Classifying evolution equations.

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1 Introduction.

In this short note we give a resumé of some work done on giving a Lie *point symmetry* classification of evolution equations in 1+1 time-space dimensions. Our intention is to give an idea of some of the arguments and the method we use. The interested reader can find full details in ([2]).

The basic idea is to combine the standard Lie algorithm for point symmetries with the equivalence group of the given type of equation in order to give a classification of evolution equations in some canonical form. It should be noted that, for the general case which we consider, the standard Lie algorithm alone fails to provide the results which we are able to obtain. The reason for this will be seen when we give the symmetry condition.

We consider evolution equations of the type

$$u_t = F(t, x, u, u_x)u_{xx} + G(t, u, u_x) \tag{1}$$

where $F(t, x, u, u_x)$ and $G(t, u, u_x)$ are real-valued functions of their arguments. The special case of equation (1) where $F = 1$ is treated in [1].

The first step in our procedure is to set up the symmetry condition for equation (1). Thus, we consider the generator Q of point symmetry transformations of equation (1) given by

$$Q = a(t,x,u)\partial_t + b(t,x,u)\partial_x + c(t,x,u)\partial_u. \tag{2}$$

We then find the following result:

Theorem 1 *If Q is given as in equation (2), then it generates a point symmetry of equation (1) if and only if $a(t,x,u) = a(t)$ is a function of t alone, and if*

$$\begin{aligned} (2D_x b - \dot{a}) &= FaF_t + bF_x + cF_u + (D_x c - u_x D_x b)F_{u_x} \\ c_t - u_x b_t + (c_u - \dot{a} - u_x b_u)G &+ (u_x b_{xx} - c_{xx} - 2u_x c_{ux} - u_x^2 c_{uu} + \\ +2u_x^2 b_{xu} + u_x^3 b_{uu})F &= aG_t + bG_x + cG_u + (D_x c - u_x D_x b)G_{u_x} \end{aligned} \tag{3}$$

where D_x is the operator of total x-differentiation.

As one can see from equation (3), unless F and G are functions of one variable, it is well-nigh impossible to make the usual step of splitting this equation in terms of powers of the derivative u_x. This is where it is necessary to add a further element into the argument. The idea is simple: given that we are not able to derive the Lie invariance algebra by first obtaining the defining equations from (3), we then begin by *specifying* a Lie algebra and then requiring for it to be a symmetry algebra of (1).

Given a Lie algebra, we then look at the possible representations of this Lie algebra within the class of operators of the form given in (2), and it is in this step that we make use of the equivalence group of equation (1). This gives us canonical representations of the symmetry algebra candidates. The final step is to calculate the allowed forms for the functions F and G for a given canonical representation of our chosen Lie algebra. This procedure yields canonical forms of evolution equations which are inequivalent under point transformations of the equivalence group of equation (1). Then, for each such canonical evolution equation, one can calculate the maximal symmetry algebra.

The above method requires a list of inequivalent Lie algebra representations (in terms of commutation relations). All simple (and therefore all semi-simple) finite-dimensional real and complex Lie algebras have been classified ([3], [4]). However, the list of *solvable* Lie algebras is far from complete, and as far as we can ascertain, they are given in the work of Morozov, Mubarakzyanov and Turkowski ([5]–[11])). The work of Morozov and

Mubarakzyanov are, as far as we are aware, not translated from the original Russian. Here, the solvable Lie algebras up to and including dimension six are classified.

Our method is constructive in the sense that we are able to use the low-dimensional Lie algebras up to dimension five in order to give our complete point-symmetry classification, and we conclude that no non-linear evolution equation of the form (1) has a point-symmetry invariance algebra of dimension greater than five. We note that a classification in terms of contact symmetries has been given by Magadeev ([12]).

2 Some previous work.

Here we give a short list of previous work done in this area. First we mention the point-symmetry classification of certain equations with specified nonlinearities.

