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# APPLICATION OF MODULES OVER A DYAD FOR THE CLASSIFICATION OF FINITE $p$-GROUPS POSSESSING AN ABELIAN SUBGROUP OF INDEX $p$ AND OF PAIRS OF MUTUALLY ANNIHILATING OPERATORS

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In [1] it was shown that the classification of finite $p$-groups possessing an Abelian subgroup of index $p$ can be reduced to a certain matrix problem ([1], p. 75). The resulting matrix problem admits of an entirely visible solution ([1], §4). However, the construction of the groups themselves (from the given matrix invariants) was not carried out. Furthermore, as was ascertained while writing the present article, two inaccuracies were admitted in [1].*

In this paper we give another simpler method of reduction to this same matrix problem and, using its solution, obtained in [1], we write out a complete catalog of the groups of the type mentioned above in terms of the defining relations. As in [1], the classification of the groups indicated is obtained from the classification of modules over a dyad of two local Dedekind rings (see §4). As the last author of the present article has noted, from the classification of modules over a dyad there directly ensues also (see §5) the classification of pairs of matrices $A$, $B$ for which $AB = BA = 0$, obtained in [2] in connection with the study of indecomposable representations of the Lorentz group.

The fact that two problems, so unlike at first glance, admit, essentially, of a like solution is, in our opinion, a very curious circumstance.

*Namely, the assertion that $\mathfrak{M}'$ can be treated like a module over the ring $D_1 \oplus D_2$ ([1], p. 67) and the assertion on the indecomposability (without the imposition of additional conditions) of module 1 ([1], pp. 80-81) are false.

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## §1. Comparison of a Finitely-Generated Module over a Dyad of a Pair of Matrices with Rows and Columns Furnished with a Weight*

We recall [1, 3] that a dyad of two local Dedekind rings $D_1$ and $D_2$ for which the residue fields are isomorphic is the ring

$$D = \{(d_1, d_2) \in D_1 \oplus D_2 \mid \varepsilon_1(d_1) = \varepsilon_2(d_2)\},$$

where $\varepsilon_1$ and $\varepsilon_2$ are given homomorphisms of rings $D_1$ and $D_2$, respectively, onto their "common" residue field.

By $\pi_1(\pi_2)$ we denote the prime element of ring $D_1\,(D_2)$. Then $R_2 = \pi_2 D$ $(R_2 = \pi_1 D_1 \oplus \pi_2 D_2)$ is the unique maximal ideal of ring $D_2\,(D)$.

If $\mathfrak{M}$ is an arbitrary finitely generated module over dyad $D$, then its submodule $\mathfrak{M}' = \mathfrak{M} R_2$ can obviously be treated as a module over $D_2$, while the quotient module $\mathfrak{M}'' = \mathfrak{M}/\mathfrak{M}'$, as a module over $D/R_2 \approx D_1$. $D_2(D_1)$ is a ring of principal ideals, therefore, the module $\mathfrak{M}\,(\mathfrak{M}'')$ splits up into a direct sum of indecomposable cyclic modules with generators $u_\alpha\ (v_\beta)$ each of which is isomorphic to either the quotient module $D_2/\pi_2^{c_\alpha} D_2$ $(D_1/\pi_1^{\kappa_\beta} D_1)$, or $D_2\,(D_1)$; in the latter case we take $c_\alpha = \infty$ $(\kappa_\beta = \infty)$. The number $c_\alpha\ (\kappa_\beta)$ will be called the weight of $u_\alpha\ (v_\beta)$.

We select a representative $\bar{v}_\beta$ from the co-set $v_\beta$ of module $\mathfrak{M}$. Then $\bar{v}_\beta \pi_2$ and $\bar{v}_\beta \pi_1^{\kappa_\beta}$ (with $\kappa_\beta \neq \infty$) belong to $\mathfrak{M}$; therefore, $\bar{v}_\beta \pi_2 = \sum_{\alpha=1}^{m} u_\alpha \bar{a}_{\alpha\beta}$ and, in view of the relation $\pi_1 \pi_2 = 0$ $\bar{v}_\beta \pi_1^{\kappa_\beta} = \sum_{\substack{\alpha=1 \\ c_\alpha \neq \infty}}^{m} u_\alpha \bar{b}_{\alpha\beta} \pi_2^{c_\alpha - 1}$, where $\bar{a}_{\alpha\beta}, \bar{b}_{\alpha\beta} \in D$.

Let $a_{\alpha\beta}\ (b_{\alpha\beta})$ be an element of field $K$, into which $\bar{a}_{\alpha\beta}\ (\bar{b}_{\alpha\beta})$ goes under the natural homomorphism of $D$ onto $K = D/R$. We set $b_{\alpha\beta} = 0$ if $c_\alpha = \infty$ or $\kappa_\beta = \infty$. It is easily verified that $a_{\alpha\beta}\ (b_{\alpha\beta})$ does not depend upon the choice of the representative $\bar{v}_\beta$ from the co-set $v_\beta$.

As a result, with finitely generated module over a dyad, for a fixed choice of $\{u_\alpha\} \subset \mathfrak{M}'$ and $\{v_\beta\} \subset \mathfrak{M}''$ (but not $\{\bar{v}_\beta\} \subset \mathfrak{M}$), we associate a pair of matrices $A = (a_{\alpha\beta})$, $B = (b_{\alpha\beta})$ of like dimension over the residue field $K$ of dyad $D$ $(\alpha = 1, \ldots, m,\ \beta = 1, \ldots, n)$. The weight $c_\alpha\ (\kappa_\beta)$ of element $u_\alpha\ (v_\beta)$ is called the weight of the $\alpha$-th row ($\beta$-th column) of this pair.

Let us now ascertain to what extent the pair of matrices $A$, $B$ depends upon the choice of generators $u_\alpha$, $v_\beta$. Let $u_1, \ldots, u_m$ be some collection of generators in the direct summands of module $\mathfrak{M}'$. We replace $u_\alpha$ by $u'_\alpha = u_\alpha d$, $d \in D$, $d \notin R$, i.e., we construct the new collection of generators $u_1, \ldots, u_{\alpha-1}, u'_\alpha, u_{\alpha+1}, \ldots, u_m$. Here, obviously, the $\alpha$-th row of matrices $A$ and $B$ is divided by the element $a \in K$ corresponding to the element $d \in D$ under the natural homomorphism of $D$ onto $K$. If we set $u'_\alpha = u_\alpha + u_\beta d$, where $d \in D$, then, as is not difficult to be convinced, a necessary and sufficient condition

*In [1] (§2) the description of the modules was reduced to a pair of matrices for which a weight was furnished only to the rows, while here, to both rows as well as columns. However, this distinction is unessential since under a trivial reduction of one of the matrices to diagonal form, relative to the other we obtain one and the same matrix problem in both cases; see p. 75 of [1] and §2 of the present paper.

that the collection obtained is a collection of generators, is $u'_\alpha D = u_\alpha D$. Therefore, the substitution $u'_\alpha = u_\alpha + u_\beta d$ is admissible if and only if $c_\alpha > c_\beta$ (for any $d \in D$) or $c_\alpha \leq c_\beta$ for $d \in \pi_2^{c_\alpha - c_\beta} D$. If $c_\alpha > c_\beta$, then from the $\beta$-th row of matrix $A$ we subtract the $\alpha$-th row multiplied by $a \in K$, corresponding to $d$, while the matrix $B$ remains unchanged. If $c_\alpha = c_\beta$, then from the $\beta$-th row we subtract the $\alpha$-th row multiplied by $a$ in matrices $A$ and $B$, simultaneously. If $c_\alpha < c_\beta$ and $d \in \pi_2^{c_\beta - c_\alpha} D$, then from the $\beta$-th row of matrix $B$ we subtract the $\alpha$-th row multiplied by $b \in K$ corresponding to $d\pi_2^{c_\alpha - c_\beta}$, while matrix $A$ is unchanged.

Analogously, if $\{v_\alpha\}$ is a collection of generators in the cyclic direct summands of module $\mathfrak{M}''$ then the substitution $v'_\alpha = v_\alpha d$, $d \in D$, $d \in R$, is admissible; here the $\alpha$-th columns of matrices $A$ and $B$ are multiplied by $a \in K$ corresponding to $d$. Also admissible is the substitution $v'_\alpha = v_\alpha + v_\beta d$, where $d \in D$ for $\kappa_\alpha > \kappa_\beta$, and here, to the $\alpha$-th column of matrix $A$ is added the $\beta$-th column multiplied by $a \in K$ corresponding to $d$; or where $d \in D$ for $\kappa_\alpha = \kappa_\beta$, and here, to the $\alpha$-th column is added the $\beta$-th column multiplied by $a \in K$ corresponding to $d$, in $A$ and $B$, simultaneously; or where $d \in \pi_1^{\kappa_\beta - \kappa_\alpha} D$ for $\kappa_\alpha < \kappa_\beta$, and here, to the $\alpha$-th column of matrix $B$ is added the $\beta$-th column multiplied by $b \in K$ corresponding to $d\pi_1^{\kappa_\alpha - \kappa_\beta}$.

Finally, we can rearrange the places of the direct summands in $\mathfrak{M}'$ and $\mathfrak{M}''$ by rearranging the rows and columns "together with their weights" in $A$ and $B$, simultaneously.

Thus, by reselecting the generators in $\mathfrak{M}'$ and $\mathfrak{M}''$ we can make the following elementary transformations in the matrices $A$ and $B$:

I. Row (columns) can be relocated "together with their weights" simultaneously in both matrices.

II. A row (column) can be multiplied by $a \in K$ simultaneously in $A$ and $B$.

III. For $c_\alpha = c_\beta$ (where $c_\alpha (c_\beta)$ is the weight of the $\alpha$-th ($\beta$-th) row) we can add the $\beta$-th row, multiplied by $a$, to the $\alpha$-th row simultaneously in $A$ and $B$.

