Higher Conditional Symmetry and Reduction of Initial Value Problems

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Abstract. We give the exposition of a generalized symmetry approach to reduction of initial value problems for nonlinear evolution equations in one spatial variable. Using this approach we classify the initial value problems for third-order evolution equations that admit reduction to Cauchy problems for systems of two ordinary differential equations. These reductions are shown to correspond to higher conditional symmetries admitted by the corresponding nonlinear evolution equations.

Keywords: Conditional symmetry, reduction, nonlinear evolution equation, Cauchy problem.

The idea of dimensional reduction of partial differential equations (PDEs) with the help of the special substitutions (invariant solutions) is the basic idea of application of the methods of Lie group analysis to solving these equations. The requirement of invariance of a given PDE with respect to some Lie transformation group ensures its reducibility. However, there are reductions which cannot be understood within the framework of the traditional Lie approach. The symmetry setting for this kind of reductions (called non-classical [1] or non-Lie [2] ones) has been provided by the concept of conditional symmetry introduced by Fushchych [3]. Recently we proved that the necessary and sufficient condition for reducibility of a given PDE to a differential equation having fewer number of independent variables is that it admits non-trivial conditional symmetry [4, 5].

However, with all the wealth of new possibilities for constructing exact solutions of nonlinear PDEs provided by conditional symmetries, there was an open problem of symmetry interpretation of the so-called "nonlinear separation of variables", introduced by Galaktionov [6, 7] and the "anti-reduction method" suggested by Fushchych [3, 8, 9]. In contrast to the classical reduction algorithm, they are based on reduction of a PDE under study to several (ordinary) differential equations. Consequently, these phenomena in principle cannot be understood within the framework of first-order symmetry formalism. The idea of a symmetry interpretation comes from the traditional variable separation in linear PDEs. Indeed, a possibility of separation of variables in a linear PDE is closely related to its symmetry within the class of second-order differential operators. So one may think about characterizing the methods of nonlinear separation of variables and anti-reduction in terms of higher (Lie-Bäcklund) symmetries. However, the majority of the models studied in [3], [6]–[9] admit no Lie-Bäcklund symmetry. The final solution to this problem has become available after introduction by Fushchych and Zhdanov [9, 10] of the concept of conditional Lie-Bäcklund symmetry (see, also [11]–[14]). Since then, the idea of higher conditional symmetry has been successfully exploited to construct new exact solutions of a number of nonlinear evolution PDEs, that have poor Lie symmetry (see, e.g., [15]–[17] and the references therein).

Recently, we have established the one-to-one correspondence between reducibility of a given evolution PDE in one spatial dimension to several ordinary differential equations and its higher conditional symmetry [18]. Furthermore, it has been suggested to apply this kind of symmetry for dimensional reduction of initial value problems for nonlinear evolution equations [19]. The standard group approach to reduction and solution of initial value problem has been developed by Ovsjannikov [20]. It is based on the concept of invariant solution and by this very reason has rather restricted applicability. Indeed, the requirement for the initial value problem to be invariant with respect to some Lie transformation group can rarely be met, since the group in question must leave invariant not only the PDE itself but also the initial condition. So there is an evident need for more general symmetry techniques enabling to extend the class of initial value problems that admit dimensional reduction. It turns out [18, 19], that higher conditional symmetry is a very efficient tool for treating the problem of dimensional reductions of initial value problems for evolution equations in one spatial variable x

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \tag{1}$$

where $u \in C^n(\mathbf{R^2}, \mathbf{R^1})$, $u_k = \partial^k u/\partial x^k$, $1 \leq k \leq n$.

Using the symmetry approach to dimensional reduction of evolution PDEs suggested in [18], we have classified in [19] initial value problems for nonlinear evolution PDE having quadratic non-linearities

$$u_t = uu_{xx} + \lambda_0 u_x^2 + \lambda_1 uu_x + \lambda_2 u^2 + \mu_0 u_x + \mu_1 u + \mu_2,$$

where $\lambda_0, \ldots, \mu_2$ are constants, that admit reduction to Cauchy problems for some systems of nonlinear ordinary differential equations.

