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Symmetry classification of KdV-type nonlinear evolution equations

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Group classification of a class of third-order nonlinear evolution equations generalizing KdV and mKdV equations is performed. It is shown that there are two equations admitting simple Lie algebras of dimension three. Next, we prove that there exist only four equations invariant with respect to Lie algebras having nontrivial Levi factors of dimension four and six. Our analysis shows that there are no equations invariant under algebras which are semi-direct sums of Levi factor and radical. Making use of these results we prove that there are three, nine, thirty-eight, fifty-two inequivalent KdV-type nonlinear evolution equations admitting one-, two-, three-, and four-dimensional solvable Lie algebras, respectively. Finally, we perform a complete group classification of the most general linear third-order evolution equation.
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I. INTRODUCTION

The purpose of this article is classifying equations of the form

$$u_t = u_{xxx} + F(x,t,u,u_x,u_{xx}), \tag{1.1}$$

which admit nontrivial Lie (point) symmetries. The standard Korteweg–de Vries (KdV) equation,

$$u_t = u_{xxx} + uu_x,$$

belongs to the family of evolution equations (1.1). Classification of the KdV equation with variable coefficients (vcKdV),

$$u_t = f(x,t)uu_x + g(x,t)u_{xxx}, \quad f\cdot g \neq 0, \tag{1.2}$$

by their symmetries is done in Ref. 1, where it is shown that the vcKdV can admit at most four-dimensional Lie point symmetry group and those having four-dimensional symmetry group can be transformed into the ordinary KdV equation by local point transformations. In Ref. 2, Eq. (1.2) is investigated from the point of view of its integrability. It is shown, in particular, that equations of the form (1.2) with a three-dimensional Lie point symmetry group have a property of "partially integrability."

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Our motivation is the same as for classifying vcKdV equations. We start with a rather general class of nonlinear equations generalizing (1.2) for $g_x = 0$. Note that any t dependent coefficient of u_{xxx} in (1.1) can be normalized by a reparametrization of time. The main advantage of this classification is that, if we know that the equation under study admits a nontrivial symmetry group, then it is usually possible to apply the whole spectrum of the methods and algorithms of Lie group analysis. This enables us to derive exact analytical solutions of equations that, under study, reveal their integrability properties, find linearizing transformations, etc. Note that the connection between Lie point symmetries and integrability was discussed in Refs. 2, 3.

Recently, a novel generic approach to group classification of low-dimensional partial differential equations (PDEs) has been developed in Ref. 4. The full account of ideas and algorithms applied can be found in the review paper[5] where the approach in question has been applied to classify the most general second-order quasi-linear heat-conductivity equations admitting nontrivial Lie point symmetries. Here we adopt the same approach which basically consists of three steps. We first construct the equivalence group, namely, the most general group of point transformations that transform any equation of the form (1.1) to a (possibly different) equation belonging to the same class. Also, we find the most general element of the symmetry group together with a determining equation for F. As a second step, we realize low-dimensional Lie algebras by vector fields of the above form up to equivalence transformations. To this end, we use various results on the structure of abstract Lie algebras.[6–9] A review of the classification results of nonisomorphic finite-dimensional Lie algebras can be found in Ref. 5. In the last step, after transforming symmetry generators to canonical forms, we proceed to classifying equations that admit nontrivial symmetries. We do this by inserting these generators into the symmetry condition and solving for F.

Let us mention that similar ideas have been used by Winternitz and co-workers for the group classification of several nonlinear partial differential equations[1,10,11] and of discrete dynamical systems.[12–14] Note also that group classification of the nonlinear wave and Schrödinger equations in the same spirit has been done in Refs. 15, 16.

The paper is organized as follows. In Sec. II we present the determining equations for the symmetries and the equivalence group. Section III is devoted to the classification of the equations invariant under low-dimensional symmetry groups. In Sec. IV we perform a classification of linear equations in the class (1.1). A discussion of results and some conclusions are presented in the final section.

II. DETERMINING EQUATIONS AND EQUIVALENCE TRANSFORMATIONS

The Lie algebra of the symmetry group of Eq. (1.1) is realized by vector fields of the form

$$X = \tau(x,t,u)\partial_t + \xi(x,t,u)\partial_x + \phi(x,t,u)\partial_u . \tag{2.1}$$

In order to implement the symmetry algorithm we need to calculate the third order prolongation of the field vector field (2.1),[17–19]

$$\mathrm{pr}^{(3)}X = X + \phi^t \partial_{u_t} + \phi^x \partial_{u_x} + \phi^{xx}\partial_{u_{xx}} + \phi^{xxx}\partial_{u_{xxx}}, \tag{2.2}$$

where

$$\phi^t = D_t\phi - u_t D_t\tau - u_x D_t\xi,$$

$$\phi^x = D_x\phi - u_t D_x\tau - u_x D_x\xi,$$

$$\phi^{xx} = D_x\phi^x - u_{xt} D_x\tau - u_{xx} D_x\xi,$$

$$\phi^{xxx} = D_x\phi^{xx} - u_{xxt} D_x\tau - u_{xxx} D_x\xi.$$

Here D_x and D_t denote the total space and time derivatives. In order to find the coefficients of the vector field we require that the prolonged vector field (2.2) annihilate Eq. (1.1) on its solution manifold,

$$\mathrm{pr}^{(3)}X(\Delta)\big|_{\Delta=0}=0, \quad \Delta=u_t-u_{xxx}-F. \tag{2.3}$$

Equating coefficients of linearly independent terms of invariance criterion (2.3) to zero yields an overdetermined system of linear PDEs (called determining equations). Solving this system we obtain the following assertion.

Proposition 2.1: The symmetry group of the nonlinear equation (1.1) for an arbitrary (fixed) function F is generated by the vector field

$$X=\tau(t)\partial_t+\left(\frac{\dot{\tau}}{3}x+\rho(t)\right)\partial_x+\phi(x,t,u)\partial_u, \tag{2.4}$$

where the functions $\tau(t)$, $\rho(t)$ and $\phi(x,t,u)$ satisfy the determining equation

$$\begin{aligned}&-3\,u_x\,\dot{\rho}-x\,u_x\,\ddot{\tau}-9\,u_x\,u_{xx}\,\phi_{uu}-3\,u_x^3\,\phi_{uuu}+3\,\phi_t-9\,u_{xx}\,\phi_{xu}-9\,u_x^2\,\phi_{xuu}-9\,u_x\,\phi_{xxu}-3\,\phi_{xxx}\\ &\quad+3(\phi_u-\dot{\tau})\,F+(2\,u_{xx}\,\dot{\tau}-3\,u_{xx}\,\phi_u-3\,u_x^2\,\phi_{uu}-6\,u_x\,\phi_{xu}-3\,\phi_{xx})\,F_{u_{xx}}\\ &\quad+(u_x\,\dot{\tau}-3\,u_x\,\phi_u-3\,\phi_x)\,F_{u_x}-3\,\phi\,F_u-3\,\tau\,F_t-(3\,\rho+x\,\dot{\tau})\,F_x=0.\end{aligned} \tag{2.5}$$

Here the dot over a symbol stands for the time derivative.

If there are no restrictions on F, then (2.5) should be satisfied identically, which is possible only when the symmetry group is a trivial group of identity transformations. Here we shall be concerned with the identification of all specific forms of F for which nontrivial symmetry groups occur. The basic idea is to utilize the fact that for an arbitrarily fixed function F all admissible vector fields form a Lie algebra. This immediately implies the idea of using the classical results on the classification of low-dimensional Lie algebras obtained mostly in the late 1960s.[6–8] Saying it another way, we need to construct a kind of representation theory on low-dimensional Lie algebras generated by Lie vector fields preserving the manifold (2.5).

Our classification is up to equivalence under a group of locally invertible point transformations,

$$\tilde{t}=T(x,t,u), \quad \tilde{x}=Y(x,t,u), \quad \tilde{u}=U(x,t,u), \tag{2.6}$$

that preserve the form of the equation (1.1), but (possibly) change function F into a new one, namely, we have

$$\tilde{u}_{\tilde{t}}=\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}+\tilde{F}(\tilde{x},\tilde{t},\tilde{u},\tilde{u}_{\tilde{x}},\tilde{u}_{\tilde{x}\tilde{x}}). \tag{2.7}$$

Inserting (2.6) into (1.1) and requiring that the form of the equation be preserved, we arrive at the following assertion.

Proposition 2.2: The maximal equivalence group $\mathcal{E}$ has the form

$$\tilde{t}=T(t), \quad \tilde{x}=\dot{T}^{1/3}x+Y(t), \quad \tilde{u}=U(x,t,u), \tag{2.8}$$

where $\dot{T}\neq 0$, $U_u\neq 0$.

We note that the Lie infinitesimal technique can also be used to obtain the equivalence group (2.8). It is straightforward to prove that both approaches produce the same results.

We make use of equivalence transformations (2.8) to transform vector field X into a convenient (canonical) form.

Proposition 2.3: Vector field (2.4) *is equivalent within a point transformation of the form* (2.6) *to one of the following vector fields:*

$$X=\partial_t,\quad X=\partial_x,\quad X=\partial_u. \tag{2.9}$$

Proof: Transformation (2.8) transforms vector field (2.4) into

$$X\to\tilde{X}=\tau(t)\dot{T}(t)\partial_{\tilde{t}}+[\tfrac{1}{3}(\tau\dot{T}^{-1}\ddot{T}+\dot{\tau})(\tilde{x}-Y)+\tau\dot{Y}+\rho\dot{T}^{1/3}]\partial_{\tilde{x}}$$
$$+[\tau U_t+(\tfrac{1}{3}\dot{\tau}x+\rho)U_x+\phi U_u]\partial_{\tilde{u}}. \tag{2.10}$$

There are two cases to consider.

(I) $\phi=0$. Choose $U=U(u)$ so that we have

$$\tilde{X}=\tau(t)\dot{T}(t)\partial_{\tilde{t}}+[\tfrac{1}{3}(\tau\dot{T}^{-1}\ddot{T}+\dot{\tau})(\tilde{x}-Y)+\tau\dot{Y}+\rho\dot{T}^{1/3}]\partial_{\tilde{x}}+\phi U_u\partial_{\tilde{u}}. \tag{2.11}$$

Now if $\tau=0$, then $\rho\neq 0$ (otherwise X would be zero), and we choose $T(t)$ to satisfy

$$\dot{T}=\rho^{-3}.$$

In this case $\tilde{X}$ is transformed into $\partial_{\tilde{x}}$.

If $\tau\neq 0$, then we choose T and Y to satisfy

$$\dot{T}=\tau^{-1},\quad \tau\dot{Y}+\rho\dot{T}^{1/3}=0.$$

With this choice of T and Y vector field $\tilde{X}$ is transformed into $\partial_{\tilde{t}}$.

(II) $\phi\neq 0$. If $\tau=\rho=0$ then we can choose U to satisfy $\phi U_u=1$ so that we have $\tilde{X}=\partial_{\tilde{u}}$. Otherwise, U can be chosen to satisfy

$$\tau U_t+(\tfrac{1}{3}\dot{\tau}x+\rho)U_x+\phi U_u=0.$$

Hence we recover Case I.

Summing up, the vector field (2.4) is equivalent, up to equivalence under $\mathcal{E}$, to one of the three standard vector fields $\partial_x,\partial_t,\partial_u$. This completes the proof.

III. GROUP CLASSIFICATION OF LINEAR EQUATIONS

To the best of our knowledge no group classification of the most general linear third-order PDE appears in the literature. So we devote this section to the group classification of third-order PDEs:

$$u_t=f_1(x,t)u_{xxx}+f_2(x,t)u_{xx}+f_3(x,t)u_x+f_4(x,t)u+f_5(x,t). \tag{3.1}$$

If we perform the local change of variables $(x,t,u)\to(\tilde{x},\tilde{t},\tilde{u})$ preserving the form of (3.1),

$$\tilde{t}=t,\quad \tilde{x}=F(x,t),\quad u=V(x,t)v(\tilde{x},\tilde{t})+G(x,t),\quad V\neq 0,\ F_x\neq 0, \tag{3.2}$$

we obtain

$$v_{\tilde{t}}=f_1F_x^3v_{\tilde{x}\tilde{x}\tilde{x}}+\{3f_1V^{-1}[V_xF_x^2+VF_xF_{xx}]+f_2F_x^2\}v_{\tilde{x}\tilde{x}}+\{f_1V^{-1}[3V_{xx}F_x+3V_xF_{xx}+VF_{xxx}]$$
$$+f_2V^{-1}[2V_xF_x+Vf_{xx}]+f_3F_x-F_t\}v_{\tilde{x}}+\{f_1V^{-1}V_{xxx}+f_2V^{-1}V_{xx}+f_3V^{-1}V_x+f_4-V^{-1}V_t\}v$$
$$+V^{-1}[f_1G_{xxx}+f_2G_{xx}+f_3G_x+f_4G+f_5-G_t].$$

Now we choose the functions $F,V,$ and G in (3.2) to satisfy constraints,

$$f_1F_x^3=1,$$

$$G_t=f_1G_{xxx}+f_2G_{xx}+f_3G_x+f_4G+f_5,$$

$$3f_1F_x^2V_x+[3f_1F_xF_{xx}+f_2F_x^2]V=0,$$

and thus normalize $f_1(x,t)\to 1$, and set $f_2(x,t)\to 0$, $f_5(x,t)\to 0$.

Thus (3.1) reduces to the following particular form:

$$u_t=u_{xxx}+A(x,t)u_x+B(x,t)u. \tag{3.3}$$

Here A,B are arbitrary smooth functions of x and t.