IV. For $\kappa_\alpha = \kappa_\beta$ (where $\kappa_\alpha (\kappa_\beta)$ is the weight of the $\alpha$-th ($\beta$-th) column) we can add the $\beta$-th column, multiplied by $a$, to the $\alpha$-th column simultaneously in $A$ and $B$.

V. For $c_\alpha < c_\beta$ we can add the $\beta$-th ($\alpha$-th) row of matrix $A (B)$, multiplied by $a$, to the $\alpha$-th ($\beta$-th) row, not changing $B (A)$.

VI. For $\kappa_\alpha > \kappa_\beta$ we can add the $\beta$-th ($\alpha$-th) column of matrix $A (B)$, multiplied by $a$, to the $\alpha$-th ($\beta$-th) column, not changing $B (A)$.

Conversely, if two pairs of matrices are obtained from one module $\mathfrak{M}$, then we can pass from one pair to the other by a finite number of transformations of type I-VI.

The pair of matrices $A$, $B$ will be called decomposable if after several transformations I-VI we can find a proper subset of the set of rows and columns, one and the same for both matrices, such that the nonzero elements of the rows and columns selected can occur only at the intersections of the rows and columns of this subset. Otherwise, the pair is called indecomposable.

It is obvious that a finitely generated module over a dyad is indecomposable into a direct sum if and only if the corresponding pair of matrices is indecomposable.

The aim of the next section is to ascertain the form to which an indecomposable pair of matrices, whose rows and columns are furnished with weight, can be led by transformations I-VI.

## §2. Reduction of a Pair of Matrices Whose Rows and Columns Are Furnished with Weight

Let us rearrange the rows and columns of matrices $A$ and $B$ so that every row with lesser weight occurs below a row of greater weight, while every column with lesser weight occurs to the left of a column of greater weight. Matrices $A$ and $B$ are divided into horizontal and vertical strips within which the weight of the rows (columns) is equal. Starting from the upper strip we reduce $A$ to the form:

$$A = \left(\begin{array}{ccccc|ccccc|cc|ccccc}
E & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & E & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & E & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\hline
0 & E & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & E & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & E & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & E & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & E & 0 \\
0 & 0 & \cdot & 0 & 0 & 0 & 0 & \cdot & 0 & 0 & \cdot & \cdot & 0 & 0 & \cdot & 0 & 0
\end{array}\right) \tag{1}$$

We partition matrix $B$ into cells in accordance with the partitioning (1). Retaining the terminology in [1], a cell of matrix $B$, corresponding to a zero (unit) cell of matrix $A$, will be called a cell of type I (II). A horizontal and a vertical strip of matrix $B$, intersecting in a cell of type II, are said to be dual.

On $B$ we shall make only those sequences of transformations I-VI from §1 which do not spoil the form of $A$ in (1):

1-2) any elementary row (column) transformation within each horizontal (vertical) strip not containing a cell of type II;

3) the $\beta$-th row of a horizontal strip containing a cell of type II, multiplied by $\alpha \in K$, can be added to the $\alpha$-th row of this same strip, but here the $\alpha$-th column of the dual vertical strip, multiplied by $\alpha$, must be subtracted from the $\beta$-th column of this same vertical strip;

4-5) the rows (columns) of any lower-standing (right-standing) horizontal (vertical) strip can be added to the rows (columns) of any higher-standing (left-standing) strip.

Thus, our problem is reduced to the problem of the reduction, by transformations of form 1)-5), of one matrix $B$ divided into cells of two types, I, II; moreover, cells of type II are square and no more than one cell of type II occurs in each horizontal and vertical strip. The reduction problem for such a matrix $B$ by transformations 1)-5) has been solved in [1]. For the reader's convenience we cite this solution here.

We shall take $B$ to be decomposable if in the set $u$ of rows of $B$ and in the set $v$ of columns of $B$ we can find subsets $u'$ and $v'$ satisfying the following conditions:

1) $u'$ is a proper subset in $u$ or $v'$ is a proper subset in $v$ ;

2) if $u'(v')$ contains the $\alpha$ -th row (column) from the horizontal (vertical) strip passing through a cell of type II, then $v'(u')$ contains the $\alpha$ -th column from the dual vertical (horizontal) strip;

3) zero elements in the rows of $u'$ (columns of $v'$ ) can occur only at the intersections with the columns of $v'$ (rows of $u'$ ).

Otherwise we shall say that $B$ is indecomposable.

It is easily verified that the pair of matrices $A$ , $B$ is indecomposable in the sense of the definition in §1 ifandonly if $B$ is indecomposable in the sense of the definition given above.

Proposition 1. Let $B$ be indecomposable. Then, after applying certain transformations 1)-5) to $B$ , we can make additional horizontal and vertical partitions in it such that the new cells (obtained as a result of subdividing the old ones) satisfy the following conditions:

1) all cells are square;

2) each cell is either a zero matrix or a nonsingular matrix;

3) no more than one nonzero cells occurs in each (new) horizontal and vertical strip.

The proof of the proposition will be carried out by induction on the dimension of $B$ . The base is trivial.

We select the lowest one of all the horizontal strips containing nonzero elements. Next, from the cells of this strip we select the rightmost nonzero cell. We denote it $B_1$ .

Our arguments will depend essentially on what type this cell is.

At first we assume that $B_1$ is a cell of type I. By admissible transformations we lead $B_1$ to the form

$$B_1=\left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right).$$

In accordance with the partitioning of $B_1$ we make new partitions in the horizontal and vertical strips of $B$ containing cell $B_1$ . Next, by transformations 4), 5) we make zero the parts of the vertical and horizontal strips being considered, standing opposite the nonzero part of $B_1$ .

It is not difficult to see that there is a cell of type II in the horizontal or in the vertical strip containing $B_1$ , since otherwise $B$ is decomposable. If $B_2$ is a cell of type II occurring in one horizontal strip with $B_1$ , then in the horizontal strip containing $B_2$ we have already made a partition in accordance with the partition in $B_1$ . We now partition the vertical strip containing $B_2$ into two strips such that the upper left cell of the four cells into which $B_2$ has been divided, is square. In exactly the same way, if $B_3$ is a cell of type II occurring in one vertical strip with $B_1$ , then besides the partitions of the vertical strip containing them, we partition the horizontal strip containing $B_3$ such that the upper-left part of $B_3$ is square.

Note that if even one of the cells $B_2$, $B_3$ (or both at once) were zero and occurred, respectively, to the right of or below $B_1$, then all the arguments would be preserved.

We now examine the matrix $B$ obtained from $B$ after the deletion of the rows and columns containing the nonzero elements of $B_1$.

It is easy to establish that, not changing the form of cell $B_1$, we can make the very same elementary transformations 1)-5) on $B'$ as well, if we take the following cells to be cells of type II in matrix $B'$: cells of type II of matrix $B$, which do not touch the new partitions; the lower-right cell of $B_2$; the lower-right cell of $B_3$; the upper-left cell of the matrix occurring in one horizontal strip with $B_3$ and in one vertical strip with $B_2$. If, as a matter of fact, a cell of type II exists only in the horizontal (vertical) strip, i.e., there is only $B_2$ ($B_3$) but no $B_3$ ($B_2$), then, naturally, we obtain only one new cell of type II, namely, the lower-right part of $B_2$ ($B_3$).

Here, if some of the transformations 4), 5) in $B'$ change the form of $B$, then we can restore it by making transformations on the deleted rows and columns of $B$.

Thus, we have constructed a matrix $B'$ of lesser dimension than $B$, admitting of elementary transformations of the same form. It is easy to be convinced that the decomposability of $B$ follows from the decomposability of $B'$. We apply the induction assumption to $B'$ and, if necessary, we make additional divisions corresponding to the division of $B_1$ in the vertical and horizontal strips to be deleted. In this case the theorem is proved.

We now assume that $B_1$ is a cell of type II. Making simultaneous elementary transformations in the horizontal and vertical strips containing this cell, we decompose $B$ into a direct sum of matrices each of which is an indecomposable Frobenius cell.

If even one of the Frobenius cells is a nonsingular matrix (and it does not coincide with the whole $B$), then by transformations 4), 5) we can isolate its direct summands from $B$. Hence, $B_1$ consists only of singular Frobenius cells and, as is well known, they can be reduced to a normal Jordan form with a zero eigenvalue. Obviously, by an appropriate renumbering of the rows and columns we can reduce $B_1$ to the form:

$$B_1 = \left(\begin{array}{c|cc|ccc|c|cc|c|cc}
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & E & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & E & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & E & \cdots & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & E & & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \ddots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \ddots & \ddots & \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & & 0 & E \\
0 & 0 & 0 & 0 & 0 & 0 & & 0 & 0 & & 0 & 0
\end{array}\right).$$

As before, we make new partitions in $B$ corresponding to the partitions in $B_1$ and next we make zero all the elements occurring in the rows and columns of $B$, containing the nonzero elements of $B_1$.

Let us examine the matrix $B'$ obtained from $B$ after the deletion of the rows and columns of $B$, containing nonzero elements of $B_1$.