In the present paper we give the self-contained exposition of the symmetry approach to reduction of initial value problems for PDEs (1) developed in [10, 18, 19, 24]. Next, we apply this approach to classify initial value problems

$$u_t = u_{xxx} + F(t,\, x,\, u,\, u_x,\, u_{xx}), \tag{2}$$

$$(\alpha(x)u_1 + \beta(x)u)|_{t=t_0} = \gamma(x), \tag{3}$$

where F, α, β, γ are smooth functions of the given arguments, that admit reduction to Cauchy problems for system of two ordinary differential equations.

1 Theoretical Background

As we have already mentioned in Introduction, reducibility of (1) to a system of ordinary differential equations is in one-to-one correspondence with its invariance under a non-point (Lie-Bäcklund) group of infinitesimal transformations

$$\begin{aligned} u' &= u + \varepsilon\, \eta(t, x, u, u_1, \ldots, u_N), \\ u_1' &= u_1 + \varepsilon\, D_x\, \eta(t, x, u, u_1, \ldots, u_N), \ldots \end{aligned} \tag{4}$$

provided some smoothness requirements are satisfied. This group is generated by the Lie-Bäcklund vector field

$$Q = \sum_{k=0}^{\infty} \left(D_x^k\, \eta\right) \frac{\partial}{\partial u_k} \equiv \eta \frac{\partial}{\partial u} + (D_x\, \eta) \frac{\partial}{\partial u_1} + \left(D_x^2\, \eta\right) \frac{\partial}{\partial u_2} + \cdots. \tag{5}$$

In formulas (4), (5) we use the following notation:

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_x^{j+1} = D_x\,(D_x^j), \quad D_x^0 = 1.$$

Note, that the function η in (5) may also depend on the derivatives of the function u with respect to t. However, we can express all the derivatives of u with respect to t in terms of $t, x, u, u_1, u_2, \ldots$ on the solution manifold of PDE (1). Furthermore, if the function η has the structure

$$\eta = \tilde{\eta}(t, x, u) - \xi_0(t, x, u)u_t - \xi_1(t, x, u)u_x,$$

then Lie-Bäcklund vector field (5) is equivalent to the Lie vector field (see, e.g., [21]):

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}.$$

As the form of a Lie-Bäcklund vector field is fully defined by the form of the coefficient η of $\dfrac{\partial}{\partial u}$, we will use the shorthand notation

$$Q = \eta\frac{\partial}{\partial u} + \dots$$

instead of its full version (5).

Definition 1. *We say that PDE (1) is invariant under the Lie-Bäcklund vector field (5), if the condition*

$$Q(u_t - F)\Big|_{\mathcal{M}} = 0$$

holds, where $\mathcal{M}$ is the set of all differential consequences of the equation $u_t - F = 0$, that is $D_x^j D_t^k (u_t - F) = 0$, $j, k = 0, 1, 2, \dots$.

This definition is a re-formulation of the standard definition of invariance with respect to Lie-Bäcklund group [21]. And now we give its generalization.

Definition 2. *PDE (1) is conditionally invariant under the Lie-Bäcklund vector field (5), if the following condition holds*

$$Q(u_t - F)\Big|_{\mathcal{M}\cap\mathcal{L}_x} = 0. \tag{6}$$

Here the symbol $\mathcal{L}_x$ stands for the set of all differential consequences of the equation $\eta = 0$ with respect to x, that is $D_x^j \eta = 0$, $j = 0, 1, 2, \dots$.

