The lemma is proved.

The proof of the estimate
\[ \sum_{m_i=r_i}^{r} \sum_{m_j=0}^{r_i} S_{m_i m_j} = O(r/d) \]
is almost a verbatim repetition of the proof of Lemma 4.

Combining the results of the last three lemmas, we obtain the estimate
\[ \sum_{m_i=r_i}^{r} \sum_{m_j=0}^{r_i} S_{m_i m_j} = O\left( \frac{r}{d} + \frac{r}{d^2} \right) \]
Consequently, if \( u \leq N_r \), then
\[ |R_u| = O\left( \frac{r}{d} + \frac{r}{d^2} + \frac{n \log r}{d} \right) \]
It is now easy to see that under the conditions of the theorem we have
\[ |R_u| = o\left( \frac{n^2}{r} \right) \]
At the same time, according to Lemma 1,
\[ r - |N_r| = o(r) \]
The theorem is proved.

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**SYMMETRIC REPRESENTATIONS OF ALGEBRAS WITH INVOLUTION**

V. V. Sergeichuk

Suppose \( K \) is a field of characteristic \( \neq 2 \) with involution \( k \rightarrow \bar{k} \) (possibly the identity mapping) and \( \Lambda \) is an algebra over \( K \) with involution, i.e., a mapping \( \iota: \Lambda \rightarrow \Lambda \) such that \( (\lambda + \mu)^\iota = \lambda^\iota + \mu^\iota \), \( (\lambda \mu)^\iota = \mu^\iota \lambda^\iota \) for all \( \lambda, \mu \in \Lambda, k \in K \).

By a **representation** of the algebra \( \Lambda \) by operators of a vector space \( V \) over \( K \) we mean a homeomorphism \( \varphi: \Lambda \rightarrow \text{End}(V) \). The representation is **symmetric** if to a conjugate element there is assigned the conjugate linear operator relative to a fixed scalar product in \( V \): \( \varphi(\lambda^\iota) = \overline{\varphi(\lambda)} \). If we introduce in \( V \) the multiplication \( \Lambda \times K \times (v, w) = \epsilon F(v, w) \) we obtain an \( \epsilon \)-Hermitian module defined as follows.

**Definition.** By an \( \epsilon \)-Hermitian module \((M, F), (M', F')\) we mean a pair \((M, F)\), where \( M \) is a module over \( \Lambda \) that is finite-dimensional over \( K \), \( F(v, w) = \epsilon F(w, v) \) is a nondegenerate \( \epsilon \)-Hermitian form on the vector space \( pM \) of the module \( M \), and
\[ F(\lambda v, w) = F(v, \lambda w), \quad \lambda \in \Lambda, \quad v, w \in M. \] \( (1) \)

Two \( \epsilon \)-Hermitian modules \((M, F), (M', F')\) are **isomorphic** if there exists a \( \Lambda \)-isomorphism \( \varphi \): \( M \cong M' \), preserving the forms:
\[ F(v, w) = F'(\varphi v, \varphi w), \quad v, w \in M. \] \( (2) \)

Examples. 1) \( \Lambda = \mathbb{K}[x], x^t = x \). The module \( M \) over \( \Lambda \) is the vector space \( \mathbb{K}^M \) with fixed linear operator \( v \mapsto xv \). The problem of classifying \( \varepsilon \)-Hermitian modules is that of classifying self-conjugate linear operators in a finite-dimensional vector space with a nondegenerate \( \varepsilon \)-Hermitian form.

2) \( \Lambda = \mathbb{K}[x, x^t], x^t = x \). The problem is to classify isometric operators in a space with a nondegenerate \( \varepsilon \)-Hermitian form.

3) \( \Lambda = \mathbb{K}G \) is the group algebra of a group \( G \) with involution \( (kgg)^* = g^*gg^{-1} \). The problem is to classify representations of \( G \) by isometric operators in a space with a nondegenerate \( \varepsilon \)-Hermitian form.

4) \( K = \mathbb{C} \) with a nonidentity involution. Then a \( 1 \)-Hermitian module \( (M, F) \), where \( F \) is a positive definite Hermitian form, defines a symmetric representation of the algebra \( \Lambda \) by operators of the unitary space \( (cM, F) \) (see [1, Chap. 2, Sec. 2.6]). In particular, if \( \Lambda = CG \) with involution \( (\sum_k g_k) = \sum_k g_k^{-1} \), then such a module defines a unitary representation of \( G \) (see [1, Chap. 2, Sec. 2.8]).

