y')} \sim \varphi_{\mathcal{B}(x)}$, we write $\partial_{y'}(\mathcal{B}, x) = \partial_{\varphi_{\mathcal{D}(y')}}(\mathcal{B}, x)$.

Let us introduce a relation \leq on $C(S)$. First, we add the maximal element 1 and the minimal element 0 to $C(S)$ (the relation \leq on $C(S)$ is induced by the relation \leq on $C(S) \sqcup \{0, 1\}$). Consider the S -graphs (\mathcal{B}, x) and $(\mathcal{D}, y) \in C(S)$. The relation \leq is defined by induction on the common number of vertices of both S -graphs \mathcal{B} and \mathcal{D} .

- (a) If $\varphi_{\mathcal{B}(x)} \approx \varphi_{\mathcal{D}(y)}$, we set $(\mathcal{B}, x) \approx (\mathcal{D}, y)$;
- (b) If $\varphi_{\mathcal{B}(x)} \triangleleft \varphi_{\mathcal{D}(y)}$, we set $(\mathcal{B}, x) \leq (\mathcal{D}, y)$;
- (c) If $\varphi_{\mathcal{B}(x)} = \varphi_{\mathcal{D}(y)} = s$ and $s \triangleleft s$, then $(\mathcal{B}, x) \leq (\mathcal{D}, y)$ if and only if $\partial_t^2(\mathcal{B}, x) \leq \partial_t^2(\mathcal{D}, y)$ for every $t \in S^-(s)$;
- (d) If $s_1 = \varphi_{\mathcal{B}(x)} \Rightarrow s_2 = \varphi_{\mathcal{D}(y)}$, then $(\mathcal{B}, x) \leq (\mathcal{D}, y)$ if and only if there exist $t_1, t_2 \in S$ such that $s_1 \Rightarrow s_2, t_1 \Rightarrow t_2, s_1 \sim t_1, s_2 \sim t_2$, and at least one of the following conditions is satisfied:
 - (d1) $\partial_{t_1}^2(\mathcal{B}, x) \leq \partial_{t_2}^2(\mathcal{D}, y)$;
 - (d2) $\partial_{t_1}^1(\mathcal{B}, x) \leq \partial_{t_2}^2(\mathcal{D}, y)$;
 - (d3) $\partial_{t_1}^2(\mathcal{B}, x) \leq \partial_{t_2}^1(\mathcal{D}, y)$.

The reason for introducing the relation \leq is clarified by the following construction: Let \mathcal{V} be a chain vectroid and let $S = S(\mathcal{V})$. We define a *spectroid* El of elementary representations with marked zero column as follows: The objects of the spectroid El are the elements of the set $C(\mathcal{V}) = C(S(\mathcal{V}))$. For $(\mathcal{B}, x), (\mathcal{D}, y) \in C(\mathcal{V}) = \text{El}$, morphisms $(\varphi, \xi) \in \text{Rep } \mathcal{V}(R\mathcal{B}, R\mathcal{D})$ such that $x\xi \in U_{\mathcal{D}} + yk$ form the set of morphisms $\text{El}((\mathcal{B}, x), (\mathcal{D}, y))$. Note that $y \notin U_{\mathcal{D}}$ and $U_{\mathcal{D}} + yk$ is a subspace of $\mathcal{X}_{\mathcal{D}}$. Obviously, El is a category because $\text{El}((\mathcal{B}, x), (\mathcal{D}, y)) \circ \text{El}((\mathcal{D}, y), (\mathcal{E}, z)) \subset \text{El}((\mathcal{B}, x), (\mathcal{E}, z))$.

Lemma 9. *If a vectroid \mathcal{V} is acyclic, then El is a spectroid.*

Proof. The indecomposability of the representation $R(\mathcal{B})$ proved in Proposition 4 (b) implies that the object $(\mathcal{B}, x) \in \text{El}$ is indecomposable.

Let us show that different objects of the category El are not isomorphic. Let $(\mathcal{B}, x), (\mathcal{D}, y) \in \text{El}$. If $\mathcal{B} \neq \mathcal{D}$, then $R(\mathcal{B}) \neq R(\mathcal{D})$ by virtue of Proposition 4 (a). Hence, $(\mathcal{B}, x) \neq (\mathcal{D}, y)$. If $\mathcal{B} = \mathcal{D}$ but $x \neq y$, then the isomorphism $(\varphi, \xi): (\mathcal{B}, x) \rightarrow (\mathcal{D}, x)$ induces the isomorphism $(\varphi, \xi): R(\mathcal{B}') \approx R(\mathcal{D}')$, where $\mathcal{B}' = (\mathcal{B}, \Gamma \cup \{x\}, -, \sim, \varphi_{\mathcal{B}})$ and $\mathcal{D}' = (\mathcal{B}, \Gamma \cup \{y\}, -, \sim, \varphi_{\mathcal{B}})$. Recall that $\mathcal{B} = \mathcal{D}$ and $x, y \notin \Gamma$; the relation \sim on \mathcal{B}' and \mathcal{D}' is a trivially extended relation \sim on Γ . Thus, it suffices to prove that the S -graphs \mathcal{B}' and \mathcal{D}' are not isomorphic if $x \neq y$.