Clearly, the second definition is less restrictive and as such it brings new reduction possibilities. The detailed description of the procedure for calculating Lie-Bäcklund group of PDEs and numerous examples of equations possessing these symmetries can be found in [21]–[23]. The way for obtaining the determining equations that leads to higher conditional symmetries is essentially the same as for usual Lie-Bäcklund symmetries. The only difference is the necessity of taking into account not only the differential consequences of Equation (1) but also the differential consequences $D_x^j \eta = 0$, $j = 1, 2, \dots$ of the side condition $\eta = 0$. This additional restriction, on the one hand, extends considerably the scope of invariant equations (as the number of variables to be split, decreases) but, on the other hand, yields a nonlinear system of determining equations.

Now, we consider the nonlinear PDE

$$\eta(t, x, u, u_1, \ldots, u_N) = 0 \tag{7}$$

as an N-th order ordinary differential equation with respect to variable x. Its general integral can be given (locally) in the form

$$u(t, x) = U\left(t,\, x,\, \varphi_1(t),\, \varphi_2(t), \ldots, \varphi_N(t)\right), \tag{8}$$

where $\varphi_j(t)$, $(j = 1, \ldots N)$ are arbitrary smooth functions. In the following, we call expression (8) the ansatz invariant under Lie-Bäcklund vector field (5). The following result (see [18]) establishes the connection between reducibility of a given PDE (1) to ordinary differential equations and its higher-order conditional symmetry.

Theorem 1. *Let Equation (1) with $F \in C^{N+1}(\mathcal{D})$, where $\mathcal{D}$ is an open domain in $\mathbf{R}^{n+3}$, be conditionally invariant under Lie-Bäcklund vector field (5) with $\eta \in C^{ren2}(\mathcal{D}')$, where $\mathcal{D}'$ is an open domain in $\mathbf{R}^{N+3}$ and $\partial\eta/\partial u_N \neq 0$ on $\mathcal{D}'$. Then ansatz (8) invariant under Lie-Bäcklund vector field (5) reduces PDE (1) to a system of N ordinary differential equations for N functions $\varphi_j(t)$*

$$\dot{\varphi}_j = F_j(t, \varphi_1, \ldots, \varphi_N), \qquad j = 1, \ldots, N. \tag{9}$$

Now suppose the converse. Namely, that ansatz (8), where the function U and its derivatives $\partial U^{k+1}/\partial\varphi_j \partial x^k$, $(j = 1, \ldots, N,\ k = 0, \ldots, N)$ exist and are continuous on an open domain $\mathcal{D}_1$ in $\mathbf{R}^{N+2}$, reduces (1) to a system of ordinary differential Equations (9) with $F_i \in C^1(\mathcal{D}'_1)$, where $\mathcal{D}'_1$ is an open domain in $\mathbf{R}^{N+2}$. Then there exists a Lie-Bäcklund vector field (5) such that Equation (1) is conditionally invariant with respect to it.

Turn now to the initial value problem for the evolution Equation (1)

$$\begin{aligned} &u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \\ &\left(\alpha(x)u_1 + \beta(x)u\right)\big|_{t=t_0} = \gamma(x), \end{aligned} \tag{10}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$ are smooth functions.

According to Ovsjannikov [20], if both the equation and initial condition in (10) are invariant under a one-parameter transformation group, then it is possible to apply the symmetry reduction technique to solve the problem (10). However, a choice of initial conditions, that are invariant with respect to a Lie group can be very limited. Indeed, the transformation group in question must leave invariant the initial surface $t = t_0$, and this imposes rather strong limitations on the exploitation of symmetry reduction in its classical setting.

We will show that using higher-order conditional symmetries enables us to extend the scope of applicability of the idea of dimensional reduction to solving initial value problems for nonlinear PDEs. To this end we rewrite a generator of a one-parameter transformation group

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}$$

as a Lie-Bäcklund field of the form (5), with $\eta = \tilde{\eta}(t,x,u) - \xi_0(t,x,u)u_t - \xi_1(t,x,u)u_1$, and then eliminate all t derivatives with the use of equation (1). Since vector field (5) generates an infinitesimal transformation group of the form (4), the initial surface at $t = t_0$ is invariant with respect to it. So that we can use an arbitrary Lie symmetry for dimensional reduction of the initial value problem (10). The price to be paid for this possibility is that the order of the symmetry operator is no longer equal to one, and the standard procedure of symmetry reduction has to be modified according to [18, 24].