We will show (see the theorem) that the classification of \( \varepsilon \)-Hermitian modules reduces to that of ordinary modules over \( \Lambda \) and Hermitian forms over a skew field. This follows from [2, Chap. 7, Theorem 10.9], but we will use [3, 4] in order to obtain the reduction in a more explicit form. We will apply the reduction to symmetric representations of algebras with involution in pseudo-unitary and pseudo-Euclidean spaces (see Corollary 1) and in unitary, Euclidean, and complex Euclidean spaces (see Corollary 2).

By the orthogonal sum of \( \varepsilon \)-Hermitian modules we mean the \( \varepsilon \)-Hermitian module \((M', F') = (M \oplus M', F \oplus F')\).

Suppose \( M \) is a module over \( \Lambda \). We define the dual module \( M^\ast \) over \( \Lambda \) as the module whose vector space is the space of semilinear forms \( f: \mathbb{K}M \rightarrow \mathbb{K} \), with multiplication by elements \( \lambda \in \Lambda \) defined by \( \lambda f = f \lambda \). We also define the \( \varepsilon \)-Hermitian module \( M^{(\varepsilon)} = (M \oplus M^\ast, F) \), where \( F(v \oplus f, w \oplus g) = g(v) + \overline{f(w)} \) (3)

(all sesquilinear forms are regarded as semilinear in the first argument and linear in the second).

Let \( \text{ind}(\Lambda) \) be a fixed complete system of nonisomorphic modules over \( \Lambda \) that are indecomposable into a direct sum and finite-dimensional over \( \mathbb{K} \). Let \( \text{ind}_\varepsilon^s(\Lambda) \) denote the set of all \( N \in \text{ind}(\Lambda) \) for which there exists an \( \varepsilon \)-Hermitian module \((N, F)\), and fix one such module \((N, F_N)\). In the set \( \text{ind}_\varepsilon^\ast(\Lambda) \) we include all \( M \in \text{ind}(\Lambda), M^\ast \cong M \cong \text{ind}_\varepsilon^\ast(\Lambda) \), and one module from each pair \( \{M, N\} \in \text{ind}(\Lambda) \), \( M \neq M^\ast \cong N \).

Suppose \( N \in \text{ind}_\varepsilon^\ast(\Lambda) \). In the algebra \( \text{End}(N) \) of endomorphisms we define an involution \( \varphi \mapsto \varphi^t \), where \( \varphi^t \) is the conjugate endomorphism relative to \( F_N \):

\[ F_N(\varphi v, w) = F_N(v, \varphi^t w), \quad v, w \in N. \]

The algebra of endomorphisms of an indecomposable module is local, hence the quotient algebra by the radical, \( T(N) = \text{End}(N)/R \), is a skew field with involution \( (\psi + R)^t = \psi^t + R \). For each \( 0 \neq t = t^t \in T(N) \) we fix \( \varphi_t = \varphi_t^t \in t \) (we can take \( \varphi_t = 1/2 (\varphi + \varphi^t) \), where \( \varphi (\equiv t) \) and define an \( \varepsilon \)-Hermitian form \( F_N^t(v, w) = F_N(v, \varphi_t^t w) \). For each Hermitian form \( \varphi (x) = x_1 t_1 x_1 + \ldots + x_n t_n x_n \), over the skew field \( T(N) \) \( 0 \neq t_i = t_i^t \in T(N) \) we put

\[ N^{(\varphi(x))} = (N, F_N^t) \perp \ldots \perp (N, F_N^t). \]

THEOREM. Each \( \varepsilon \)-Hermitian module over \( \Lambda \) is isomorphic to an orthogonal sum

\[ M_1^{(\varepsilon)} \perp \ldots \perp M_n^{(\varepsilon)} \perp N_1^{\varphi_1(x)} \perp \ldots \perp N_n^{\varphi_n(x)}, \]

where \( M_i \in \text{ind}_\varepsilon^s(\Lambda), N_j \in \text{ind}_\varepsilon^\ast(\Lambda), N_j \neq N_j \) for \( j \neq j' \). This orthogonal sum is uniquely determined to within a rearrangement of the summands and the replacement of \( N_j^{\varphi_j(x)} \) by \( N_j^{\varphi_j(x)} \), where \( \varphi_j(x), \varphi_j(x) \) are equivalent Hermitian forms over the skew field \( T(N_j) \).