If $\mathcal{B}' \approx \mathcal{D}'$, then $\varphi_{\mathcal{B}(x)} = \varphi_{\mathcal{B}(y)}$. By Lemma 8, there exists a node R in $K(\mathcal{B})$ such that $\varphi_{\mathcal{B}(R)} \neq \varphi_{\mathcal{B}(Q)}$ for any node $Q \in K(\Gamma)$. The connectedness of \mathcal{B} implies that there exist paths $(x'_0, x_1, x'_1, \dots, x_{n-1}, x'_{n-1}, x_n)$ and $(y'_0, y_1, y'_1, \dots, y_{m-1}, y'_{m-1}, y_m)$ that join $\mathcal{B}^-(x)$ and $\mathcal{B}^-(y)$, respectively, with R for which $x'_0 \approx x, y'_0 \approx y, x_n \in R$, and $y_m \in R$.

If $f: \mathcal{B}' \rightarrow \mathcal{D}'$ is an isomorphism, then, for any $z \in R \cap \Gamma$, we have $f(r_z \mathcal{B}') \subset r_z \mathcal{D}'$. This readily follows from the facts that $f|_R$ is an identity and f transforms the path joining some node with R into the path that joins the image of this node with R . Therefore, f induces the isomorphism $r_z \mathcal{B}' \simeq r_z \mathcal{D}'$ for every $z \in \Gamma \cap R$.

If $x_n = y_m$, then, passing to the S -graph $x_n \mathcal{B}$, we can assume that $x_n \neq y_m$. If $x_n \neq y_m$, then, as proved above, we get $r_{y_m} \mathcal{B}' \simeq r_{y_m} \mathcal{D}'$, which is not true because x is a nondegenerate vertex in $r_{y_m} \mathcal{D}'$ and a degenerate vertex in $r_{y_m} \mathcal{B}'$. Lemma 9 is proved.

There is a one-dimensional (nonfaithful) module N , $N(\mathcal{B}, x) = a_{\mathcal{B}, x} k (\simeq U_{\mathcal{B}} + xk/U_{\mathcal{B}})$, over the category El. Every morphism $(\varphi, \xi) \in \text{El}((\mathcal{B}, x), (\mathcal{D}, y))$ induces a linear map $\bar{\xi}: N(\mathcal{B}, x) \rightarrow N(\mathcal{D}, y)$, which establishes the structure of the El-module on N . The correctness of the definition is obvious.

We set $\overline{\text{El}} = \text{El} / \text{Ann}_{\text{El}} N$.

Lemma 10. *Let a vectroid \mathcal{V} be quasifinite. Then $\overline{\text{El}}((\mathcal{B}, x), (\mathcal{D}, y)) \neq 0$ if and only if $(\mathcal{B}, x) \leq (\mathcal{D}, y)$.*

Proof. 1. For the S -graph \mathcal{B} , we denote by $\{\tau_z, z \in \mathcal{B}\}$ the basis of the space $D\mathcal{X}_{\mathcal{B}} (= \text{mod } k(\mathcal{X}_{\mathcal{B}}, k))$ dual to the basis $\{z, z \in \mathcal{B}\}$. In this case, it follows from step I of the proof of Lemma 2 that $\varphi_{\mathcal{B}}(x) \leq \varphi_{\mathcal{D}}(y)$ for $x \in \mathcal{B}$ and $y \in \mathcal{D}$ if and only if there exists $\xi \in \mathcal{V}(V_{\mathcal{B}^{\neq}(x)}, V_{\mathcal{D}^{\neq}(y)})$ such that $(x)\xi\tau_y \neq 0$.

In particular, if $\overline{\text{El}}((\mathcal{B}, x), (\mathcal{D}, y)) \neq 0$, then $\varphi_{\mathcal{B}}(x) \leq \varphi_{\mathcal{D}}(y)$.