To proceed any further, we need the following definitions.

Definition 3. *Ansatz (8) reduces the initial-value condition*

$$(\alpha(x)u_1 + \beta(x)u)|_{t=t_0} = \gamma(x). \tag{11}$$

if substituting it into (11) yields an expression that vanishes identically when

$$\varphi_i(t_0) = C_i, \quad i = 1, 2, \ldots, N,$$

where $C_1, \ldots, C_N$ are some constants.

Definition 4. *We say that Equation (7), considered as an ordinary differential equation with respect to x, is conditionally invariant under the operator*

$$X = \xi(t,x,u)\frac{\partial}{\partial x} + \zeta(t,x,u)\frac{\partial}{\partial u} \tag{12}$$

with $\xi \neq 0$, if the following relation hold:

$$\widetilde{X}\eta|_{\mathcal{M}} = 0, \tag{13}$$

Here, $\tilde{X}$ is the Nth prolongation of the operator X. The symbol $\mathcal{M}$ stands for the intersection of the surface defined in the space $\mathbf{R}^{N+4}$ of the variables t, x, u, u_t, u_x, ..., u_N by the equation $\eta = 0$ and differential consequences (with respect to x) of the equation $\xi(t,x,u)u_1 - \zeta(t,x,u) = 0$ up to the order $N-1$. When $\xi = 0$, we say that (7) is conditionally invariant under the operator (12), if solving the relation $\zeta(t,x,u) = 0$ with respect to u yields an exact solution of (7).

In [24] we have proved the theorem establishing connection between reducibility of an initial value condition and its conditional symmetry. We give it without proof.

Theorem 2. *Ansatz (8), invariant with respect to Lie-Bäcklund field (5), reduces the initial value condition (11) if and only if*

a) Equation (7) is conditionally invariant under the operator (12) with $\xi = \alpha(x)$, $\zeta = -\beta(x)u + \gamma(x)$, and

b) solution of the system

$$\eta(t, x, u, u_1, \ldots, u_N) = 0, \quad \alpha(x)u_1 + \beta(x)u - \gamma(x) = 0$$

exists and is obtained from the general integral (8) of the equation $\eta = 0$, considered as ordinary differential equation with respect to x.

In view of the above theorem, the necessary condition for the reducibility of a Cauchy problem by solutions invariant under some Lie-Bäcklund field (5) is the conditional invariance of the equation $\eta = 0$ with respect to the operator (12) with $\xi = \alpha(x), \zeta = -\beta(x)u + \gamma(x)$. With this fact in hand we can formulate the three-step symmetry approach to the reduction of initial value problem (10) admitting higher conditional symmetries

- rewrite the symmetry operator in the canonical form (5),
- calculate the classical or conditional symmetry of the equation $\eta = 0$ within the class of operators

$$X = \alpha(x)\frac{\partial}{\partial x} + (-\beta(x)u + \gamma(x))\frac{\partial}{\partial u}, \tag{14}$$

- carry out the reduction of the initial value problem (10) to a Cauchy problem for a system of ordinary differential equations using ansatz (8), invariant with respect to the Lie-Bäcklund vector field (5).

2 Conditionally Invariant Third-Order PDEs

There are three principal restrictions on the scope of applicability of higher conditional symmetries for reduction of initial value problems, namely,

1. Evolution equation under investigation must admit non-trivial higher conditional symmetry.

2. We need to be able to integrate high order nonlinear ordinary differential Equation (7) by quadratures in order to get ansatz (8).

