Preliminary group classification of a class of fourth-order evolution equations*

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Abstract

We perform preliminary group classification of a class of fourth-order evolution equations in one spatial variable. Following the approach developed in [1] we construct all inequivalent partial differential equations belonging to the class in question which admit semi-simple Lie groups. In addition, we describe all fourth-order evolution equations from the class under consideration which are invariant under solvable Lie groups of dimension $n <= 4$. We have constructed all Galilei-invariant equations belonging to the class of evolution differential equations under study. The list of so obtained invariant equations contains both the well-known fourth-order evolution equations and a variety of new ones possessing rich symmetry and as such may be used to model nonlinear processes in physics, chemistry and biology.

Key words: Group classification, Symmetry group, Solvable Lie algebra, Semi-simple algebra, Fourth-order parabolic type equation.

1. Introduction

In the present paper we study the class of fourth-order parabolic type equations

$$u_t = -u_{xxxx} + F(t, x, u, u_x, u_{xx}, u_{xxx}) \tag{1}$$

admitting nontrivial Lie symmetries. Hereafter we adopt the following notations, $u = u(t, x),\ u_t = \partial u/\partial t,\ u_x = \partial u/\partial x,\ u_{xx} = \partial^2 u/\partial x^2$ and so on, and F is a smooth function of the indicated variables.

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The class of partial differential equations (1) is a generalization of a number of important mathematical physics equations. It contains, in particular, the Kuramoto-Sivashinsky equation (KS)

$$u_t = -u_{xxxx} - u_{xx} - \frac{1}{2}u_x^2.$$

KS equation has been derived by Kuramoto [2, 3] to model phase turbulence in the Belousov-Zhabotinsky reaction. Later on Sivashinsky generalized KS equation for the case of two and more spatial dimensions while studying propagation of a flame front for the case of mild combustion [4, 5]. The equation in question has also been utilized for describing long-wave motions of the liquid thin film over a vertical plane, bifurcation and chaos, and reaction-diffusion problems. The extended Fisher-Kolmogorov equation (eFK)

$$u_t = -u_{xxxx} + u_{xx} - u^3 + u$$

is applied to model phase transitions and other bistable phenomena [6], What is more, it has important applications in the theory of instability in nematic liquid crystals [7]. Another typical example is the Swift-Hohenberg equation (SH)

$$u_t = -u_{xxxx} - 2u_{xx} - u^3 + (\kappa - 1)u, \quad \kappa \in \mathbf{R},$$

which has been suggested by Swift and Hohenberg in studies of Rayleigh-Bénard convection [8], and has been used for describing various non-equilibrium systems in optics, for modeling chemical reactions with diffusion, and in biology. In recent years, Eq.(1) has attracted a lot of interest within a mathematical community and has been studied intensively (see, for example, [9]–[13]).

In the present paper we perform preliminary group classification of the class of partial differential equations (1). To this end we describe all possible forms of the function $F(t, x, \ldots, u_{xxx})$ such that Eq.(1) admits Lie symmetry groups of the dimension $n <= 4$.

Group classification has already become a universal and convenient tool for analysis of partial differential equations (PDEs). There exist a wealth of excellent references on different aspects of group classification ranging from the classical papers by Lie [14, 15] and Ovsyannikov [16] to the more recent papers by Akhatov, Gazizov and Ibragimov [19, 20], Torrisi *et al* [21, 22] and Ibragimov and Torrisi [23, 24]. A more detailed account of the results on group classification of differential equations can be found in [1, 25].

Ovsyannikov introduced the concept of equivalence group which is in the core of the majority of approaches to the problem of group classification of PDEs. His method works at its best when the equivalence group of an equation under study is finite-dimensional and is not very efficient otherwise. So that the equations with arbitrary elements depending on two and more arguments cannot be efficiently handled within the Ovsyannikov's approach. To overcome this difficulty Zhdanov and Lahno [26] developed a different purely algebraic approach enabling to classify classes of PDEs having infinite-dimensional equivalence groups.

The above mentioned paper was followed by numerous publications classifying the broad classes of heat conductivity [1], Schrödinger [27], KdV-type evolution [28], nonlinear wave [29] and general second-order quasi-linear evolution [25] equations. Here we adopt the approach of [26] in order to classify the class of PDEs (1).

Our classification algorithm is implemented as three major steps. The first step is calculating the maximal equivalence group admitted by Eq.(1). We remind that equivalence group of Eq.(1) is the Lie transformation group which preserves the class of PDEs (1). In addition, we compute the most general symmetry group of (1) together with the classifying equations for F.

The second step is essentially based on the fact that the explicit forms of commutation relations defining low dimensional abstract Lie algebras are well known (see., e.g., [30]–[32]). Using these we describe all inequivalent realizations of symmetry algebras by basis operators admitted by Eq.(1).

At the third step, inserting the canonical forms of symmetry generators into the classifying equations and solving them, we derive the explicit forms of invariant equations. Also we need to make sure that the corresponding symmetry algebras are maximal in Lie's sense.

Note that some elements of this approach have been utilized to perform group classification of the nonlinear d'Alembert (Fushchych and Serov [33]), nonlinear Schrödnger (Gazeau, Gagnon and Winternitz [34, 35]), generalized Burgers (Qu [36]) and multi-component wave (Zhdanov, Fushchych and Marko [37]) equations.

The paper is organized as follows. In Section 2, we derive the classifying equation for the function F and compute the maximal equivalence group admitted by Eq.(1). In Section 3 we provide a complete description of equations of the form (1) invariant under semi-simple algebras. Section 4 is devoted to the classification of equations invariant with respect to solvable symmetry 1-, 2- and 3-parameter groups. In Section 5, we obtain the equations invariant under four-parameter symmetry groups. The next section is devoted to Galilei-invariant equations of the form (1). The final section contains a summary and conclusions.

2. The classifying equation and equivalence transformation

It is a common knowledge that the most general Lie transformation group admitted by Eq.(1) is generated by vector fields of first-order differential operators

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

where τ, ξ and η are arbitrary smooth functions. Following the infinitesimal Lie approach we construct the fourth-order prolongation $\mathrm{pr}^{(4)}X$ of the infinitesimal operator X, keeping only the necessary terms

$$\mathrm{pr}^{(4)}X = X + \eta^t\partial_{u_t} + \eta^x\partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{xxx}\partial_{u_{xxx}} + \eta^{xxxx}\partial_{u_{xxxx}} + \dots,$$

where

$$
\begin{aligned}
\eta^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi),\\
\eta^{xxxx} &= D_x(\eta^{xxx}) - u_{xxxt} D_x(\tau) - u_{xxxx} D_x(\xi)
\end{aligned}
$$

and the symbols D_t and D_x stand for the total differentiation operators with respect to t and x, correspondingly,

$$
\begin{aligned}
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \dots,\\
D_x &= \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \dots.
\end{aligned}
$$

The infinitesimal invariance criterion of Eq.(1) with respect to the Lie symmetry X requires that the prolonged field $\mathrm{pr}^{(4)}X$ annihilates (1) on its solution manifold, namely,

$$
\mathrm{pr}^{(4)}X(\Delta)|_{\Delta} = 0, \quad \Delta = u_t + u_{xxxx} - F.
$$

Splitting the above relation by independent variables yields an over-determined system of linear PDEs, whose general solution is presented below.

Proposition 1. *The most general invariance group of* (1) *is generated by the infinitesimal operators*

$$
X = \tau(t)\partial_t + \Big(\frac{\dot\tau}{4}x + \rho(t)\Big)\partial_x + \eta(t,x,u)\partial_u, \tag{2}
$$

where τ, ρ and η are real-valued functions satisfying the classifying equation

$$
\begin{aligned}
&\Big(- \eta_u u_{xxx} + \frac{3}{4}\dot\tau u_{xxx} - 3\eta_{xu}u_{xx} - 3\eta_{uu}u_x u_{xx} - \eta_{uuu}u_x^3 - 3\eta_{xuu}u_x^2 - 3\eta_{xxu}u_x\\
&\quad - \eta_{xxx}\Big)F_{u_{xxx}} + \Big(- \eta_u u_{xx} + \frac{1}{2}\dot\tau u_{xx} - \eta_{uu}u_x^2 - 2\eta_{xu}u_x - \eta_{xx}\Big)F_{u_{xx}}\\
&\quad + \Big(- \eta_u u_x + \frac{1}{4}\dot\tau u_x - \eta_x\Big)F_{u_x} - \Big(\rho + \frac{1}{4}\dot\tau x\Big)F_x - \tau F_t - \eta F_u\\
&\quad + (\eta_u - \dot\tau)F + 4\eta_{xu}u_{xxx} + 4\eta_{uu}u_x u_{xxx} + 3\eta_{uu}u_{xx}^2 + 6\eta_{xxu}u_{xx}\\
&\quad + 6\eta_{uuu}u_x^2 u_{xx} + 12\eta_{xuu}u_x u_{xx} + \eta_{uuuu}u_x^4 + 4\eta_{xuuu}u_x^3 + 6\eta_{xxuu}u_x^2\\
&\quad - \dot\rho u_x - \frac{1}{4}\ddot\tau x u_x + 4\eta_{xxxu}u_x + \eta_t + \eta_{xxxx} = 0,
\end{aligned} \tag{3}
$$

where and hereafter the dot over a symbol denotes differentiation with respect to its argument.

Now the group classification problem of Eq.(1) reduces to constructing all possible solutions of the determining equation (3). The problem however is that (3) is the under-determined system of one PDE for three unknown functions τ, ρ and η. In addition, unknown function F is to be determined, as well. To overcome this difficulty we adopt the approach developed in [26] and utilize the classical results on classification of abstract low order Lie algebras for constructing Lie symmetries of the form (2). With explicit forms of symmetries in hand we can proceed to integrating the classifying equation and get the explicit forms of F.

We derive the maximal equivalence group of Eq.(1) directly. Let

$$\tilde{t} = T(t,x,u), \quad \tilde{x} = Y(t,x,u), \quad \tilde{u} = U(t,x,u), \qquad \frac{D(T,Y,U)}{D(t,x,u)} \neq 0 \tag{4}$$

be an invertible point transformation preserving the form of (1), so that the transformed equations has the same form (1)

$$\tilde{u}_{\tilde{t}} = -\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{F}(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}). \tag{5}$$

In the sequel, we use the notation $\epsilon = \pm 1$.

Proposition 2. *The maximal equivalence group of Eq.*(1) *reads as*

$$\tilde{t} = T(t), \quad \tilde{x} = \epsilon(\dot{T})^{\frac{1}{4}}x + Y(t), \quad \tilde{u} = U(t,x,u), \tag{6}$$

where $\dot{T} > 0$ *and* $U_u \neq 0$.

Proof. Computing u_x according to (4), we obtain

$$u_x = \frac{T_x \tilde{u}_{\tilde{t}} + Y_x \tilde{u}_{\tilde{x}} - U_x}{U_u - T_u \tilde{u}_{\tilde{t}} - Y_u \tilde{u}_{\tilde{x}}}.$$

Since $\tilde{F}$ is an arbitrary function of its arguments and is independent of $\tilde{u}_{\tilde{t}}$, we necessarily have

$$u_x = f(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}})$$

for some f. This implies that $T_x = T_u = 0$. Consequently, $T = T(t)$ and $\dot{T} \neq 0$.