Let us show that for the matrix $B'$ the admissible transformations (i.e., transformations not spoiling the form of $B_1$) will be transformations satisfying conditions 1)-5) if only we can find cells of type II in $B'$ in an appropriate manner. Namely, as cells of type II we shall take the cells on the main diagonal in the matrix obtained from $B_1$ after the deletion of the above-mentioned rows and columns (and, of course, of the cells of type II in matrix $B$, which do not touch the partitions). Indeed, let us number from left to right (from top to bottom) the vertical (horizontal) strips into which the vertical (horizontal) strip containing $B_1$ has been split after the new partition (but before the deletion).

Suppose that the $\alpha$-th one of these strips is left in the matrix after the deletion. Making elementary transformations on the columns of this strip, we should (according to condition 3) for $B$) make the inverse transformation on the rows of the $\alpha$-th horizontal strip. This transformation spoils the form of the (unit) matrix occurring at the intersection of the $\alpha$-th horizontal and $(\alpha+1)$st vertical strip. In order to "restore" this form we need to make transformations in the $(\alpha+1)$st vertical and, hence, also the horizontal strips (we remark that both these strips will be deleted). This spoils the form of $E$ at the location $\alpha+1$, $\alpha+2$. We "restore" it by making a transformation in the $(\alpha+2)$nd strips, etc. This process terminates after $\beta$ stages. Here, on the $(\alpha+\beta)$th horizontal strip (which is the only one of the strips being transformed here to remain after the deletion) we make an elementary transformation inverse to that made on the $\alpha$-th vertical strip (while the part of the matrix to be deleted is not changed). Therefore, as a cell of type II we should take the cell occurring at the intersection of the $\alpha$-th vertical and $(\alpha+\beta)$th horizontal strips (before deletion), but this cell exactly falls on the diagonal in the matrix obtained from $B$ after the deletion.

By analogous arguments we can show that transformations 4), 5) are admissible for cells appearing as a result of the deletion (no other transformations besides transformations 1)-5) are admissible).

Thus, we have once again led the reduction of matrix $B$ (in the case when $B_1$ is a cell of type II) to the reduction of a matrix $B'$ of lesser dimension by precisely those same transformations 1)-5); moreover, yet again it is not difficult to see that $B$ and $B'$ are simultaneously decomposable and indecomposable. Applying the induction assumption to $B'$ completes the proof of the proposition.

Remark. It is easy to verify that if two matrices $B$ and $\bar{B}$ go into each other by transformations 1)-5), then during the proof of the proposition they are reduced to one and the same form.

Using Proposition 1 it is not difficult to write out all the indecomposable pairs of matrices with rows and column furnished with a weight. Indeed, first of all, taking the proposition into account we can reduce any pair of matrices, using only a rearrangement of rows and columns, to one of the following forms (depending on the ratio of the numbers of rows and columns and on the singularity or nonsingularity of matrices $A$ and $B$):

$$\text{I.}\quad A=\left(\begin{array}{c|c|c|c} E & \cdots & 0 & 0 \\ \hline 0 & \ddots & 0 & 0 \\ \hline & & \ddots & \\ \hline 0 & \cdots & 0 & E \end{array}\right),\qquad B=\left(\begin{array}{c|c|c|c} 0 & \cdots & 0 & B_q \\ \hline B_1 & & 0 & 0 \\ \hline & \ddots & & \\ \hline 0 & \cdots & B_{q-1} & 0 \end{array}\right);$$

$$\text{II.}\quad A=\left(\begin{array}{c|c|c|c} E & \cdots & 0 & 0 \\ \hline 0 & \ddots & 0 & 0 \\ \hline & & \ddots & \\ \hline 0 & \cdots & 0 & E \end{array}\right), \qquad B=\left(\begin{array}{c|c|c|c} 0 & \cdots & 0 & 0 \\ \hline B_1 & \cdots & 0 & 0 \\ \hline & \ddots & & \\ \hline 0 & \cdots & B_q & 0 \end{array}\right);$$

$$\text{III.}\quad A=\left(\begin{array}{c|c|c|c} 0 & \cdots & 0 & 0 \\ \hline E & \cdots & 0 & 0 \\ \hline & \ddots & & \\ \hline 0 & \cdots & E & 0 \end{array}\right), \qquad B=\left(\begin{array}{c|c|c|c} B_1 & \cdots & 0 & 0 \\ \hline 0 & \ddots & 0 & 0 \\ \hline & & \ddots & \\ \hline 0 & \cdots & 0 & B_q \end{array}\right);$$

$$\text{IV.}\quad A=\left(\begin{array}{c|c|c} 0 & \cdots & 0 \\ \hline E & \cdots & 0 \\ \hline & \ddots & \\ \hline 0 & \cdots & E \end{array}\right); \qquad B=\left(\begin{array}{c|c|c} B_1 & \cdots & 0 \\ \hline & \ddots & \\ \hline 0 & \cdots & B_q \\ \hline 0 & \cdots & 0 \end{array}\right);$$

$$\text{V.}\quad A=\left(\begin{array}{c|c|c|c} 0 & E & \cdots & 0 \\ \hline & & \ddots & \\ \hline 0 & 0 & \cdots & E \end{array}\right), \qquad B=\left(\begin{array}{c|c|c|c} B_1 & \cdots & 0 & 0 \\ \hline & \ddots & & \\ \hline 0 & \cdots & B_q & 0 \end{array}\right),$$

where $B_1, \ldots, B_q$ are nonsingular matrices. Further, using elementary transformations on the rows (columns) lying in one strip, we can, obviously, reduce all cells $B_1, \ldots, B_q$ to unit form in cases II-V.

From the indecomposability of the pair it follows that all cells $B_\alpha$ are one-dimensional. By a direct examination of all possible cases of decomposition of pairs of form II-V it is easy to be convinced that pairs with one-dimensional $B_\alpha$ are indeed indecomposable. It is completely obvious also that different pairs of matrices of form II-V cannot be led one to the other or to a pair of form I by transformations I-VI.

Let us consider case I. We denote the weight of the $\lambda$ -th horizontal strip of matrices $A$ and $B$ by $r_\lambda$ , the weight of the columns of the $\lambda$ -th vertical strip, by $s_\lambda$ . In this way we associate with each pair $A$, $B$ of form I the sequence of pairs

$$(r_1 s_1), \ldots, (r_q s_q). \tag{2}$$

Let us present (2) in the form $(r_1 s_1), \ldots, (r_t s_t), (r_1 s_1), \ldots, (r_t \cdot s_t)$. The smallest $t$ for which this presentation holds is called the period of (2). A sequence (2) of period $q$ is called aperiodic.

Let us show that each pair $A$ , $B$ of form I is reduced by a rearrangement of strips and a consolidation of strips to a form I with an aperiodic sequence (2). Indeed, let a sequence (2) of the pair $A$ , $B$ have a period $t < q$ , $q = t\kappa$ . We rearrange the strips (horizontal and vertical simultaneously) of matrices $A$ and $B$ so that each $(\alpha t+\beta)$th strip $(0 \leq \alpha < \kappa)$, $(0 < \beta \leq t)$ occupies the place of the $(\alpha+1+(\beta-1)\kappa)$th strip. Next, for each $\lambda = 1, \ldots, t$ we combine the strips with numbers $\kappa(\lambda-1)+1, \ldots, \kappa\lambda$ (having equal weight) into one. As a result the pair $A$ , $B$ is reduced to form I where in the place of cells $B_1, \ldots, B_{q-1}, B_q$ there stand the cells $\bar{B}_1, \ldots, \bar{B}_{t-1}, \bar{B}_t$:

$$\bar{B}_\lambda = \begin{pmatrix} B_\lambda & & & 0 \\ & B_{\lambda+t} & & \\ & & \ddots & \\ 0 & & & B_{\lambda+(\kappa-1)t} \end{pmatrix}, \quad \bar{B}_\lambda = \begin{pmatrix} 0 & \cdots & 0 & \cdot & B_{\kappa t} \\ B_t & \ddots & 0 & & 0 \\ & & \ddots & & \\ 0 & \cdots & B_{(\kappa-1)t} & & 0 \end{pmatrix},$$

where $\lambda = 1, \ldots, t-1$. Here the sequence of pairs of weights of the new strips $(r_1, s_1), \ldots, (r_t, s_t)$ is already aperiodic, which is what we had to show.

Let us reduce the cells $B_1, \ldots, B_{q-1}$ of the pair $A$, $B$ of form I with an aperiodic sequence (2) to unit form. Cell $B_q$ can be reduced by similarity transformations (without changing the form of the cells already reduced). From the indecomposability of $B$ it follows that $B_q$ can be made an indecomposable Frobenius cell:

$$B_q = \Phi = \begin{pmatrix} 0 & \cdots & \cdots & 0 & b_t \\ 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & 0 & b_2 \\ 0 & \cdots & & 1 & b_1 \end{pmatrix}, \tag{3}$$

whose characteristic polynomial

$$\xi(x) = x^t - b_1 x^{t-1} - \ldots - b_t \qquad (b_t \neq 0)$$

is a power of an irreducible polynomial over field $K$.