The most general equivalence transformation preserving the class of equations (3.3), which is a subset of (2.8), reads as

$$\tilde{t}=T(t),\quad \tilde{x}=\dot{T}^{1/3}x+Y(t),\quad \tilde{u}=V(t)u, \tag{3.4}$$

with $\dot{T}\neq 0$, $V\neq 0$.

Performing change of variables (3.4) transforms Eq. (3.3) to become

$$\tilde{u}_{\tilde{t}}=\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}+\tilde{A}\tilde{u}_{\tilde{x}}+\tilde{B}\tilde{u}, \tag{3.5}$$

where the coefficients $\tilde{A},\tilde{B}$ are expressed in terms of the functions A,B and their derivatives as follows:

$$\begin{aligned}\tilde{A}&=\dot{T}^{-1}(A\dot{T}^{1/3}-\tfrac{1}{3}\ddot{T}\dot{T}^{-2/3}x-\dot{Y}),\\ \tilde{B}&=\dot{T}^{-1}(B+V^{-1}\dot{V}).\end{aligned} \tag{3.6}$$

As Eq. (3.3) is linear, it admits trivial infinite-parameter group having the generator

$$X(\beta)=\beta(x,t)\partial_u,\quad \beta_t=\beta_{xxx}+A\beta_x+B\beta,$$

and the one-parameter group generated by the operator $u\partial_u$. These symmetries give no nontrivial information about the solution structure of the equation under study and therefore are neglected in the sequel.

The nontrivial invariance group of Eq. (3.3) is generated by operators of the form

$$X=\tau(t)\partial_t+(\tfrac{1}{3}\dot{\tau}x+\rho(t))\partial_x+\alpha(t)u\partial_u, \tag{3.7}$$

functions τ,ρ,α,A and B satisfying equations

$$\begin{aligned}&3\dot{\alpha}-3B\dot{\tau}-3\tau B_t-B_x(3\rho+x\dot{\tau})=0,\\ &-3\dot{\rho}-x\ddot{\tau}-2A\dot{\tau}-3\tau A_t-A_x(3\rho+x\dot{\tau})=0.\end{aligned} \tag{3.8}$$

Provided $A=A(x,t)$, $B=B(x,t)$ are arbitrary functions, $\tau=\rho=0$, $\dot{\alpha}=0$. So in this case Eq. (3.3) admits trivial symmetries only.

Transformation (3.4) leaves operator $X_1=u\partial_u$ invariant while transforming operator (3.5) to become

$$X\xrightarrow{(3.4)}\tilde{X}=\tau\dot{T}\partial_{\tilde{t}}+[\tau(\tfrac{1}{3}\ddot{T}\dot{T}^{-2/3}x+\dot{Y})+\dot{T}^{1/3}(\tfrac{1}{3}\dot{\tau}x+\rho)]\partial_{\tilde{x}}+(\tau\dot{V}+\alpha V)u\partial_{\tilde{u}}. \tag{3.9}$$

That is why, if $\tau\neq 0$ in (3.7), then putting

$$\dot{T}=\tau^{-1}, \quad Y=-\int^{t}\rho(\xi)\tau^{-4/3}(\xi)d\xi,$$

and taking V as a nonzero solution of the equation

$$\tau\dot{V}+\alpha V=0,$$

in (3.4) transforms (3.9) to the canonical form of the generator of time displacements

$$\tilde{X}=\partial_{\tilde{t}}.$$

Next, if $\tau=0$, $\rho\neq 0$ in (3.7), then putting $\dot{T}=\rho^{-3}$ in (3.4) yields the operator

$$\tilde{X}=\partial_{\tilde{x}}+\alpha\tilde{u}\partial_{\tilde{u}}.$$

Finally, if $\tau=\rho=0$, $\dot{\alpha}\neq 0$ in (3.7), we put $T=\alpha$ in (3.4) thus getting the operator

$$\tilde{X}=\tilde{t}\tilde{u}\partial_{\tilde{u}}.$$

Taking into account the above considerations, we see that there are transformations (3.4), that transform operator (3.7) to one of the following inequivalent forms:

$$\partial_t, \quad \partial_x, \quad \partial_x+f(t)u\partial_u \ \ (\dot{f}\neq 0), \quad tu\partial_u.$$

In what follows, we analyze each of the above operators separately.

Operator $X_1=\partial_t$. The system of determining Eqs. (3.8) for this operator reads as

$$B_t=A_t=0,$$

whence it follows that $A=A(x)$, $B=B(x)$. Inserting these functions into (3.8) yields

$$3\dot{\alpha}-3B\dot{\tau}-B_x(3\rho+x\dot{\tau})=0,$$

$$-3\dot{\rho}-x\ddot{\tau}-2A\dot{\tau}-A_x(3\rho+x\dot{\tau})=0.$$

Analyzing the above system of ordinary differential equations shows that for the case under consideration Eq. (3.3) admits an invariance group whose dimension is higher than one if and only if the following occurs.

(1) $A=mx^{-2}$, $B=nx^{-3}$, $|m|+|n|\neq 0$ with the additional symmetry operator $t\partial_t+\frac{1}{3}x\partial_x$;
(2) $A=0$, $B=\varepsilon x$ ($\varepsilon=\pm 1$) with the additional symmetry operator $\partial_t+\varepsilon tu\partial_u$;
(3) $A=B\equiv 0$ with the additional symmetry operators $\partial_x, t\partial_t+\frac{1}{3}x\partial_x$.

Operator $X_2=\partial_x$. If Eq. (3.3) is invariant under X_2, then $A=A(t)$, $B=B(t)$. What is more, it follows from (3.6) that there are transformations (3.4), which reduce equation (3.3) to the form (3.5) with $\tilde{A}=\tilde{B}\equiv 0$. So we arrive at the already known case.

Operator $X_3=\partial_x+f(t)u\partial_u$ ($\dot{f}\neq 0$). If Eq. (3.3) admits operator X_3, then we have $A=0$, $B=\dot{f}x$. Inserting these expressions into (3.8) yields

$$\dot{\rho}=0, \quad \ddot{\tau}=0, \quad \dot{\alpha}=\rho\dot{f},$$
$$3\tau\ddot{f}+4\dot{\tau}\dot{f}=0. \tag{3.10}$$

From the first three equations it follows that $\rho=C_1$, $\tau=C_2t+C_3$, $\alpha=C_1f+C_4$, $C_1,C_2,C_3,C_4\in\mathbb{R}$. Hence we conclude that the last equation of system (3.10) takes the form

TABLE I. Symmetry classification of 3.3.

N	A	B	Symmetry operators
1	$A(x)$	$B(x)$	∂_t
2	0	$\dot{f}(t)x$	$\partial_x+f(t)u\partial_u$, $\dot{f}\neq 0$
3	mx^{-2}, $m\in\mathbb{R}$	nx^{-3}, $n\in\mathbb{R}$, $\lvert m\rvert+\lvert n\rvert\neq 0$	$\partial_t, t\partial_t+\frac{1}{3}x\partial_x$
4	0	εx, $\varepsilon=\pm 1$	$\partial_t, \partial_x+\varepsilon tu\partial_u$
5	0	$-mt^{-4/3}x$, $m\in\mathbb{R}$, $m\neq 0$	$\partial_x+3mt^{-1/3}u\partial_u$, $t\partial_t+\frac{1}{3}x\partial_x$
6	$a\in\mathbb{R}$	0	$\partial_t, \partial_x, t\partial_t+\frac{1}{3}(x-2at)\partial_x$

$$3(C_2t+C_3)\ddot{f}+4C_2\dot{f}=0.$$

Analyzing this equation we see that extension of the symmetry algebra of Eq. (3.3) with $A=0$, $B=\dot{f}x$ is only possible when

$$f=3mt^{-1/3}, \quad m\neq 0;$$

$$f=\varepsilon t, \quad \varepsilon=\pm 1.$$

The second case has already been considered. In the first case the basis of nontrivial invariance algebra is formed by the operators $\partial_x+3mt^{-1/3}u\partial_u$, $t\partial_t+\frac{1}{3}x\partial_x$.

Operator $X_4=tu\partial_u$. Inserting the coefficients of this operator into (3.8) leads to the contradiction $3=0$, whence it follows that the operator X_4 cannot be a symmetry operator of Eq. (3.3).

We summarize the above classification results of in Table I, where we give the forms of the functions A and B and basis operators of the nontrivial symmetry algebras of the corresponding equations (3.3).

So the equation $u_t=u_{xxx}$ has the highest symmetry within the class of equations (3.3). Its maximal finite-dimensional symmetry algebra is four-dimensional.

Note that according to Ref. 5 the class of nonlinear equations of the form

$$u_t=F(t,x,u,u_x)u_{xx}+G(t,x,u,u_x), \quad F\neq 0, \tag{3.11}$$

contains five nonlinear equations admitting five-dimensional symmetry algebras. Furthermore, an equation admitting six-dimensional symmetry algebra is equivalent to the heat equation. It is the linear heat conductivity equation $u_t=u_{xx}$ that possess the largest symmetry group within the class of second-order equations (3.11).

This is not the case for the class of third-order PDEs under consideration in the present paper. We shall see that there are examples of nonlinear equations that admits higher symmetry algebras than does the linear equation. For instance, the nonlinear Schwarzian KdV equation (4.14) admits a six-dimensional symmetry algebra.

IV. CLASSIFICATION OF EQUATIONS INVARIANT UNDER SEMI-SIMPLE ALGEBRAS AND ALGEBRAS HAVING NONTRIVIAL LEVI DECOMPOSITIONS

In order to describe equations (1.1) that admit Lie algebras isomorphic to the Lie algebras having nontrivial Levi decomposition, we need, first of all, to describe equations whose invariance algebras are semi-simple.

The lowest order semi-simple Lie algebras are isomorphic to one of the following three-dimensional algebras:

$$\mathrm{sl}(2,\mathbb{R}):[X_1,X_3]=-2X_2,\quad [X_1,X_2]=X_1,\quad [X_2,X_3]=X_3;$$

$$\mathrm{so}(3):[X_1,X_2]=X_3,\quad [X_2,X_3]=X_1,\quad [X_3,X_1]=X_2.$$

Taking into account our preliminary classification we conclude that one of the basis operators reduces to one of the canonical forms $\partial_t, \partial_x, \partial_u$.

First, we study realizations of the algebra so(3) within the class of operators (2.4).

Let $X_1=\partial_t$ and let the operators X_2, X_3 be of the form (2.4). Checking commutation relations $[X_1,X_2]=X_3, [X_3,X_1]=X_2$ we see that

$$X_2=3\alpha\cos t\partial_t+[-\alpha x\sin t+\beta\cos(t+\gamma)]\partial_x+\varphi(x,u)\cos(t+\psi(x,u))\partial_u,$$

$$X_3=-3\alpha\sin t\partial_t-[\alpha x\cos t+\beta\sin(t+\gamma)]\partial_x-\varphi(x,u)\sin(t+\psi(x,u))\partial_u.$$

Here α, β, γ are arbitrary real constants and φ, ψ are arbitrary real-valued smooth functions.

The third commutation relation $[X_2,X_3]=X_1$ implies that $9\alpha^2=-1$. As this equation has no real solutions, there are no realizations of so(3) with $X_1=\partial_t$.

The same assertion holds for the cases when $X_1=\partial_x$ and $X_1=\partial_u$. So the class of operators (2.4) contains no realizations of the algebra so(3). This means that there are no so(3)-invariant equations of the form (1.1).

Theorem 4.1: *There exist no realizations of the algebra* so(3) *in terms of vector fields* (2.4). *Hence no equation of the form* (1.1) *is invariant under* so(3) *algebra.*

Similar reasoning yields that there are three inequivalent realizations of the algebra sl(2,$\mathbb{R}$) by operators of the form (2.4),

$$\{\partial_t, t\partial_t+\tfrac{1}{3}x\partial_x, -t^2\partial_t-\tfrac{2}{3}tx\partial_x\},$$

$$\{\partial_t, t\partial_t+\tfrac{1}{3}x\partial_x, -t^2\partial_t-\tfrac{2}{3}tx\partial_x-x^3\partial_u\},$$

$$\{\partial_u, u\partial_u, -u^2\partial_u\}.$$

Inserting the coefficients of basis operators of the first realization of the algebra sl(2,$\mathbb{R}$) into invariance criterion yields the following classifying equations:

$$2u_{xx}F_{u_{xx}}+u_xF_{u_x}-xF_x-3F=0,$$

$$t(2u_{xx}F_{u_{xx}}+u_xF_{u_x}-xF_x-3F)-xu_x=0,$$

from which we get the equation $xu_x=0$. Consequently, the realization in question cannot be invariance algebra of the equation under study.

The two remaining realizations of sl(2,$\mathbb{R}$) do yield invariance algebras of equation (1.1). The forms of the function F in the corresponding invariant equations read as

$$\{\partial_t, t\partial_t+\tfrac{1}{3}x\partial_x, -t^2\partial_t-\tfrac{2}{3}tx\partial_x-x^3\partial_u\}\quad:\quad F=-x^{-3}[2xu_x+\tfrac{1}{9}x^2u_x^2-G(\omega_1,\omega_2)],$$

$$\omega_1=3u-xu_x,\quad \omega_2=6u-x^2u_{xx};$$

$$\{\partial_u, u\partial_u, -u^2\partial_u\}\quad:\quad F=-\tfrac{3}{2}u_x^{-1}u_{xx}^2+u_xG(x,t).$$

As any semi-simple or simple algebra contains either so(3) or sl(2,$\mathbb{R}$) (or both) as subalgebra(s),[20] the above result can be utilized to perform the classification of equations (1.1) admitting invariance algebras isomorphic to one having a nontrivial Levi decomposition.