Remarks. 1) Suppose \( M \) is a module over \( \Lambda \) and \( A_i (\Lambda \in \Lambda) \) is the matrix of the linear operator \( v \mapsto \lambda v (v \in M) \) in the basis \( e_1, \ldots, e_n \) of the space \( \mathbb{K}^M \). Then in the dual basis \( e_1^\ast, \ldots, e_n^\ast \) of the space of the module \( M^\ast \) the operator \( f \mapsto \lambda f (f \in M^\ast) \) is defined by the matrix

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A \ast_1^* \text{ for each matrix } A = (a_{ij}) \text{ we define the matrix } A^* = (a_{ji}^\ast). \text{ In the basis } e_1, \ldots, e_n, e_1^\ast, \ldots, e_n^\ast \text{ of the space of the module } M^\ast = (M \oplus M^\ast, \rho), \text{ the linear operator } \varphi \mapsto \lambda \varphi (\varphi \in M \oplus M^\ast) \text{ and the } \varepsilon \text{-Hermitian form } \mathcal{F} \text{ are defined by the matrices }

\begin{pmatrix}
A & 0 \\
0 & A^* \\
\end{pmatrix}

\text{ and }

\begin{pmatrix}
0 & E \\
E & 0 \\
\end{pmatrix}.

2) \text{ (See [2, Chap. 7, Theorem 4.5].) For each } N^\ast \simeq N \in \text{ind}(\Lambda) \text{ there exists a } \varepsilon \text{-Hermitian or } (-\varepsilon) \text{-Hermitian module } (N, \mathcal{F}). \text{ Indeed, suppose } \varphi_0 : N \simeq N^\ast, N \in \text{ind}(\Lambda). \text{ Consider the dual isomorphism } \varphi^* : N = N^{**} \simeq N^*, \varphi = \varphi^{**} \mapsto \varphi^{**}. \text{ Since the algebra } \text{End}(\Lambda) \text{ of endomorphisms is local, the invertibility of } 2\varphi = (\varphi + \varphi^*) + (\varphi - \varphi^*) \text{ implies the invertibility of } \varphi + \varphi^* \text{ or } \varphi - \varphi^*. \text{ Consequently, there exists an isomorphism } \psi = \pm \varphi^* : N \simeq N^*, \varepsilon \in \{1, -1\}, \text{ hence the module } (N, \mathcal{F}), \mathcal{F}(v, w) = \psi(w)(v) \text{ is } \varepsilon \text{-Hermitian.}

3) \text{ If } K \text{ is a field with a nonidentity involution, then } \text{ind}_0(\Lambda) \text{ consists of all } N \in \text{ind}(\Lambda), N = N^\ast. \text{ It suffices to use the preceding remark and the fact that over the field } K \text{ each } \varepsilon \text{-Hermitian form can be made Hermitian by multiplying it by } 1 + \epsilon \text{ if } \varepsilon = -1, \text{ or by } k - k = 0 (k \in K) \text{ if } \varepsilon = 1.

\text{Proof of the Theorem. It is only in proving the theorem that we will assume as known the definitions and notation of [4].}

We represent } \Lambda \text{ as a quotient algebra of a free algebra with generators } x_1, x_2, \ldots:

\Lambda = K \langle x_1, x_2, \ldots \rangle / K \langle f_\lambda, x_2, \ldots \rangle,

where the } f_\lambda(x_1, x_2, \ldots) \text{ are certain noncommutative polynomials. Then the } \lambda_j = x_j + K \langle f_\lambda, x_2, \ldots \rangle \text{ are generators of } \Lambda. \text{ The involution in } \Lambda \text{ is defined by certain relations}

\lambda_j = g_j (\lambda_2, \ldots) . \quad (4)

Suppose } (M, \mathcal{F}) \text{ is an } \varepsilon \text{-Hermitian module over } \Lambda. \text{ Fix a basis of the vector space } K M. \text{ Let } A_j \text{ be the matrix of the linear operator } \varphi \mapsto \lambda \varphi (\varphi \in M), \text{ and } B = \varepsilon B^\ast \text{ the matrix of the } \varepsilon \text{-Hermitian form } \mathcal{F}. \text{ The set of matrices } A_j \text{ must satisfy the relations satisfied by the elements } \lambda_j \text{ of } \Lambda, \text{ hence}

f_\lambda (A_1, A_2, \ldots) = 0. \quad (5)

It follows from these relations \cite{1, 4} that

A_j^* B = B g_j (A_1, A_2, \ldots). \quad (6)

Conversely, any set consisting of a nondegenerate } \varepsilon \text{-Hermitian matrix } B = \varepsilon B^\ast \text{ and square matrices } A_j \text{ of the same size satisfying relations (5) and (6) defines some } \varepsilon \text{-Hermitian module } (M, \mathcal{F}).