The pair $A$, $B$ takes the form:

$$A = \left(\begin{array}{c|c|c|c} E & & 0 & 0 \\ \hline 0 & \ddots & 0 & 0 \\ \hline & & \ddots & \\ \hline 0 & & 0 & E \end{array}\right), \qquad B = \left(\begin{array}{c|c|c|c} 0 & & 0 & \Phi \\ \hline E & & 0 & 0 \\ \hline & \ddots & & \\ \hline 0 & & E & 0 \end{array}\right) \tag{4}$$

with an aperiodic sequence (2). Let us show that pair (4) is indecomposable. For this we convince ourselves that the partitioning of $A$ and $B$ into cells is unique, i.e., the partitions in the matrices arise as a result of the process of reducing $B$ used in the proof of the proposition. Indeed, it is easy to see that by applying this reduction process to matrix $B$ ($A$ has been reduced to form (1) beforehand), it is sufficient for use from the elementary transformations 1)-5) only a rearrangement of strips (taking into account that a Frobenius cell can always be driven to such a location that it is reduced in the last turn). Consequently, a natural subdivision could be obtained from subdivision (4) only by a rearrangement of strips and a consolidation of cells. But, as is easy to be convinced, it is impossible to reduce pair (4) to form I by these transformations. From the remark to the proposition and from the naturalness of the partitions indicated in (4), it follows that the decomposition of pair (4) should be consistent with the partition of this pair into cells, but this is not possible in view of the indecomposability of $\Phi$. The indecomposability of pair (4) is proved.

Analogously we can show that two pairs of matrices $A$, $B$ and $A'$, $B'$ of form (3) with an aperiodic sequence of weights $(r_1\ s_1), \ldots, (r_q\ s_q)$ and $(r'_1\ s'_1), \ldots, (r'_q\ s'_q)$ go one into the other by transformations I-VI only in the case when $A = A'$, $B = B'$ and the sequence $(r'_1 s'_1), \ldots, (r'_q\ s'_q)$ has the form $(r_{k+1}, s_{k+1}), \ldots, (r_q\ s_q)$, $(r_1 s_1), \ldots, (r_k\ s_k)$.

Conclusion. We have shown that every indecomposable pair of matrices with rows and columns furnished with weight is uniquely reduced to one of the following forms by transformations I-VI:

$$A=\left(\begin{array}{c|c|c|c} E & & 0 & 0 \\ \hline 0 & \ddots & & \\ & & \ddots & \\ & & & \ddots \\ \hline 0 & & 0 & E \end{array}\right) \qquad B=\left(\begin{array}{c|c|c|c} 0 & & 0 & \Phi \\ \hline E & & 0 & 0 \\ & \ddots & & \\ & & \ddots & \\ \hline 0 & & E & 0 \end{array}\right),$$

where $\Phi$ is the indecomposable Frobenius cell (3). The weights of all rows (columns) belonging to the $\lambda$ -th vertical (horizontal) strip are equal to each other and equal to some $r_\lambda (s_\lambda)$, $(1 \leq r_\lambda \leq \infty)$, $1 \leq s_\lambda \leq \infty$, $\lambda = 1, \ldots, q$). The sequence of pairs $(r_1, s_1), \ldots, (r_q\, s_q)$ is aperiodic and is determined to within any cyclic permutation.

II. $\quad A=E, \qquad B=\begin{pmatrix} 0 & \ldots & 0 & 0 \\ 1 & \ldots & 0 & 0 \\ & \ddots & & \\ 0 & \ldots & 1 & 0 \end{pmatrix}$

III. $\quad A=\begin{pmatrix} 0 & \ldots & 0 & 0 \\ 1 & \ldots & 0 & 0 \\ & \ddots & & \\ 0 & & 1 & 0 \end{pmatrix}, \qquad B=E.$

IV. $\quad A=\begin{pmatrix} 0 & \ldots & 0 \\ 1 & \ldots & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & \ldots & 1 \\ 0 & \ldots & 0 \end{pmatrix}.$

V. $\quad A=\begin{pmatrix} 0 & 1 & \ldots & 0 \\ & & \ddots & \\ 0 & 0 & \ldots & 1 \end{pmatrix}, \qquad B=\begin{pmatrix} 1 & \ldots & 0 & 0 \\ & \ddots & & \\ 0 & \ldots & 1 & 0 \end{pmatrix}.$

In cases II-V the weight $c_\alpha$ of the $\alpha$ -th row ( $\kappa_\beta$ of the $\beta$ -th column) is any positive integer or the symbol $\infty$ . The weights are uniquely determined by the pair (independently of the reduction method).

Pairs of form I-V are indecomposable.

## §3. Generating and Defining Relations for Finitely Generated Modules over a Dyad

Thus, to every finitely generated module $\mathfrak{M}$ over a dyad $D$ , indecomposable into a direct sum, there is uniquely associated a pair of matrices $A$ , $B$ of dimension $m \times n$ of one of the forms I-V and two sequences of weights $c_1, \ldots, c_m$ and $\kappa_1, \ldots, \kappa_n$ (in case I the aperiodic sequence $(r_1, s_1), \ldots, (r_q, s_q)$ is determined to within any cyclic permutation). From the method of construction of the pair of matrices $A=(a_{\alpha\beta})$ , $B=(b_{\alpha\beta})$ from the module $\mathfrak{M}$ (see §1) it follows that $b_{\alpha\beta}=0$ as soon as $c_\alpha=\infty$ or $\kappa_\alpha=\infty$ . Therefore, $c_\alpha$ can equal $\infty$ only for $\alpha=1$ for a pair of type II and for $\alpha=n$ for a pair of type IV; $\kappa_\beta$ can equal $\infty$ only for $\beta=m$ for pairs II and V. In all the remaining cases the weights are finite.

Conversely, we can restore module $\mathfrak{M}$ from the pair $A=(a_{\alpha\beta})$ , $B=(b_{\alpha\beta})$ corresponding to it. We select an element $a'_{\alpha\beta}(b'_{\alpha\beta}) \in D$, going into $a_{\alpha\beta}(b_{\alpha\beta}) \in K$ under the natural homomorphism of $D$ onto

$K = D/R$ ( $R$ is the radical of the dyad); moreover, we agree to choose always $1 \in D$ as the representative of $1 \in K$. We construct the module $\mathfrak{M}$ with generators $u_\alpha$ and $v_\beta$ by specifying the relations:

$$u_\alpha \pi_2^{c_\alpha} = 0 \quad (\text{ absent for } c_\alpha = \infty \ ),$$
$$v_\beta \pi_2 = \sum_{\alpha=1}^{m} u_\alpha a'_{\alpha\beta}, \qquad v_\beta \pi_1^{\kappa_\beta} = \sum_{\substack{\alpha=1 \\ c_\alpha \neq \infty}}^{m} u_\alpha b'_{\alpha\beta} \pi_2^{c_\alpha - 1} \ (\text{ absent for } \kappa_\beta = \infty \ ), \tag{5}$$

where $\alpha = 1, \ldots, m$, $\beta = 1, \ldots, n$.

It is not difficult to verify that in §1 precisely the pair $A$ , $B$ is constructed from the new module $\mathfrak{M}$ if and only if $\mathfrak{M} R_2 = \sum_{\alpha=1}^{m} u_\alpha D$ $(R_2 = \pi_2 D)$ , in other words, if for every $v_\beta$ we can find elements $d_\alpha$ of the dyad such that $v_\beta \pi_2 = \sum_{\alpha=1}^{m} u_\alpha d_\alpha$. Therefore, pairs of types I, II, V, and only they, can be obtained from an indecomposable finitely generated module over a dyad.

Thus, we have established a one-to-one correspondence between, on the one hand, finitely generated modules over a dyad, indecomposable into a direct sum, and, on the other hand, pairs of matrices of types I, II, V for which the above-mentioned restrictions on the weights are fulfilled.

Let us remove (in cases I, II, V) the elements $u_\alpha$ from the system of generators $\{u_\alpha, v_\beta\}$ of module $\mathfrak{M}$ and the relations $u_\alpha \pi_2^{c_\alpha} = 0$ (which follow from the remaining relations and from the equality $\pi_1 \pi_2 = 0$ ) and $v_\beta \pi_2 = \sum_{\alpha=1}^{m} u_\alpha a'_{\alpha\beta}$ from the system (5) of generating relations, having replaced in all the remaining relations the element $u_\alpha$ by $v_\alpha \pi_2$ in cases I, II and by $v_{\alpha+1} \pi_2$ in case y . In case I we denote by $v_{\mu\nu}$ the element $v_{\mu t + \nu}$ , $0 \leq \nu < t$ ( $t$ is the degree of the characteristic polynomial $\xi(x) = x^t - b'_1 x^{t-1} - \ldots - b'_t$ of the indecomposable Frobenius cell in matrix $B$ ) and we renumber the sequences $\{v_{\mu\nu}\}$, $\{(r_\mu, s_\mu)\}$, $\{b'_\nu\}$ , replacing everywhere the indices $\mu$ , $\nu$ by $q - \mu + 1$ , $t - \nu + 1$ . In cases II and V we introduce the notation: $\ell_\alpha = c_\alpha$ in case II, $\ell_1 = 0$ , $\ell_{\alpha+1} = c_\alpha$ in case V (by this change the relations in cases II and V reduce to one series); we replace $\kappa_n$ by $\kappa_n + 1$ and then we renumber the sequences $\{v_\alpha\}$ and $\{(\kappa_\alpha , \ell_\alpha)\}$ , replacing everywhere the index $\alpha$ by $n - \alpha + 1$ .

Finally, every finitely generated module, indecomposable into a direct sum, over a dyad $D$ of local Dedekind rings with prime elements $\pi_1$ and $\pi_2$ is given by defining relations of one of the following two forms:

I. $v_1 \pi_1^{\kappa_1 + 1} = 0$ (absent for $\kappa_1 = \infty$ ), $v_{\beta+1} \pi_1^{\kappa_\beta + 1} = v_\beta \pi_2^{\ell_\beta}$ $(\beta = 1, \ldots, n-1)$, $v_n \pi_2^{\ell_n + 1} = 0$ (absent for $\ell_n = \infty$ ).