Next, making the change of variables (4) with account of the fact that $\tilde{t} = T(t)$ yields

$$\begin{aligned} u_t =& \dot{T}(U_u - Y_u \tilde{u}_{\tilde{x}})^{-1} \tilde{u}_{\tilde{t}} + \theta_1(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}), \\ u_{xxxx} =& -(Y_u \tilde{u}_{\tilde{x}} - U_u)^{-5}(Y_x U_u - Y_u U_x)^4 \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \theta_2(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}), \end{aligned}$$

where θ_1 and θ_2 are smooth functions of given arguments. Substitutions the obtained expressions into (5), we get

$$\dot{T}(Y_u \tilde{u}_{\tilde{x}} - U_u)^4 - (Y_x U_u - Y_u U_x)^4 = 0.$$

As T, Y and U do not depend on u_x, coefficients of $\tilde{u}_{\tilde{x}}$ in the above polynomial must vanish, separately, which yields the system of determining equations for functions T, Y and U:

$$\dot{T}Y_u = 0, \quad \dot{T}U_u^4 - (Y_xU_u - Y_uU_x)^4 = 0.$$

Since $\dot{T} \neq 0$, it follows from the first equation that $Y_u = 0$ and the original system reduces to a single equation

$$(\dot{T} - Y_x^4)U_u^4 = 0.$$

In view of the fact that $U_u \neq 0$ (otherwise (4) is not invertible) we get $\dot{T} - Y_x^4 = 0$. Therefore, $\dot{T} > 0$ and $Y = \epsilon\dot{T}^{\frac{1}{4}}x + Y(t)$. The theorem is proved. □

Consider next the one-dimensional Lie algebras generated by (2). The transformation (6) reduce vector field (2) to the form

$$\begin{aligned} \tilde{X} &= \tau\dot{T}\partial_{\tilde{t}} + \left[\frac{1}{4}(\tau\ddot{T}\dot{T}^{-1} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \epsilon\rho\dot{T}^{\frac{1}{4}}\right]\partial_{\tilde{x}} \\ &\quad + \left[\tau U_t + \left(\frac{\epsilon}{4}\dot{\tau}\dot{T}^{-\frac{1}{4}}(\tilde{x} - Y) + \rho\right)U_x + \eta U_u\right]\partial_{\tilde{u}}. \end{aligned}$$

We consider the cases $\eta = 0$ and $\eta \neq 0$ separately.

Case 1. $\eta = 0$. Choosing $U = U(u)$, we have

$$\tilde{X} = \tau\dot{T}\partial_{\tilde{t}} + \left[\frac{1}{4}(\tau\ddot{T}\dot{T}^{-1} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \epsilon\rho\dot{T}^{\frac{1}{4}}\right]\partial_{\tilde{x}}.$$

If $\tau = 0$, then inequality $\rho \neq 0$ holds, since otherwise operator (2) vanishes identically. Choosing as T a solution of equation $|\rho|\dot{T}^{\frac{1}{4}} = 1$ yields the operator $\tilde{X} = \epsilon\partial_{\tilde{x}}$. Making use of the space reflection $x \to -x$ we get $\tilde{X} = \partial_{\tilde{x}}$.

In the case when $\tau \neq 0$, we can put $\epsilon = 1$ in (6). Taking solutions of

$$|\tau|\dot{T} = 1, \quad \tau\dot{Y} + \rho\dot{T}^{\frac{1}{4}} = 0, \quad \dot{T} > 0$$

as T and Y, yields the operator $\tilde{X} = \epsilon\partial_{\tilde{t}}$.

Case 2. $\eta \neq 0$. Provided $\tau = \rho = 0$, we choose U as a solution of $\eta U_u = 1$ and thus arrive at the operator $\tilde{X} = \partial_{\tilde{u}}$. Otherwise, we choose U to satisfy

$$\tau U_t + \left(\frac{\dot{\tau}}{4}x + \rho\right)U_x + \eta U_u = 0, \quad U_u \neq 0$$

and we get the operator from Case 1.

It is straightforward to verify that $\epsilon\partial_t$, ∂_x, ∂_u cannot be transformed one into another with the transformation (6).

We summarize the above results in the following lemma.

Lemma 1. *Within the point transformation* (6), *the vector field* (2) *is equivalent to one of the following canonical operators*

$$\epsilon\partial_t, \quad \partial_x, \quad \partial_u. \tag{7}$$

Thus there are three inequivalent realizations of one-dimensional Lie algebras by operators (2). Integrating the classifying equation for each symmetry operator yield the corresponding inequivalent equations from the class (1) which admit one-dimensional Lie algebras. In a sequel we adopt the notation $A_{k.i} = \langle X_1, X_2, \cdots, X_k\rangle$ to denote an ith realization of a k-dimensional Lie algebra with X_i $(i = 1, 2, \cdots, k)$ being its basis elements.

Theorem 1. *There are three inequivalent equations of the form* (1) *invariant under one-dimensional Lie algebras:*

$$\begin{aligned}
A_1^1 = \langle\partial_t\rangle : &\quad u_t = -u_{xxxx} + F(x, u, u_x, u_{xx}, u_{xxx}),\\
A_1^2 = \langle\partial_x\rangle : &\quad u_t = -u_{xxxx} + F(t, u, u_x, u_{xx}, u_{xxx}),\\
A_1^3 = \langle\partial_u\rangle : &\quad u_t = -u_{xxxx} + F(t, x, u_x, u_{xx}, u_{xxx})
\end{aligned}$$

where F is an arbitrary function. Furthermore, the corresponding symmetry algebra is maximal in Lie's sense.

3. Classification of equations invariant under semi-simple Lie algebras

The lowest dimension semi-simple Lie algebras are isomorphic to one of the following two algebras

$$\begin{aligned}
\mathfrak{so}(3) : &\quad [X_1, X_2] = X_3,\ [X_1, X_3] = -X_2,\ [X_2, X_3] = X_1;\\
\mathfrak{sl}(2, \mathbf{R}) : &\quad [X_1, X_2] = 2X_2,\ [X_1, X_3] = -2X_3,\ [X_2, X_3] = X_1.
\end{aligned}$$

The following assertion holds.

Theorem 2. *There exists no realization of the algebra $\mathfrak{so}(3)$ that can be a symmetry algebra of* (1). *Furthermore there are at most two inequivalent realizations of $\mathfrak{sl}(2, \mathbf{R})$ by operators* (2), *which are admitted by Eq.*(1). *The realizations and the corresponding invariant equations read as*

$$\mathfrak{sl}^1(2, \mathbf{R}) = \langle -2t\partial_t - \frac{1}{2}x\partial_x, \partial_t, -t^2\partial_t - \frac{1}{2}tx\partial_x - x^4\partial_u\rangle :$$

$$u_t = -u_{xxxx} + \frac{1}{x^4}[u^2 + (24 - \frac{1}{2}xu_x)u] + \frac{1}{x^4}G(\omega_1, \omega_2, \omega_3),$$

$$\omega_1 = xu_x - 4u,\ \omega_2 = x^2u_{xx} - 12u, \omega_3 = x^3u_{xxx} - 24u,$$

$$\mathfrak{sl}^2(2, \mathbf{R}) = \langle -2u\partial_u, \partial_u, -u^2\partial_u\rangle :$$

$$u_t = -u_{xxxx} + 4\frac{u_{xx}u_{xxx}}{u_x} - 3\frac{u_{xx}^3}{u_x^2} + u_x G(t, x, 2\frac{u_{xxx}}{u_x} - 3\frac{u_{xx}^2}{u_x^2}).$$

The algebras $\mathfrak{sl}^1(2,\mathbf{R})$ and $\mathfrak{sl}^2(2,\mathbf{R})$ are the maximal invariance algebras of the corresponding PDEs, provided that the functions G are arbitrary.

Proof. Consider first inequivalent realizations of the algebra $\mathfrak{so}(3)$. Taking operators (2) as its basis X_i (i=1,2,3), inserting the latter into the commutation relations of $\mathfrak{so}(3)$ and solving equations obtained we obtain all possible realizations of the algebra under study.

In view of Lemma 1, we can assume without any loss of generality that one of the basis operators, say X_1, is of the one of three canonical forms $\epsilon\partial_t$, ∂_x and ∂_u.

Let $X_1 = \partial_t$ and X_2, X_3 be of the general form (2). Inserting them into the first two commutation relations of $\mathfrak{so}(3)$ yields

$$X_2 = 4a\cos t\partial_t + [-ax\sin t + b\cos(t+c)]\partial_x + \phi(x,u)\cos(t+\psi(x,u))\partial_u,$$

$$X_3 = -4a\sin t\partial_t - [ax\cos t + b\sin(t+c)]\partial_x - \phi(x,u)\sin(t+\psi(x,u))\partial_u.$$

Here $a,\ b,\ c$ are arbitrary real constants and $\phi,\ \psi$ are arbitrary real-valued smooth functions. Substituting $X_2,\ X_3$ into the last commutation relation implies that $-16a^2 = 1$ which has no real solution a. Thus, there are no realizations of $\mathfrak{so}(3)$ in the case when $X_1 = \partial_t$.

The same assertion holds for $X_1 = -\partial_t,\ \partial_x$ or ∂_u. Consequently, there are no realizations of algebra $\mathfrak{so}(3)$ within the class of operators (2) and, therefore, Eq.(1) does not admit $\mathfrak{so}(3)$.

The case of the algebra $\mathfrak{sl}(2,\mathbf{R})$ is treated in a similar way which yields two inequivalent realizations given in the formulation of the theorem.

Solving the determining equations for each of the above obtained realizations we arrive at the invariant equations presented in the formulation of the theorem. The assertion is proved. □

Corollary 1. *The invariant equations listed in Theorem 2 exhaust the list of all possible inequivalent PDEs (1), whose invariance algebras are semi-simple.*

Proof. It is a common knowledge that there exist four types of classical semi-simple Lie algebras, $A_{n-1},\ B_n,\ C_n,\ D_n$, and five exceptional semi-simple Lie algebras, $G_2,\ F_4,\ E_6,\ E_7,\ E_8$ [31].

• A_{n-1} $(n > 1)$ has four real forms of the algebra $\mathfrak{sl}(n,\mathbf{C})$: $\mathfrak{su}(n)$, $\mathfrak{sl}(n,\mathbf{R})$, $\mathfrak{su}(p,q)$ $(p+q=n, p\geq q)$, $\mathfrak{su}^*(2n)$.

• B_n $(n\geq 1)$ contains two real forms of the algebra $\mathfrak{so}(2n+1,\mathbf{C})$: $\mathfrak{so}(2n+1)$, $\mathfrak{so}(p,q)$ $(p+q=2n+1, p>q)$.

• C_n $(n\geq 1)$ contains three real forms of the algebra $\mathfrak{sp}(n,\mathbf{C})$: $\mathfrak{sp}(n)$, $\mathfrak{sp}(n,\mathbf{R})$, $\mathfrak{sp}(p,q)$ $(p+q=n, p\geq q)$.

• D_n $(n>1)$ has three real forms of the algebra $\mathfrak{so}(2n,\mathbf{C})$: $\mathfrak{so}(2n)$, $\mathfrak{so}(p,q)$ $(p+q=2n, p\geq q)$, $\mathfrak{so}^*(2n)$.

The lowest dimensional semi-simple Lie algebras admit the following isomorphisms

$$\mathfrak{so}(3) \sim \mathfrak{su}(2) \sim \mathfrak{sp}(1), \qquad \mathfrak{sl}(2, \mathbf{R}) \sim \mathfrak{su}(1,1) \sim \mathfrak{so}(2,1) \sim \mathfrak{sp}(1, \mathbf{R}).$$

Hence, realizations of the algebra $\mathfrak{sl}(2, \mathbf{R})$ exhaust the set of all possible realizations of three-dimensional semi-simple Lie algebras admitted by (1).

The next admissible dimension for classical semi-simple Lie algebras is six. There are four non-isomorphic semi-simple Lie algebras, namely, $\mathfrak{so}(4)$, $\mathfrak{so}^*(4)$, $\mathfrak{so}(3,1)$, and $\mathfrak{so}(2,2)$. As $\mathfrak{so}(4) \sim \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, $\mathfrak{so}^*(4) \sim \mathfrak{so}(3) \oplus \mathfrak{sl}(2, \mathbf{R})$, and $\mathfrak{so}(3,1)$ contains $\mathfrak{so}(3)$ as a subalgebra, these algebras can not be invariance algebras of Eq.(1). Thus $\mathfrak{so}(2,2)$ is the only possible six-dimensional semi-simple algebra that can be admitted by (1).