2. Let $s = \varphi_{\mathcal{B}}(x) \triangleleft t = \varphi_{\mathcal{D}}(y)$. Then there exists a morphism $\xi \in \mathcal{V}(V_{\mathcal{B}^{\neq}(x)}, V_{\mathcal{D}^{\neq}(y)})$ such that $x\xi = y$. Therefore, $0 \neq \overline{(0, \xi)} \in \overline{\text{El}}((\mathcal{B}, x), (\mathcal{D}, y))$.

3. Let $s = \varphi_{\mathcal{B}}(x) = \varphi_{\mathcal{D}}(y)$ and let $s \triangleleft s$. Let us prove that $\overline{\text{El}}((\mathcal{B}, x), (\mathcal{D}, y)) \neq 0$ if $(\mathcal{B}, x) \leq (\mathcal{D}, y)$. By the definition of \leq , we have $\partial_t^2(\mathcal{B}, x) \leq \partial_t^2(\mathcal{D}, y)$ for every $t \in S^{\sim}(s)$. In particular, for an arbitrary nondegenerate vertex $x' \in \Gamma^{\sim}(x)$, there is a unique nondegenerate vertex $y'_{x'} \in \Delta^{\sim}(y)$ such that $\varphi_{\mathcal{B}}(x') = \varphi_{\mathcal{D}}(y'_{x'})$. We set $R_x = \{x' \in \Gamma^{\sim}(x) \mid \partial_{x'}^2(\mathcal{B}, x) \neq 1\}$.

By the induction hypothesis, we can assume that there exist nonzero morphisms $f_{x'} = \overline{(y_{x'}, \xi_{x'})} \in \overline{\text{El}}(\partial_{x'}^2(\mathcal{B}, x), \partial_{x'}^2(\mathcal{D}, y))$ defined for every $x' \in R_x$. Let us decompose the vector spaces

$$\mathcal{X}_{\mathcal{B}} = V_{\mathcal{B}^{\neq}(x)} \oplus \left(\bigoplus_{x' \in R_x} \mathcal{X}_{\partial_{x'}^2(\mathcal{B}, x)} \right),$$

$$\mathcal{X}_{\mathcal{D}} = V_{\mathcal{D}^{\neq}(y)} \oplus \left(\bigoplus_{x' \in R_x} \mathcal{X}_{\partial_{x'}^2(\mathcal{D}, y)} \right) \oplus \left(\bigoplus_{y' \in R_y, y' \neq y'_{x'}} \mathcal{X}_{\partial_{x'}^2(\mathcal{D}, y)} \right).$$

Taking into account that $V_{\mathcal{B}^{\neq}(x)} = V_{\mathcal{D}^{\neq}(y)}$, we define the morphism ξ as follows:

$$\xi = \begin{pmatrix} 1_{V_{\mathcal{B}^{\neq}(x)}} & 0 & 0 \\ 0 & \bigoplus_{x' \in R_x} \xi_{x'} & 0 \end{pmatrix}: \mathcal{X}_{\mathcal{B}} \rightarrow \mathcal{X}_{\mathcal{D}}$$

We also define $\varphi: U_{\mathcal{B}} \rightarrow U_{\mathcal{D}}$ as a unique linear map such that $(\varphi, \xi) \in \text{Rep } \mathcal{V}(R(\mathcal{B}), R(\mathcal{D}))$. One can directly verify that $0 \neq \overline{(\varphi, \xi)} \in \overline{\text{El}}$.

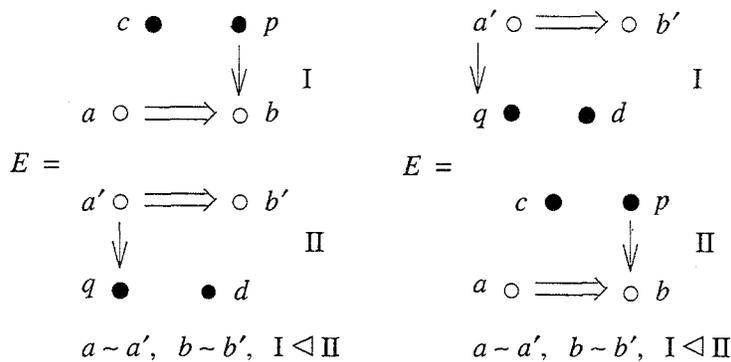
with the representation $(W, g, Z) \in \text{Rep } \mathcal{U}$ with the matrix $M(g)$. Here, $\overline{M}(\lambda(s_i))$ is the matrix of the elementary representation $P(\lambda(s))$ in which the (zero) column corresponding to the marked vertex $\lambda(s_i)$ of the $S(\mathcal{V})$ -graph is omitted and replaced in the matrix $M(\text{Ex}(\lambda)g)$ by the column of the matrix $M(g)$ marked by the element s_i .