3. We need to ensure that the ordinary differential equation $\eta = 0$ has a non-trivial Lie symmetry.

Clearly, the last two requirements are closely inter-related and this fact is used in [19] to classify second-order evolution PDEs that are conditionally invariant with respect to second-order Lie-Bäcklund vector fields. The idea is to take as η the second-order differential invariants of the N-parameter ($N \geq 2$) transformation groups on plane and then to construct evolution PDEs that are conditionally invariant under the operator $Q = \eta\frac{\partial}{\partial u} + \cdots$. To this end the famous Lie's classification of invariant ordinary differential equations on plane [25, 26] have been exploited.

Here we use the same idea in order to classify initial value problems for third-order PDEs (2) that admit reduction to Cauchy problems for system of two ordinary differential equations.

In Table 1 we present the complete list of invariant real second-order ordinary differential equations, whose invariance algebras have dimension no less than two [25, 26]. Note that a, k are arbitrary real parameters and f is an arbitrary function with $f' \neq 0$. As classification has been done to within an arbitrary reversible transformation of the variables x, y, the equations given in Table 1 are representatives of the conjugacy classes of invariant ordinary differential equations.

We exclude from further consideration case 3, since the symmetry algebra admitted by the corresponding ordinary differential equation within the class (14) is equivalent to that of the more general equation given in case 2 of Table 1. The same argument applies to case 7. Consequently, we will consider the cases 1, 2, 4–6, 8 only.

We take as the function η in operator (5) the expression $y'' - f(x, y, y')$, where f is one of the right-hand sides of equations listed in the second column of Table 1 and make the replacements $y \to u$, $y' \to u_x$ and $y'' \to u_{xx}$. Next, we classify the third-order evolution PDEs (2) admitting the corresponding Lie-Bäcklund vector fields. Remarkably, this programme can be implemented in full generality. Indeed, given the above choice of higher conditional symmetry operators, we can describe the most general PDEs (2) admitting these symmetries. The form of the function F is the same for all the cases, namely:

$$u_t = u_{xxx} + rF_1(t,\, \omega_1,\, \omega_2,\, \omega_3) + pF_2(t,\, \omega_1,\, \omega_2,\, \omega_3) + q.$$

Table 1. Lie's classification of invariant differential equations

No.	Equation	Symmetry algebra
1	$y'' = (f'(y'))^{-1}$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$
2	$y'' = (xf'(y'))^{-1}$	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
3	$y'' = 2(y-x)^{-1}((y')^2 + ay'\sqrt{y'} + y')$	$\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$ $x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$
4	$y'' = ay^{-3}$	$\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$ $x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$
5	$y'' = a\exp(-y')$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$
6	$y'' = a(y')^{\frac{k-2}{k-1}},\ k \neq 1, 2$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$
7	$y'' = a(1 + (y')^2)^{\frac{3}{2}}\exp(k\arctan y')$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y},$ $(kx+y)\frac{\partial}{\partial x} + (ky-x)\frac{\partial}{\partial y}$
8	$y'' = 0$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x},$ $y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$ $xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

In the above formula, F_1, F_2 are arbitrary smooth functions and the forms of the coefficients r, p, q and of the "invariants" $\omega_1, \omega_2, \omega_3$ are given below

1. $r = 1, \quad p = u_x, \quad q = \dfrac{f''(u_x)u_{xx}}{(f'(u_x))^2},$

$$\omega_1 = u - \int^{f(u_x)} g(z)dz, \quad \omega_2 = f(u_x) - x, \quad \omega_3 = u_{xx} - \frac{1}{f'(u_x)};$$