Ovsjannikov (1959)	$F = F(u)$,	$G = \frac{dF}{du}u_x^2$ [13];
Akhatov et al (1987)	$F = F(u_x)$,	$G = 0$ [19];
Dorodnitsyn (1982)	$F = F(u)$,	$G = \frac{dF}{du}u_x^2 + g(u)$ [14];
Oron & Rosenau (1986), Edwards (1994)	$F = F(u)$,	$G = \frac{dF}{du}u_x^2 + f(u)u_x$ [15, 16];
Gandarias (1996)	$F = u^n$,	$G = \frac{dF}{du}u_x^2 + g(x)u^m u_x$ $+f(x)u^s$ [18];
Cherniha & Serov (1998)	$F = F(u)$,	$G = \frac{dF}{du}u_x^2 + f(u)u_x + g(u)$ [17];
Zhdanov & Lahno (1999)	$F = 1$,	$G = G(t, x, u, u_x)$ [1].

Work on classification using the equivalence group has been done by Torrisi and his co-workers ([20]–[23]). The methods of Torrisi *et al* are based on the infinitesimal representation of the equivalence transformations, in contrast to our approach which involves finite forms of the equivalence transformations.

3 Results.

In this section we give some results and show how the equivalence group of the equation plays its role in the classification of representations of a given Lie algebra. First, we begin with the description of the equivalence group of equation (1).

Lemma 1 *The equivalence group of equation (1) is given by the following group of transformations (changes of variable):*

$$\bar{t} = T(t), \quad \overline{x} = X(t,x,u), \quad \overline{u} = U(t,x,u) \tag{4}$$

where $\dot{T} \neq 0$ *and* $\dfrac{\partial(X,U)}{\partial(x,u)} \neq 0.$

The proof of this is by direct calculation, and details are given in ([2]).

The next step is to give a canonical form (linearization) for a vector field, with respect to the allowed transformations (4). This is given in the following result:

Lemma 2 *The operator*

$$Q = a(t)\partial_t + +b(t,x,u)\partial_x + c(t,x,u)\partial_u$$

is equivalent, under the changes of variable (4) to one of the following forms:

$$Q = \partial_t \quad \textit{if} \;\; a(t) \neq 0$$

or

$$Q = \partial_x \quad \textit{if}\; a(t) = 0.$$

We refer to ([2]) for the proof.

Example: $sl(2,\mathbf{R})$. Now we give an example of the calculations involved. We do this using the Lie algebra $sl(2,\mathbf{R})$ which is the Lie algebra with basis $\langle Q_1, Q_2, Q_3 \rangle$ satisfying the commutation relations

$$[Q_1, Q_2] = 2Q_2, \;\; [Q_1, Q_3] = -2Q_3, \;\; [Q_2, Q_3] = 2Q_1. \tag{5}$$

First, we take one operator from the basis and put it into canonical form. So let us choose Q_3 for this purpose (this is done for purely practical reasons).

By Lemma 2, we can choose Q_3 in one of two canonical forms: $Q_3 = \partial_t$ or $Q_3 = \partial_x$.

We consider in detail the case $Q_3 = \partial_t$. With this choice of Q_3 we proceed to find the allowable forms for Q_1 and Q_2. So, put

$$Q_1 = A(t)\partial_t + B(t, x, u)\partial_x + C(t, x, u)\partial_u.$$

The second commutation relation in (5) then implies that

$$\dot{A} = 2, \ \ \dot{B} = \dot{C} = 0.$$

Therefore, $A(t) = 2t + \text{const.}$ and B, C are independent of t. So we take

$$Q_1 = 2t\partial_t + B(x, u)\partial_x + C(x, u)\partial_u.$$

The next step is to find a canonical form for Q_1 under the equivalence transformations (4). However, we must now use only those equivalence transformations of (4) which preserve the form of $Q_3 = \partial_t$. Thus we require that $Q_3 \rightarrow \overline{Q}_3$ with

$$\overline{Q}_3 = \dot{T}\partial_{\overline{t}} + X_t\partial_{\overline{x}} + U_t\partial_{\overline{u}} = \partial_{\overline{t}}$$

which yields $\dot{T} = 1, \ X_t = U_t = 0$. Thus we take $T(t) = t, \ X = X(x, u), \ U = U(x, u)$. Under this type of transformation we find

$$Q_1 \rightarrow \overline{Q}_1 = 2\overline{t}\partial_{\overline{t}} + (BX_x + CX_u)\partial\overline{x} + (BU_x + CU_u)\partial_{\overline{u}}.$$

We now choose X and U so that

$$BX_x + CX_u = X, \ \ BU_x + CU_u = 0.$$

This gives us the canonical form

$$\overline{Q}_1 = 2\overline{t}\partial_{\overline{t}} + \overline{x}\partial\overline{x}.$$