As $\mathfrak{so}(2,2) \sim \mathfrak{sl}(2, \mathbf{R}) \oplus \mathfrak{sl}(2, \mathbf{R})$, we can choose $\mathfrak{so}(2,2) = \langle Q_i, K_i | i = 1,2,3 \rangle$, where $\langle Q_1, Q_2, Q_3 \rangle = \mathfrak{sl}(2, \mathbf{R})$, $\langle K_1, K_2, K_3 \rangle = \mathfrak{sl}(2, \mathbf{R})$ and $[Q_i, K_j] = 0$, $(i, j = 1,2,3)$. Taking $Q_1,\ Q_2,\ Q_3$ to be the basis operators of the realizations of $\mathfrak{sl}(2, \mathbf{R})$ given in the formulation of the theorem, and $K_1,\ K_2,\ K_3$ be of the general form (2), after some algebra we prove that realizations of $\mathfrak{sl}(2, \mathbf{R})$ cannot be extended to a realization of $\mathfrak{so}(2,2)$. Consequently, Eq.(1) does not admit six-dimensional semi-simple Lie algebras.

The same assertion holds true for eight-dimensional semi-simple Lie algebras $\mathfrak{sl}(3, \mathbf{R})$, $\mathfrak{su}(3)$ and $\mathfrak{su}(2,1)$.

As $\mathfrak{su}^*(4) \sim \mathfrak{so}(5,1) \supset \mathfrak{so}(4)$, the algebra A_{n-1} $(n > 1)$ has no realizations by operators (2) except for those given in the formulation of the theorem. There are also no realizations of the algebra D_n $(n > 1)$, since the lowest dimensional algebras of this type $\mathfrak{so}(4)$, $\mathfrak{so}(2,2)$, $\mathfrak{so}^*(4)$ have no realizations within the class of differential operators (2).

The similar reasoning yields that there are no new realizations of the algebras B_n $(n \geq 1)$ and C_n $(n \geq 1)$, that can be symmetry algebras of Eq.(1). This follows from the fact that the algebra B_2 contains subalgebras isomorphic to $\mathfrak{so}(4)$ and $\mathfrak{so}(1,3)$, and from the following properties of the algebra C_2

$$\mathfrak{sp}(2) \sim \mathfrak{so}(5) \supset \mathfrak{so}(4),\ \mathfrak{sp}(2, \mathbf{R}) \sim \mathfrak{so}(3,2) \supset \mathfrak{so}(3,1),\ \mathfrak{sp}(1,1) \sim \mathfrak{so}(4,1) \supset \mathfrak{so}(4).$$

To complete the proof, we need to consider the exceptional semi-simple Lie algebras G_2, F_4, E_6, E_7, E_8. Here we only consider the first two algebras and others are handled in the same way. The algebra G_2 has a compact real form g_2 and one noncompact real form $\tilde{g}_2$, where $g_2 \cap \tilde{g}_2 \sim \mathfrak{su}(2) \oplus \mathfrak{su}(2) \sim \mathfrak{so}(4)$. Since Eq.(1) cannot admit $\mathfrak{so}(4)$ as an invariance algebra, G_2 has no realization by operators (2). F_4 has a compact real form f_4 and two noncompact real forms $\tilde{f}_4, \hat{f}_4$ with $f_4 \cap \tilde{f}_4 \sim \mathfrak{sp}(3) \oplus \mathfrak{su}(2) \supset \mathfrak{so}(3)$, $f_4 \cap \hat{f}_4 \sim \mathfrak{sp}(9)$. Thus Eq.(2) cannot admit a realization of an algebra of the type F_4. The same assertion holds for the remaining exceptional semi-simple Lie algebras E_6, E_7, E_8.

The corollary is proved. □

4. Classification of equations invariant under low-dimensional solvable Lie algebras

Using the concept of compositional series for a solvable algebra we can construct all possible realizations of solvable Lie algebras admitted by Eq.(1) starting from the one-dimensional ones and proceeding to the solvable Lie algebras of the dimension $2, 3, 4, \ldots$ (for more details, see [1]).

In this section we describe inequivalent Eqs.(1) which are invariant under solvable Lie algebras of the dimension up to four. Since equations invariant with respect to one-dimensional algebras have already been constructed, we start by analyzing two-dimensional solvable algebras.

4.1. Equations with two-dimensional Lie algebras

There are two non-isomorphic two-dimensional Lie algebras,

$$A_{2.1}: \quad [X_1, X_2] = 0, \qquad A_{2.2}: \quad [X_1, X_2] = X_2.$$

As both $A_{2.1}$ and $A_{2.2}$ contain the algebra A_1, we can assume that the basis operator of the latter is reduced to a canonical form.

We consider in detail the case of $A_{2.1}$, while for the case of algebra $A_{2.2}$ we present the final results only.

Let $X_1 = \partial_t$ and X_2 be the operator of the most generic form (2)

$$X_2 = \tau(t)\partial_t + [\dot{\tau}/4x + \rho(t)]\partial_x + \eta(t, x, u)\partial_u.$$

Then the commutation relation implies that $\dot{\tau} = \dot{\rho} = \eta_t = 0$. Therefore τ, ρ are constants and $\eta = \eta(x, u)$. So without any loss of generality we can put $X_2 = c\partial_x + \eta(x, u)\partial_u$, where and hereafter $c = const$. Before simplifying X_2 with equivalence transformations (6), we seek those equivalent transformations which preserve the basis operator of the algebra $A_{1.1}$

$$X_1 \to \tilde{X}_1 = \dot{T}\partial_{\tilde{t}} + \left[\frac{1}{4}\dot{T}^{-1}\ddot{T}(\tilde{x} - Y) + \dot{Y}\right]\partial_{\tilde{x}} + U_t\partial_{\tilde{u}} = \partial_{\tilde{t}}.$$

Hence, $\dot{T} = 1$ and $\dot{Y} = U_t = 0$. Consequently, we can take $T = t$, $U = U(x, u)$ and $Y = const$. Performing this transformation yields

$$X_2 \to \tilde{X}_2 = \epsilon c\partial_{\tilde{x}} + (cU_x + \eta U_u)\partial_{\tilde{u}}.$$

If $\eta = 0$, we choose $U = U(u)$ thus getting $\tilde{X}_2 = \partial_{\tilde{x}}$. Provided $\eta \neq 0$ and $c = 0$, we can take U as a solution of $\eta U_u = 1$ and get $\partial_{\tilde{u}}$. When $\eta \neq 0$ and $c \neq 0$, we select U to satisfy $cU_x + \eta U_u = 0, U_u \neq 0$, whence we conclude that the operator $\partial_{\tilde{x}}$ is admitted by Eq.(1). Thus within the action of equivalence group of Eq.(1), we get the following two-dimensional invariance algebras, $\langle\partial_t, \partial_x\rangle$, $\langle\partial_t, \partial_u\rangle$.

Consider now the case when $X_1 = \partial_x$ and X_2 is an operator of the form (2). Inserting X_1, X_2 into the commutation relation yields that $X_2 = c\partial_t + \rho(t)\partial_x + \eta(t,u)\partial_u$. The equivalence transformation, which leaves X_1 invariant, reads as

$$\tilde{t} = t,\ \tilde{x} = x + Y(t),\ \tilde{u} = U(t,u)$$

with $U_u \neq 0$. This transformation reduces X_2 to the form

$$\tilde{X}_2 = c\partial_{\tilde{t}} + [c\dot{Y} + \rho(t)]\partial_{\tilde{x}} + (cU_t + \eta U_u)\partial_{\tilde{u}}.$$

We consider the cases $\eta = 0$ and $\eta \neq 0$ separately. If $\eta = 0$ and $c = 0$, then $\tilde{X}_2 = \rho(t)\partial_{\tilde{x}}$. It is straightforward to verify that the so obtained two-dimensional Lie algebra, $\langle \partial_x, g(t)\partial_x \rangle$, cannot be invariance algebras of an equation of the form (1). Provided $\eta = 0$ and $c \neq 0$, we can choose solutions of PDEs $U_t = 0$ and $c\dot{Y} + \rho = 0$ as U and Y and thus get $\tilde{X}_2 = \epsilon\partial_{\tilde{t}}$.

Similarly, the case $\eta \neq 0$ and $c = 0$ leads to the operator $\tilde{X}_2 = \rho(t)\partial_{\tilde{x}} + \partial_{\tilde{u}}$. Next, if $\eta \neq 0$ and $c \neq 0$, then the operator $\tilde{X}_2 = \epsilon\partial_{\tilde{t}}$ is obtained.

Turn now to the remaining case $X_1 = \partial_u$. With the choice of X_2 of the form (2), the commutation relation implies $X_2 = \tau(t)\partial_t + [\dot{\tau}/4x + \rho(t)]\partial_x + \eta(t,x)\partial_u$. Note that the transformation

$$\tilde{t} = T(t), \quad \tilde{x} = \epsilon\dot{T}^{\frac{1}{4}}x + Y(t), \quad \tilde{u} = u + U(t,x)$$

belongs to the equivalence group of (1), which preserves X_1 and converts X_2 into

$$\tau\dot{T}\partial_{\tilde{t}} + \left[\frac{1}{4}(\tau\dot{T}^{-1}\ddot{T} + \dot{\tau})(\tilde{x} - Y) + \tau\dot{Y} + \epsilon\rho\dot{T}^{\frac{1}{4}}\right]\partial_{\tilde{x}}$$
$$[\tau U_t + \left(\frac{1}{4}\dot{\tau}x + \rho\right)U_x + \eta]\partial_{\tilde{u}}.$$

Choosing T, Y and U suitably, we can reduce the operator X_2 to one of the forms, $\partial_{\tilde{x}}$, $\epsilon\partial_{\tilde{t}}$ and $\eta(t,x)\partial_{\tilde{u}}$. This completes the analysis of the realizations of the algebra $A_{2.1}$. The case of $A_{2.2}$ is treated similarly.

Substituting the so obtained basis operators into the classifying equation and solving the latter yields the corresponding invariant equations.

Theorem 3. *There exist four Abelian and five non-Abelian two-dimensional symmetry algebras admitted by* (1). *These algebras and the corresponding invariant equations are given below*

$A^1_{2.1} = \langle \partial_t, \partial_x \rangle$: $\quad u_t = -u_{xxxx} + G(u, u_x, u_{xx}, u_{xxx})$,

$A^2_{2.1} = \langle \partial_t, \partial_u \rangle$: $\quad u_t = -u_{xxxx} + G(x, u_x, u_{xx}, u_{xxx})$,

$A^3_{2.1} = \langle \partial_x, g(t)\partial_x + \partial_u \rangle$: $\quad u_t = -u_{xxxx} + g'uu_x + G(t, u_x, u_{xx}, u_{xxx})$,

$A^4_{2.1} = \langle \partial_u, g(t,x)\partial_u \rangle,\ g_x \neq 0$: $\quad u_t = -u_{xxxx} + (g_{xxxx} - g_t)g_x^{-1}u_x + G(t, x, \omega_1, \omega_2)$,

$$\omega_1 = (g_x u_{xx} - g_{xx} u_x) g_x^{-1},$$

$$\omega_2 = (g_x u_{xxx} - g_{xxx} u_x) g_x^{-1},$$

$$A^1_{2.2} = \langle -t\partial_t - \frac{x}{4}\partial_x, \partial_t \rangle : \qquad u_t = -u_{xxxx} + x^{-4} G(u, xu_x, x^2 u_{xx}, x^3 u_{xxx}),$$

$$A^2_{2.2} = \langle -4t\partial_t - x\partial_x, \partial_x \rangle : \qquad u_t = -u_{xxxx} + t^{-1} G(u, t^{\frac{1}{4}} u_x, t^{\frac{1}{2}} u_{xx}, t^{\frac{3}{4}} u_{xxx}),$$

$$A^3_{2.2} = \langle -u\partial_u, \partial_u \rangle : \qquad u_t = -u_{xxxx} + u_x G(t, x, u_x^{-1} u_{xx}, u_x^{-1} u_{xxx}),$$

$$A^4_{2.2} = \langle \partial_x - u\partial_u, \partial_u \rangle : \qquad u_t = -u_{xxxx} + \mathrm{e}^{-x} G(t, \mathrm{e}^x u_x, \mathrm{e}^x u_{xx}, \mathrm{e}^x u_{xxx}),$$

$$A^5_{2.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u \rangle : \qquad u_t = -u_{xxxx} + u_x G(x, \mathrm{e}^{\epsilon t} u_x, \mathrm{e}^{\epsilon t} u_{xx}, \mathrm{e}^{\epsilon t} u_{xxx}).$$

4.2. Equations admitting three-dimensional solvable Lie algebras

We consider the cases of decomposable and non-decomposable Lie algebras separately.