First we turn to the equation

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2 + u_x G(x,t). \tag{4.1}$$

Applying the Lie infinitesimal algorithm we see that the maximal invariance algebra of Eq. (4.1) is spanned by the operators $X_1 = \partial_u$, $X_2 = u\partial_u$, $X_3 = -u^2\partial_u$, and

$$X_4 = \tau(t)\partial_t + (\tfrac{1}{3}\dot{\tau}x + \rho(t))\partial_x, \tag{4.2}$$

functions τ, ρ and G satisfying the equation

$$(x\dot{\tau} + 3\rho)G_x + 3\tau G_t + 2\dot{\tau}G + x\ddot{\tau} + 3\dot{\rho} = 0. \tag{4.3}$$

By direct verification we ensure that the form of basis operators of the realization of sl$(2,\mathbb{R})$ under study is not altered by the transformations

$$\tilde{t} = T(t), \quad \tilde{x} = \dot{T}^{1/3}x + Y(t), \quad \tilde{u} = \gamma u, \quad \dot{T} \neq 0, \quad \gamma \neq 0. \tag{4.4}$$

As transformation (4.4) reduces (4.2) to the form

$$X_4 \xrightarrow{(4.4)} \tilde{X}_4 = \tau(t)\dot{T}(t)\partial_{\tilde{t}} + [\tfrac{1}{3}(\tau\dot{T}^{-1}\ddot{T} + \tau)(\tilde{x} - Y) + \tau\dot{Y} + \rho\dot{T}^{1/3}]\partial_{\tilde{x}},$$

we can put $X_4 = \partial_t$ or $X_4 = \partial_x$ within the equivalence relation.

Provided $X_4 = \partial_t$, it follows from (4.3) that $G = \tilde{G}(x)$ in (4.1). Next, if $X_4 = \partial_x$, then necessarily $G = \tilde{G}(t)$. Consequently, the class of Eqs. (4.1) contains two inequivalent equations:

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2 + u_x \tilde{G}(x) \tag{4.5}$$

and

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2 + u_x \tilde{G}(t), \tag{4.6}$$

which are invariant under extensions of the algebra sl$(2,\mathbb{R})$. Namely, they admit algebras sl$(2,\mathbb{R}) \oplus \{\partial_t\}$ and sl$(2,\mathbb{R}) \oplus \{\partial_x\}$, correspondingly. What is more, if the function $\tilde{G} = \tilde{G}(x)$ in (4.5) is arbitrary, the given algebra is maximal (in Lie sense) invariance algebra of Eq. (4.5).

Equation (4.6) is reduced to PDE (4.5) with $\tilde{G}(x) = 0$ with the help of the change of variables,

$$\tilde{t} = t, \quad \tilde{x} = x + \int^t \tilde{G}(\xi) \, d\xi, \quad u = v(\tilde{x}, \tilde{t}).$$

Therefore, we can restrict our further considerations to Eq. (4.5), where we need to differentiate between the cases $\tilde{G} = 0$ and $\tilde{G} \neq 0$.

Classifying Eq. (4.3) with $G = \tilde{G}(x)$ reads as

$$(x\dot{\tau} + 3\rho)\tilde{G}_x + 2\dot{\tau}\tilde{G} + x\ddot{\tau} + 3\dot{\rho} = 0.$$

Hence it follows that there are two cases providing for extension of the symmetry algebra. Namely, the case when $\tilde{G} = 0$, which gives rise to two additional symmetry operators $X_5 = t\partial_t + \frac{1}{3}x\partial_x$ and $X_6 = \partial_x$. Another case of the extension of symmetry of Eq. (4.5) is when $\tilde{G} = \lambda x^{-2}$ $(\lambda \neq 0)$. If this is the case, (4.5) admits the additional operator $X_5 = t\partial_t + \frac{1}{3}x\partial_x$.

Now we turn to the equation

$$u_t=u_{xxx}-2x^{-2}u_x-\tfrac{1}{9}x^{-1}u_x^2+x^{-3}G(\omega_1,\omega_2),$$
$$\omega_1=3u-xu_x,\quad \omega_2=6u-x^2u_{xx}. \tag{4.7}$$

First of all, we ensure that the class of PDEs (4.7) does not contain equations whose invariance algebras possess semi-simple subalgebras of the dimension $n>3$.

It is common knowledge[20] that there are four types of abstract simple Lie algebras over the field of real numbers:

- The type A_{n-1} $(n>1)$ contains four real forms of the algebras $\mathrm{sl}(n,\mathbb{C})$: $\mathrm{su}(n)$, $\mathrm{sl}(n,\mathbb{R})$, $\mathrm{su}(p,q)$ $(p+q=n,p\geqslant q)$, $\mathrm{su}^*(2n)$.
- The type B_n $(n>1)$ contains two real forms of the algebra $\mathrm{so}(2n+1,\mathbb{C})$: $\mathrm{so}(2n+1)$, $\mathrm{so}(p,q)$ $(p+q=2n+1,p>q)$.
- The type C_n $(n\geqslant 1)$ contains three real forms of the algebra $\mathrm{sp}(n,\mathbb{C})$: $\mathrm{sp}(n)$, $\mathrm{sp}(n,\mathbb{R})$, $\mathrm{sp}(p,q)$ $(p+q=n,p\geqslant q)$.
- The type D_n $(n>1)$ contains three real forms of the algebra $\mathrm{so}(2n,\mathbb{C})$: $\mathrm{so}(2n)$, $\mathrm{so}(p,q)$ $(p+q=2n,\ p\geqslant q)$, $\mathrm{so}^*(2n)$.

The lowest order classical semi-simple Lie algebras are three-dimensional. The next admissible dimension for classical semi-simple Lie algebras is six. There are four nonisomorphic semi-simple Lie algebras: so(4), so(3,1), so(2,2) and $\mathrm{so}^*(4)$. As $\mathrm{so}(4)=\mathrm{so}(3)\oplus\mathrm{so}(3)$, $\mathrm{so}^*(4)\sim\mathrm{so}(3)\oplus\mathrm{sl}(2,\mathbb{R})$, and the algebra so(3,1) contains so(3) as a subalgebra, the algebra so(2,2) is the only possible six-dimensional semi-simple algebra that might be invariance algebra of Eq. (4.7). Taking into account that $\mathrm{so}(2,2)\sim\mathrm{sl}(2,\mathbb{R})\oplus\mathrm{sl}(2,\mathbb{R})$ and choosing $\mathrm{so}(2,2)=\{X_1,X_2,X_3\}\oplus\{\tilde{X}_1,\tilde{X}_2,\tilde{X}_3\}$, where X_1,X_2,X_3 form a basis of $\mathrm{sl}(2,\mathbb{R})$, which is invariance algebra of (4.7) and $\tilde{X}_1,\tilde{X}_2,\tilde{X}_3$ are of the form (2.4), we require the commutation relations

$$[X_i,\tilde{X}_j]=0\quad (i,j=1,2,3)$$

to hold, whence

$$\tilde{X}_j=\lambda_j\partial_u\quad (j=1,2,3),$$

where λ_j are arbitrary real constants. Hence we conclude that the class of operators (2.4) does not contain a realization of so(2,2).

The same result holds for eight-dimensional semi-simple Lie algebras $\mathrm{sl}(3,\mathbb{R})$, su(3), su(2,1).

As $\mathrm{su}^*(4)\sim\mathrm{so}(5,1)$ and the algebra so(5,1) contains so(4) as a subalgebra, the class of operators (2.4) contains no realizations of A_n and D_n $(n>1)$ type algebras that are inequivalent to the algebra $\mathrm{sl}(2,\mathbb{R})$.

The same assertion holds true for B_n $(n>1)$ and C_n $(n\geqslant 1)$ type Lie algebras. Indeed, B_2 type algebras contain so(4) and so(3,1) and what is more,

$$\mathrm{sp}(2,\mathbb{R})\sim\mathrm{so}(3,2)\supset\mathrm{so}(3,1),\quad \mathrm{sp}(1,1)\sim\mathrm{so}(4,1)\supset\mathrm{so}(4),\quad \mathrm{sp}(2)\sim\mathrm{so}(5)\supset\mathrm{so}(4).$$

What remains to be done is to consider the exceptional semi-simple Lie algebras that belong to one of the following five types:[20] G_1,F_4,E_6,E_7,E_8. We consider in some detail G_1 type Lie algebras.

The type G_1 contains one compact real form g_2 and one noncompact real form g_2'. As $g_2\cap g_2'\sim\mathrm{su}(2)\oplus\mathrm{su}(2)\sim\mathrm{so}(4)$ and the algebra so(4) has no realization within the class of operators (2.4), the latter contains no realizations of type G_1.

Summing up, we conclude that class of PDEs (4.7) contains no equations, whose invariance algebras are isomorphic to n-dimensional semi-simple Lie algebras (or contains the latter as subalgebras) under $n>3$.

Consider now Eqs. (4.7), whose invariance algebras has nontrivial Levi factor. First, we turn to equations which are invariant with respect to the Lie algebras that can be decomposed into a direct sum of semi-simple Levi factor and radical, $\mathrm{sl}(2,\mathbb{R})\oplus L$, L being a radical. To this end, we will study possible extensions of the algebra $\mathrm{sl}(2,\mathbb{R})$ by operators (2.4).

Let $\mathrm{sl}(2,\mathbb{R})=\{X_1,X_2,X_3\}$, where X_1,X_2,X_3, form a basis of the invariance algebra of Eq. (4.7). Then it follows from

$$[X_i,Y]=0 \quad (i=1,2,3),$$

Y being an operator of the form (2.4), that $Y=\lambda\partial_u$, $\lambda=\mathrm{const}$. Hence L is the one-dimensional Lie algebra spanned by the operator ∂_u. For Eq. (4.7) to admit the algebra $\mathrm{sl}(2,\mathbb{R})\oplus\{\partial_u\}$, the equation

$$G_{\omega_1}+2G_{\omega_2}=0,$$

has to be satisfied, whence

$$G=\tilde{G}(\sigma), \quad \sigma=x^2u_{xx}-2xu_x.$$

Consequently, an equation of the form (4.7) admits invariance algebra which is the direct sum of semi-simple Levi factor and radical iff it reads as

$$u_t=u_{xxx}-2x^{-2}u_x-\tfrac{1}{9}x^{-1}u_x^2+x^{-3}\tilde{G}(\sigma), \quad \sigma=x^2u_{xx}-2xu_x. \tag{4.8}$$

As Eq. (4.8) contains an arbitrary function of one variable, we can perform direct group classification by a straightforward application of the Lie infinitesimal algorithm. The determining equation for coefficients of the infinitesimal symmetry operator are of the form

(a) $\phi_{uuu}=0;$

(b) $3\phi_{uu}\tilde{G}_\sigma+18\phi_{uu}+9x\phi_{xuu}+\tfrac{1}{3}(x^{-1}\rho-\phi_u)=0;$

(c) $$6x^{-1}(\phi_{xu}+x^{-2}\rho)\tilde{G}_\sigma+9x^{-2}\sigma\phi_{uu}+3\rho_t+x\tau_{tt}+6x^{-1}(3\phi_{xu}+2x^{-2}\rho)$$
$$+9\phi_{xxu}-\tfrac{2}{3}x^{-1}\phi_x=0;$$

(d) $$[-3x^{-3}(\phi_u+2x^{-1}\rho)\sigma+6x^{-2}\phi_x-3x^{-1}\phi_{xx}]\tilde{G}_\sigma+3x^{-3}(\phi_u+3x^{-1}\rho)\tilde{G}-9x^{-2}\phi_{xu}\sigma$$
$$+3[\phi_t-\phi_{xxx}+2x^{-2}\phi_x]=0.$$

It follows from (a) that

$$\phi=f(x,t)u^2+g(x,t)u+h(x,t), \tag{4.9}$$

where f,g,h are arbitrary smooth functions. Inserting (4.9) into (b) yields

$$6f\tilde{G}_\sigma+36f+18xf_x+\tfrac{1}{3}(x^{-1}\rho-g)-\tfrac{2}{3}fu=0.$$

Taking into account that functions $f,\rho,g,\tilde{G}$ do not depend on u, we get

$$f=0,\ \ g=x^{-1}\rho.$$

So that equation (c) reduces to

$$3\rho_t+x\tau_{tt}+12x^{-3}\rho+\tfrac{2}{3}x^{-3}\rho u-\tfrac{2}{3}x^{-1}h_x=0.$$

Hence it follows that

$$\rho=0,\ \ h=\tfrac{1}{2}x^3\tau_{tt}+\tilde{h}(t).$$

Finally, inserting the obtained expression for φ into equation (d) gives

$$\tau_{ttt}=0,\ \ \tilde{h}_t=0,$$

whence

$$\tau=C_1t^2+C_2t+C_3,$$

$$\tilde{h}=C_4.$$

Here C_1,C_2,C_3,C_4 are arbitrary (integration) constants.

Summing up, we conclude that the algebra $\mathrm{sl}(2,\mathbb{R})\oplus\{\partial_u\}$ is the maximal invariance algebra admitted by Eq. (4.8). It cannot be extended by specifying the form of an arbitrary function $\tilde{G}(\sigma),\sigma=x^2u_{xx}-2xu_x$.