This means that we can, up to an equivalence transformation of equation (1), take

$$Q_3 = \partial_t, \ \ Q_1 = 2t\partial_t + x\partial_x.$$

Finally, we need to determine Q_2. We put

$$Q_2 = \alpha(t)\partial_t + \beta(t,x,u)\partial_x + \gamma(t,x,u)\partial_u.$$

The commutation relation $[Q_2, Q_3] = Q_1$ gives

$$\alpha = -t^2 + \text{const.}, \quad \beta = -xt + b(x,u), \quad \gamma = \gamma(x,u).$$

Thus we may take

$$Q_2 = -t^2\partial_t + (b(x,u) - xt)\partial_x + \gamma(x,u)\partial_u.$$

Then we use the relation $[Q_1, Q_2] = 2Q_2$. From this we obtain

$$b(x,u) = m(u)x^3, \quad \gamma(x,u) = n(u)x^2$$

and we have

$$Q_2 = -t^2\partial_t + (m(u)x^3 - xt)\partial_x + n(u)x^2\partial_u.$$

All that remains to be done is to find a canonical form for Q_2. We do this using equivalence transformations (4) which leave invariant the form of Q_1, Q_3. These are given by

$$T(t) = t, \quad X(u,x) = q(u)x, \quad U = p(u)$$

with $q(u) \neq 0$ and $\dot{p}(u) \neq 0$. Under this transformation we find

$$\overline{Q}_2 = -\bar{t}^2\partial\bar{t} + \left(\frac{q(u)m(u) + n(u)\dot{q}(u)}{q(u)^3}\overline{x}^3 - \overline{x}\bar{t}\right)\partial_{\overline{x}} + \frac{n(u)\dot{p}(u)}{q(u)^2}\overline{x}^2\partial_{\overline{u}}.$$

There are two cases: $n(u) \neq 0$ and $n(u) = 0$. If $n(u) \neq 0$ then we may choose $p(u)$ and $q(u)$ such that

$$\frac{q(u)m(u) + n(u)\dot{q}(u)}{q(u)^3} = 1, \quad \frac{n(u)\dot{p}(u)}{q(u)^2} = 1$$

which gives us

$$\overline{Q}_2 = -\bar{t}^2\partial\bar{t} + (\overline{x}^3 - \overline{x}\bar{t})\partial_{\overline{x}} + \overline{x}^2\partial_{\overline{u}}.$$

If, however, $n(u) = 0$ then we have

$$\overline{Q}_2 = -\overline{t}^2\partial\overline{t} + \left(\frac{q(u)m(u)+n(u)\dot{q}(u)}{q(u)^3}\overline{x}^3 - \overline{x}\overline{t}\right)\partial_{\overline{x}}.$$

If now $m(u) > 0$ we may choose $m(u) = q(u)^2$ and we find

$$\overline{Q}_2 = -\overline{t}^2\partial\overline{t} + (\overline{x}^3 - \overline{x}\overline{t})\partial_{\overline{x}}.$$

If $m(u) < 0$ then we choose $m(u) = -q(u)^2$ and we find

$$\overline{Q}_2 = -\overline{t}^2\partial\overline{t} + (-\overline{x}^3 - \overline{x}\overline{t})\partial_{\overline{x}}.$$

Finally, we note that $t \to -t$, $x \to x$, $u \to u$ is an equivalence transformation of equation (1), and the last two canonical forms for Q_2 are equivalent under this transformation.

Summarising this calculation, we find that the algebra $sl(2,\mathbf{R})$ has two canonical forms with $Q_3 = \partial_t$:

$$\begin{gathered}\langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + (x^3 - xt)\partial_x,\ \partial_t\rangle,\\ \langle 2t\partial_t + x\partial_x,\ -t^2\partial_t + (x^3 - xt)\partial_x + x^2\partial_u,\ \partial_t\rangle,\end{gathered}$$

and these are inequivalent under the equivalence group given by (4).