Assume that the map λ satisfies the following conditions:

- (i) If $t_1, t_2 \in S(\mathcal{U})$ and $t_1 \approx t_2$, then $P\lambda(t_1) \approx P\lambda(t_2)$;
- (ii) if $\lambda(t_1) < \lambda(t_2)$ (i.e., $\lambda(t_1) \neq \lambda(t_2)$ and $\overline{\text{El}}(\lambda(t_1), \lambda(t_2)) \neq 0$), then $t_1 < t_2$.

In this case, the representation $\text{Ex}(\lambda)g$ is called the λ -extension of the representation g . In particular, let \mathcal{V} be a quasifinite vectroid. Then the partial ordering \leq is defined on $C(\mathcal{V})$ (see Sec. 5). Let \mathcal{U} be a one-dimensional vectroid constructed on the basis of $C(\mathcal{V})$. For this vectroid, $S(\mathcal{U}) = (C(\mathcal{V}), \leq, \leq, \Delta)$, where Δ is the diagonal. The image of the map $\text{Mul} = \text{Ex}(\text{id}_{C(\mathcal{V})}) : \text{Ob Rep } \mathcal{U} = \text{Ob Rep } C(\mathcal{V}) \rightarrow \text{Ob Rep } \mathcal{V}$ consists of representations of the vectroid \mathcal{V} called *multielementary representations*.

Example 6 (Nonmultielementary representation). Let



be completed biordered sets and let $\mathcal{E} = \text{Vect}(E)$ and $\mathcal{E}^* = \text{Vect}(E^*)$ (see Sec. 2). Then E and E^* can naturally be regarded as bipartite completed posets in the sense of [11, 12]. It was proved in [11, 12] that \mathcal{E} and \mathcal{E}^* are finitely represented and each of them admits a faithful indecomposable representation unique to within an isomorphism, namely, the representation g with the matrix

$$M(g) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}.$$

The columns of this matrix correspond to a, b, c, p, a', b', d, q . The representation g is not multielementary. It is called the *exceptional representation* (of E or E^*).

Proposition 6. Let \mathcal{V} be a quasifinite vectroid and let g_1 and g_2 be representations of the poset $C(\mathcal{V})$. Then the following assertions are true:

(i) $Mul(g_1 \oplus g_2) \simeq Mul(g_1) \oplus Mul(g_2)$;

(ii) if $g_1 \simeq g_2$ [in $Rep C(\mathcal{V})$], then $Mul(g_1) \simeq Mul(g_2)$ (in $Rep \mathcal{V}$).

To prove the proposition we perform the following construction: Let $(A, M) \in \mathcal{M}$ be a module over the aggregate and let $Der(A, M) = (Rep M, E_M)$ be a derivative module. Assume that another module $(B, N) \in \mathcal{M}$ over the aggregate and the morphism $(F, \Phi): (B, N) \rightarrow Der(A, M) \in Mor \mathcal{M}$ are given. In this case, we can define a morphism $(I, \Pi): Der(B, N) \rightarrow Der(A, M)$. Here, $I: Rep N \rightarrow Rep M$ is the functor that takes the value

$$I(W, g, Z) = \left(W \oplus V, \begin{pmatrix} g \circ \Phi(Z) \\ f \end{pmatrix}, X \right)$$

on the representation $(W, g, Z) \in Rep N$ with $F(Z) = (V, f, X)$, and $\Pi: E_N \rightarrow I^* E_M$ is the homomorphism of B -modules defined by the following diagram:

$$\begin{array}{ccc} E_N(W, g, Z) & \xrightarrow{(W, g, Z)} & E_M(I(W, g, Z)) = E_M(FZ) \\ \parallel & & \parallel \\ N(Z) & \xrightarrow{\Phi(Z)} & M(FZ) \end{array}$$

Under certain additional conditions, which can easily be written in the explicit form, $I(W, g, Z)$ is the λ -extension of the representation g .