4.2.1 Three-dimensional decomposable algebras

There exist two non-isomorphic three-dimensional decomposable Lie algebras, $A_{3.1}$ and $A_{3.2}$,

$$A_{3.1}: \quad [X_i, X_j] = 0 \ (i, j = 1, 2, 3) \quad (A_{3.1} = A_1 \oplus A_1 \oplus A_1),$$
$$A_{3.2}: \quad [X_1, X_2] = X_2, \ [X_1, X_3] = [X_2, X_3] = 0, \quad (A_{3.2} = A_{2.2} \oplus A_1).$$

It is a common knowledge that any three-dimensional solvable Lie algebra contains two-dimensional solvable algebra. So that to describe all possible realizations of three-dimensional solvable algebras admitted by Eq.(1) it suffices to consider all possible extensions of two dimensional algebras listed in Theorem 3 by vector fields, X_3, of the form (2). Then for each of the so obtained realization we simplify X_3 using equivalence transformations which preserve the operators X_1 and X_2. Having performed these two steps we obtain the following list of invariant equations (1):

$A_{3.1}-$ invariant equations

$$A^1_{3.1} = \langle \partial_t, \partial_x, \partial_u \rangle :$$

$$F = G(u_x, u_{xx}, u_{xxx}),$$

$$A^2_{3.1} = \langle \partial_t, \partial_u, g(x)\partial_u \rangle, \ g' \neq 0 :$$

$$F = \frac{g^{(4)}}{g'} u_x + G(x, \frac{g''}{g'} u_x - u_{xx}, \frac{g'''}{g'} u_x - u_{xxx}).$$

$A_{3.2}-$ invariant equations

$$A^1_{3.2} = \langle -t\partial_t - \frac{x}{4}\partial_x, \partial_t, \partial_u \rangle :$$

$$F = \frac{1}{x^4} G(xu_x, x^2 u_{xx}, x^3 u_{xxx}),$$

$$A^2_{3.2} = \langle -4t\partial_t - x\partial_x, \partial_x, \partial_u \rangle :$$

$$F = \frac{1}{t} G(|t|^{\frac{1}{4}} u_x, |t|^{\frac{1}{2}} u_{xx}, |t|^{\frac{3}{4}} u_{xxx}),$$

$$A^3_{3.2} = \langle -4t\partial_t - x\partial_x, \partial_x, |t|^{\frac{1}{4}}\partial_x + \partial_u \rangle :$$

$$F = -\frac{uu_x}{4|t|^{\frac{3}{4}}} + \frac{1}{t} G(|t|^{\frac{1}{4}} u_x, |t|^{\frac{1}{2}} u_{xx}, |t|^{\frac{3}{4}} u_{xxx}),$$

$$A^4_{3.2} = \langle -u\partial_u, \partial_u, \partial_x \rangle :$$

$$F = u_x G(t, \frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}),$$

$$A^5_{3.2} = \langle -u\partial_u, \partial_u, \partial_t \rangle :$$

$$F = u_x G(x, \frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}),$$

$$A^6_{3.2} = \langle \partial_x - u\partial_u, \partial_u, \mathrm{e}^{-x} g(t)\partial_u \rangle, \ g \neq 0 :$$

$$F = -(1 + \frac{g'}{g}) u_x + \mathrm{e}^{-x} G(t, \mathrm{e}^x (u_{xx} + u_x), \mathrm{e}^x (u_{xxx} - u_x)),$$

$$A^7_{3.2} = \langle \partial_x - u\partial_u, \partial_u, g(t)\partial_x \rangle, \ g \neq 0 :$$

$$F = -\frac{g'}{g}(x + \ln|u_x|) u_x + u_x G(t, \frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}),$$

$$A^8_{3.2} = \langle \partial_x - u\partial_u, \partial_u, \partial_t \rangle :$$

$$F = \mathrm{e}^{-x} G(\mathrm{e}^x u_x, \mathrm{e}^x u_{xx}, \mathrm{e}^x u_{xxx}),$$

$$A^9_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \mathrm{e}^{-\epsilon t} g(x)\partial_u \rangle, \ g' \neq 0 :$$

$$F = \frac{g^{(4)} - \epsilon g}{g'} u_x + \mathrm{e}^{-\epsilon x} G(x, \mathrm{e}^{\epsilon t}(\frac{g''}{g'} u_x - u_{xx}), \mathrm{e}^{\epsilon t}(\frac{g'''}{g'} u_x - u_{xxx})),$$

$$A^{10}_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_x \rangle :$$

$$F = \mathrm{e}^{-\epsilon t} G(\mathrm{e}^{\epsilon t} u_{xx}, \mathrm{e}^{\epsilon t} u_x, \mathrm{e}^{\epsilon t} u_{xxx}),$$

$$A^{11}_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_t + \lambda\partial_x \rangle, \ \lambda \neq 0 :$$

$$F = \mathrm{e}^{\epsilon(\frac{x}{\lambda} - t)} G(\mathrm{e}^{\epsilon(t - \frac{x}{\lambda})} u_x, \mathrm{e}^{\epsilon(t - \frac{x}{\lambda})} u_{xx}, \mathrm{e}^{\epsilon(t - \frac{x}{\lambda})} u_{xxx}),$$

$$A^{12}_{3.2} = \langle \epsilon\partial_t - u\partial_u, \partial_u, \partial_t \rangle :$$

$$F = u_x G(x, \frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}).$$

4.2.2 Three-dimensional non-decomposable algebras

There are seven non-isomorphic non-decomposable three-dimensional real solvable Lie algebras. The list of these algebras is exhausted by one nilpotent Lie algebra,

$$A_{3.3}: \quad [X_2, X_3] = X_1, \quad [X_1, X_2] = [X_1, X_3] = 0,$$

and six solvable Lie algebras (only nonzero commutation relations are given),

$$\begin{aligned}
&A_{3.4}: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_1 + X_2,\\
&A_{3.5}: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2,\\
&A_{3.6}: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = -X_2,\\
&A_{3.7}: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = qX_2, \ (0 < |q| < 1),\\
&A_{3.8}: \quad [X_1, X_3] = -X_2, \quad [X_2, X_3] = X_1,\\
&A_{3.9}: \quad [X_1, X_3] = qX_1 - X_2, \quad [X_2, X_3] = X_1 + qX_2, \ (q > 0).
\end{aligned}$$

All these algebras contain a two-dimensional Abelian ideal as a subalgebra. Thus we can use our classification of $A_{2.1}-$ invariant equations to construct Eqs.(1), which admit non-decomposable three-dimensional solvable Lie algebras. We skip intermediate calculations and present the final result, the list of invariant equations and the corresponding symmetry algebras.

$A_{3.3}-$ invariant equations

$$A^1_{3.3} = \langle \partial_x, \partial_t, t\partial_x + \partial_u \rangle :$$

$$F = -uu_x + G(u_x, u_{xx}, u_{xxx}),$$

$$A^2_{3.3} = \langle \partial_u, \partial_t, \lambda\partial_x + t\partial_u \rangle, \ \lambda > 0 :$$

$$F = \frac{x}{\lambda} + G(u_x, u_{xx}, u_{xxx}),$$

$$A^3_{3.3} = \langle \partial_u, \partial_x, g(t)\partial_x + x\partial_u \rangle :$$

$$F = -\frac{1}{2} g' u_x^2 + G(t, u_{xx}, u_{xxx}),$$

$$A^4_{3.3} = \langle \partial_u, \partial_x, \lambda\partial_t + x\partial_u \rangle, \ \lambda \neq 0 :$$

$$F = G(\lambda u_x - t, u_{xx}, u_{xxx}),$$

$$A^5_{3.3} = \langle \lambda t^{\frac{1}{3}} \partial_x + \partial_u, \partial_x, 3\lambda t^{\frac{4}{3}} \partial_t + \lambda t^{\frac{1}{3}} x\partial_x + (x - \lambda t^{\frac{1}{3}} u)\partial_u \rangle, \ \lambda \neq 0 :$$

$$F=-\frac{\lambda}{3}t^{-\frac{2}{3}}uu_x+t^{-\frac{5}{3}}G(\lambda t^{\frac{2}{3}}u_x-t^{\frac{1}{3}},tu_{xx},t^{\frac{4}{3}}u_{xxx}),$$

$A^6_{3.3}=\langle\partial_u,[g(x)-t]\partial_u,\partial_t\rangle,\ g'\neq 0:$

$$F=\frac{g^{(4)}-1}{g'}u_x+G(x,\frac{g''}{g'}u_x-u_{xx},\frac{g'''}{g'}u_x-u_{xxx}),$$

$A^7_{3.3}=\langle\partial_u,-x\partial_u,\partial_x\rangle:$

$$F=G(t,u_{xx},u_{xxx}),$$

$A^8_{3.3}=\langle\partial_u,(\epsilon t-x)\partial_u,\partial_x\rangle:$

$$F=-\epsilon u_x+G(t,u_{xx},u_{xxx}),$$

$A^9_{3.3}=\langle-\frac{1}{x}\partial_u,\partial_u,\partial_x-\frac{u}{x}\partial_u\rangle:$

$$F=-\frac{4}{x}u_{xxx}+\frac{1}{x}G(t,xu_{xx}+2u_x,xu_{xxx}+3u_{xx}).$$

$A_{3.4}-$ invariant equations

$A^1_{3.4}=\langle\partial_u,\partial_t,t\partial_t+\frac{1}{4}x\partial_x+(u+t)\partial_u\rangle:$

$$F=4\ln|u_x|+G(\frac{u_x}{x^3},\frac{u_{xx}}{x^2},\frac{u_{xxx}}{x}),$$

$A^2_{3.4}=\langle\partial_u,\partial_x,4t\partial_t+x\partial_x+(u+x)\partial_u\rangle:$

$$F=t^{-\frac{3}{4}}G(u_x-\frac{1}{4}\ln|t|,|t|^{\frac{1}{4}}u_{xx},|t|^{\frac{1}{2}}u_{xxx}),$$

$A^3_{3.4}=\langle\alpha\partial_x+\partial_u,\partial_x,\frac{\alpha^2}{\alpha'}\partial_t+(1+\alpha)x\partial_x+[x+(1-\alpha)u]\partial_u\rangle:$

$$F=-\alpha' uu_x+\alpha^{-5}\mathrm{e}^{\frac{3}{\alpha}}G(\alpha^2u_x-\alpha,\alpha^3\mathrm{e}^{-\frac{1}{\alpha}}u_{xx},\alpha^4\mathrm{e}^{-\frac{2}{\alpha}}u_{xxx}),$$

where $\alpha=\alpha(t),\ \alpha'\neq 0$ and $\alpha^2\alpha''+2(2+\alpha)\alpha'^2=0,$