What remains to be done is classifying Eqs. (4.7), whose invariance algebras are isomorphic to *semi-direct* sums of a semi-simple Levi factor and radical, i.e., whose invariance algebras have the following structure: $\mathrm{sl}(2,\mathbb{R})\Subset L$. To perform this classification we utilize the classification of these type of Lie algebras obtained by Turkowski.[21]

We choose $\mathrm{sl}(2,\mathbb{R})=\{v_1,v_2,v_3\}$ with

$$v_1=-2t\partial_t-\tfrac{2}{3}x\partial_x,\ \ v_2=\partial_t,\ \ v_3=-t^2\partial_t-\tfrac{2}{3}tx\partial_x-x^3\partial_u.$$

According to Ref. 21, there is only one five-dimensional Lie algebra of the desired form $\mathrm{sl}(2,\mathbb{R})\Subset L$ with $L=\{e_1,e_2\}$, operators e_1,e_2 satisfying the commutation relations:

$$[e_1,e_2]=0,\ \ [v_1,e_1]=e_1,\ \ [v_1,e_2]=-e_2,$$

$$[v_2,e_1]=0,\ \ [v_2,e_2]=e_1,$$

$$[v_3,e_1]=e_2,\quad [v_3,e_2]=0.$$

As operators e_1,e_2 are necessarily of the form (2.4), we easily get that

$$e_1=|x|^{-3/2}\partial_u,\quad e_2=t|x|^{-3/2}\partial_u.$$

However, checking the invariance criterion for the above realization we find that the algebra in question cannot be invariance algebra of an equation of the form (4.7).

According to Ref. 21, there exist three six-dimensional Lie algebras that are semi-direct sums of semi-simple Levi factor and radical, algebra L being of the form $L=\{e_1,e_2,e_3\}$. Nonzero commutation relations for e_1,e_2,e_3 read as

$$(1)\quad [v_1,e_1]=2e_1,\ \ [v_2,e_2]=2e_1,\ \ [v_3,e_1]=e_2,$$

$$[v_1,e_3]=-2e_3,\ \ [v_2,e_3]=e_2,\ \ [v_3,e_2]=2e_3;$$

$$(2)\quad [v_1,e_1]=e_1,\ \ [v_2,e_2]=e_1,\ \ [v_3,e_1]=e_2,$$

$$[v_1,e_2]=-e_2,\ \ [e_1,e_2]=e_3;$$

$$(3)\quad [v_1,e_1]=e_1,\ \ [v_2,e_2]=e_1,\quad [v_3,e_1]=e_2,$$

$$[v_1,e_2]=-e_2,\ \ [e_1,e_3]=e_1,\ \ [e_2,e_3]=e_2.$$

Solving the above relations we see that the corresponding realizations cannot be invariance algebras of Eq. (4.7).

Next, we consider seven-dimensional algebras from the Turkowski's classification. According to Ref. 21 there are five inequivalent algebras of the targeted dimension. Four of them contain the above five- and six-dimensional algebras as subalgebras. So we need to consider only the fifth algebra $\mathrm{sl}(2,\mathbb{R}) \Subset L$, where $L=\{e_1,e_2,e_3,e_4\}$ and the following commutation relations hold:

$$[v_1,e_1]=3e_1, \quad [v_2,e_2]=3e_1,$$

$$[v_3,e_1]=e_2, \quad [v_1,e_2]=e_2,$$

$$[v_2,e_3]=2e_2, \quad [v_3,e_2]=2e_3,$$

$$[v_1,e_3]=-e_3, \quad [v_2,e_4]=e_3,$$

$$[v_3,e_3]=3e_4, \quad [v_1,e_4]=-3e_4.$$

The most general form of operators e_1,e_2,e_3,e_4 satisfying the above relations is as follows:

$$e_1=|x|^{-9/2}\partial_u, \quad e_2=3t|x|^{-9/2}\partial_u,$$

$$e_3=3t^2|x|^{-9/2}\partial_u, \quad e_4=t^3|x|^{-9/2}\partial_u.$$

However, verifying the invariance criterion yields that this algebra cannot be the symmetry algebra of Eq. (4.7).

Thus we proved that the class of PDEs (4.7) contains no equations admitting symmetry algebras of the dimension $n\leqslant 7$, which are semi-direct sums of the Levi factor and radical. It is natural to conjecture that the same assertion holds for an arbitrary n. To prove this fact we need to consider in full details classification of nonlinear equations (1.1), whose invariance algebras are solvable.

Let us sum up the above results as theorems.

Theorem 4.2: *The class of PDEs (1.1) contains two inequivalent equations whose invariance algebra are semi-simple* ($\mathrm{sl}(2,\mathbb{R})$),

$$u_t=u_{xxx}-\tfrac{3}{2}u_x^{-1}u_{xx}^2+u_xG(x,t);$$

$$u_t=u_{xxx}-x^{-3}[2xu_x+\tfrac{1}{9}x^2u_x^2-G(\omega_1,\omega_2)],$$

$$\omega_1=3u-xu_x, \quad \omega_2=6u-x^2u_{xx}.$$

The maximal invariance algebras of the above equations under arbitrary G *read as*

$$\mathrm{sl}^1(2,\mathbb{R})=\{\partial_u,u\partial_u,-u^2\partial_u\};$$

$$\mathrm{sl}^2(2,\mathbb{R})=\{\partial_t,t\partial_t+\tfrac{1}{3}x\partial_x,-t^2\partial_t-\tfrac{2}{3}tx\partial_x-x^3\partial_u\}.$$

Theorem 4.3: *Nonlinear equation (1.1) whose invariance algebra is isomorphic to a Lie algebra having nontrivial Levi decomposition is represented by one of the following equations:*

$$u_t=u_{xxx}-\tfrac{3}{2}u_x^{-1}u_{xx}^2+u_x\tilde{G}(x), \quad \mathrm{sl}^1(2,R)\oplus\{\partial_t\}; \tag{4.10}$$

$$u_t=u_{xxx}-\tfrac{3}{2}u_x^{-1}u_{xx}^2+\lambda x^{-2}u_x, \quad \lambda\neq 0, \quad \mathrm{sl}^1(2,\mathbb{R})\oplus\{\partial_t,t\partial_t+\tfrac{1}{3}x\partial_x\}; \tag{4.11}$$

$$u_t = u_{xxx} - \tfrac{3}{2} u_x^{-1} u_{xx}^2, \quad \mathrm{sl}^1(2,\mathbb{R}) \oplus \{\partial_t, \partial_x, t\partial_t + \tfrac{1}{3} x \partial_x\}; \tag{4.12}$$

$$u_t = u_{xxx} - 2x^{-2} u_x - \tfrac{1}{9} x^{-1} u_x^2 + x^{-3} \tilde{G}(\sigma), \quad \sigma = x^2 u_{xx} - 2x u_x, \quad \mathrm{sl}^2(2,\mathbb{R}) \oplus \{\partial_u\}, \tag{4.13}$$

where $\tilde{G}$ is an arbitrary function of x or σ. Moreover, the associated symmetry algebras are maximal.

Note that Eq. (4.12) can be expressed in the form

$$\frac{u_t}{u_x} = \{u;x\}, \tag{4.14}$$

where $\{u;x\}$ denotes the Schwarzian derivative of u with respect to x. It is known that a nonpoint transformation taking this equation into the usual KdV exists.

V. CLASSIFICATION OF EQUATIONS INVARIANT UNDER LOW-DIMENSIONAL SOLVABLE SYMMETRY ALGEBRAS

In this section we apply the strategy summarized in the Introduction to identify representative classes of equations of the form (1.1) invariant under one-, two-, and three-dimensional solvable symmetry algebras. In order to approach this task in a systematic manner we realize all possible inequivalent algebras in terms of vector fields (2.4) under the action of the equivalence group $\mathcal{E}$.

A. Equations with one-dimensional symmetry algebras

We assume that for a given F, Eq. (1.1) is invariant under a one-parameter symmetry group, generated by the vector field (2.4) with coefficients subject to the constraint (2.5). We make use of Proposition 2.3 which characterizes the canonical forms of the vector field X of (2.4). We then substitute the coefficients of the canonical vector field into the determining equation (2.5), which is a first order linear homogeneous PDE for F, and solve the latter in order to construct invariant equations.

According to Proposition 2.3 we have three types of one-dimensional symmetry algebras:

$$A_{1,1}: \quad X_1 = \partial_t, \quad A_{1,2}: X_1 = \partial_x, \quad A_{1,3}: X_1 = \partial_u. \tag{5.1}$$

The corresponding invariant equations will have the form

$$A_{1,1}: \quad u_t = u_{xxx} + F(x,u,u_x,u_{xx}), \tag{5.2a}$$

$$A_{1,2}: \quad u_t = u_{xxx} + F(t,u,u_x,u_{xx}), \tag{5.2b}$$

$$A_{1,3}: \quad u_t = u_{xxx} + F(x,t,u_x,u_{xx}). \tag{5.2c}$$

Theorem 5.1: *There are three inequivalent classes of Eqs. (1.1) invariant under one-parameter symmetry group. Their representatives are given by (5.2).*

B. Equations with two-dimensional symmetry algebras

There are two isomorphy classes of two-dimensional Lie algebras, Abelian and non-Abelian, satisfying the commutation relations $[X_1, X_2] = \kappa X_2$, $\kappa = 0,1$. We denote them by $A_{2,1}$ and $A_{2,2}$.

1. Abelian

We start from each of the one-dimensional cases obtained in (5.1) and add to it vector fields X_2 of the form (2.4) commuting with X_1. We then simplify X_2 by equivalence transformations leaving the vector field X_1 invariant. For further details we refer the reader to Ref. 4. The standardized X_2 and the restricted form of F in (5.2) are then substituted into (2.5). Solving this equation will further restrict the form of the function F. The number of variables of F reduces by one, three variables in this case. Thus, we find that there exist precisely four classes of two-dimensional Abelian symmetry algebras represented by the following ones:

$$A_{2,1}^1: \quad X_1=\partial_t, \quad X_2=\partial_x, \quad F=F(u,u_x,u_{xx}); \tag{5.3}$$

$$A_{2,1}^2: \quad X_1=\partial_t, \quad X_2=\partial_u, \quad F=F(x,u_x,u_{xx}); \tag{5.4}$$

$$A_{2,1}^3: \quad X_1=\partial_x, \quad X_2=\alpha(t)\partial_x+\partial_u, \quad F=-\dot{\alpha}uu_x+\tilde{F}(t,u_x,u_{xx}); \tag{5.5}$$

$$A_{2,1}^4: \quad X_1=\partial_u, \quad X_2=g(x,t)\partial_u, \quad g_x\neq \text{const}; \tag{5.6}$$

$$F=(g_t-g_{xxx})g_x^{-1}u_x+\tilde{F}(x,t,\omega), \quad \omega=g_{xx}u_x-g_xu_{xx}.$$

2. Non-Abelian

Imposing that X_1 reads as (5.1) and X_2 is in generic form and that they satisfy $[X_1,X_2]=X_2$, we find that five classes of symmetry algebras exist. Those algebras and nonlinear functions F are represented by

$$A_{2,2}^1: \quad X_1=\partial_t, \quad X_2=-t\partial_t-\frac{x}{3}\partial_x,$$

$$F=x^{-3}\tilde{F}(u,\omega_1,\omega_2), \quad \omega_1=xu_x, \quad \omega_2=x^2u_{xx};$$

$$A_{2,2}^2: \quad X_1=-3t\partial_t-x\partial_x, \quad X_2=\partial_x,$$

$$F=t^{-1}\tilde{F}(u,\omega_1,\omega_2), \quad \omega_1=t^{1/3}u_x, \quad \omega_2=t^{2/3}u_{xx};$$

$$A_{2,2}^3: \quad X_1=-u\partial_u, \quad X_2=\partial_u, \quad F=u_x\tilde{F}(x,t,\omega), \quad \omega=u_x^{-1}u_{xx};$$

$$A_{2,2}^4: \quad X_1=\partial_x-u\partial_u, \quad X_2=\partial_u;$$

$$F=e^{-x}\tilde{F}(t,\omega_1,\omega_2), \quad \omega_1=e^xu_x, \quad \omega_2=e^xu_{xx};$$

$$A_{2,2}^5: \quad X_1=\partial_t-u\partial_u, \quad X_2=\partial_u,$$

$$F=u_x\tilde{F}(x,\omega_1,\omega_2), \quad \omega_1=e^tu_x, \quad \omega_2=e^tu_{xx}.$$

Theorem 5.2: *There exist nine classes of two-dimensional symmetry algebras admitted by Eq. (1.1). They are represented by the algebras $A_{2,1}^1,\ldots,A_{2,1}^4$ and $A_{2,2}^1,\ldots,A_{2,2}^5$.*

C. Equations with three-dimensional symmetry algebras

1. Decomposable algebras

A Lie algebra is decomposable if it can be written as a direct sum of two or more Lie algebras $L=L_1\oplus L_2$ with $[L_1,L_2]=0$. There are two types of 3-dimensional decomposable Lie algebras: $A_{3,1}=3A_1=A_1\oplus A_2\oplus A_3$ with $[X_i,X_j]=0$ for $i,j=1,2,3$ and $A_{3,2}=A_{2,2}\oplus A_1$ with $[X_1,X_2]=X_2$, $[X_1,X_3]=0$, $[X_2,X_3]=0$.