Proof of Proposition 6. Assume that (B, N) is a module over the aggregate obtained from the module over the spectroid (El, N) , i.e., $B = \oplus El$ (see Sec. 5). We define a morphism $(F, \Phi): (B, N) \rightarrow Der(\mathcal{V})$, setting $F(\mathcal{B}, x) = R(\mathcal{B})$, as follows:

$$\begin{aligned} \Phi(\mathcal{B}, x): N(\mathcal{B}, x) &= a_{\mathcal{B}, x} k \rightarrow \mathcal{X}_{\mathcal{B}} = E_{\mathcal{V}}(U_{\mathcal{B}}, i_{\mathcal{B}}, \mathcal{X}_{\mathcal{B}}), \\ a_{\mathcal{B}, x} \Phi(\mathcal{B}, x) &= x \in \mathcal{X}_{\mathcal{B}}. \end{aligned}$$

By applying the construction described above, we obtain a functor $I: Rep N \rightarrow Rep \mathcal{V}$. On the other hand, the category $Rep N$ is naturally epivalent to the category $Rep C(\mathcal{V})$ (see Corollary 3). It is easy to see that every pre-image f of a representation $g \in C(\mathcal{V})$ in $Rep N$ has the same matrix as g , and $I(f)$ is the representation of $Mul(g)$. This obviously proves Proposition 6.

If $\dim \mathcal{V} = 2$, then the poset $C(\mathcal{V})$ coincides (by definition) with the poset of “flagged sequences” $St(\mathcal{V})$ defined in ([1], 5.8). It was proved in [7–9] that a vectroid \mathcal{V} of dimension ≤ 2 is finitely represented if and only if the poset $C(\mathcal{V})$ is finitely represented. However, in this case, not all indecomposable representations are multielementary [11, 12].

Hypothesis 1. A vectroid \mathcal{V} is finitely represented if and only if \mathcal{V} is a chain vectroid, $\text{def } \mathcal{V} \leq 1$, $\dim \mathcal{V} \leq 3$, and the poset $C(\mathcal{V})$ is finitely represented.

Hypothesis 2. Suppose that a vectroid \mathcal{V} is finitely represented and g is its faithful representation. Then g satisfies one of the following conditions:

- (i) g is decomposable;
- (ii) g is indecomposable and multielementary;
- (iii) g is the λ -extension of the exceptional representation of E or E^* for proper $\lambda : E \rightarrow C(\mathcal{V})$ or $\lambda : E^* \rightarrow C(\mathcal{V})$ (in particular, g is indecomposable).

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REFERENCES

1. P. Gabriel and A. V. Roiter, "Representations of finite-dimensional algebras," in: *Encyclopaedia of Mathematical Sciences*, Vol. 73, Algebra 8, Springer-Verlag, New York (1992).
2. L. A. Nazarova and A. V. Roiter, *Category Matrix Problems and the Brauer–Thrall Problem* [in Russian], Preprint No. 09.73, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1973).
3. P. Gabriel, L. A. Nazarova, A. V. Roiter, V. V. Sergeichuk, and D. Vossieck, "Tame and wild subspace problems," *Ukr. Mat. Zh.*, **46**, No. 3, 313–352 (1993).
4. A. V. Roiter and V. V. Sergeichuk, "Existence of a multiplicative basis for a finitely spaced module over an aggregate," *Ukr. Mat. Zh.*, **46**, No. 5, 567–579 (1994).
5. L. A. Nazarova and A. V. Roiter, "Representations of partially ordered sets," *Zap. Nauchn. Sem. Leningrad. Otd. Mat. Inst. Akad. Nauk SSSR*, **28**, 5–31 (1972).
6. M. M. Kleiner, "Partially ordered sets of finite type," *Zap. Nauchn. Sem. Leningrad. Otd. Mat. Inst. Akad. Nauk SSSR*, **28**, 32–41 (1972).
7. L. A. Nazarova and A. V. Roiter, *Representations of Biinvolutive Partially Ordered Sets* [in Russian], Preprint No. 91.34, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1991).
8. L. A. Nazarova and A. V. Roiter, *Representations of Biinvolutive Partially Ordered Sets. II* [in Russian], Preprint No. 94.02, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1994).
9. L. A. Nazarova and A. V. Roiter, "Representations and forms of weakly completed partially ordered sets," in: *Linear Algebra and Representation Theory* [in Russian], Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1983), pp. 19–54.
10. D. Vossieck, "Représentations de bifoncteurs et interprétation en termes de modules," *Comptes-Rendus Acad. Sc. Paris.*, **307**, 713–716 (1988).
11. L. A. Nazarova and A. V. Roiter, *Representations of Bipartite Completed Sets* [in Russian], Preprint No. 87.05, Institute of Mathematics, Ukrainian Academy of Sciences, Kiev (1987).
12. L. A. Nazarova and A. V. Roiter, "Representations of bipartite completed posets," *Comment. Helv. Math.*, **63**, 498–526 (1988).
13. G. Birkhoff, *Lattice Theory*, Providence, Rhode Island (1967).
14. K. Kuratowski and A. Mostowski, *Theory of Sets* [Russian translation], Mir, Moscow (1970).