$A^4_{3.4}=\langle\partial_x,-\frac{1}{4}\ln|t|\partial_x+\partial_u,4t\partial_t+x\partial_x+u\partial_u\rangle:$

$$F=\frac{1}{4}t^{-1}uu_x+|t|^{-\frac{3}{4}}G(u_x,|t|^{\frac{1}{4}}u_{xx},|t|^{\frac{1}{2}}u_{xxx}),$$

$A^5_{3.4}=\langle\partial_u,-x\partial_u,\partial_x+u\partial_u\rangle:$

$$F=\mathrm{e}^xG(t,\mathrm{e}^{-x}u_{xx},\mathrm{e}^{-x}u_{xxx}),$$

$A^6_{3.4}=\langle\partial_u,[g(x)-\epsilon t]\partial_u,\epsilon\partial_t+u\partial_u\rangle,\ g'\neq 0:$

$$F=\frac{g^{(4)}-\epsilon}{g'}u_x+\mathrm{e}^{\epsilon t}G(x,\mathrm{e}^{-\epsilon t}(\frac{g''}{g'}u_x-u_{xx}),\mathrm{e}^{-\epsilon t}(\frac{g'''}{g'}u_x-u_{xxx})).$$

$A_{3.5}-$ invariant equations

$$A^1_{3.5}=\langle\partial_u,\partial_t,t\partial_t+\frac{1}{4}x\partial_x+u\partial_u\rangle:$$

$$F=G(\frac{u_x}{x^3},\frac{u_{xx}}{x^2},\frac{u_{xxx}}{x}),$$

$$A^2_{3.5}=\langle\partial_u,\partial_x,4t\partial_t+x\partial_x+u\partial_u\rangle:$$

$$F=t^{-\frac{3}{4}}G(u_x,|t|^{\frac{1}{4}}u_{xx},|t|^{\frac{1}{2}}u_{xxx}),$$

$$A^3_{3.5}=\langle\partial_u,g(x)\partial_u,\epsilon\partial_t+u\partial_u\rangle,\ \ g'\neq 0:$$

$$F=\frac{g^{(4)}}{g'}u_x+\mathrm{e}^{\epsilon t}G(x,\mathrm{e}^{-\epsilon t}(\frac{g''}{g'}u_x-u_{xx}),\mathrm{e}^{-\epsilon t}(\frac{g'''}{g'}u_x-u_{xxx})).$$

$A_{3.6}-$ invariant equations

$$A^1_{3.6}=\langle\partial_t,\partial_u,t\partial_t+\frac{1}{4}x\partial_x-u\partial_u\rangle:$$

$$F=x^{-8}G(x^5u_x,x^6u_{xx},x^7u_{xxx}),$$

$$A^2_{3.6}=\langle\partial_x,\lambda|t|^{\frac{1}{2}}\partial_x+\partial_u,4t\partial_t+x\partial_x-u\partial_u\rangle,\ \ \lambda\in\mathbf{R}:$$

$$F=-\frac{\lambda}{2}t^{-\frac{1}{2}}uu_x+t^{-\frac{5}{4}}G(|t|^{\frac{1}{2}}u_x,|t|^{\frac{3}{4}}u_{xx},tu_{xxx}),$$

$$A^3_{3.6}=\langle\partial_u,\mathrm{e}^{2x}g(t)\partial_u,\partial_x+u\partial_u\rangle,\ \ g\neq 0:$$

$$F=(\frac{g'}{2g}+8)u_x+\mathrm{e}^{x}G(t,\mathrm{e}^{-x}(u_{xx}-2u_x),\mathrm{e}^{-x}(u_{xxx}-4u_x)),$$

$$A^4_{3.6}=\langle\partial_u,\mathrm{e}^{2\epsilon t}g(x)\partial_u,\epsilon\partial_t+u\partial_u\rangle,\ \ g'\neq 0:$$

$$F=\frac{g^{(4)}+2\epsilon g}{g'}u_x+\mathrm{e}^{\epsilon t}G(x,\mathrm{e}^{-\epsilon t}(\frac{g''}{g'}u_x-u_{xx}),\mathrm{e}^{-\epsilon t}(\frac{g'''}{g'}u_x-u_{xxx})).$$

$A_{3.7}-$ invariant equations

$$A^1_{3.7}=\langle\partial_t,\partial_x,t\partial_t+\frac{1}{4}x\partial_x\rangle:$$

$$F=u_x^4G(u,\frac{u_{xx}}{u_x^2},\frac{u_{xxx}}{u_x^3}),$$

$$A^2_{3.7}=\langle\partial_t,\partial_x,t\partial_t+\frac{1}{4}x\partial_x+u\partial_u\rangle:$$

$$F=G(\frac{u_x^4}{u^3},\frac{u_{xx}^4}{u^2},\frac{u_{xxx}^4}{u}),$$

$$A^3_{3.7}=\langle\partial_t,\partial_u,t\partial_t+\frac{1}{4}x\partial_x+qu\partial_u\rangle,\ q\neq 0,\ \pm 1:$$

$$F=x^{4(q-1)}G(x^{(1-4q)}u_x,x^{(2-4q)}u_{xx},x^{(3-4q)}u_{xxx}),$$

$$A^4_{3.7}=\langle\partial_x,\lambda t^{\frac{1-q}{4}}\partial_x+\partial_u,4t\partial_t+x\partial_x+qu\partial_u\rangle,\ q\neq 0,\ \pm 1,\ \lambda\in\mathbf{R}:$$

$$F=\frac{1}{4}\lambda(q-1)|t|^{-\frac{q+3}{4}}uu_x+|t|^{\frac{1}{4}(q-4)}G(|t|^{\frac{1-q}{4}}u_x,|t|^{\frac{2-q}{4}}u_{xx},|t|^{\frac{3-q}{4}}u_{xxx}),$$

$$A^5_{3.7}=\langle\partial_u,\mathrm{e}^{(1-q)x}g(t)\partial_u,\partial_x+u\partial_u\rangle,\ g\neq 0,\ q\neq 0,\ \pm 1:$$

$$F=[\frac{g'}{(1-q)g}+(1-q)^3]u_x+\mathrm{e}^xG(t,\omega_1,\omega_2),$$

where $\omega_1=\mathrm{e}^{-x}[u_{xx}-(1-q)u_x],\ \omega_2=\mathrm{e}^{-x}[u_{xxx}-(1-q)^2u_x],$

$$A^6_{3.7}=\langle\partial_u,\mathrm{e}^{\epsilon(1-q)t}g(x)\partial_u,\epsilon\partial_t+u\partial_u\rangle,\ g'\neq 0,\ q\neq 0,\ \pm 1:$$

$$F=\frac{g^{(4)}+\epsilon(1-q)g}{g'}u_x+\mathrm{e}^{\epsilon t}G(x,\mathrm{e}^{-\epsilon t}(\frac{g''}{g'}u_x-u_{xx}),\mathrm{e}^{-\epsilon t}(\frac{g'''}{g'}u_x-u_{xxx})).$$

$A_{3.8}-$ invariant equations

$$A^1_{3.8}=\langle\partial_x,\alpha\partial_x+\partial_u,-\frac{1+\alpha^2}{\alpha'}\partial_t-\alpha x\partial_x+(\alpha u-x)\partial_u\rangle:$$

$$F=-\alpha' uu_x+(1+\alpha^2)^{-\frac{5}{2}}G(\omega_1,\omega_2,\omega_3),$$

where $\alpha=\alpha(t)$ with $\alpha'\neq 0$ satisfies $2\alpha\alpha'^2+(1+\alpha^2)\alpha''=0$,

$$\omega_1=(1+\alpha^2)u_x-\alpha,\ \omega_2=(1+\alpha^2)^{\frac{3}{2}}u_{xx},\ \omega_3=(1+\alpha^2)^2u_{xxx}.$$

$A_{3.9}-$ invariant equations

$$A^1_{3.9}=\langle\partial_x,\alpha\partial_x+\partial_u,-\frac{1+\alpha^2}{\alpha'}\partial_t+(q-\alpha)x\partial_x+[(\alpha+q)u-x]\partial_u\rangle,\ q>0:$$

$$F=-\alpha' uu_x+(1+\alpha^2)^{-\frac{5}{2}}\mathrm{e}^{3q\arctan\alpha}G(\omega_1,\omega_2,\omega_3),$$

where $\alpha=\alpha(t)$ with $\alpha'\neq 0$ satisfies $(2\alpha-4q)\alpha'^2+(1+\alpha^2)\alpha''=0$,

$$\omega_1=(1+\alpha^2)u_x-\alpha,\ \omega_2=(1+\alpha^2)^{\frac{3}{2}}\mathrm{e}^{-q\arctan\alpha}u_{xx},$$

$$\omega_3=(1+\alpha^2)^2\mathrm{e}^{-2q\arctan\alpha}u_{xxx}.$$

Note that the nonlinear ordinary differential equation

$$(2\alpha-4q)\alpha'^2+(1+\alpha^2)\alpha''=0$$

can be solved by quadratures. Its general solution is given by the following implicit formula:

$$\int^{\alpha(t)} \mathrm{e}^{-4q \arctan z}(1+z^2)\mathrm{d}z + C_1 t + C_2 = 0,\ \{C_1, C_2\} \subset \mathbf{R}.$$

5. Classification of equations invariant under four-dimensional solvable Lie algebras

Now we perform group classification of Eqs.(1) admitting four-dimensional solvable Lie algebras. To this end we use the realizations of three-dimensional solvable algebras obtained in the previous section and the fact that any four-dimensional solvable Lie algebra contains a three-dimensional solvable algebra.

5.1. Equations with four-dimensional decomposable algebras

The list of non-isomorphic four-dimensional decomposable algebras contains following ten algebras:

$$\begin{aligned}
&A_{2.2} \oplus A_{2.2} = 2A_{2.2},\\
&A_{3.1} \oplus A_1 = 4A_1,\\
&A_{3.2} \oplus A_1 = A_{2.2} \oplus 2A_1,\\
&A_{3.i} \oplus A_1\ (i = 3, 4, \cdots, 9).
\end{aligned}$$

The complete list of Eqs.(1) invariant with respect to the above algebras is given below.