II. $v_{11} \pi_1^{r_q} = (v_{q1} b_1 + \ldots + v_{qt} b_t) \pi_2^{s_q}$, $v_{1,\nu+1} \pi_1^{r_1} = v_{q\nu} \pi_2^{s_q}$ $(\nu = 1, \ldots, t-1)$ $v_{\mu+1,\nu} \pi_1^{r_{\mu+1}} = v_{\mu\nu} \pi_2^{s_\mu}$ $(\mu = 1, \ldots, q-1,$ $\nu = 1, \ldots, t)$.

A module of type I is uniquely determined by the sequence $(\kappa_1, \ell_1), \ldots, (\kappa_n, \ell_n)$ , where $\kappa_1$ , $\ell_n$ are nonnegative integers of the symbol $\infty$ ; $\kappa_2, \ldots, \kappa_n$ , $\ell_1, \ldots, \ell_{n-1}$ are positive integers.

A module of type II is uniquely determined by the following invariants: $(r_1, s_1), \ldots, (r_q, s_q)$ is an aperiodic sequence of pairs of positive integers, determined to within any cyclic permutation; $\xi(x) = x^t - b'_1 x^{t-1} - \ldots - b'_t$ $(b'_t \neq 0)$ is the power of an irreducible polynomial over the residue field $K$ of dyad $D$ .

Then $b_\nu$ is some element of the dyad, which goes into $b'_\nu$ under the natural homomorphism of $D$ onto $K$ $(\nu = 1, \ldots, t)$.

Let $\mathfrak{M} = \mathfrak{M}_0 \oplus \ldots \oplus \mathfrak{M}_{m_0} \oplus \mathfrak{N}_1 \oplus \ldots \oplus \mathfrak{N}_{m_1}$ be the decomposition of a finitely generated module $\mathfrak{M}$ over a dyad into a direct sum of indecomposable ones. Each module $\mathfrak{M}_\alpha$ of type I ( $\mathfrak{N}_\lambda$ of type II) is given by the generator system $\{v_{\alpha\beta}\}$ ($\{v_{\lambda\mu\nu}\}$) and by the system of defining relations $\Phi_\alpha$ of type I ( $\Psi_\lambda$ of type II). Then the set $\{v_{\alpha\beta}, v_{\lambda\mu\nu}\}$ will be the generator system for module $\mathfrak{M}$ , while the union $\{\Phi_\alpha, \Psi_\lambda\}$ of all sets $\Phi_\alpha$ and $\Psi_\lambda$ will be its system of defining relations. It can be shown that for a finitely generated module over a dyad there holds a unique decomposition into indecomposable ones (for modules of finite length the uniqueness of the decomposition follows immediately from the Krull–Schmidt theorem). Therefore, the set $\{(\kappa_{\alpha\beta}, \ell_{\alpha\beta})\}$ , $(\kappa_{\lambda\mu}, s_{\lambda\mu})$ , $f_\lambda(x)\}$ does not depend upon the method of decomposition of module $\mathfrak{M}$ into a direct sum of indecomposable ones and is its complete system of invariants.

If we consider only finite modules over a dyad, the only change will be that in this case the parameters $\kappa_1$ , $\ell_n$ cannot equal $\infty$ .

## §4. Finite $p$ -Groups Possessing an Abelian Subgroup of Index $p$

In this section we aim to obtain the defining relations for finite $p$ -groups possessing an Abelian subgroup of index $p$ . We also find the complete system of invariants.

Let $G$ be a finite $p$ -group, $A$ be an Abelian subgroup of index $p$ , $g$ be some element of the group, not contained in $A$ . $A$ is a normal divisor of $G$ (as a maximal subgroup of a finite $p$-group ([4], Corollary 4.2.2)); therefore, it is natural to treat it as a module over $Z_p(\mu)$ where $Z_p$ is the ring of $p$ -adic integers, $\mu$ is the root of the polynomial $x^p - 1$ , having defined for any $h \in A$ and $z(\mu) = z_0 + z_1\mu + \ldots + z_{p-1}\mu^{p-1} \in Z_p(\mu)$:

$$h^{z(\mu)} = h^{z_0} \cdot g^{-1} h^{z_1} g \cdot g^{-2} h^{z_2} g^2 \ldots g^{-p+1} h^{z_{p-1}} g^{p-1},$$
$$h^{z_i} = h^{c_{i0} + c_{i1}p + \ldots + c_{i,n-1}p^{n-1}},$$

where $p^n$ is the order of $h$ , $z_i = c_{i0} + c_{i1}p + \ldots + c_{in}p^n + \ldots$, $0 \leq c_{ij} < p$.

It is easy to verify that the equality

$$z(\mu) = (z(\varepsilon),\ z(1)),$$

where $z(x)$ is a polynomial with coefficients in $Z_p$ , $\varepsilon$ is the root of the polynomial $1 + x + \ldots + x^{p-1}$ , irreducible over the field of $p$ -adic numbers, yields the composition of $Z_p(\mu)$ into a dyad (§1) of local Dedekind rings $Z_p(\varepsilon)$ and $Z_p$ with prime elements $\varepsilon - 1$ and $p$ . To these prime elements corresponds the dyad's elements $\pi_1 = \mu - 1 = (\varepsilon - 1, 0)$ and $\pi_2 = 1 + \mu + \ldots + \mu^{p-1} = (0, p)$.

Let us derive one relation between elements $\pi_1$ and $\pi_2$ which we need subsequently. As is easily shown, $z(\mu) = z_0 + z_1\mu + \ldots + z_{p-1}\mu^{p-1} \in Z_p(\mu)$ is invertible if and only if $z(1) = z_0 + z_1 + \ldots + z_{p-1} \notin pZ_p$. Furthermore, $z(\mu)\pi_2 = z(1)\pi_2$ for every $z(\mu) \in Z_p(\mu)$. Therefore, in the obvious equality $\pi_2 - \pi_1^{p-1} = p\tau$ the element $\tau$ is invertible and

$$\pi_1^{p-1}\tau^{-1} = \pi_2 - p, \qquad \pi_2\tau^{-1} = \pi_2. \tag{5}$$

Thus, with the group $G$ we associate, with respect to fixed $A$ and $g$ , a module $\mathfrak{M}$ over the dyad $Z_p(\mu)$ with a preferred element $v \in \mathfrak{M}$ which is identified with the element $g^p \in A$ under the natural identification of the elements of $\mathfrak{M}$ with the elements of $A$ . It is clear that the pair $(\mathfrak{M}, v)$ completely defines the group $G$ . Modules over a dyad have been described in §3. Therefore, for a complete description of all groups of the form being considered by us it is sufficient to ascertain what pairs $(\mathfrak{M}, v)$ can indeed be obtained from such groups, as well as to study when different pairs $(\mathfrak{M}, v)$ and $(\mathfrak{M}', v')$ defines one groups $G$ (with respect to different $A$ , $g$ ).

Suppose, as a matter of fact, that with the groups $G$ we have associated, with respect to $A$ and $g$ , a module $\mathfrak{M}$ and, with respect to $A$ and $g_1 \notin A$ , a module $\mathfrak{M}'$ . It is easy to see that if $gA = g_1^n A$ $(0 < n < p)$, then, having replace the dyad's element $\mu$ by $\mu^n$ in the defining relations $\{\Phi_\alpha, \Psi_\lambda\}$ of module $\mathfrak{M}$ (see §3) (here $\pi_1 = \mu - 1$ is replaced by $\pi_1' = \mu^n - 1 = \pi_1 \; (1+\mu+\ldots+\mu^{n-1})$, $\pi_2 = 1+\mu+\ldots+\mu^{p-1}$ by $\pi_2' = 1+\mu^n+\ldots+\mu^{n(p-1)} = \pi_2$ ), we obtain the defining relations $\{\Phi_\alpha', \Psi_\lambda'\}$ of module $\mathfrak{M}'$. By the change of generators

$$v_{\alpha\beta}' = v_{\alpha\beta}(1+\mu+\ldots+\mu^{n-1})^{1+\kappa_{\alpha1}+\kappa_{\alpha2}+\ldots+\kappa_{\alpha\beta}},$$

$$v_{\lambda\mu\nu}' = v_{\lambda\mu\nu}(1+\mu+\ldots+\mu^{n-1})^{(\nu-1)(\tau_{\lambda1}+\ldots+\tau_{\lambda q_\lambda})+(\tau_{\lambda1}+\ldots+\tau_{\lambda\mu})}$$

(admissible in view of the invertibility of $1+\mu+\ldots+\mu^{n-1}$ in $Z_p(\mu)$) we bring $\{\Phi_\alpha', \Psi_\lambda'\}$ to the standard form with the system of invariants $\{(\kappa_{\alpha\beta}, \ell_{\alpha\beta}), (\tau_{\lambda\mu}, s_{\lambda\mu}), \xi_\lambda'(x)\}$ (see §3), where

$$\xi_\lambda'(x) = \xi_\lambda(x\kappa^{\tau_{\lambda1}+\ldots+\tau_{\lambda q_\lambda}})\kappa^{-t_\lambda(\tau_{\lambda1}+\ldots+\tau_{\lambda q_\lambda})} \tag{6}$$

( $\kappa$ is the image of the number $n$ under the natural homomorphism of the integer ring $Z$ into the field $Z/Z_p$).

We have shown that two modules $\mathfrak{M}$ and $\mathfrak{M}'$ can be obtained from one group $G$ with respect to a fixed $A$ if and only if in the field of $P$ elements we can find a nonzero element $\kappa$ such that as a result of replacing all $\xi_\lambda(x)$ from the system of invariants of module $\mathfrak{M}$ by the polynomials $\xi_\lambda'(x)$ from (6) we obtain the system of invariants of module $\mathfrak{M}'$.