Consequently, an } \varepsilon \text{-Hermitian module } (M, \mathcal{F}) \text{ defines a representation of a digraph with relations (cf. [4, digraph (9)])}

\begin{align*}
\lambda_1 & \quad \beta \\
\lambda_2 & \quad \gamma
\end{align*}

and each such representation defines an } \varepsilon \text{-Hermitian module.}

The quiver with involution of the digraph \mathcal{S} is

\begin{align*}
\lambda_1 & \quad \beta \\
\lambda_2 & \quad \gamma
\end{align*}

\text{We do not include in (7) the conjugate relations, but they follow from the relations (7) since the involution } \lambda \mapsto \lambda^\ast \text{ in } \Lambda \text{ is compatible with addition and multiplication.}
Consider the quiver

\[
\begin{array}{c}
\lambda_1 \\
\lambda_2 \\
\vdots \\
f_i (\lambda_1, \lambda_2, \ldots) = 0
\end{array}
\]

defining modules over \( \Lambda \). We extend each representation \( \Lambda \equiv \text{ind} (Q) \) to a representation of the quiver \( \mathcal{S} \) by putting \( A^\alpha = A^\beta = A^\gamma = 1, A^\lambda = g_j (A_1, A_2, \ldots) \), the resulting representations form a set \( \text{ind} (\mathcal{S}) \). We can therefore identify \( \text{ind} (Q) \) and \( \text{ind} (\mathcal{S}) \). Furthermore, the dual module \( M^* \) can be identified with the conjugate representation \( A^0 \), the module \( M^i (C) \) with the representation \( \Lambda^i \), and the set \( \text{ind}^i (\Lambda) \) with the set \( \text{ind}^i (\mathcal{S}) \), \( i = 0, 1 \). To prove the theorem we need only use \([4, \text{Theorem } 1]\).

**COROLLARY 1.** Suppose \( K \) is one of the following fields of characteristic \( \neq 2 \):

a) an algebraically closed field with the identity involution;

b) an algebraically closed field with a nonidentity involution;

c) a maximal ordered field \( \{i \mid K_{\text{alg}} : K < \infty \} \), where \( K_{\text{alg}} \) is an algebraic closure of \( K \), e.g., \( K = \mathbb{R} \);

d) a finite field.

Then each \( \varepsilon \)-Hermitian module is isomorphic to a uniquely defined (to within a rearrangement of the summands) orthogonal sum of \( \varepsilon \)-Hermitian modules of the form \( (M \equiv \text{ind}^i (\Lambda), N \equiv \text{ind}^i (\Lambda)) \)

a) \( M^\varepsilon \), \( (N, F_N) \);

b) \( M^\varepsilon \), \( (N, F_N), (N, -F_N) \);

c) \( M^\varepsilon \), \( (N, tF_N) \), where \( t = 1 \) if \( T(N) \) is an algebraically closed field with the identity involution or the skew field of quaternions with involution different from \( a + bi + cj + dk \rightarrow a - bi - cj - dk \), and \( t \in \{-1, 1\} \) otherwise;

d) \( M^\varepsilon \), \( (N, tF_N) \), where \( t = 1 \) for a nonidentity involution on the field \( T(N) \), \( t \) is equal to 1 or a fixed nonsquare in \( T(N) \) for the identity involution, and for each \( N \) the orthogonal sum contains at most one summand \( (N, tF_N) \) with \( t \neq 1 \).

The proof follows from the theorem and \([2, \text{Theorem } 2]\).

**COROLLARY 2.** Suppose \((M, F), (M', F')\) are \(1\)-Hermitian modules in which \((K_M, F), (K_M', F')\) are Euclidean, or unitary, or complex Euclidean spaces \((K = \mathbb{R}, K = \mathbb{C} \text{ with a nonidentity involution, or } K = \mathbb{C} \text{ with the identity involution, respectively})\).

1) \((M, F) \equiv (M', F')\) if and only if \( M = M' \).

2) \((M, F)\) is uniquely (to within isomorphism of summands) decomposable into an orthogonal sum of orthogonally indecomposable \(1\)-Hermitian modules.

3) If \((M, F)\) is indecomposable into an orthogonal sum, then either \( M \) is indecomposable into a direct sum, or (only in the case of a complex Euclidean space) \( M \equiv N \oplus N^* \), where \( N \) is indecomposable into a direct sum.

The proof follows easily from the law of inertia for Hermitian forms and Corollary 1.

**LITERATURE CITED**