We start from the two-dimensional algebras in (5.3) and add a further linearly independent vector field X_3 in the form (2.4) and impose the above commutation relations. We simplify X_3 using equivalence transformations leaving the space $\{X_1,X_2\}$ invariant. We present the following result without proof. We emphasize that there exist several realizations that do not produce invariant equations of the form (1.1):

$$A^1_{3,1}:\quad X_1=\partial_t,\quad X_2=\partial_x,\quad X_3=\partial_u,\quad F=F(u_x,u_{xx});$$

$$A^2_{3,1}:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=f(x)\partial_u,$$

$$F=-\frac{f'''}{f'}u_x+\tilde F(x,\omega),\quad \omega=f''u_x-f'u_{xx};$$

$$A^1_{3,2}:\quad X_1=-t\partial_t-\frac{x}{3}\partial_x,\quad X_2=\partial_t,\quad X_3=\partial_u,$$

$$F=x^{-3}\tilde F(\omega_1,\omega_2),\quad \omega_1=xu_x,\quad \omega_2=x^2u_{xx};$$

$$A^2_{3,2}:\quad X_1=-3t\partial_t-x\partial_x,\quad X_2=\partial_x,\quad X_3=\partial_u,$$

$$F=t^{-1}\tilde F(\omega_1,\omega_2),\quad \omega_1=tu_x^3,\quad \omega_2=t^2u_{xx}^3;$$

$$A^3_{3,2}:\quad X_1=-3t\partial_t-x\partial_x,\quad X_2=\partial_x,\quad X_3=t^{1/3}\partial_x+\partial_u,$$

$$F=-\tfrac{1}{3}t^{-2/3}uu_x+t^{-1}\tilde F(\omega_1,\omega_2),\quad \omega_1=tu_x^3,\quad \omega_2=t^2u_{xx}^3;$$

$$A^4_{3,2}:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=e^{-x}f(t)\partial_u,\quad f\neq 0,$$

$$F=-\left(1+\frac{\dot f}{f}\right)u_x+e^{-x}\tilde F(t,\omega),\quad \omega=e^x(u_x+u_{xx});$$

$$A^5_{3,2}:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\alpha(t)\partial_x,\quad \alpha\neq 0,$$

$$F=-\frac{\dot\alpha}{\alpha}u_x\ln(e^xu_x)+u_x\tilde F(t,\omega),\quad \omega=u_x^{-1}u_{xx};$$

$$A^6_{3,2}:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t,$$

$$F=e^{-x}\tilde F(\omega_1,\omega_2),\quad \omega_1=e^xu_x,\quad \omega_2=e^xu_{xx};$$

$$A^7_{3,2}:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\quad X_3=e^{-t}f(x)\partial_u,\quad f'\neq 0,$$

$$F=-\frac{f'''+f}{f'}u_x+e^{-t}\tilde F(x,\omega),\quad \omega=e^t(f''u_x-f'u_{xx});$$

$$A^8_{3,2}:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_x,$$

$$F=e^{-t}\tilde F(\omega_1,\omega_2),\quad \omega_1=e^tu_x,\quad \omega_2=e^tu_{xx};$$

$$A^9_{3,2}:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t+\lambda\partial_x,\quad \lambda\neq 0,$$

$$F=\exp(x/t-\lambda)\tilde F(\omega_1,\omega_2),$$

$$\omega_1=\exp(t-x/\lambda)u_x,\quad \omega_2=\exp(t-x/\lambda)u_{xx};$$

$$A^{10}_{3,2}:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t,$$

$$F=u_x\tilde{F}(x,\omega),\quad \omega=u_x^{-1}u_{xx}.$$

2. Nondecomposable algebras

The isomorphy classes of these algebras are represented by the following list:

$$A_{3,3}:\quad [X_2,X_3]=X_1,\quad [X_1,X_2]=[X_1,X_3]=0;$$

$$A_{3,4}:\quad [X_1,X_3]=X_1,\quad [X_2,X_3]=X_1+X_2;$$

$$A_{3,5}:\quad [X_1,X_3]=X_1,\quad [X_2,X_3]=X_2;$$

$$A_{3,6}:\quad [X_1,X_3]=X_1,\quad [X_2,X_3]=-X_2;$$

$$A_{3,7}:\quad [X_1,X_3]=X_1,\quad [X_2,X_3]=qX_2\quad (0<|q|<1);$$

$$A_{3,8}:\quad [X_1,X_3]=-X_2,\quad [X_2,X_3]=X_1;$$

$$A_{3,9}:\quad [X_1,X_3]=qX_1-X_2,\quad [X_2,X_3]=X_1+qX_2,\quad q>0.$$

Remark: Solvable nondecomposable algebras can be written as semidirect sums of a one-dimensional subalgebra $\{X_3\}$ and an Abelian ideal $\{X_1,X_2\}$. Note that the algebras $A_{3,6}$ and $A_{3,8}$ are isomorphic to $\mathfrak{e}(1,1)$, and $\mathfrak{e}(2)$, respectively. The algebra $A_{3,3}$ is a non-Abelian nilpotent algebra (Heisenberg algebra).

The commutation relations of the algebras in question can be represented in the matrix notation

$$\begin{pmatrix}[X_1,X_3]\\ [X_2,X_3]\end{pmatrix}=J\begin{pmatrix}X_1\\ X_2\end{pmatrix},\quad [X_1,X_2]=0,$$

where J is a 2×2 real matrix that can be taken in Jordan canonical form.

A solvable three-dimensional Lie algebra always possesses a two-dimensional Abelian ideal. We assume that the ideal $\{X_1,X_2\}$ is already of the form (5.3) and add a third element X_3 in the form (2.4) acting on the ideal. Imposing commutation relations and simplifying with equivalence transformations (2.6) (we consider each canonical form of the matrix individually) yield the realizations of solvable Lie algebras together with the corresponding invariant equations.

There exist nine classes of realizations of nilpotent algebras which give rise to invariant equations:

$$A_{3,3}:\quad J=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{5.7}$$

$$A_{3,3}^1:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=t\partial_u+\lambda\partial_x,\quad \lambda>0,$$

$$F=\frac{x}{\lambda}+\tilde{F}(u_x,u_{xx});$$

$$A_{3,3}^2:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=x\partial_u+b(t)\partial_x,\quad \dot{b}\neq 0,$$

$$F=-\frac{\dot{b}}{2}u_x^2+\tilde{F}(t,u_{xx});$$

$$A_{3,3}^3:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=x\partial_u+\lambda\partial_t,\quad \lambda\neq 0,$$

$$F=\tilde{F}(t-3\lambda u_x,u_{xx});$$

$$A_{3,3}^4:\quad X_1=\partial_u+3\lambda t^{1/2}\partial_x,\quad X_2=\partial_x,$$

$$X_3=6\lambda t^{3/2}\partial_t+3\lambda t^{1/2}x\partial_x+(x-3\lambda t^{1/2}u)\partial_u,\quad \lambda\neq 0,$$

$$F=-\tfrac{3}{2}\lambda t^{-1/2}uu_x+t^{-2}\tilde{F}(\omega_1,\omega_2),\quad \omega_1=tu_x-\frac{1}{3\lambda}t^{1/2},\quad \omega_2=t^{3/2}u_{xx};$$

$$A_{3,3}^5:\quad X_1=\partial_x,\quad X_2=\partial_t,\quad X_3=t\partial_x+\partial_u,$$

$$F=-uu_x+\tilde{F}(u_x,u_{xx});$$

$$A_{3,3}^6:\quad X_1=\partial_u,\quad X_2=(f(x)-t)\partial_u,\quad X_3=\partial_t,\ (f'\neq 0),$$

$$F=-(1+f''')(f')^{-1}u_x+\tilde{F}(x,\omega),\quad \omega=f''u_x-f'u_{xx};$$

$$A_{3,3}^7:\quad X_1=\partial_u,\quad X_2=(t-x)\partial_u,\quad X_3=\partial_x,$$

$$F=u_x+\tilde{F}(t,u_{xx});$$

$$A_{3,3}^8:\quad X_1=\partial_u,\quad X_2=-x\partial_u,\quad X_3=\partial_x,$$

$$F=-\tilde{F}(t,u_{xx});$$

$$A_{3,3}^9:\quad X_1=-x^{-1}\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_x-x^{-1}u\partial_u,$$

$$F=3x^{-1}u_{xx}+x^{-1}\tilde{F}(t,\omega),\quad \omega=2u_x+xu_{xx};$$

$$A_{3,4}:\quad J=\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}; \tag{5.8}$$

$$A_{3,4}^1:\quad X_1=\partial_u,\quad X_2=\partial_t,\quad X_3=t\partial_t+\frac{x}{3}\partial_x+(u+t)\partial_u,$$

$$F=3\ln x+\tilde{F}(\omega_1,\omega_2),\quad \omega_1=x^{-2}u_x,\quad \omega_2=x^{-1}u_{xx};$$

$$A_{3,4}^2:\quad X_1=\partial_x,\quad X_2=\partial_u-\tfrac{1}{3}\ln t\,\partial_x,\quad X_3=3t\partial_t+x\partial_x+u\partial_u,$$

$$F=\frac{1}{3t}uu_x+t^{-2/3}\tilde{F}(u_x,\omega),\quad \omega=t^{1/3}u_{xx};$$

$$A_{3,4}^3:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=3t\partial_t+x\partial_x+(u+x)\partial_u,$$

$$F=t^{-2/3}\tilde{F}(\omega_1,\omega_2),\quad \omega_1=u_x-\tfrac{1}{3}\ln t,\quad \omega_2=t^{1/3}u_{xx};$$

$$A_{3,4}^4:\quad X_1=\alpha(t)\partial_x+\partial_u,\ X_2=\partial_x,$$

$$X_3=(\alpha')^{-1}\alpha^2\partial_t+(1+\alpha)x\partial_x+[x+(1-\alpha)u]\partial_u,\quad \alpha'\neq 0,$$

$$\alpha^2\alpha''+(3+\alpha)(\alpha')^2=0,$$

$$F=-\alpha' u u_x+\alpha^{-4}\exp(2\alpha^{-1})\tilde{F}(\omega_1,\omega_2),$$

$$\omega_1=\alpha^3\exp(-\alpha^{-1})u_{xx},\quad \omega_2=\alpha^2 u_x-\alpha;$$

$$A_{3,4}^5:\quad X_1=\partial_u,\quad X_2=(-t+f(x))\partial_u,\quad X_3=\partial_t+u\partial_u,\quad f'\neq 0,$$

$$F=-(1+f''')(f')^{-1}u_x+e^t\tilde{F}(x,\omega),\quad \omega=e^{-t}(f''u_x-f'u_{xx});$$

$$A_{3,4}^6:\quad X_1=\partial_u,\quad X_2=-x\partial_u,\quad X_3=\partial_x+u\partial_u,$$

$$F=e^x\tilde{F}(t,\omega),\quad \omega=e^{-x}u_{xx};$$

$$A_{3,5}:\quad J=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix};\tag{5.9}$$

$$A_{3,5}^1:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=t\partial_t+\frac{x}{3}\partial_x+u\partial_u,$$

$$F=\tilde{F}(\omega_1,\omega_2),\quad \omega_1=x^{-2}u_x,\quad \omega_2=x^{-1}u_{xx};$$

$$A_{3,5}^2:\quad X_1=\partial_x,\quad X_2=\partial_u,\quad X_3=3t\partial_t+x\partial_x+u\partial_u,$$

$$F=t^{-2/3}\tilde{F}(u_x,t^{1/3}u_{xx});$$

$$A_{3,5}^3:\quad X_1=\partial_u,\ X_2=f(x)\partial_u,\quad X_3=\partial_t+u\partial_u,\ f'\neq 0,$$

$$F=-f'''(f')^{-1}u_x+e^t\tilde{F}(x,\omega),$$

$$\omega=e^{-t}[f''u_x-f'u_{xx}];$$

$$A_{3,6}:\quad J=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix};\tag{5.10}$$

$$A_{3,6}^1:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=t\partial_t+\frac{x}{3}\partial_x-u\partial_u,$$

$$F=x^{-6}\tilde{F}(x^4u_x,x^5u_{xx});$$

$$A_{3,6}^2:\quad X_1=\partial_x,\quad X_2=\partial_u+\lambda t^{2/3}\partial_x,\quad X_3=3t\partial_t+x\partial_x-u\partial_u,$$

$$F=-\frac{2\lambda}{3}t^{-1/3}uu_x+t^{-4/3}\tilde{F}(t^{2/3}u_x,tu_{xx});$$

$$A_{3,6}^3:\quad X_1=\partial_u,\quad X_2=e^{2t}f(x)\partial_u,\quad X_3=\partial_t+u\partial_u,\quad f'\neq 0,$$

$$F=(2f-f''')(f')^{-1}u_x+e^t\tilde{F}(x,\omega),\quad \omega=e^{-t}(f''u_x-f'u_{xx});$$

$$A_{3,6}^4:\quad X_1=\partial_u,\quad X_2=e^{2f^{-1}x}h(t)\partial_u,\quad X_3=f(t)\partial_x+u\partial_u,\quad fh\neq 0,$$

$$F=-[4f^{-2}-\tfrac{1}{2}\dot{h}h^{-1}f+f^{-1}f'x]u_x+e^{f^{-1}x}\tilde{F}(t,\omega),$$

$$\omega = e^{-f^{-1}x}(2u_x - f u_{xx});$$

$$A_{3,7}: \quad J = \begin{pmatrix} 1 & 0 \\ 0 & q \end{pmatrix}, \quad 0 < |q| < 1; \tag{5.11}$$

$$A_{3,7}^1: \quad X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t + \frac{x}{3}\partial_x, \quad q = 1/3,$$

$$F = u_x^3 \tilde{F}(u, u_x^{-2} u_{xx});$$

$$A_{3,7}^2: \quad X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = t\partial_t + \frac{x}{3}\partial_x + u\partial_u, \quad q = 1/3,$$

$$F = \tilde{F}(\omega_1, \omega_2), \quad \omega_1 = u^{-2/3} u_x, \quad \omega_2 = u^{-1/3} u_{xx};$$

$$A_{3,7}^3: \quad X_1 = \partial_t, \quad X_2 = \partial_u, \quad X_3 = t\partial_t + \frac{x}{3}\partial_x + qu\partial_u, \quad q \neq 0, \pm 1,$$

$$F = x^{3(q-1)} \tilde{F}(\omega_1, \omega_2), \quad \omega_1 = x^{1-3q} u_x, \quad \omega_2 = x^{2-3q} u_{xx};$$