$2A_{2.2}-$ invariant equations

$$2A^1_{2.2} = \langle -t\partial_t - \frac{x}{4}\partial_x, \partial_t, \partial_u, \mathrm{e}^u\partial_u\rangle :$$

$$F = u_x^4 - 6u_x^2 u_{xx} + 4u_x u_{xxx} + 3u_{xx}^2 + \frac{u_x}{x^3} G(\omega_1, \omega_2),$$

$$\omega_1 = \frac{x}{u_x}(u_x^2 - u_{xx}),\ \omega_2 = \frac{x^2}{u_x}(u_x^3 + u_{xxx} - 3u_x u_{xx}),$$

$$2A^2_{2.2} = \langle -4t\partial_t - x\partial_x, \partial_x, \lambda|t|^{\frac{1}{4}}\partial_x - u\partial_u, \partial_u\rangle$$

$$F = \frac{\lambda}{4|t|}\omega \ln|\omega| + \frac{\omega}{|t|} G(|t|^{\frac{1}{4}}\frac{u_{xx}}{u_x}, |t|^{\frac{1}{2}}\frac{u_{xxx}}{u_x}),$$

$$\omega = |t|^{\frac{1}{4}} u_x,$$

$$2A^3_{2.2} = \langle -u\partial_u, \partial_u, \epsilon\partial_t, \mathrm{e}^{\epsilon t}\partial_t + \epsilon \mathrm{e}^{\epsilon t}(\frac{x}{4} + 1)\partial_x\rangle :$$

$$F=-\frac{x+4}{4}u_x+\frac{1}{(x+4)^3}u_xG((x+4)\frac{u_{xx}}{u_x},(x+4)^2\frac{u_{xxx}}{u_x}),$$

$2A^4_{2.2}=\langle\partial_x-u\partial_u,\partial_u,\lambda\partial_t,\mathrm{e}^{\frac{t}{\lambda}}\partial_x\rangle,\ \lambda\neq 0:$

$$F=-\frac{x+\ln|u_x|}{\lambda}u_x+u_xG(\frac{u_{xx}}{u_x},\frac{u_{xxx}}{u_x}),$$

$2A^4_{2.2}=\langle\partial_x-u\partial_u,\partial_u,\lambda\partial_t,\mathrm{e}^{\frac{t}{\lambda}-x}\partial_u\rangle,\ \lambda\neq 0:$

$$F=-\frac{\lambda+1}{\lambda}u_x+\mathrm{e}^{-x}G(\mathrm{e}^x(u_{xx}+u_x),\mathrm{e}^x(u_{xxx}-u_x)),$$

$2A^5_{2.2}=\langle\epsilon\partial_t-u\partial_u,\partial_u,\beta\partial_t+\gamma\partial_x,\mathrm{e}^{\frac{\epsilon\beta+1}{\gamma}x-\epsilon t}\partial_u\rangle,\ \beta\neq-\epsilon,\ \gamma\neq 0:$

$$F=\frac{(\epsilon\beta+1)^4-\epsilon\gamma^4}{\gamma^3(\epsilon\beta+1)}u_x+\mathrm{e}^{\epsilon(\frac{\beta}{\gamma}x-t)}G(\omega_1,\omega_2),$$

$$\omega_1=\mathrm{e}^{\epsilon(t-\frac{\beta}{\gamma}x)}[\gamma u_{xx}-(\epsilon\beta+1)u_x],\ \omega_2=\mathrm{e}^{\epsilon(t-\frac{\beta}{\gamma}x)}[\gamma^2u_{xx}-(\epsilon\beta+1)^2u_x].$$

$A_{3.2}\oplus A_1-$ invariant equations

$A^4_{3.2}\oplus A_1=\langle-u\partial_u,\partial_u,\partial_x,\partial_t\rangle:$

$$F=u_xG(\frac{u_{xx}}{u_x},\frac{u_{xxx}}{u_x}),$$

$A^8_{3.2}\oplus A_1=\langle\partial_x-u\partial_u,\partial_u,\partial_t,\mathrm{e}^{-x}\partial_u\rangle:$

$$F=-u_x+\mathrm{e}^{-x}G(\mathrm{e}^x(u_{xx}+u_x),\mathrm{e}^x(u_{xxx}-u_x),$$

$A^9_{3.2}(g=\mathrm{e}^{\lambda x})\oplus A_1=\langle\epsilon\partial_t-u\partial_u,\partial_u,\mathrm{e}^{\lambda x-\epsilon t}\partial_u,\lambda\partial_t+\partial_x\rangle,\ \lambda\neq 0:$

$$F=\frac{\lambda^4-\epsilon}{\lambda}u_x+\mathrm{e}^{\epsilon(\lambda x-t)}G(\mathrm{e}^{\epsilon(t-\lambda x)}(u_{xx}-\lambda u_x),\mathrm{e}^{\epsilon(t-\lambda x)}(u_{xxx}-\lambda^2u_x)).$$

$A_{3.3}\oplus A_1-$ invariant equations,

$A^2_{3.3}\oplus A_1=\langle\partial_u,\partial_t,\lambda\partial_x+t\partial_u,\partial_t+\beta\partial_x+\frac{x}{\lambda}\partial_u\rangle,\ \lambda>0,\ \beta\in\mathbf{R}:$

$$F=-\beta u_x+\frac{x}{\lambda}+G(u_{xx},u_{xxx}),$$

$A^4_{3.3}\oplus A_1=\langle\partial_u,\partial_x,\lambda\partial_t+x\partial_u,\beta\partial_t+\partial_x+\frac{t}{\lambda}\partial_u\rangle,\ \lambda\beta\neq 0:$

$$F=-\frac{1}{\beta}u_x+\frac{t}{\beta\lambda}+G(u_{xx},u_{xxx}),$$

$A^6_{3.3}(g=\lambda x)\oplus A_1=\langle\partial_u,(\lambda x-t)\partial_u,\partial_t,\partial_x+\lambda\partial_t\rangle,\ \lambda\neq 0:$

$$F=-\frac{1}{\lambda}u_x+G(u_{xx},u_{xxx}),$$

$$A^7_{3.3}\oplus A_1=\langle\partial_u,-x\partial_u,\partial_x,\partial_t\rangle:$$

$$F=G(u_{xx},u_{xxx}),$$

$$A^9_{3.3}\oplus A_1=\langle-\frac{1}{x}\partial_u,\partial_u,\partial_x-\frac{u}{x}\partial_u,\partial_t+\frac{g(t)}{x}\partial_u\rangle:$$

$$F=\frac{g-4u_{xxx}}{x}+\frac{1}{x}G(xu_{xx}+2u_x,xu_{xxx}+3u_{xx}).$$

$A_{3.4}\oplus A_1-$ invariant equations,

$$A^1_{3.4}\oplus A_1=\langle\partial_u,\partial_t,t\partial_t+\frac{1}{4}x\partial_x+(u+t)\partial_u,x^4\partial_u\rangle:$$

$$F=4\ln|u_x|+\frac{6}{x^3}u_x+G(\frac{xu_{xx}-3u_x}{x^3},\frac{x^2u_{xxx}-6u_x}{x^3}),$$

$$A^5_{3.4}\oplus A_1=\langle\partial_u,-x\partial_u,\partial_x+u\partial_u,\mathrm{e}^xg(t)\partial_u\rangle,\ g\neq0:$$

$$F=(\frac{g'}{g}+1)u_{xx}+\mathrm{e}^xG(t,\mathrm{e}^{-x}(u_{xxx}-u_{xx})),$$

$$A^5_{3.4}\oplus A_1=\langle\partial_u,-x\partial_u,\partial_x+u\partial_u,\partial_t\rangle:$$

$$F=\mathrm{e}^xG(\mathrm{e}^{-x}u_{xx},\mathrm{e}^{-x}u_{xxx}),$$

$$A^6_{3.4}(g=\epsilon\lambda x)\oplus A_1=\langle\partial_u,\epsilon(\lambda x-t)\partial_u,\epsilon\partial_t+u\partial_u,\lambda\partial_t+\partial_x\rangle,\ \lambda\neq0:$$

$$F=-\frac{1}{\lambda}u_x+\mathrm{e}^{\epsilon(t-\lambda x)}G(\mathrm{e}^{\epsilon(\lambda x-t)}u_{xx},\mathrm{e}^{\epsilon(\lambda x-t)}u_{xxx}).$$

$A_{3.5}\oplus A_1-$ invariant equations

$$A^1_{3.5}\oplus A_1=\langle\partial_u,\partial_t,t\partial_t+\frac{1}{4}x\partial_x+u\partial_u,x^4\partial_u\rangle:$$

$$F=6\frac{u_x}{x^3}+G(\frac{xu_{xx}-3u_x}{x^3},\frac{x^2u_{xxx}-6u_x}{x^3}),$$

$$A^3_{3.5}\oplus A_1=\langle\partial_u,g(x)\partial_u,\epsilon\partial_t+u\partial_u,\partial_t\rangle,\ g'\neq0:$$

$$F=\frac{g^{(4)}}{g'}u_x+(g''u_x-g'u_{xx})G(x,\frac{g'u_{xxx}-g'''u_x}{g'(g'u_{xx}-g''u_x)}).$$

$A_{3.6}\oplus A_1-$ invariant equations,

$$A^1_{3.6}\oplus A_1=\langle\partial_t,\partial_u,t\partial_t+\frac{1}{4}x\partial_x-u\partial_u,\frac{1}{x^4}\partial_u\rangle:$$

$$F=-210x^{-3}u_x+x^{-8}G(x^5(xu_{xx}+5u_x),x^5(x^2u_{xxx}-30u_x)),$$

$A^3_{3.6}(g=1)\oplus A_1=\langle\partial_u,\mathrm{e}^{2x}\partial_u,\partial_x+u\partial_u,\partial_t\rangle:$

$$F=8u_x+\mathrm{e}^xG(\mathrm{e}^{-x}(u_{xx}-2u_x),\mathrm{e}^{-x}(u_{xxx}-4u_x)),$$

$A^4_{3.6}(g=\mathrm{e}^{-\frac{2\epsilon}{\beta}x})\oplus A_1=\langle\partial_u,\mathrm{e}^{2\epsilon(t-\frac{x}{\beta})}\partial_u,\epsilon\partial_t+u\partial_u,\partial_t+\beta\partial_x\rangle,\ \beta\neq 0:$

$$F=+\mathrm{e}^{\epsilon(t-\frac{x}{\beta})}G(\mathrm{e}^{\epsilon(\frac{x}{\beta}-t)}(\beta u_{xx}+2\epsilon u_x),\mathrm{e}^{\epsilon(\frac{x}{\beta}-t)}(\beta^2u_{xxx}-4u_x))-$$
$$(\frac{8\epsilon}{\beta^3}+\beta)u_x.$$

$A_{3.7}\oplus A_1-$ invariant equations

$A^1_{3.7}\oplus A_1=\langle\partial_t,\partial_x,t\partial_t+\frac{1}{4}x\partial_x,\partial_u\rangle:$

$$F=u_x^4G(\frac{u_{xx}}{u_x^2},\frac{u_{xxx}}{u_x^3}),$$

$A^2_{3.7}\oplus A_1=\langle\partial_t,\partial_x,t\partial_t+\frac{1}{4}x\partial_x+u\partial_u,u\partial_u\rangle:$

$$F=\frac{u_x^4}{u^3}G(\frac{uu_{xx}}{u_x^2},\frac{u^2u_{xxx}}{u_x^3}),$$

$A^3_{3.7}\oplus A_1=\langle\partial_t,\partial_u,t\partial_t+\frac{1}{4}x\partial_x+qu\partial_u,x^{4q}\partial_u\rangle,\ q\neq 0,\ \pm 1:$

$$F=\frac{(4q-3)(4q-2)(4q-1)}{x^3}u_x+x^{4(q-1)}G(\omega_1,\omega_2),$$
$$\omega_1=x^{(1-4q)}[xu_{xx}-(4q-1)u_x],$$
$$\omega_2=x^{(1-4q)}[x^2u_{xxx}-(4q-2)(4q-1)u_x],$$

$A^5_{3.7}(g=1)\oplus A_1=\langle\partial_u,\mathrm{e}^{(1-q)x}\partial_u,\partial_x+u\partial_u,\partial_t\rangle,\ q\neq 0,\ \pm 1:$

$$F=(1-q)^3u_x+\mathrm{e}^xG(\mathrm{e}^{-x}(u_{xx}-(1-q)u_x),\mathrm{e}^{-x}(u_{xxx}-(1-q)^2u_x)),$$

$A^6_{3.7}(g=\mathrm{e}^{\frac{\epsilon(q-1)}{\beta}x})\oplus A_1=\langle\partial_u,\mathrm{e}^{\epsilon(1-q)(t-\frac{x}{\beta})}\partial_u,\epsilon\partial_t+u\partial_u,\partial_t+\beta\partial_x\rangle,\ q\neq 0,\ \pm 1:$

$$F=[\frac{\epsilon(q-1)^3}{\beta^3}-\beta]u_x+\mathrm{e}^{\epsilon(t-\frac{x}{\beta})}G(\omega_1,\omega_2),$$
$$\omega_1=\mathrm{e}^{\epsilon(\frac{x}{\beta}-t)}[\beta u_{xx}-\epsilon(q-1)u_x],\ \omega_2=\mathrm{e}^{\epsilon(\frac{x}{\beta}-t)}[\beta^2u_{xxx}-(q-1)^2u_x].$$