2. $r=1, \quad p=xu_x-u, \quad q=\dfrac{f'(u_x)+xu_{xx}f''(u_x)}{(xf'(u_x))^2},$

$$\omega_1=u-x\exp(-f(u_x))\int^{f(u_x)} g(z)\exp(z)dz, \quad \omega_2=f(u_x)-\ln x,$$

$$\omega_3=u_{xx}-\frac{1}{xf'(u_x)};$$

4. $r=\dfrac{u_x}{u}(a+u^2u_x^2)^{\frac{1}{2}} \quad p=\dfrac{u^3(2xu_x-u)}{(a+u^2u_x^2)^{\frac{3}{2}}} \quad q=\dfrac{3au_x}{u^4},$

$$\omega_1=\frac{2u^3u_x}{a+u^2u_x^2}-2x, \quad \omega_2=\frac{u_x^2u^2-a}{u^2}, \quad \omega_3=u_{xx}-\frac{a}{u^3};$$

5. $r=1, \quad p=u_x, \quad q=a\exp(-u_x)u_{xx},$

$$\omega_1=au+(1-u_x)\exp(u_x), \quad \omega_2=-ax+\exp(u_x),$$

$$\omega_3=u_{xx}-a\exp(-au_x);$$

6.1 $r=1, \quad p=u_x, \quad q=\dfrac{a(2-k)}{k-1}(u_x)^{\frac{1}{1-k}}u_{xx},$

$$\omega_1=u+\frac{1-k}{ak}(u_x)^{\frac{k}{k-1}}, \quad \omega_2=\frac{ax}{1-k}+(u_x)^{\frac{1}{k-1}},$$

$$\omega_3=u_{xx}-a(u_x)^{\frac{k-2}{k-1}}, \quad (k\neq 1,2);$$

6.2 $r=1, \quad p=u_x, \quad q=-2au_xu_{xx},$

$$\omega_1=u-a^{-1}\ln(u_x), \quad \omega_2=ax+(u_x)^{-1}, \quad \omega_3=u_{xx}-au_x^2;$$

8. $r=1, \quad p=x, \quad q=0,$

$$\omega_1=xu_x-u, \quad \omega_2=u_x, \quad \omega_3=u_{xx}.$$

Next, we calculate initial conditions (11) using Theorem 2 and following the procedure described in [24]. First of all, we note that for Equations 1, 2 of Table 2, the condition b of Theorem 2 fails to hold. For this reason, the corresponding initial value problem is not reduced to a Cauchy problem within the framework of our approach. For the remaining cases 4, 5, 6.1, 6.2 and 8 we get the following initial conditions:

Case 4. $(C_1+2C_2x+C_3x^2)u_x(t_0,x)-(C_2+C_3x)u(t_0,x)=0,$
$C_3\neq 0, \quad C_1C_3\neq C_2^2;$

Case 5. $(C_1+C_3x)u_x(t_0,x)-C_3u(t_0,x)=C_2+C_3x, \quad C_3\neq 0;$

Case 6.1. $(C_1+C_3x)u_x(t_0,x)-kC_3u(t_0,x)=C_2, \quad C_3\neq 0;$

Case 6.2. $(C_1+C_3x)u_x(t_0,x)=C_2, \quad C_2\neq 0;$

Case 8. $(C_1 + C_4x + C_6x^2)u_x(t_0, x) - (C_5 + C_6x)u(t_0, x) = C_2 + C_3x,$
$\Delta = C_1C_6 + C_5(C_5 - C_4) \neq 0, \quad \Delta_1 = C_2(C_4 - C_5) - C_1C_3,$
$\Delta_2 = C_2C_6 - C_3C_5.$

In the above formulae $C_1, C_2, \dots, C_6$ are arbitrary real constants satisfying the given constraints. These constraints ensure that the corresponding initial value problem fulfill both the conditions of Theorem 2.

The last step of the reduction algorithm is the construction of ansatzes for the function $u(t, x)$ and reduction of both invariant equation and initial conditions to systems of ordinary differential equations and algebraic equations, correspondingly. We present the list of obtained results in Table 2, where $F_i = F_i(t, \varphi_1(t), \varphi_2(t)), i = 1, 2$, the functions $\varphi_1(t), \varphi_2(t)$ are new dependent variables, the symbol g stands for the converse of the function $f(u_x)$, i.e., $g(f(u_x)) \equiv u_x$ and ODEs is the abbreviation for ordinary differential equations.