The canonical form $Q_3 = \partial_x$ gives rise to a similar calculation, and three inequivalent representations for the Lie algebra $sl(2,\mathbf{R})$ are found. These results are summarised in the following:

Theorem 2 *There exist five inequivalent realizations of the algebra $sl(2,\mathbf{R})$ by operators (2), which are admitted by partial differential equations of the form (1)*

$$\langle 2t\,\partial_t + x\,\partial_x,\ -t^2\,\partial_t - tx\,\partial_x + x^2\,\partial_u,\ \partial_t\rangle, \tag{6}$$

$$\langle 2t\,\partial_t + x\,\partial_x,\ -t^2\,\partial_t + x(x^2 - t)\,\partial_x,\ \partial_t\rangle, \tag{7}$$

$$\langle 2x\,\partial_x - u\,\partial_u, -x^2\,\partial_x + xu\,\partial_u,\ \partial_x\rangle, \tag{8}$$

$$\langle 2x\,\partial_x - u\,\partial_u,\ (u^{-4} - x^2)\,\partial_x + xu\,\partial_u,\ \partial_x\rangle, \tag{9}$$

$$\langle 2x\,\partial_x - u\,\partial_u,\ -(u^{-4} + x^2)\,\partial_x + xu\,\partial_u,\ \partial_x\rangle. \tag{10}$$

The forms of the functions F, G determining the corresponding invariant equations are given in Table 1.

Table 1. *Equations invariant under the group $sl(2,\mathbf{R})$*

$sl(2,\mathbf{R})$	F	G
(6)	$\tilde{F}(\omega)$	$x^{-2}\left[\tilde{G}(\omega)-2u\tilde{F}(\omega)+u^2-u\omega\right]$, $\omega=2u-xu_x$
(7)	ω^{-3}	$x^{-2}\left[-\frac{1}{4}\omega+3\omega^{-2}+\omega^{-1}\tilde{G}(u)\right],\ \omega=xu_x$
(8)	u^{-4}	$-2u^{-5}u_x^2$
(9)	$u^{-4}\left(1+4\omega^2\right)^{-1}$	$u\left[\sqrt{1+4\omega^2}\tilde{G}(t)-\frac{10\omega^2+1}{8\omega^2+2}\right],\ \omega=u^{-3}u_x$
(10)	$u^{-4}\left(1-4\omega^2\right)^{-1}$	$u\left[\sqrt{\lvert 1-4\omega^2\rvert}\tilde{G}(t)+\frac{10\omega^2-1}{8\omega^2-2}\right],\ \omega=u^{-3}u_x$

The other real simple Lie algebra of dimension three is $so(3)$. This Lie algebra yields only one representation which is a symmetry:

Theorem 3 *There exists only one realization of the algebra $so(3)$ by operators of the form (2) which is an invariance algebra of (1):*

$$\langle \partial_x,\ \tan u\,\sin x\,\partial_x+\cos x\,\partial_u,\ \tan u\,\cos x\,\partial_x-\sin x\,\partial_u\rangle, \tag{11}$$

Furthermore, the most general form of the functions F, G allowing for equation (1) to be invariant under the above realization is given by

$$F=\frac{\sec^2 u}{1+\omega^2},\quad G=\frac{2\omega^2+1}{1+\omega^2}\tan u+\sqrt{1+\omega^2}\tilde{G}(t),\quad \omega=u_x\sec u. \tag{12}$$

Provided the function $\tilde{G}$ is arbitrary, the above realization is the maximal symmetry algebra of the corresponding equation (1).

There are of course many more possible semi-simple Lie algebras. However, they play no role as symmetry algebras of equation (1). In fact, one can prove the following:

Theorem 4 *The realizations of the Lie algebra* $so(3)$ *and* $sl(2, \mathbf{R})$ *given in Theorems 2 and 3 above, exhaust all possible realizations of sem-simple Lie algebras (by the given type of vector fields) which are admitted by equations of the form of equation (1).*

The details of the proofs of Theorems 2, 3 and 4 are to be found in ([2]).

4 Comments.

The above results deal with semi-simple Lie algebras. Solvable Lie algebras and semi-direct sums of semi-simple and solvable Lie algebras are treated in ([2]), to which we refer the reader for tables of inequivalent canonical forms of evolution equations (1).

Our approach is, as we have demonstrated, a combination of the Lie point symmetry analysis and an exploitation of the equivalence group of the equation to give possible canonical forms for the various Lie algebras. Each representation of the Lie algebra is then tested as a symmetry algebra, and is discarded if it gives no result (by which we mean, amongst other things, that the equation admitting a given representation as symmetry algebra must have $F \neq 0$ in (1)).