5.2. Equations with four-dimensional non-decomposable algebras

There exist ten non-isomorphic four-dimensional non-decomposable Lie algebras, $A_{4.i}$ $(i=1,2,\cdots,10)$:

$A_{4.1}:\qquad [X_2,X_4]=X_1,\quad [X_3,X_4]=X_2;$

$A_{4.2}:\qquad [X_1,X_4]=qX_1,\quad [X_2,X_4]=X_2,\quad [X_3,X_4]=X_2+X_3,\ q\neq 0;$

$A_{4.3}:\qquad [X_1,X_4]=X_1,\quad [X_3,X_4]=X_2;$

$A_{4.4}:\qquad [X_1,X_4]=X_1,\quad [X_2,X_4]=X_1+X_2,\quad [X_3,X_4]=X_2+X_3;$

$A_{4.5}:\qquad [X_1,X_4]=X_1,\quad [X_2,X_4]=qX_2,\quad [X_3,X_4]=pX_3,$

$-1\leqslant p\leqslant q\leqslant 1,\ pq\neq 0;$

$A_{4.6}:\qquad [X_1,X_4]=qX_1,\quad [X_2,X_4]=pX_2-X_3,\quad [X_3,X_4]=X_2+pX_3,$

$q\neq 0,\ p\geqslant 0;$

$A_{4.7}:\qquad [X_2,X_3]=X_1,\quad [X_1,X_4]=2X_1,\quad [X_2,X_4]=X_2,$

$[X_3,X_4]=X_2+X_3;$

$A_{4.8}:\qquad [X_2,X_3]=X_1,\quad [X_1,X_4]=(1+q)X_1,\quad [X_2,X_4]=X_2,$

$[X_3,X_4]=qX_3,\ |q|\leqslant 1;$

$A_{4.9}:\qquad [X_2,X_3]=X_1,\quad [X_1,X_4]=2qX_1,\quad [X_2,X_4]=qX_2-X_3,$

$[X_3,X_4]=X_2+qX_3,\ q\geqslant 0;$

$A_{4.10}:\qquad [X_1,X_3]=X_1,\quad [X_2,X_3]=X_2,\quad [X_1,X_4]=-X_2,\quad [X_2,X_4]=X_1.$

Each of the above algebras can be decomposed into a semi-direct sum of a three-dimensional ideal N and a one-dimensional Lie algebra. Analysis of the commutation relations above shows that N is of the type $A_{3.1}$ for the algebras $A_{4.i}$ $(i=1,2,\cdots,6)$, of the type $A_{3.3}$ for the algebras $A_{4.7},\ A_{4.8},\ A_{4.9}$, and of the type $A_{3.5}$ for the algebra $A_{4.10}$. Thus we can extend the already known realizations of three-dimensional Lie algebras to obtain exhaustive description of the four-dimensional non-decomposable solvable Lie algebras admitted by Eq.(1). Below we present the final result, the lists of invariant equations together with the corresponding symmetry algebras.

$A_{4.1}-$ invariant equations

$$A^1_{4.1}=\langle\partial_u,\partial_x,\partial_t,t\partial_x+x\partial_u\rangle:$$

$$F=-\frac{1}{2}u_x^2+G(u_{xx},u_{xxx}),$$

$$A^2_{4.1}=\langle\partial_u,x\partial_u,\partial_t,\partial_x+tx\partial_u\rangle:$$

$$F=\frac{1}{2}x^2+G(u_{xx},u_{xxx}).$$

$A_{4.2}-$ invariant equations

$$A^1_{4.2}=\langle\partial_t,\partial_u,\partial_x,4t\partial_t+x\partial_x+(x+u)\partial_u\rangle:$$

$$F = u_{xx}^3 G(\mathrm{e}^{u_x} u_{xx}, \mathrm{e}^{2u_x} u_{xxx}),$$

$$A_{4.2}^2 = \langle \partial_x, \partial_u, \partial_t, t\partial_t + \frac{1}{4}x\partial_x + (t+u)\partial_u \rangle :$$

$$F = \frac{4}{3}\ln|u_x| + G(\frac{u_{xx}^3}{u_x^2}, \frac{u_{xxx}^3}{u_x}),$$

$$A_{4.2}^3 = \langle \partial_t, \partial_u, -\frac{4}{q}\ln|x|\partial_u, qt\partial_t + \frac{q}{4}x\partial_x + u\partial_u \rangle,\ q \neq 0 :$$

$$F = -6\frac{u_x}{x^3} + x^{4(\frac{1}{q}-1)}G(\omega_1, \omega_2),$$

$$\omega_1 = x^{1-\frac{4}{q}}(xu_{xx} + u_x),\ \omega_2 = x^{1-\frac{4}{q}}(x^2 u_{xxx} - 2u_x),$$

$$A_{4.2}^4 = \langle x^{4(1-q)}\partial_u, \partial_u, \partial_t, t\partial_t + \frac{1}{4}x\partial_x + (t+u)\partial_u \rangle,\ q \neq 0,\ 1 :$$

$$F = -\frac{(4q-3)(4q-2)(4q-1)}{x^3}u_x + 4\ln|x| + G(\omega_1, \omega_2),$$

$$\omega_1 = \frac{xu_{xx} - (3-4q)u_x}{x^3},\ \omega_2 = \frac{x^2 u_{xxx} - (3-4q)(2-4q)u_x}{x^3}.$$

$A_{4.3}-$ invariant equations

$$A_{4.3}^1 = \langle \partial_u, \partial_x, \partial_t, t\partial_x + u\partial_u \rangle :$$

$$F = -u_x \ln|u_x| + u_x G(\frac{u_{xx}}{u_x}, \frac{u_{xxx}}{u_x}),$$

$$A_{4.3}^2 = \langle \partial_t, \partial_u, -4\ln|x|\partial_u, t\partial_t + \frac{1}{4}x\partial_x \rangle :$$

$$F = -\frac{6}{x^3}u_x + \frac{1}{x^4}G(x^2 u_{xx} + xu_x, x^3 u_{xxx} - 2xu_x),$$

$$A_{4.3}^3 = \langle \partial_u, \mathrm{e}^x\partial_u, \partial_t, \partial_x + (t\mathrm{e}^x + u)\partial_u \rangle :$$

$$F = u_x + x\mathrm{e}^x + \mathrm{e}^x G(\mathrm{e}^{-x}(u_{xx} - u_x), \mathrm{e}^{-x}(u_{xxx} - u_{xx})).$$

$A_{4.4}-$ invariant equations

$$A_{4.4}^1 = \langle \partial_u, -4\ln|x|\partial_u, \partial_t, t\partial_t + \frac{1}{4}x\partial_x + (u - 4t\ln|x|)\partial_u \rangle :$$

$$F = -8\ln^2|x| - 6\frac{u_x}{x^3} + G(\frac{xu_{xx} + u_x}{x^3}, \frac{x^2 u_{xxx} - 2u_x}{x^3}).$$

$A_{4.5}-$ invariant equations,

$$A_{4.5}^1 = \langle \partial_t, \partial_x, \partial_u, t\partial_t + \frac{1}{4}x\partial_x + pu\partial_u \rangle,\ -1 \leqslant p < \frac{1}{4},\ p \neq 0 :$$

$$F = u_x^{4\frac{1-p}{1-4p}} G(u_x^{\frac{2-4p}{4p-1}} u_{xx}, u_x^{\frac{3-4p}{4p-1}} u_{xxx}),$$

$$A_{4.5}^2 = \langle \partial_t, \partial_x, \partial_u, t\partial_t + \frac{1}{4}x\partial_x + \frac{1}{4}u\partial_u \rangle :$$

$$F = u_{xx}^3 G(u_x, \frac{u_{xxx}}{u_{xx}^2}),$$

$$A_{4.5}^3 = \langle \partial_t, \partial_u, x^{4(q-p)}\partial_u, t\partial_t + \frac{1}{4}x\partial_x + qu\partial_u \rangle,\ -1 \leqslant p < q \leqslant 1,\ pq \neq 0 :$$

$$F = \frac{(4q-4p-1)(4q-4p-2)(4q-4p-3)}{x^3} u_x + x^{4(q-1)} G(\omega_1, \omega_2),$$

$$\omega_1 = x^{1-4q}[xu_{xx} - (4q-4p-1)u_x],$$

$$\omega_2 = x^{1-4q}[x^2 u_{xxx} - (4q-4p-2)(4q-4p-1)u_x].$$

$A_{4.7}-$ invariant equations

$$A_{4.7}^1 = \langle \partial_u, \partial_x, -\frac{1}{4}\ln|t|\partial_x + x\partial_u, 4t\partial_t + x\partial_x + 2u\partial_u \rangle :$$

$$F = \frac{u_x^2}{8t} + \frac{1}{|t|^{\frac{1}{2}}} G(u_{xx}, |t|^{\frac{1}{4}} u_{xxx}),$$

$$A_{4.7}^2 = \langle \partial_u, \alpha\partial_x + x\partial_u, -\partial_x, -\frac{\alpha^2}{\alpha'}\partial_t + (1-\alpha)x\partial_x + (2u - \frac{1}{2}x^2)\partial_u \rangle :$$

$$F = -\frac{1}{2}\alpha' u_x^2 + \alpha^{-4} \mathrm{e}^{-\frac{2}{\alpha}} G(\omega_1, \omega_2),$$

$$\omega_1 = \alpha^2 u_{xx} - \alpha,\ \omega_2 = \mathrm{e}^{\frac{1}{\alpha}} \alpha^3 u_{xxx},$$

where $\alpha = \alpha(t)$ with $\alpha' \neq 0$ satisfies $\alpha^2\alpha'' + 2(\alpha - 2)\alpha'^2 = 0$.

$$A_{4.7}^3 = \langle \partial_u, (\lambda x^4 - t)\partial_u, \partial_t, t\partial_t + \frac{1}{4}x\partial_x + (2u + \lambda t x^4 - \frac{1}{2}t^2)\partial_u \rangle,\ \lambda \neq 0 :$$

$$F = 4\lambda x^4 \ln|x| + \frac{24\lambda - 1}{4\lambda x^3} u_x + x^4 G(\frac{xu_{xx} - 3u_x}{x^7}, \frac{x^2 u_{xxx} - 6u_x}{x^7}),$$

$$A_{4.7}^4 = \langle \partial_u, (\epsilon t - x)\partial_u, \partial_x, 4t\partial_t + (x + 3\epsilon t)\partial_x + (\epsilon x t + 2u - \frac{1}{2}x^2)\partial_u \rangle :$$

$$F = -\epsilon u_x + \frac{1}{6}t + \frac{1}{|t|^{\frac{1}{2}}} G(u_{xx} + \frac{1}{4}\ln|t|, |t|^{\frac{1}{4}} u_{xxx}),$$

$$A_{4.7}^5 = \langle \partial_u, -x\partial_u, \partial_x, 4t\partial_t + x\partial_x + (2u - \frac{1}{2}x^2)\partial_u \rangle :$$

$$F = \frac{1}{|t|^{\frac{1}{2}}} G(u_{xx} + \frac{1}{4}\ln|t|, |t|^{\frac{1}{4}} u_{xxx}),$$

$$A^6_{4.7} = \langle -\frac{1}{x}\partial_u, \partial_u, \partial_x - \frac{u}{x}\partial_u, 4t\partial_t + x\partial_x + (u + \frac{1}{2}x)\partial_u\rangle :$$

$$F = -\frac{4}{x}u_{xxx} + \frac{1}{x|t|^{\frac{1}{2}}}G(\omega_1, \omega_2),$$

$$\omega_1 = xu_{xx} + 2u_x - \frac{1}{4}\ln|t|,\ \omega_2 = |t|^{\frac{1}{4}}(xu_{xxx} + 3u_{xx}).$$