Note that the so constructed nonlinear evolution equations, for arbitrary functions F_1, F_2, admit no Lie symmetry. Consequently, the symmetry reduction algorithm cannot be applied to reduce these equations. This means that the reductions obtained here are purely non-Lie reductions and cannot be obtained within the framework of the standard Lie's approach.

3 Concluding Remarks

When classifying conditionally invariant evolution equations we use essentially the famous Lie's classification of invariant differential equations on the plane. In this paper we have restricted our considerations to the second-order differential equations. However the method allows for using higher order invariant ordinary differential equations to generate Lie-Bäcklund vector fields, that lead to reducible initial value problems.

The exposed approach is easily generalized to the case of systems of evolution equations in one spatial dimension. This kind of generalization might be especially efficient for the hydro-dynamic type equations, that often have broad symmetry within the classes of Lie and Lie-Bäcklund vector fields (see, e.g., [27]).

Much more difficult is to modify the approach to make it applicable to the case of initial value problems for multi-dimensional evolution equations. This is a really challenging problem and it certainly deserves further investigation.

These and the related problems are under study now and will be reported in our future publications.

Table 2. *Ansatzes and reduced equations*

No.	Ansatz for $u(t,x)$	ODEs	Cauchy data
1	$\varphi_1(t) + g(x + \varphi_2(t))$	$\varphi_1' = F_1$ $\varphi_1' = F_2$	No reduction
2	$\varphi_1(t) + \exp(-\varphi_2(t))\times$ $\times \int^{\ln(x)+\varphi_2(t)} \exp(y)g(y)dy$	$\varphi_1' = F_1-$ $-\varphi_1 F_2$ $\varphi_2' = F_2$	No reduction
4	$\frac{1}{2\sqrt{\varphi_2(t)}}\Big(4a + \varphi_2^2(t)\times$ $\times(2x + \varphi_1(t))^2\Big)^{\frac{1}{2}}$	$\varphi_1' = 2\varphi_1^{\frac{1}{2}}F_1-$ $-2\varphi_1\varphi_2^{\frac{1}{2}}F_2$ $\varphi_2' = 2\varphi_2^{\frac{1}{2}}F_2$	$\varphi_1(t_0) = \frac{2C_2}{C_3}$ $\varphi_2(t_0) = \frac{C_3\sqrt{a}}{\sqrt{C_1C_3 - C_2^2}}$
5	$\varphi_1(t) + a^{-1}(ax + \varphi_2(t))\times$ $\times(\ln(ax + \varphi_2(t)) - 1)$	$\varphi_1' = F_1$ $\varphi_2' = aF_2$	$\varphi_1(t_0) = \frac{C_2 - C_1}{C_3}$ $\varphi_2(t_0) = \frac{aC_1}{C_3}$
6.1	$\varphi_1(t) + \frac{k-1}{ak}\left(\frac{ax}{k-1} + \varphi_2(t)\right)^k$	$\varphi_1' = F_1$ $\varphi_2' = \frac{a}{k-1}F_2$	$\varphi_1(t_0) = \frac{C_2}{kC_3}$ $\varphi_2(t_0) = \frac{aC_1}{(k-1)C_3}$
6.2	$\varphi_1(t) - a^{-1}\ln(\varphi_2(t) - ax)$	$\varphi_1' = F_1$ $\varphi_2' = -aF_2$	$C_3 = -aC_2$ $\varphi_2(t_0) = \frac{C_1}{C_2}$
8	$\varphi_1(t) + \varphi_2(t)x$	$\varphi_1' = F_1$ $\varphi_2' = F_2$	$\varphi_1(t_0) = \frac{\Delta_1}{\Delta}$ $\varphi_2(t_0) = \frac{\Delta_2}{\Delta}$

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