The present method gives a complete point-symmetry classification of (1), so that any evolution equation of that form is necessarily *point-equivalent* under the transformations (3) to one of the canonical forms for equation (1).

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References

[1] Zhdanov, R. and Lahno, V. Group classification of the heat conductivity equations with a nonlinear source. *J. Phys. A.* 1999, **26** 7061-7076.

[2] Basarab-Horwath P., Lahno V. and Zhdanov R. The structure of Lie algebras and the classification problem for partial differential equations. Linköping University preprint LiTH-MAT-R-2000-10. (http://xxx.lanl.gov/math-ph/0005013).

[3] Goto, M. and Grosshans, F.D. Semisimple Lie Algebras. Marcel Dekker (1978).

[4] Helgason, S. Differential Geometry, Lie Groups and Symmetric Spaces. Academic Press (1978).

[5] Morozov V.V. Classification of six-dimensional nilpotent Lie algebras (in Russian), *Izv. Vys. Ucheb. Zaved.*, 1958, no. 5 (5), 161–171.

[6] Mubarakzyanov G.M. On solvable Lie algebras (in Russian), *Izv. Vys. Ucheb. Zaved.*, 1963, no. 1 (32),114–123.

[7] Mubarakzyanov G.M. The classification of the real structure of five-dimensional Lie algebras (in Russian), *Izv. Vys. Ucheb. Zaved.*, 1963, no. 3 (34), 99–105.

[8] Mubarakzyanov G.M. The classification of six-dimensional Lie algebras with one nilpotent basis element (in Russian), *Izv. Vys. Ucheb. Zaved.*, 1963, no. 4 (35), 104–116.

[9] Mubarakzyanov G.M. Some theorems on solvable Lie algebras (in Russian), *Izv. Vys. Ucheb. Zaved.*, 1966, no. 3 (55), 95–98.

[10] Turkowski P. Solvable Lie algebras of dimensional six, *J. Math. Phys.*, 1990, **31**, 1344–1350.

[11] Turkowski P. Low–dimensional real Lie algebras, *J. Math. Phys.*, 1988, **29**, 2139–2144.

[12] Magadeev B.A. On group classification of nonlinear evolution equations (in Russian), *Algebra i Analiz*, 1993, **5**, 141–156.

[13] Ovsiannikov L.V. Group properties of nonlinear heat equation (in Russian), *Dokl. AN SSSR,* 1959, **125**, N3, 492–495.

[14] Dorodnitsyn V.A. On invariant solutions of non-linear heat equation with a source (in Russian), *Zhurn. Vych. Matemat. Matem. Fiziki*, 1982, **22**, 1393–1400.

[15] Oron A. and Rosenau P. Some symmetries of the nonlinear heat and wave equations, *Phys. Lett. A.*, 1986, **118**, 172–176.

[16] Edwards M.P. Classical symmetry reductions of nonlinear diffusion-convection equations, 1994, *Phys. Lett. A*, **190**, 149–154.

[17] Cherniha R. and Serov M. Symmetries, ansätze and exact solutions of nonlinear second-order evolution equations with convection terms, *Euro. J. of Applied Mathematics*, 1998, **9**, 527–542.

[18] Gandarias M.L. Classical point symmetries of a porous medium equation, *J. Phys. A: Math. Gen.*, 1996, **29**, 607–633.

[19] Akhatov I.S., Gazizov R.K. and Ibragimov N.K. Group classification of equations of nonlinear filtration (in Russian), *Proc. Acad. Sci. USSR*, 1987, **293**, 1033–1035.

[20] Torrisi M., Tracina R. and Valenti A. A group analysis approach for a nonlinear differential system arising in diffusion phenomena, *J. Math. Phys.*, 1996, **37**, 4758–4767.

[21] Torrisi M. and Tracina R. Equivalence transformations and symmetries for a heat conduction model, *Int. J. of Non-Linear Mechanics*, 1998, **33**, 473–487.

[22] Ibragimov N.H., Torrisi M. and Valenti A., Preliminary group classification of equation $v_{tt} = f(x, v_x)v_{xx} + g(x, v_x)$, *J. Math. Phys.*, 1991, **32**, 2988–2995.

[23] Ibragimov N.K. and Torrisi M. A simple method for group analysis and its applications to a model of detonation, *J. Math. Phys.*, 1992, **33**, 3931–3937.