$A_{4.8}-$ invariant equations

$$A^1_{4.8} = \langle \partial_x, \partial_t, t\partial_x + \partial_u, t\partial_t + \frac{1}{4}x\partial_x - \frac{3}{4}u\partial_u\rangle :$$

$$F = -uu_x + |u_x|^{\frac{7}{4}}G(\frac{u_{xx}}{|u_x|^{\frac{5}{4}}}, \frac{u_{xxx}}{|u_x|^{\frac{3}{2}}}),$$

$$A^2_{4.8} = \langle \partial_u, \partial_t, \lambda\partial_x + t\partial_u, t\partial_t + \frac{1}{4}x\partial_x + \frac{5}{4}u\partial_u\rangle,\ \lambda > 0 :$$

$$F = \frac{1}{\lambda}x + |u_x|^{\frac{1}{4}}G(\frac{u_{xx}}{|u_x|^{\frac{3}{4}}}, \frac{u_{xxx}}{|u_x|^{\frac{1}{2}}}),$$

$$A^3_{4.8} = \langle \partial_u, \partial_x, \lambda t^{\frac{1-q}{4}}\partial_x + x\partial_u, 4t\partial_t + x\partial_x + (1+q)u\partial_u\rangle,\ \lambda \neq 0,\ |q| \leqslant 1 :$$

$$F = \frac{\lambda(q-1)}{8}t^{-\frac{q+3}{4}}u_x^2 + t^{\frac{q-3}{4}}G(t^{\frac{1-q}{4}}u_{xx}, t^{\frac{2-q}{4}}u_{xxx}),$$

$$A^4_{4.8} = \langle \partial_u, \partial_x, \lambda\partial_t + x\partial_u, 4t\partial_t + x\partial_x + 5u\partial_u\rangle,\ \lambda \neq 0 :$$

$$F = |\lambda u_x - t|^{\frac{1}{4}}G(\frac{u_{xx}}{|\lambda u_x - t|^{\frac{3}{4}}}, \frac{u_{xxx}}{|\lambda u_x - t|^{\frac{1}{2}}}),$$

$$A^5_{4.8} = \langle \partial_u, (\lambda x^4 - t)\partial_u, \partial_t, qt\partial_t + \frac{q}{4}x\partial_x + (1+q)u\partial_u\rangle,\ \lambda q \neq 0,\ |q| \leqslant 1 :$$

$$F = \frac{24\lambda - 1}{4\lambda x^3}u_x + x^{\frac{4}{q}}G(x^{-3-\frac{4}{q}}(xu_{xx} - 3u_x), x^{-3-\frac{4}{q}}(x^2u_{xxx} - 6u_x)),$$

$$A^6_{4.8} = \langle \partial_u, (\lambda x^4 - t)\partial_u, \partial_t, u\partial_u\rangle,\ \lambda \neq 0 :$$

$$F = \frac{24\lambda - 1}{4\lambda x^3}u_x + (xu_{xx} - 3u_x)G(x, \frac{x^2u_{xxx} - 6u_x}{xu_{xx} - 3u_x}),$$

$$A^7_{4.8} = \langle \partial_u, -x\partial_u, \partial_x, 4qt\partial_t + qx\partial_x + (1+q)u\partial_u\rangle,\ |q| \leqslant 1,\ q \neq 0 :$$

$$F = t^{\frac{1-3q}{4q}}G(t^{\frac{q-1}{4q}}u_{xx}, t^{\frac{2q-1}{4q}}u_{xxx}),$$

$$A^8_{4.8} = \langle \partial_u, -x\partial_u, \partial_x, u\partial_u\rangle :$$

$$F = u_{xx}G(t, \frac{u_{xxx}}{u_{xx}}),$$

$A^9_{4.8} = \langle \partial_u, (\epsilon t - x)\partial_u, \partial_x, 4qt\partial_t + q(x+3\epsilon t)\partial_x + (1+q)u\partial_u\rangle,\ |q| \leqslant 1, q \neq 0:$

$$F = -\epsilon u_x + t^{\frac{1-3q}{4q}} G(t^{\frac{q-1}{4q}} u_{xx}, t^{\frac{2q-1}{4q}} u_{xxx}),$$

$A^{10}_{4.8} = \langle \partial_u, (\epsilon t - x)\partial_u, \partial_x, u\partial_u\rangle:$

$$F = -\epsilon u_x + u_{xx} G(t, \frac{u_{xxx}}{u_{xx}}),$$

$A^{11}_{4.8} = \langle -\frac{1}{x}\partial_u, \partial_u, \partial_x - \frac{u}{x}\partial_u, 4qt\partial_t + qx\partial_x + u\partial_u\rangle,\ |q| \leqslant 1,\ q \neq 0:$

$$F = -\frac{4}{x}u_{xxx} + \frac{t^{\frac{1-3q}{4q}}}{x} G(t^{\frac{q-1}{4q}}(xu_{xx} + 2u_x), t^{\frac{2q-1}{4q}}(xu_{xxx} + 3u_{xx})),$$

$A^{12}_{4.8} = \langle -\frac{1}{x}\partial_u, \partial_u, \partial_x - \frac{u}{x}\partial_u, u\partial_u\rangle:$

$$F = -\frac{4}{x}u_{xxx} + (u_{xx} + \frac{2}{x}u_x) G(t, \frac{xu_{xxx} + 3u_{xx}}{xu_{xx} + 2u_x}).$$

$A_{4.9}-$ invariant equations

$A^1_{4.9} = \langle \partial_u, \partial_x, \alpha\partial_x + x\partial_u, -\frac{1+\alpha^2}{\alpha'}\partial_t + (q-\alpha)x\partial_x + (2qu - \frac{1}{2}x^2)\partial_u\rangle,\ q \geqslant 0:$

$$F = -\frac{1}{2}\alpha' u_x^2 + \frac{\mathrm{e}^{2q\arctan\alpha}}{(1+\alpha^2)^2} G(\omega_1, \omega_2),$$

$$\omega_1 = (1+\alpha^2)u_{xx} - \alpha,\ \omega_2 = (1+\alpha^2)^{\frac{3}{2}} \mathrm{e}^{-q\arctan\alpha} u_{xxx},$$

where $\alpha = \alpha(t)$ with $\alpha' \neq 0$ satisfies $(1+\alpha^2)\alpha'' + 2(\alpha - 2q)\alpha'^2 = 0$.

$A_{4.10}-$ invariant equations

$A^1_{4.10} = \langle \partial_u, -\tan x\partial_u, \partial_t + u\partial_u, \lambda\partial_t + \partial_x + u\tan x\partial_u\rangle,\ \lambda \in \mathbf{R}:$

$$F = 8\tan x u_x + 4\tan x u_{xxx} + \mathrm{e}^{t-\lambda x}\sec x G(\omega_1, \omega_2),$$

$$\omega_1 = \mathrm{e}^{\lambda x - t}(\cos x u_{xx} - 2\sin x u_x),$$

$$\omega_2 = \mathrm{e}^{\lambda x - t}(\cos x u_{xxx} - 3\sin x u_{xx} - 2\cos x u_x).$$

6. Galilei-invariant equations

One of the important requirements in the classical mechanics is the requirement for the motion equation to satisfy the Galilei relativity principle. From the mathematical standpoint this requirement means that the motion equation has to admit Galilei group. Our approach enables to handle the problem of mathematical description of all possible PDEs of the form (1) that admit Galilei transformation group or, equivalently, the Galilei algebra of first-order differential operators (vector fields).

The Lie algebra $g(1) = \langle T, P, G \rangle$ is called the classical Galilei algebra if its basis operators satisfy the following commutation relations:

$$[T, P] = 0, \quad [T, G] = -P, \tag{8}$$
$$[P, G] = 0. \tag{9}$$

The Lie algebra $\tilde{g}(1) = \langle T, P, G, M \rangle$ is called the extended classical Galilei algebra if the commutation relations (9) hold and what is more the basis operator M obeys the following commutation relations: Let an operator M satisfy the following commutation relations:

$$[M, T] = [M, P] = [M, G] = 0, \quad [G, P] = M.$$

Below we give without derivation expressions for the most general PDEs of the form (1) which are invariant under the Galilei algebras.

The list of inequivalent Eqs.(1) invariant under the classical Galilei algebra is exhausted by the following five classes of PDEs:

$$\begin{aligned}
1.\quad & u_t = -u_{xxxx} + uu_x + f(u_x, u_{xx}, u_{xxx}), \\
& T = \partial_t, \quad P = \partial_x, \quad G = -t\partial_x + \partial_u; \\
2.\quad & u_t = -u_{xxxx} - u_x + f(x, u_{xx}, u_{xxx}), \\
& T = \partial_t, \quad P = \partial_u, \quad G = (x - t)\partial_u; \\
3.\quad & u_t = -u_{xxxx} + f(t + u_x, u_{xx}, u_{xxx}), \\
& T = \partial_x, \quad P = \partial_u, \quad G = \partial_t - x\partial_u; \\
4.\quad & u_t = -u_{xxxx} - \frac{1}{2}u_x^2 + f(t, u_{xx}, u_{xxx}), \\
& T = \partial_x, \quad P = \partial_u, \quad G = -t\partial_x - x\partial_u; \\
5.\quad & u_t = -u_{xxxx} + f(t, u_{xx}, u_{xxx}), \\
& T = \partial_x, \quad P = \partial_u, \quad G = -x\partial_u.
\end{aligned}$$

Here f is an arbitrary real-valued smooth function.

There are three inequivalent equations of the form (1) invariant under the extended classical Galilei algebra $\tilde{g}(1)$:

$$\begin{aligned}
1.\quad & u_t = -u_{xxxx} - u_x^2 + f(u_{xx}, u_{xxx}), \\
& T = \partial_t, \quad P = -\partial_x, \quad M = \frac{1}{2}\partial_u, \quad G = t\partial_x + \frac{1}{2}x\partial_u; \\
2.\quad & u_t = -u_{xxxx} + \frac{1}{2\lambda}x - 4\frac{u_{xxx}}{x} + \frac{1}{x}f(xu_{xx} + 2u_x, xu_{xxx} + 3u_{xx}), \\
& T = \partial_t, \quad X = -\partial_u, \quad M = -\frac{\lambda}{x}\partial_u, \quad G = \lambda\partial_x + (t - \lambda\frac{u}{x})\partial_u\rangle \quad \lambda \neq 0;
\end{aligned}$$

3. $u_t = -u_{xxxx} - \frac{1}{4\lambda^2}x^2 + f(u_{xx}, u_{xxx});$

$$T = \partial_t, \quad P = \frac{1}{2\lambda}x\partial_u, \quad M = \frac{1}{2}\partial_u, \quad G = \lambda\partial_x - \frac{1}{2\lambda}xt\partial_u, \quad \lambda \neq 0.$$

where f is an arbitrary smooth function.

Note that the corresponding symmetry algebras are maximal invariance algebras provided f is arbitrary.

7. Concluding remarks

In this paper, we perform symmetry classification of the fourth-order parabolic equation of the very general form (1). As a result, the broad classes of invariant equations (1) are constructed together with their maximal symmetry algebras. Symmetry properties of these equations can be briefly summarized as follows,

• there are three inequivalent equations admitting one-dimensional Lie algebra.

• two equations which admit semi-simple Lie algebras isomorphic to $\mathfrak{sl}(2, \mathbf{R})$.

• there are nine equations admitting two-dimensional Lie algebras, among them, four equations admitting Abelian algebras and five admitting non-Abelian algebras.

• there are forty-four equations admitting three-dimensional solvable Lie algebras.

• there are fifty-seven equations admitting four-dimensional solvable Lie algebras.

We have also established that the class of evolution equations (1) contains at most eight inequivalent classes of PDEs obeying the Galilei relativity principle. The corresponding equations are presented in the previous section.

Finally we mention that in order to complete group classification of the general class of PDEs one needs to consider solvable Lie algebras of the dimension higher than 4. This work is in progress now and will be reported elsewhere.

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