$$A_{3,7}^4: \quad X_1 = \partial_x, \quad X_2 = \partial_u + \lambda t^{(1-q)/3}\partial_x, \quad X_3 = 3t\partial_t + x\partial_x + qu\partial_u,$$

$$q \neq 0, \pm 1, \ \lambda \in \mathbb{R},$$

$$F = \frac{\lambda}{3}(q-1) t^{-(q+2)/3} u u_x + \tilde{F}(\omega_1, \omega_2),$$

$$\omega_1 = t^{-(q-1)/3} u_x, \quad \omega_2 = t^{-(q-2)/3} u_{xx};$$

$$A_{3,7}^5: \quad X_1 = \partial_u, \quad X_2 = e^{(1-q)t} f(x)\partial_u, \quad X_3 = \partial_t + u\partial_u, \quad f' \neq 0, \quad q \neq 0, \pm 1,$$

$$F = [(1-q)f - f'''](f')^{-1} u_x + e^t \tilde{F}(x, \omega),$$

$$\omega = e^{-t}[f'' u_x - f' u_{xx}];$$

$$A_{3,7}^6: \quad X_1 = \partial_u, \quad X_2 = e^{(1-q)f^{-1}(t)x} h(t)\partial_u,$$

$$X_3 = f(t)\partial_x + u\partial_u, \quad f \cdot h \neq 0, \quad q \neq 0, \ \pm 1,$$

$$F = -[(1-q)^2 f^2 + f^{-1} f' x - (1-q)^{-1} f h^{-1} h'] u_x + e^{f^{-1}x} \tilde{F}(t, \omega),$$

$$\omega = e^{-f^{-1}x}[(1-q)u_x - f u_{xx}].$$

Remark: The algebra $A_{3,7}$ has another realization,

$$\{X_1 = \partial_t, X_2 = \partial_x, X_3 = t\partial_t + \tfrac{1}{3}(x + b_0 t)\partial_x + u\partial_u\},$$

that is isomorphic to $A_{3,7}^2$ under the change of basis

$$X_1 \to X_1 + \frac{b_0}{2} X_2, \quad X_2 \to X_2, \quad X_3 \to X_3.$$

Note that its equivalence to the latter is established by the change of variables,

$$\tilde{t}=t, \quad \tilde{x}=x-\tfrac{1}{2}b_0 t, \quad \tilde{u}=u.$$

That is why we have excluded it from the above list:

$$A_{3,8}: \quad J=\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}; \tag{5.12}$$

$$A_{3,8}^1: \quad X_1=\partial_x, \quad X_2=\alpha(t)\partial_x+\partial_u,$$

$$X_3=-\frac{1}{\dot{\alpha}}(1+\alpha^2)\partial_t-\alpha x\partial_x+(\alpha u-x)\partial_u,$$

$$F=-\dot{\alpha}uu_x+(1+\alpha^2)^{-2}\tilde{F}(\omega_1,\omega_2),$$

$$\omega_1=(1+\alpha^2)u_x-\alpha, \quad \omega_2=(1+\alpha^2)^{3/2}u_{xx},$$

where $\alpha(t)$, $\dot{\alpha}\neq 0$ satisfies

$$(1+\alpha^2)\ddot{\alpha}+\alpha\dot{\alpha}^2=0; \tag{5.13}$$

$$A_{3,9}: \quad J=\begin{pmatrix} q & -1 \\ 1 & q \end{pmatrix}, \quad q>0; \tag{5.14}$$

$$A_{3,9}^1: \quad X_1=\partial_x, \quad X_2=\alpha(t)\partial_x+\partial_u,$$

$$X_3=-\frac{1}{\dot{\alpha}}(1+\alpha^2)\partial_t+(q-\alpha)x\partial_x+[(q+\alpha)u-x]\partial_u,$$

$$F=-\dot{\alpha}uu_x+\exp\{2q\arctan\alpha\}(1+\alpha^2)^{-2}\tilde{F}(\omega_1,\omega_2),$$

$$\omega_1=(1+\alpha^2)u_x-\alpha, \quad \omega_2=(1+\alpha^2)^{3/2}\exp\{-q\arctan\alpha\}u_{xx},$$

where $\alpha(t)$, $\dot{\alpha}\neq 0$ satisfies

$$(1+\alpha^2)\ddot{\alpha}+(\alpha-3q)\dot{\alpha}^2=0. \tag{5.15}$$

Remark: $\alpha(t)$ can be obtained implicitly by quadratures as

$$\int^{\alpha}\exp(-3q\arctan\xi)(1+\xi^2)^{1/2}d\xi=c_1 t+c_0.$$

Theorem 5.3: *There are thirty-eight inequivalent three-dimensional solvable symmetry algebras admitted by Eq. (1.1).*

VI. EQUATIONS WITH FOUR-DIMENSIONAL SOLVABLE ALGEBRAS

For $\dim L=4$, we proceed exactly in the same manner as above. We start from the already standardized three-dimensional algebras, and add a further linearly independent element X_4, and require that they form a Lie algebra.

A. Decomposable algebras

The list of decomposable four-dimensional Lie algebras consists of the twelve algebras: $4A_1=A_{3,1}\oplus A_1$, $A_{2,2}\oplus 2A_1=A_{3,2}\oplus A_1$, $2A_{2,2}=A_{2,2}\oplus A_{2,2}$, $A_{3,i}\oplus A_1$ $(i=3,4,\ldots,9)$. We preserve the notations of the previous section.

There are four inequivalent realizations of the algebra $2A_{2,2}$ which are invariance algebras of PDEs of the form (1.1). We give these realizations together with the corresponding invariant equations:

$$2A_{2,2}^1:\quad X_1=-t\partial_t-\frac{x}{3}\partial_x,\quad X_2=\partial_t,\quad X_3=\partial_u,\quad X_4=e^u\partial_u,$$

$$F=u_x^3-3u_xu_{xx}+x^{-2}u_x\tilde{F}(\omega),\quad \omega=x(u_x^{-1}u_{xx}-u_x);$$

$$2A_{2,2}^2:\quad X_1=-3t\partial_t-x\partial_x,\quad X_2=\partial_x,\quad X_3=-u\partial_u+\lambda t^{1/3}\partial_x,\quad X_4=\partial_u,$$

$$F=\frac{\lambda}{3t}\omega_1\ln|\omega_1|+\frac{\omega_1}{t}\tilde{F}(\omega),\quad \omega_1=t^{1/3}u_x,\quad \omega=t^{1/3}u_x^{-1}u_{xx};$$

$$2A_{2,2}^3:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\frac{1}{\lambda}\partial_t,\quad X_4=\exp(\lambda t)\partial_x,$$

$$F=-\lambda xu_x-\lambda u_x\ln|u_x|+u_x\tilde{F}(\omega),\quad \omega=u_x^{-1}u_{xx};$$

$$2A_{2,2}^4:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\lambda\partial_t,\quad X_4=e^{\lambda^{-1}t-x}\partial_u,\quad \lambda\neq 0,$$

$$F=(1+\lambda^{-1})u_x+e^{-x}\tilde{F}(\omega),\quad \omega=e^x(u_x+u_{xx});$$

$$2A_{2,2}^5:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,\quad X_3=\beta(\partial_x+\gamma\partial_t)-\partial_t,$$

$$X_4=e^{\gamma x-t}\partial_u,\quad \beta\gamma\neq 0,$$

$$F=e^{(\gamma-\beta^{-1})x-t}\tilde{F}(\omega)-\gamma^{-1}(1+\gamma^3)u_x,$$

$$\omega=e^{t+(\beta^{-1}-\gamma)x}(\gamma u_x-u_{xx}).$$

Equations invariant under the algebra $A_{2,2}\oplus 2A_1=A_{3.2}\oplus A_1$:

$$A_{3,2}^6\oplus\{X_4\}:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t,\quad X_4=e^{-x}\partial_u,$$

$$F=-u_x+e^{-x}\tilde{F}(\omega),\quad \omega=e^x(u_x+u_{xx});$$

$$A_{3,2}^6\oplus\{X_4\}:\quad X_1=\partial_x-u\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t,\quad X_4=\partial_x,$$

$$F=u_x\tilde{F}(\omega),\quad \omega=u_{xx}u_x^{-1};$$

$$A_{3,2}^7(f=e^{\lambda x},\ \lambda\neq 0)\oplus\{X_4\}:\quad X_1=\partial_t-u\partial_u,\quad X_2=\partial_u,$$

$$X_3=e^{\lambda x-t}\partial_u,\quad X_4=\partial_x+\lambda\partial_t,\quad \lambda\neq 0,$$

$$F=-(\lambda^3+1)\lambda^{-1}u_x+e^{-t+\lambda x}\tilde{F}(\omega),$$

$$\omega=e^{t-\lambda x}(\lambda u_x-u_{xx});$$

$$A_{3.3}^1 \oplus \{X_4\}: \quad X_1 = \partial_t, \quad X_2 = \partial_u, \quad X_3 = t\partial_u + \lambda \partial_x,$$

$$X_4 = \partial_t + \lambda^{-1} x \partial_u + \beta \partial_x, \quad \lambda > 0, \quad \beta \in \mathbb{R},$$

$$F = \lambda^{-1} x - \beta u_x + \tilde{F}(u_{xx});$$

$$A_{3,5}^3 \oplus \{X_4\}: \quad X_1 = \partial_u, \quad X_2 = \partial_x, \quad X_3 = x\partial_u + \lambda \partial_t, \quad X_4 = \partial_t + \beta(\partial_x + \lambda^{-1} t \partial_u),$$

$$\lambda \neq 0, \quad \beta \in \mathbb{R}, \quad F = \beta(\lambda^{-1} t - 3u_x) + \tilde{F}(u_{xx});$$

$$A_{3,5}^6(f = \lambda^{-1} x, \ \lambda \neq 0) \oplus \{X_4\}: \quad X_1 = \partial_u, \quad X_2 = (\lambda^{-1} x - t)\partial_u,$$

$$X_3 = \partial_t, \quad X_4 = \partial_t + \lambda \partial_x, \quad \lambda \neq 0,$$

$$F = -u_x + \tilde{F}(u_{xx});$$

$$A_{3,5}^9 \oplus \{X_4\}: \quad X_1 = -x^{-1}\partial_u, \quad X_2 = \partial_u, \quad X_3 = \partial_x - x^{-1} u \partial_u, \quad X_4 = \partial_t,$$

$$F = 3x^{-1} u_{xx} + x^{-1} \tilde{F}(\omega), \quad \omega = 2u_x + x u_{xx};$$

$$A_{3,4}^1 \oplus \{X_4\}: \quad X_1 = \partial_u, \ X_2 = \partial_t,$$

$$X_3 = t\partial_t + \tfrac{1}{3} x \partial_x + (u+t)\partial_u, \quad X_4 = x^3 \partial_u,$$

$$F = 3 \ln x - 2x^{-2} u_x + \tilde{F}(\omega), \ \omega = x^{-1} u_{xx} - 2x^{-2} u_x;$$

$$A_{3,4}^5 \ (f = \lambda x, \ \lambda \neq 0) \oplus \{X_4\}: \quad X_1 = \partial_u,$$

$$X_2 = (-t + \lambda x)\partial_u, \quad X_3 = \partial_t + u\partial_u,$$

$$X_4 = \partial_x + \lambda \partial_t, \quad \lambda \neq 0,$$

$$F = -\lambda^{-1} u_x + e^{t - \lambda x} \tilde{F}(\omega), \ \omega = e^{-t + \lambda x} u_{xx};$$

$$A_{3,4}^6 \oplus \{X_4\}: \quad X_1 = \partial_u, \quad X_2 = -x\partial_u,$$

$$X_3 = \partial_x + u\partial_u, \quad X_4 = \partial_t,$$

$$F = e^x \tilde{F}(\omega), \quad \omega = e^{-x} u_{xx};$$

$$A_{3,5}^1 \oplus \{X_4\}: \quad X_1 = \partial_t, \quad X_2 = \partial_u,$$

$$X_3 = t\partial_t + \frac{x}{3}\partial_x + u\partial_u, \quad X_4 = x^3 \partial_u,$$

$$F = -2x^{-2} u_x + \tilde{F}(\omega), \quad \omega = x^{-1} u_{xx} - 2x^{-2} u_x;$$

$$A_{3,6}^1 \oplus \{X_4\}: \quad X_1 = \partial_t, \quad X_2 = \partial_u,$$

$$X_3 = t\partial_t + \frac{x}{3}\partial_x - u\partial_u, \quad X_4 = x^{-3}\partial_u,$$

$$F=-20x^{-2}u_x+x^{-6}\tilde{F}(\omega), \quad \omega=4x^4u_x-x^5u_{xx},$$

$$A_{3,6}^3(f=e^{-2\beta^{-1}x}, \ \beta\neq 0)\oplus\{X_4\}: \quad X_1=\partial_u, \quad X_2=e^{2(t-\beta^{-1}x)}\partial_u,$$

$$X_3=\partial_t+u\partial_u, \quad X_4=\partial_t+\beta\partial_x, \quad \beta\neq 0,$$

$$F=-(\beta+4\beta^{-2})u_x+e^{t-\beta^{-1}x}\tilde{F}(\omega),$$

$$\omega=e^{-t+\beta^{-1}x}(2u_x+\beta u_{xx});$$

$$A_{3,6}^4(f=h=1)\oplus\{X_4\}: \quad X_1=\partial_u, \quad X_2=e^{2x}\partial_u,$$