Suppose that with group $G$ we have associated, with respect to $A$ and $g$ , the pair $(\mathfrak{M}, v)$ ($v \in \mathfrak{M}$ is identified with $g^p \in A$ ). Let us show that $g$ and the generator system $\{v_{\alpha\beta}, v_{\lambda\mu\nu}\}$ of module $\mathfrak{M}$ can be chosen so that $v = v_{01}\pi_1^{\kappa_{01}}$ or $v = 0$ (for the same $\mathfrak{M}$ ).

Indeed, it is not difficult to verify (for example, by the method of undetermined coefficients) that the set $K = \{w \in \mathfrak{M} \mid v\pi_1 = 0\}$ equals

$$K = \mathfrak{M}\pi_2 + \sum_{\alpha=0}^{m_1} v_{\alpha1}\pi_1^{\kappa_{\alpha1}} Z_p(\mu). \tag{7}$$

Since $v \in K$ ( $v$ is identified with $g^p$ ), where $v = w\pi_2 + \sum_{\alpha=0}^{m_1} v_{\alpha1}\pi_1^{\kappa_{\alpha1}} d_\alpha$, where $w \in \mathfrak{M}$ , $d_\alpha$ is an invertible element of $Z_p(\mu)$ or is zero.

Suppose that not all $d_\alpha = 0$. It is clear that then the set $\{v_{\alpha\beta}\}$ can be renumbered over the index $\alpha$ do that $d_\alpha \neq 0$ for $\alpha \leq \varepsilon_0$ , $d_\alpha = 0$ for $\alpha > \varepsilon_0$ , and the following conditions are fulfilled for the parameters $\kappa_{\alpha\beta}$, $\ell_{\alpha\beta}$, $n_\alpha$ $(\alpha = 0, \ldots, \varepsilon_0, \; \beta = 0, \ldots, n_0)$:

1) $\kappa_{0\beta} = \kappa_{1\beta} = \ldots = \kappa_{\sigma_\beta\beta} < \kappa_{\sigma_\beta+1,\beta}, \ldots, \kappa_{\varepsilon_\beta\beta}$,

2) $\ell_{0\beta} = \ell_{1\beta} = \ldots = \ell_{\varepsilon_\beta \beta} > \ell_{\varepsilon_\beta+1,\beta}, \ldots, \ell_{r_\beta \beta}$,

3) if $n_\alpha = \beta$ for some $\alpha \leq \varepsilon_\beta$, then $n_\alpha = n_\beta$.

It is easily shown that when these conditions are fulfilled the set $\{v_{\alpha\beta}, v_{\lambda\mu\nu} \mid \alpha \neq 0\}$ and all the elements $v'_{0\beta} = \sum_{\alpha=0}^{\varepsilon_\beta - 1} v_{\alpha\beta} \pi_1^{\kappa_{\alpha\beta} - \kappa_{0\beta}} d_\alpha$ together constitute a new generator system, while the relations obtained from the system of defining relations $\{\Phi_\alpha, \Psi_\lambda\}$ of module $\mathfrak{M}$ by replacing each element $v_{0\beta}$ occurring in $\Phi_0$ by $v'_{0\beta}$ constitute a new system of defining relations of standard form for $\mathfrak{M}'$.

Thus, $v = w\pi_2 + v'_{01} \pi_1^{\kappa_{01}} d$, where $d = 0$ or $d = 1$. Let $w \in \mathfrak{M}$ be identified with $h \in A$. Then, as we are easily convinced, $v - w\pi_2$ is identified with $(gh^{-1})^p$. Therefore, with the group $G$ we associate, with respect to $A$ and $gh^{-1}$, the pair $(\mathfrak{M}, v'_{01} \pi_1^{\kappa_{01}})$ for $d = 1$ or the pair $(\mathfrak{M}, 0)$ for $d = 0$, which is what we had to show.

By $h_{\alpha\beta}$ $(h_{\lambda\mu\nu})$ we denote the element of $A$, which is identified with $v_{\alpha\beta} (v_{\lambda\mu\nu}) \in \mathfrak{M}$, where $\mathfrak{M}$ is a finite module over $Z_p(\mu)$, $\mu^p = 1$; therefore, every element of module $\mathfrak{M}$ is written as the sum of elements $\pm v_{\alpha\beta} \mu^\kappa$, $\pm v_{\lambda\mu\nu} \mu^\tau$ $(0 \leq \kappa < p,\ 0 \leq \tau < p)$. Hence it follows that the set $\{h_{\alpha\beta}^{g^\kappa}, h_{\lambda\mu\nu}^{g^\tau}\}$ $(h^{g^\kappa} = g^{-\kappa} h g^\kappa)$ generates a subgroup $A$ in $G$. Let us write the defining relations $\{\Phi_\alpha, \Psi_\lambda\}$ of module $\mathfrak{M}$ in terms of the elements of group $G$. The new relations obtained, together with the relations $h_{\alpha\beta}^{g^\kappa} h_{r\delta}^{g^\ell} = h_{r\delta}^{g^\ell} h_{\alpha\beta}^{g^\kappa}$, $h_{\varepsilon\zeta}^{g^m} h_{\lambda\mu\nu}^{g^\tau} = h_{\lambda\mu\nu}^{g^\tau} h_{\varepsilon\zeta}^{g^m}$, $h_{\xi\pi\rho}^{g^s} h_{\sigma\tau\chi}^{g^t} = h_{\sigma\tau\chi}^{g^t} h_{\xi\pi\rho}^{g^s}$, ensuring the Abelianness of $A$, and the relation $g^p = h$ $(h \in A)$ yield, as we easily are convinced, the system of defining relations of group $G$.

We shall use the usual notation for commutators:

$$[x, y] = x^{-1} y^{-1} x y, \quad [x, {}_0 g] = x, \quad [x, {}_{\kappa+1} g] = [[x, {}_\kappa g], g] \quad (\kappa = 0, 1, 2, \ldots).$$

We have proved that a finite $p$-group $G$ possessing an Abelian subgroup $A$ of index $p$ is given by the generators $g \notin A$, $h_{\alpha\beta}, h_{\lambda\mu\nu} \in A$ and the following defining relations:

$$\text{I.} \quad g^p = [h_{01}, {}_{\kappa_{01}} g], \quad [h_{\alpha 1}, {}_{\kappa_{\alpha 1}+1} g] = 1 \qquad (\text{for } \alpha > 0)$$

$$[h_{\alpha,\beta+1}, {}_{\kappa_{\alpha,\beta+1}} g] = g^{-p^{\ell_{\alpha\beta}}} (g h_{\alpha\beta})^{p^{\ell_{\alpha\beta}}} \ (\beta = 1, \ldots, n_\alpha - 1), \quad g^{-p^{\ell_{\alpha n_\alpha}+1}} (g h_{\alpha n_\alpha})^{p^{\ell_{\alpha n_\alpha}+1}} = 1,$$

where

$$\alpha = 0, \ldots, m_1.$$

$$\text{II.} \quad [h_{\lambda 11}, {}_{\tau_{\lambda 1}} g] = g^{-p^{s_{\lambda q_\lambda}}} (g h_{\lambda q_\lambda 1}^{b_{\lambda 1}} \ldots h_{\lambda q_\lambda t_\lambda}^{b_\lambda t_\lambda})^{p^{s_{\lambda q_\lambda}}},$$

$$[h_{\lambda 1,\nu+1}, {}_{\tau_{\lambda 1}} g] = g^{-p^{s_{\lambda q_\lambda}}} (g h_{\lambda q_\lambda \nu})^{p^{s_{\lambda q_\lambda}}} \qquad (\nu = 1, \ldots, t_\lambda - 1),$$

$$[h_{\lambda,\mu+1,\nu}, {}_{\tau_{\lambda,\mu+1}} g] = g^{-p^{s_{\lambda\mu}}} (g h_{\lambda\mu\nu})^{p^{s_{\lambda\mu}}} \quad (\mu = 1, \ldots, q_\lambda - 1,\ \nu = 1, \ldots, t_\lambda)$$

where

$$\lambda = 1, \ldots, m_2.$$

$$\text{III.} \quad [h_{\alpha\beta}, h_{r\delta}^{g^\kappa}] = 1, \quad [h_{\varepsilon\xi}, h_{\lambda\mu\nu}^{g^\tau}] = 1, \qquad [h_{\varepsilon\pi\rho}, h_{\sigma\tau\chi}^{g^s}] = 1,$$

where $\alpha < r$ or $\alpha = r$ but $\beta < \delta$; $\xi < \sigma$, or $\xi = \sigma$ but $\pi < \tau$ or $\xi = \sigma$; $\pi = \tau$ but $\rho < \chi$; $\kappa, \tau, s = 0, \ldots, p-1$.

Let us write the defining relations I-III in another, in a certain sense simpler, form. For a positive integer $\kappa$ we denote by $\kappa'$ and $\kappa''$ numbers such that $\kappa = \kappa'(p-1)+\kappa''$, $0 < \kappa'' < p$. Having multiplied the relation $v_{\alpha,\beta+1}\pi_1^{\kappa_{\alpha,\beta+1}} = v_{\alpha\beta}\pi_2^{\ell_{\alpha\beta}}$ by $\tau^{-\kappa'_{\alpha\beta}}$, using (5) and the equalities $\pi_1\pi_2 = 0$ and $\pi_2^{\ell} = p^{\ell-1}\pi_2$, we obtain

$$v_{\alpha\beta+1}(-p)^{\kappa'_{\alpha,\beta+1}}\pi_1^{\kappa''_{\alpha,\beta+1}} = v_{\alpha\beta}\,p^{\ell_{\alpha\beta}-1}\pi_2. \tag{8}$$

As the new $h_{01}$ we select the element which is identified with $v_{01}\tau^{(\kappa_{01}+1)'}$ by the natural identification of elements of $A$ with the elements of $\mathfrak{M}$. For $\kappa''_{01} = p-1$, as the new $g$ we select the element $gh_{01}^{-(-p)^{(\kappa_{01}+1)'}-1}$.