$$X_3=\partial_x+u\partial_u, \quad X_4=\partial_t,$$

$$F=-4u_x+e^x\tilde{F}(\omega), \quad \omega=e^{-x}(2u_x-u_{xx});$$

$$A_{3,7}^1\oplus\{X_4\}: \quad X_1=\partial_t, \quad X_2=\partial_x,$$

$$X_3=t\partial_t+\tfrac{1}{3}x\partial_x, \quad X_4=u\partial_u,$$

$$F=u^{-2}u_x^3\tilde{F}(\omega), \quad \omega=u_x^{-2}uu_{xx};$$

$$A_{3,7}^3\oplus\{X_4\}: \quad X_1=\partial_t, \quad X_2=\partial_u,$$

$$X_3=t\partial_t+\frac{x}{3}\partial_x+qu\partial_u, \quad X_4=x^{3q}\partial_u,$$

$$q\neq 0, \ \pm 1,$$

$$F=-(3q-1)(3q-2)x^{-2}u_x+x^{3(q-1)}\tilde{F}(\omega),$$

$$\omega=x^{1-3q}[(3q-1)u_x-xu_{xx}];$$

$$A_{3,7}^5(f=e^{-1(1-q)\beta^{-1}x},\beta\neq 0)\oplus\{X_4\}: \quad X_1=\partial_u, \quad X_2=e^{(1-q)(t-\beta^{-1}x)}\partial_u, \quad X_3=\partial_t+u\partial_u,$$

$$X_4=\partial_t+\beta\partial_x, \quad \beta\neq 0, \ q\neq 0, \ \pm 1,$$

$$F=-[\beta+(1-q)^2\beta^{-2}]u_x+e^{t-\beta^{-1}x}\tilde{F}(\omega),$$

$$\omega=e^{-t+\beta^{-1}x}[(1-q)u_x+\beta u_{xx}];$$

$$A_{3,7}^6(f=h=1)\oplus\{X_4\}: \quad X_1=\partial_u, \quad X_2=e^{(1-q)x}\partial_u,$$

$$X_3=\partial_x+u\partial_u, \quad X_4=\partial_t, \quad q\neq 0, \ \pm 1,$$

$$F=-(1-q)^2u_x+e^x\tilde{F}(\omega),$$

$$\omega=e^{-x}[(1-q)u_x-u_{xx}].$$

Remark: The $A_{3,7}^1\oplus A_1$ invariant equation is

$$u_t = u_{xxx} + \frac{u_x^3}{u^2}\tilde{F}(\omega), \quad \omega = \frac{u u_{xx}}{u_x^2}. \tag{6.1}$$

If, in particular, $\tilde{F} = c\omega^2$, $c = \text{const}$, namely,

$$F = c u_x^{-1} u_{xx}^2, \tag{6.2}$$

the symmetry algebra is further extended by $X_5 = \partial_u$ to a five-dimensional one.

B. Nondecomposable algebras

The set of inequivalent abstract four-dimensional Lie algebras contains ten real nondecomposable Lie algebras $A_{4,i} = \{X_1, X_2, X_3, X_4\}$ $(i = 1,\ldots,10)$.[4,5] They are all solvable and therefore can be written as semidirect sums of a one-dimensional Lie algebra $\{X_4\}$ and a three-dimensional ideal $N = \{X_1, X_2, X_3\}$. For $A_{4,i}$ $(i = 1,\ldots,6)$, N is Abelian, for $A_{4,7}, A_{4,8}, A_{4,9}$ it is of type $A_{3,3}$ (nilpotent), and for $A_{4,10}$ it is of the type $A_{3,5}$. The nonzero commutation relations read as

$$A_{4,1}: \quad [X_2, X_4] = X_1, \quad [X_3, X_4] = X_2;$$

$$A_{4,2}: \quad [X_1, X_4] = qX_1, \quad [X_2, X_4] = X_2,$$

$$[X_3, X_4] = X_2 + X_3, \quad q \neq 0;$$

$$A_{4,3}: \quad [X_1, X_4] = X_1, \quad [X_3, X_4] = X_2;$$

$$A_{4,4}: \quad [X_1, X_4] = X_1, \quad [X_2, X_4] = X_1 + X_2,$$

$$[X_3, X_4] = X_2 + X_3;$$

$$A_{4,5}: \quad [X_1, X_4] = X_1, \quad [X_2, X_4] = qX_2,$$

$$[X_3, X_4] = pX_3, \quad -1 \leqslant p \leqslant q \leqslant 1, \quad pq \neq 0;$$

$$A_{4,6}: \quad [X_1, X_4] = qX_1, \quad [X_2, X_4] = pX_2 - X_3,$$

$$[X_3, X_4] = X_2 + pX_3, \quad q \neq 0, \quad p \geqslant 0;$$

$$A_{4,7}: \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = 2X_1,$$

$$[X_2, X_4] = X_2, \quad [X_3, X_4] = X_2 + X_3;$$

$$A_{4,8}: \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = (1+q)X_1,$$

$$[X_2, X_4] = X_2, \quad [X_3, X_4] = qX_3, \quad |q| \leqslant 1;$$

$$A_{4,9}: \quad [X_2, X_3] = X_1, \quad [X_1, X_4] = 2qX_1,$$

$$[X_2, X_4] = qX_2 - X_3, \quad [X_3, X_4] = X_2 + qX_3, \quad q \geqslant 0;$$

$$A_{4,10}: \quad [X_1, X_3] = X_1, \quad [X_2, X_3] = X_2,$$

$$[X_1, X_4] = -X_2, \quad [X_2, X_4] = X_1.$$

In order to obtain realizations of solvable four-dimensional symmetry algebras of PDEs that belong to the class (1.1), we add X_4 in the generic form (2.4) to the already constructed three-dimensional symmetry algebras and impose the above commutation relations. Once the algebra is found we insert X_4 into Eq. (2.5) and solve it for the function F. The form of F which is invariant under a three-dimensional algebra is further restricted:

$$A_{4,1}^1:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=\partial_t,\quad X_4=t\partial_x+x\partial_u,$$

$$F=-\tfrac{1}{2}u_x^2+\tilde{F}(u_{xx});$$

$$A_{4,1}^2:\quad X_1=\partial_u,\quad X_2=x\partial_u,\quad X_3=\partial_t,\quad X_4=\partial_x+tx\partial_u,$$

$$F=\tfrac{1}{2}x^2+\tilde{F}(u_{xx});$$

$$A_{4,2}^1:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=\partial_x,\quad X_4=3t\partial_t+x\partial_x+(x+u)\partial_u,$$

$$F=u_{xx}^2\tilde{F}(e^{u_x}u_{xx});$$

$$A_{4,2}^2:\quad X_1=\partial_x,\quad X_2=\partial_u,\quad X_3=\partial_t,\quad X_4=t\partial_t+\frac{x}{3}\partial_x+(t+u)\partial_u,$$

$$F=\frac{3}{2}\ln|u_x|+\tilde{F}(\omega),\quad \omega=\frac{u_{xx}^2}{u_x};$$

$$A_{4,2}^3:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=-3q^{-1}\ln x\partial_u,$$

$$X_4=qt\partial_t+\tfrac{1}{3}qx\partial_x+u\partial_u,\quad q\neq 0,$$

$$F=-2x^{-2}u_x+x^{3(q^{-1}-1)}\tilde{F}(\omega),\quad \omega=x^{1-3q^{-1}}u_x+x^{2-3q^{-1}}u_{xx};$$

$$A_{4,2}^4:\quad X_1=x^{3(1-q)}\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_t,$$

$$X_4=t\partial_t+\tfrac{1}{3}x\partial_x+(u+t)\partial_u,\quad q\neq 0,1,$$

$$F=-(2-3q)(1-3q)x^{-2}u_x+3\ln x+\tilde{F}(\omega),\quad \omega=(2-3q)x^{-2}u_x-x^{-1}u_{xx};$$

$$A_{4,3}^1:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=\partial_t,\quad X_4=t\partial_x+u\partial_u,$$

$$F=-u_x\ln|u_x|+u_x\tilde{F}\left(\frac{u_{xx}}{u_x}\right);$$

$$A_{4,3}^2:\quad X_1=\partial_t,\quad X_2=\partial_u,\quad X_3=-3\ln x\partial_u,$$

$$X_4=t\partial_t+\tfrac{1}{3}x\partial_x,$$

$$F=-2x^{-2}u_x+x^{-3}\tilde{F}(\omega),\quad \omega=xu_x+x^2u_{xx};$$

$$A_{4,3}^3:\quad X_1=\partial_u,\quad X_2=e^x\partial_u,\quad X_3=\partial_t,$$

$$X_4=\partial_x+(u+te^x)\partial_u;$$

$$F=-u_x+xe^x+e^x\tilde{F}(\omega),\quad \omega=e^{-x}(u_x-u_{xx});$$

$$A_{4,4}^1:\quad X_1=\partial_u\,,\ \ X_2=-3\ln x\,\partial_u\,,\ \ X_3=\partial_t\,,$$

$$X_4=t\partial_t+\tfrac{1}{3}x\partial_x+(u-3t\ln x)\partial_u\,;$$

$$A_{4,5}^1:\quad X_1=\partial_t\,,\quad X_2=\partial_x\,,\quad X_3=\partial_u\,,\quad X_4=t\partial_t+\frac{x}{3}\partial_x+ku\partial_u\,,\quad k\neq 0,\ \frac{1}{3},$$

$$F=u_x^{3(1-k)/(1-3k)}\tilde{F}(\omega),\quad \omega=u_x^{(3k-2)/(1-3k)}u_{xx}\,;$$

$$A_{4,5}^2:\quad X_1=\partial_t\,,\quad X_2=\partial_x\,,\quad X_3=\partial_u\,,$$

$$X_4=t\partial_t+\frac{x}{3}\partial_x+\frac{u}{3}\partial_u\,,$$

$$F=u_{xx}^2\tilde{F}(u_{xx});$$

$$A_{4,5}^3:\quad X_1=\partial_t\,,\quad X_2=u,\ \ X_3=x^{3(q-p)}\partial_u\,,$$

$$X_4=t\partial_t+\tfrac{1}{3}x\partial_x+qu\partial_u\,,\quad q\neq p,\quad q\cdot p\neq 0,$$

$$F=-[3(q-p)-1][3(q-p)-2]x^{-2}u_x+x^{3(q-1)}\tilde{F}(\omega),$$

$$\omega=[3(q-p)-1]x^{1-3q}u_x-x^{2-3q}u_{xx}\,;$$

$$A_{4,7}^1:\quad X_1=\partial_u\,,\quad X_2=\partial_x\,,\quad X_3=x\partial_u-\tfrac{1}{3}\ln t\,\partial_x\,,\quad X_4=3t\partial_t+x\partial_x+2u\partial_u\,,$$

$$F=\frac{1}{6t}u_x^2+t^{-1/3}\tilde{F}(u_{xx});$$

$$A_{4,7}^2:\quad X_1=\partial_u\,,\qquad X_2=x\partial_u+b\partial_x\,,\quad X_3=-\partial_x\,,$$

$$X_4=-b^2(b')^{-1}\partial_t+(1-b)x\partial_x+(2u-\tfrac{1}{2}x^2)\partial_u\,,\quad b=b(t),\ \ b'\neq 0,$$

$$b^2b''+(b-3)(b')^2=0,$$

$$F=-\tfrac{1}{2}b'u_x^2+b^{-3}e^{-b^{-1}}\tilde{F}(\omega),\quad \omega=b^2u_{xx}-b;$$

$$A_{4,7}^3:\quad X_1=\partial_u\,,\quad X_2=(\lambda x^3-t)\partial_u\,,\quad X_3=\partial_t\,,$$

$$X_4=t\partial_t+\tfrac{1}{3}x\partial_x+(2u-\tfrac{1}{2}t^2+\lambda tx^3)\partial_u\,,\quad \lambda\neq 0,$$

$$F=-\tfrac{1}{3}\lambda^{-1}(1+6\lambda)x^{-2}u_x+3\lambda x^3\ln x+x^3\tilde{F}(\omega),\quad \omega=2x^{-5}u_x-x^{-4}u_{xx}\,;$$

$$A_{4,7}^4:\quad X_1=\partial_u\,,\quad X_2=(t-x)\partial_u\,,\quad X_3=\partial_x\,,$$

$$X_4=3t\partial_t+(x+2t)\partial_x+\left(xt-\frac{x^2}{2}+2u\right)\partial_u\,,$$

$$F=-u_x+t^{-1/3}\tilde{F}(\omega)+\frac{t}{4},\quad \omega=u_{xx}+\frac{1}{3}\ln t;$$

$$A_{4,7}^5:\quad X_1=\partial_u,\quad X_2=-x\partial_u,\quad X_3=\partial_x,$$

$$X_4=3t\partial_t+x\partial_x+\left(2u-\frac{x^2}{2}\right)\partial_u,$$

$$F=t^{-1/3}\tilde{F}(\omega),\quad \omega=u_{xx}+\tfrac{1}{3}\ln t;$$

$$A_{4,7}^6:\quad X_1=-x^{-1}\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_x-x^{-1}u\partial_u,$$

$$X_4=3t\partial_t+x\partial_x+(u+\tfrac{1}{2}x)\partial_u,$$

$$F=3x^{-1}u_{xx}+x^{-1}t^{-1/3}\tilde{F}(\omega),\quad \omega=2u_x+xu_{xx}-\tfrac{1}{3}\ln t;$$

$$A_{4,8}^1:\quad X_1=\partial_x,\quad X_2=\partial_t,\quad X_3=t\partial_x+\partial_u,\quad X_4=t\partial_t+\frac{x}{3}\partial_x-\frac{2}{3}u\partial_u,$$