In view of (8) it is easy to show that the defining relations I-III of group $G$ relative to the new generators $g, h_{\alpha\beta}, h_{\lambda\mu\nu}$ can be rewritten as:

$$\mathrm{I}'.\ g^{p} = [h_{01}^{(-p)^{(\kappa_{01}+1)'}}, {}_{(\kappa_{01}+1)''-1}\, g],\ [h_{\alpha}^{(-p)^{(\kappa_{\alpha 1}+1)'}}, {}_{(\kappa_{\alpha1}+1)''}\, g] = 1 \qquad (\text{for } \alpha > 0)$$

$$[h_{\alpha,\beta+1}^{(-p)^{\kappa'_{\alpha,\beta+1}}}, {}_{\kappa''_{\alpha,\beta+1}}\, g] = g^{-p}(gh_{\alpha\beta}^{p^{\ell_{\alpha\beta}-1}}),\ (\beta = 1,\dots,n_\alpha - 1),\ g^{-p}(gh_{\alpha n_\alpha}^{p^{\ell_{\alpha n_\alpha}}})^{p} = 1,$$

where $\alpha = 0,\dots,m_1$.

$$\mathrm{II}'.\ [h_{\lambda 11}^{(-p)^{\varkappa'_{\lambda 1}}}, {}_{\varkappa''_{\lambda}}\, g] = g^{-p}(gh_{\lambda q_\lambda}^{b_{\lambda 1}p^{s_{\lambda q_\lambda}-1}} \dots h_{\lambda q_\lambda t_\lambda}^{b_{\lambda t_\lambda}p^{s_{\lambda q_\lambda}-1}})^{p},$$

$$[h_{\lambda 1,\nu+1}^{(-p)^{\varkappa'_{\lambda 1}}}, {}_{\varkappa''_{\lambda 1}}\, g] = g^{-p}(gh_{\lambda q_\lambda \nu}^{p^{s_{\lambda q_\lambda}-1}})^{p} \qquad (\nu = 1,\dots,t_{\lambda-1}),$$

$$[h_{\lambda,\mu+1,\nu}^{(-p)^{\varkappa'_{\lambda 1}}}, {}_{\varkappa''_{\lambda,\mu+1}}\, g] = g^{-p}(gh_{\lambda\mu\nu}^{p^{s_{\lambda\mu\nu}-1}})^{p} \quad (\mu = 1,\dots,q_\lambda - 1,\ \ \nu = 1,\dots,t_\lambda),$$

where $\lambda = 1,\dots,m_2$.

$$\mathrm{III}'.\ [h_{\alpha\beta}, h_{\gamma\delta}^{g^{\kappa}}] = 1,\quad [h_{\varepsilon\zeta}, h_{\lambda\mu\nu}^{g^{\varkappa}}] = 1,\ [h_{\xi\pi\rho}, h_{\theta\tau\chi}^{g^{s}}] = 1,$$

where $\alpha < \gamma$ or $\alpha = \gamma$ but $\beta < \delta$; $\xi < \theta$ or $\xi = \theta$ but $\pi < \tau$, or $\xi = \theta$, $\pi = \tau$ but $\rho < \chi$; $\kappa, \varkappa, s = 0,\dots,p-1$.

Group $G$ is completely defined by the following collection of parameters: $(\kappa_{\alpha1}, \ell_{\alpha1}),\dots,(\kappa_{\alpha n_\alpha}, \ell_{\alpha n})$ a sequence of pairs of integers $\kappa_{\alpha1} \geq 0$, $\kappa_{\alpha\beta} > 0$ for $\beta > 1$, $\ell_{on_o} \geq -1$, $\ell_{\alpha n_\alpha} \geq 0$ for $\alpha > 0$, $\ell_{\alpha\beta} > 0$ for $\beta < n_\alpha$; if $\ell_{on_o} = -1$, then $\kappa_{01} = 0$ and $n_o = 1$, $\alpha = 0,\dots,m_1$ ($m_1$ is a positive integer or zero); $(\varkappa_{\lambda1}, s_{\lambda1}),\dots,(\varkappa_{\lambda q_\lambda}, s_{\lambda q_\lambda})$, an aperiodic sequence of pairs of positive integers determined to within any cyclic permutation; $f_\lambda(x) = x^{t_\lambda} - b'_{\lambda1}x^{t_\lambda-1} - \dots - b'_{\lambda t_\lambda}$ $(b'_{\lambda t_\lambda} \neq 0)$, a power of an irreducible polynomial over a field of $p$ elements, where two sequences of polynomials $f_1(x),\dots,f_{m_2}(x)$ and $f'_1(x),\dots,f'_{m_2}(x)$ define one group $G$ if in the field of $p$ elements we can find a nonzero element $\kappa$ such that for all $\lambda$

$$f'_\lambda(x) = f_\lambda(x\kappa^{\varkappa_{\lambda1}+\dots+\varkappa_{\lambda q_\lambda}})^{-t_\lambda(\varkappa_{\lambda1}+\dots+\varkappa_{\lambda q_\lambda})}$$

$\lambda = 1,\dots,m_2$ ($m_2$ is a positive integer or zero). The collection is unordered with respect to the indices $\alpha > 0$ and $\lambda$.

Then $b_{\lambda\mu}$ is an integer, $0 \leq b_{\lambda\mu} < p$, mapping into $b'_{\lambda\mu}$ under the natural homomorpjism of the integers ring $Z$ onto the field $Z/Z_p$.

Taking into account the method by which we associated with each module $\mathfrak{M}$ over dyad $D$ the sequences of weights $c_1, \ldots, c_m$ and $\kappa_1, \ldots, \kappa_n$ (see §1), it is not difficult to calculate the order of group $G$:

$$|G| = p^{n+m_1+\sum_{\alpha\beta}(\kappa_{\alpha\beta}+\ell_{\alpha\beta})+\sum_{\lambda\mu} t_\lambda(\tau_{\lambda\mu}+s_{\lambda\mu})}. \tag{9}$$

If group $G$ is non-Abelian, then its center consists of elements of $A$ commutative with $g$. In view of (7), $Z$ is generated by the elements $[h_{\alpha 1}, {}_{\kappa_{\alpha 1}} g]$, $g^{-p}(gh_{\alpha\beta})^p$, $g^{-p}(gh_{\lambda\mu\nu})^p$. Hence it follows that the relations $[h_{\alpha 1}, {}_{\kappa_{\alpha 1}} g] = 1$, $g^{-p}(gh_{\alpha\beta})^p = 1$, $g^{-p}(gh_{\lambda\mu\nu})^p = 1$ together with I-III yield the system of defining relations for the quotient group $G/Z$. After the reduction of the system obtained to standard form I-III, using (9) we get that the center's order equals

$$|Z| = p^{1+m_1+\sum_{\alpha\beta}\ell_{\alpha\beta}+\sum_{\lambda\mu} t_\lambda s_{\lambda\mu}}. \tag{10}$$

It still remains for us to ascertain when different collections of parameters $\{(\kappa_{\alpha\beta}, \ell_{\alpha\beta}), (\tau_{\lambda\mu}, s_{\lambda\mu}), f_\lambda(x)\}$ yield one group $G$. From the preceding considerations it follows that with group $G$ there is uniquely associated, with respect to a fixed $A$, the collection of parameters $\{(\kappa_{\alpha\beta}, \ell_{\alpha\beta}), (\tau_{\lambda\mu}, s_{\lambda\mu}) f_\lambda(x)\}$. However, it can happen that $G$ possesses several Abelian subgroups $A$ of index $p$, relative to which it is specified by different collections of parameters.

If $G$ is non-Abelian and $A$, $B$ are two Abelian subgroups with index $p$ of it, then, as is not difficult to show, $A \cap B = Z$ and the index of the center $Z$ in $G$ equals $p^2$. Therefore, from (9) and (10) it follows that $\sum_{\alpha\beta}\kappa_{\alpha\beta} + \sum_{\lambda\mu} t_\lambda \tau_{\lambda\mu} = 1$.

In the general case $G$ possesses more than one Abelian subgroup of index $p$ if and only if the inequalities $\sum_{\alpha\beta}\kappa_{\alpha\beta} + \sum_{\lambda\mu} t_\lambda \tau_{\lambda\mu} > 1$ and $(m_1 + m_2 + 1)(\kappa_{01}+1) > 1$ are fulfilled for its parameters.

We exclude the following collections of parameters:

1) $m_1 \geq 1$, $n_\alpha = 1$, $\kappa_{\alpha 1} = 0$, $\ell_{01} \geq \min_{\alpha > 0}\{\ell_{\alpha 1}\}$; $m_2 = 0$,

2) $m_1 \geq 1$, $n_\alpha = 1$, $\kappa_{\alpha 1} = 0$ for $\alpha \neq 1$, $\kappa_{11} = 1$, $\ell_{01} + 1 > \ell_{11}$, $m_2 = 0$,

3) $m_1 \geq 1$, $n_\alpha = 1$ for $\alpha \neq 1$, $n_1 = 2$, $\kappa_{\alpha 1} = 0$, $\kappa_{12} = 1$, $\ell_{01} + 1 > \ell_{12}$, $m_2 = 0$,

4) $m_1$ is arbitrary, $n_\alpha = 1$, $\kappa_{\alpha 1} = 0$, $\ell_{\alpha 1}$ is arbitrary; $m_2 = q_1 = \tau_{11} = 1$, $s_{11}$ is arbitrary, $f_1(x) = x - 1$.
It is not difficult to convince ourselves that the remaining collections of parameters yield all the already mutually nonisomorphic groups $G$. Therefore, $\{(\kappa_{\alpha\beta}, \ell_{\alpha\beta}), (\tau_{\lambda\mu}, s_{\lambda\mu}, f_\lambda(x)\}$ is a complete collection of invariants of a finite $p$-group possessing an Abelian subgroup of index $p$.

Let us make several more remarks on the group $G$ given by relations I-III. For each index we denote by $A_\alpha$ $(B_\lambda)$ the smallest normal divisor of $G$, containing the elements $h_{\alpha 1}, \ldots, h_{\alpha n_\alpha}$ $(h_{\lambda 11}, \ldots, h_{\lambda q_\lambda t_\lambda})$. From the structure of the module $\mathfrak{M}$ associated with group $G$ relative to $A$ it follows that

$$A = A_0 \oplus \ldots \oplus A_{m_1} \oplus B_1 \oplus \ldots \oplus B_{m_2} \tag{11}$$

and each $A_\alpha (B_\nu)$ is now decomposed into a direct sum of normal divisors of $G$. $A$ is picked out by the semidirect factors of $G$ if and only if $\kappa_{01} = -1$.

By $H_{\alpha\beta}$ we denote the normal divisor of $G$, generated by the element $h_{\alpha\beta}$. Making use of the fact that $H_{\alpha\beta}$ is isomorphic to an additive group of the quotient ring $Z_p(\mu)/\mathrm{Ann}\ v_{\alpha\beta}$, $\mathrm{Ann}\, v_{\alpha\beta} = \{d \in Z_p(\mu) \mid v_{\alpha\beta}\, d = 0\}$, it is not difficult to show that $H_{\alpha\beta}$ expands into a direct sum of cyclic groups:

$$H_{\alpha\beta} = \begin{cases} \{h_{\alpha\beta}\} \oplus \{[h_{\alpha\beta}, g]\} \oplus \ldots \oplus \{[h_{\alpha\beta, p-1}\, g]\}, & \text{if } \left[\frac{\kappa_{\alpha\beta}}{p-1}\right] \leq \ell_{\alpha\beta}, \\ \{h_{\alpha\beta}\} \oplus \{[h_{\alpha\beta}, g]\} \oplus \ldots \oplus \{[h_{\alpha\beta, p-2}\, g]\} \oplus \{g^{-p}(g h_{\alpha\beta})^p\} & \text{if } \left[\frac{\kappa_{\alpha\beta}}{p-1}\right] \geq \ell_{\alpha\beta}, \end{cases}$$

where $[a]$ is the integer part of number $a$. The orders of the elements $h_{\alpha\beta}$, $[h_{\alpha\beta, n}\, g]$ $(n > 0)$, $g^{-p}(g h_{\alpha\beta})^p$ equal $p^{1+\max([\frac{\kappa_{\alpha\beta}}{p-1}], \ell_{\alpha\beta})}$, $p^{1+[\frac{\kappa_{\alpha\beta}-n}{p-1}]}$, $p^{\ell_{\alpha\beta}}$. The arguments remain in force if $H_{\alpha\beta}$, $h_{\alpha\beta}$, $v_{\alpha\beta}$, $\kappa_{\alpha\beta}$ $\ell_{\alpha\beta}$ are replaced by $H_{\lambda\mu\nu}$, $h_{\lambda\mu\nu}$, $v_{\lambda\mu\nu}$, $r_{\lambda\mu}$, $s_{\lambda\mu}$ for $q_\lambda t_\lambda > 1$. Using this expansion and formula (11) it is not difficult to obtain an expansion of $A$ into a direct sum of cyclic groups. We shall not do this here.

The formula for computing the number of $p$-groups of given order, possessing an Abelian subgroup of index $p$, would be rather long; however, it is easy to calculate the number $\mathfrak{f}(n)$ of such groups of order $p^n$, indecomposable into a semidirect product of subgroups and of some normal divisor contained in $A$. Obviously, groups with parameters $m_1 = m_2 = 0$, $\kappa_{01} \neq -1$ will be such groups. In view of (9), $\mathfrak{f}(n)$ equals the number of representations $n$ in the form of an ordered sum of an even number of positive integers, i.e., $\mathfrak{f}(n) = 2^{n-2}$.

## §5. Pair of Mutually Annihilating Operators

The following problem was solved in [2] (Chap. II): two nilpotent linear transformations $\pi_1$ and $\pi_2$ are given in a finite-dimensional vector space $V$ over a field $K$, such that $\pi_1 \pi_2 = \pi_2 \pi_1 = 0$.* Find in $V$ the bases in which the matrices of these transformations have a canonic form, and give also the complete system of invariants.

It is evident that $V$ can be treated as a module over a $K$-algebra $\Lambda$ of infinite rank, having the basis $1, \pi_1, \pi_2, \pi_1^2, \pi_2^2, \ldots$, where $\pi_1 \pi_2 = \pi_2 \pi_1 = 0$. It is easy to see that the equality

$$a + b_1 \pi_1 + c_1 \pi_2 + b_2 \pi_1^2 + c_2 \pi_2^2 + \ldots = (a + b_1 \pi_1 + b_2 \pi_1^2 + \ldots,\ a + c_1 \pi_2 + c_2 \pi_2^2 + \ldots),$$

$a$, $b_i$, $c_i \in K$, gives a decomposition of $\Lambda$ into a dyad) (§1) of rings of formal power series $K[[\pi_1]]$ and $K[[\pi_2]]$.†

*The requirement of nilpotency of operators $\pi_1$ and $\pi_2$ is not essential. Indeed, if one of these operators is nonnilpotent and the space $V$ is indecomposable into a direct sum of invariant spaces, then it is not difficult to show that the other operator is zero.

†Instead of a dyad of rings of power series we could take the dyad of rings of polynomials. In truth, a dyad of two nonlocal Dedekind rings would be obtained; however, it is not difficult to be convinced that the considerations of §§1-3 carry over to this case almost without change. Here nonnilpotent operators would automatically be included in the consideration.

$K[[\pi_i]]$ is a local Dedekind ring with prime element $\pi_i$.

Thus our problem is reduced to the problem, solved in §§1-3, of the classification of modules over a dyad of local Dedekind rings. From the results of §3 it is easily obtained that the vector in which the pair of nilpotent operators $\pi_1$ and $\pi_2$ such that $\pi_1\pi_2 = \pi_2\pi_1 = 0$ act, is decomposable into a direct sum of indecomposable invariant spaces of the following two types.

A space $V_1$ of type I is uniquely determined by the sequence of nonnegative integers $(\kappa_1, \ell_1), \ldots, (\kappa_n, \ell_n)$, where $\kappa_\alpha > 0$ for $\alpha > 1$, $\ell_\beta > 0$ for $\beta < n$. The space $V_1$ is constructed as a $1 + \sum(\kappa_\alpha + \ell_\alpha)$-dimensional space spanned by the vectors $v_\alpha \pi_1^\kappa$, $v_\alpha \pi_2^\ell$, $v_n \pi_2^{\ell_n}$, where $\kappa = 0, \ldots, \kappa_\alpha$, $\ell = 1, \ldots, \ell_\alpha - 1$, $\alpha = 1, \ldots, n$. The equality $\pi_1\pi_2 = \pi_2\pi_1 = 0$, the form of the basis vectors, and also the relations

$$v_1 \pi_1^{\kappa_1+1} = 0, \quad v_{\alpha+1} \pi_1^{\kappa_{\alpha+1}} = v_\alpha \pi_2^{\ell_\alpha} \quad (\alpha = 1, \ldots, n-1), \quad v_n \pi_2^{\ell_n+1} = 0$$

completely determine the action of operators $\pi_1$ and $\pi_2$ on the basis vectors.

A space $V_2$ of type II is uniquely determined by the following collection of invariants: $(\tau_1, s_1), \ldots, (\tau_q, s_q)$, an aperiodic sequence of pairs of positive integers, defined to within any cyclic permutation; $f(x) = x^t - b_1 x^{t-1} - \ldots - b_t$ $(b_t \neq 0)$, a power of an irreducible polynomial over field $K$. Space $V_2$ is spanned by the vectors $v_{\mu\nu} \pi_1^\tau$, $v_{\mu\nu} \pi_2^s$, where $\tau = 0, \ldots, \tau_\mu$, $s = 1, \ldots, s_{\mu-1}$, $\mu = 1, \ldots, q$, $\nu = 1, \ldots, t$. The operators $\pi_1$ and are completely determined by the relations:

$$v_{11} \pi_1^{\tau_1} = (v_{q1} b_1 + \ldots + v_{qt} b_t) \pi_2^{s_q}, \quad v_{1,\nu+1} \pi_1^{\tau_1} = v_{q\nu} \pi_2^{s_q} \ (\nu = 1, \ldots, t-1),$$

$$v_{\mu+1,\nu} \pi_1^{\tau_{\mu+1}} = v_{\mu\nu} \pi_2^{s_\mu} \quad (\mu = 1, \ldots, q-1, \quad \nu = 1, \ldots, t-1).$$

We easily convince ourselves that in the case field $K$ is algebraically closed spaces of types I and II coincide with the canonic modules of kind I and II defined in [2].

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