$$F=-uu_x+u_x^{5/3}\tilde{F}(u_x^{-4/3}u_{xx});$$

$$A_{4,8}^2:\quad X_1=\partial_u,\quad X_2=\partial_t,\quad X_3=t\partial_u+\lambda\partial_x,\quad X_4=t\partial_t+\frac{x}{3}\partial_x+\frac{4}{3}u\partial_u,$$

$$F=\frac{x}{\lambda}+u_x^{1/3}\tilde{F}(\omega),\quad \omega=u_x^{-2/3}u_{xx},\quad \lambda>0;$$

$$A_{4,8}^3:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=x\partial_u+\lambda t^{(1-q)/3}\partial_x,$$

$$X_4=3t\partial_t+x\partial_x+(1+q)u\partial_u,\quad q\in\mathbb{R},$$

$$F=\frac{\lambda(q-1)}{6}t^{-(2+q)/3}u_x^2+t^{(q-2)/3}\tilde{F}(\omega),\quad \omega=t^{(1-q)/3}u_{xx},\quad \lambda\neq 0,\quad |q|\neq 1;$$

$$A_{4,8}^4:\quad X_1=\partial_u,\quad X_2=\partial_x,\quad X_3=x\partial_u+\lambda\partial_t,$$

$$X_4=3t\partial_t+x\partial_x+4u\partial_u,\quad \lambda\neq 0,$$

$$F=(t-3\lambda u_x)^{1/3}\tilde{F}(\omega),\quad \omega=u_{xx}^3(t-3\lambda u_x)^2;$$

$$A_{4,8}^5:\quad X_1=\partial_u,\quad X_2=(\lambda x^3-t)\partial_u,\quad X_3=\partial_t,$$

$$X_4=qt\partial_t+\tfrac{1}{3}qx\partial_x+(1+q)u\partial_u,\quad \lambda\cdot q\neq 0,$$

$$F=-\tfrac{1}{3}\lambda^{-1}(1+6\lambda)x^{-2}u_x+x^{3q^{-1}}\tilde{F}(\omega),\quad \omega=2x^{-(2+3q^{-1})}u_x-x^{-(1+3q^{-1})}u_{xx};$$

$$A_{4,8}^6:\quad X_1=\partial_u,\quad X_2=(t-x)\partial_u,\quad X_3=\partial_x,$$

$$X_4=3qt\partial_t+q(x+2t)\partial_x+(1+q)u\partial_u,$$

$$F=-u_x+t^{(1/3)(1-2q)q^{-1}}\tilde{F}(\omega),\quad \omega=t^{(1/3)(q-1)q^{-1}}u_{xx};$$

$$A_{4,8}^7:\quad X_1=-x^{-1}\partial_u,\quad X_2=\partial_u,\quad X_3=\partial_x-x^{-1}u\partial_u,$$

$$X_4 = 3qt + qx\partial_x + u\partial_u, \quad q \neq 0,$$

$$F = 3x^{-1}u_{xx} + x^{-1}t^{(1/3)(q-1)q^{-1}}\tilde{F}(\omega), \quad \omega = t^{(1/3)(q-1)q^{-1}}(2u_x + xu_{xx});$$

$$A_{4,8}^8: \quad X_1 = \partial_u, \quad X_2 = -x\partial_u, \quad X_3 = x\partial_x, \quad X_4 = 3qt\partial_t + qx\partial_x + (1+q)u\partial_u,$$

$$F = t^{(1/3)(1-2q)q^{-1}}\tilde{F}(\omega), \quad \omega = t^{(1/3)(q-1)q^{-1}}u_{xx}.$$

Remarks:

- There exists a realization of $A_{4,6}$:

$$A_{4,6}^1: \quad X_1 = \partial_t, \quad X_2 = \tan\psi\partial_u, \quad X_3 = \partial_u,$$

$$X_4 = 2t\partial_t + \tfrac{2}{3}x\partial_x + [p + \tan\psi]u\partial_u,$$

$$\psi = \tfrac{3}{2}\ln x, \quad p \in \mathbb{R}.$$

However there are no equations that can be invariant under this algebra.

- The algebra $A_{4,8}^1$ is isomorphic to the KdV algebra which is the semidirect sum of the nilradical (maximal nilpotent ideal) $\mathfrak{h}(2) = \{X_1, X_2, X_3\}$ and the dilation $\mathrm{D} = \{X_4\}$:

$$A_{4,9}^1: \quad X_1 = \partial_u, \quad X_2 = \partial_x, \quad X_3 = \alpha(t)\partial_x + x\partial_u,$$

$$X_4 = -\frac{(1+\alpha^2)}{\dot{\alpha}}\partial_t + (q-\alpha)x\partial_x + \left(2qu - \frac{x^2}{2}\right)\partial_u, \quad q \in \mathbb{R},$$

$$F = -\tfrac{1}{2}\dot{\alpha}u_x^2 + (1+\alpha^2)^{-3/2}\exp(q\arctan\alpha)\tilde{F}(\omega), \quad \omega = (1+\alpha^2)u_{xx} - \alpha,$$

$$(1+\alpha^2)\ddot{\alpha} + (\alpha - 3q)\dot{\alpha}^2 = 0.$$

The function $\alpha(t)$, $\dot{\alpha} \neq 0$ is a solution of the ordinary differential equation (5.15):

$$A_{4,10}^1: \quad X_1 = \partial_u, \quad X_2 = -\tan x\partial_u, \quad X_3 = \partial_t + u\partial_u,$$

$$X_4 = \beta\partial_t + \partial_x + u\tan x\partial_u, \quad \beta \in \mathbb{R},$$

$$F = -2u_x - 3\tan xu_{xx} + e^{t-\beta x}\sec x\tilde{F}(\omega),$$

$$\omega = e^{\beta x - t}(\cos xu_{xx} - 2\sin xu_x).$$

We sum up the above results as a theorem.

Theorem 6.1: *There exist fifty-two inequivalent four-dimensional symmetry algebras admitted by Eq. (1.1). The explicit forms of those algebras as well as the associated invariant equations are given above*.

VII. DISCUSSION AND CONCLUSIONS

In this paper we provide a symmetry classification of the KdV type equations involving an arbitrary function of five arguments. We find that the equivalence classes of invariant equations involve an arbitrary function of four, three, two variables and one variable as soon as the symmetry algebra is one-, two-, three- and four-dimensional, respectively. In particular, we studied symmetries of the most general third order linear evolution equation. What came out from this, to our surprise, is that the symmetry group allowed is four-dimensional at most, while there are nonlinear equations with symmetry algebras greater than four. This result is in contrast to the second-order evolution equations. It is exactly the linear heat equation that allows for the maximal symmetry algebra.

To complete the classification list, it only remains to obtain the inequivalent equations invariant under solvable algebras of the dimension $\dim L \geqslant 5$. But this would require to going through a large number of isomorphism classes. To give an idea of the complexity of this task let us recall that there are sixty-six classes of nonisomorphic real, solvable Lie algebras of dimension five. For dimension six, there exist ninety-nine classes of them with a nilpotent element. We plan to devote a separate article to study equations admitting higher-dimensional symmetry algebras.

Whenever F is an arbitrary function of its arguments, the symmetry algebras given in the paper are maximal. In particular, if we impose the requirement that function F is be independent of u_{xx} then we find that $\phi = R(t)u + S(x,t)$ in (2.4). In this case, invariance under four-dimensional algebras will force F to depend on an arbitrary constant rather than on an arbitrary function. Then, they may admit symmetry groups of the dimension higher than four. We have analyzed this restricted class of equations and obtained that the only equation whose symmetry algebra is higher than four is the one corresponding to the realization $A_{4,1}^1$ for $\tilde{F} = \text{const}$. On the other hand, for the specific choices of $\tilde{F}$ involving one variable, the equations with four-dimensional symmetry algebras may be invariant under larger symmetry groups. For instance, the particular case of the equation invariant under $A_{4,1}^1$ obtained by setting $\tilde{F} = c u_{xx}^{4/3}$, $c = \text{const}$ admits an additional symmetry group generated by the dilation operator $X_5 = 3t\partial_t + x\partial_x - u\partial_u$.

We only presented representative lists of equivalence classes of invariant equations. All other invariant equations can be recovered from these lists by applying the point transformations (2.6). In other words, an equation in the class (1.1) will have a symmetry group with dimension satisfying $\dim L \leqslant 6$ if and only if it can be transformed to one in the (canonical) equations from the list.

As we mentioned, our classification is performed within point transformations of coordinates. Two equations are equivalent if one can be obtained from the other by a change of variables. On the other hand, consider a special case of (6.2) for $c = -3/4$,[24]

$$u_t = u_{xxx} - \frac{3}{4}\frac{u_{xx}^2}{u_x},$$

which additionally allows a symmetry group generated by $\{\partial_u\}$. Though this equation is equivalent to the third-order linear equation $v_t = v_{xxx}$ under the (no-point) transformation $v = \sqrt{u_x}$, we treat them as inequivalent.

To give a reader an insight into possible applications of the results of this article, we consider a subclass of Eqs. (1.1),

$$u_t = u_{xxx} + u u_x + f(t)u, \tag{7.1}$$

which arises in several physical applications such as the propagation of waves in shallow water of variable depth.

When $f(t)$ is arbitrary, (7.1) admits a two-dimensional Abelian symmetry algebra generated by

$$X_1 = \partial_x, \quad X_2 = \xi(t)\partial_x - \dot{\xi}(t)\partial_u, \quad \xi = \int \exp\left\{\int f(t) \; dt\right\} \; dt. \tag{7.2}$$

By the change of dependent variable $\tilde{u} = u/\dot{\xi}$, the generators are transformed to the realization $A_{2,1}^3$ with $\alpha = -\xi$. The corresponding invariant equation takes the form

$$\tilde{u}_t = \tilde{u}_{xxx} + \dot{\xi}\tilde{u}\tilde{u}_x,$$

which is a particular case of (1.2).

For the special case $f(t)=at^k$ $(a\neq 0)$, the algebra is larger and we have the following possibilities for the algebra to be either three- or four-dimensional.

(1) $(a,k)=(a,-1)$, $a\neq -1$: The equation admits the three-dimensional indecomposable solvable symmetry algebra spanned by

$$X_1=\partial_x,\quad X_2=t^{1+a}\partial_x-(1+a)t^a\partial_u,\quad X_3=t\partial_t+\frac{x}{3}\partial_x-\frac{2}{3}u\partial_u, \tag{7.3}$$

with nonzero commutation relations

$$[X_1,X_3]=-\frac{1}{3}X_1,\quad [X_2,X_3]=-\frac{3a+2}{3}X_2.$$

For $a=-1/3$, the algebra is isomorphic, up to the scaling of basis elements, to $A_{3,5}$, for $-1<a<-\frac{1}{3}$, to $A_{3,7}$.

For $a=-2/3$ it is isomorphic to the decomposable solvable algebra $A_{3,2}$ and a suitable basis is

$$X_1=\partial_x,\quad X_2=t^{1/3}\partial_x-\frac{1}{3}t^{-2/3}\partial_u,\quad X_3=t\partial_t+\frac{x}{3}\partial_x-\frac{2}{3}u\partial_u.$$

With the equivalence transformation

$$\tilde{t}=t,\quad \tilde{x}=x,\quad \tilde{u}=-3t^{2/3}u,$$

the basis elements are transformed, up to scaling, to the realization $A_{3,2}^3$. The transformed equation is

$$\tilde{u}_t=\tilde{u}_{xxx}-\tfrac{1}{3}t^{-2/3}\tilde{u}\tilde{u}_x.$$

This equation belongs to the class corresponding to the realization $A_{3,2}^3$.

We note that a member of (1.2) for $f=1$, $g=t^2$ (see Ref. 1) is equivalent, under appropriate point transformation, to the above equation. Similarly, the particular case $a=-\alpha/(1+\alpha)$, $\alpha\neq 0,1,2$ is equivalent to $f=1$, $g=t^\alpha$ of (1.2). In this case, the symmetry algebra is indecomposable and solvable.

(2) $(a,k)=(-1,-1)$: the spherical KdV (sKdV) equation.

In this case the equation is invariant with respect to a three-dimensional symmetry algebra. We choose its basis to be

$$X_1=\partial_x,\quad X_2=\ln t\,\partial_x-\frac{1}{t}\partial_u,\quad X_3=t\partial_t+\frac{x}{3}\partial_x-\frac{2}{3}u\partial_u, \tag{7.4}$$

with nonzero commutation relations

$$[X_3,X_1]=-\tfrac{1}{3}X_1,\quad [X_3,X_2]=X_1-\tfrac{1}{3}X_2.$$

It is easy to see that this algebra is isomorphic to $A_{3,4}$. Under the transformation $\tilde{u}=3tu$, the generators are transformed to the realization $A_{3,4}^2$. The sKdV equation takes the form

$$\tilde{u}_t=\tilde{u}_{xxx}+\frac{1}{3t}\tilde{u}\tilde{u}_x,$$

which is a particular case of the equation invariant under the algebra $A_{3,4}^2$.

We note that a member of (1.2) for $f=1$, $g=e^{3t1}$ is equivalent to the case (2), i.e. the sKdV equation.

(3) $(a,k)=(a,0)$: The basis of the symmetry algebra reads as

$$X_1=\partial_x, \qquad X_2=e^{at}(\partial_x-a\partial_u), \quad X_3=\partial_t, \tag{7.5}$$

with nonzero commutation relation $[X_3,X_2]=aX_2$. The algebra is isomorphic to $A_{3,2}$. With the transformation $\tilde{u}=e^{-at}u$ the equation is transformed to a special case of (1.2) for $f=1, g=e^{at}$.

(4) $(a,k)=(-1/2,-1)$: the cylindrical KdV (cKdV) equation.

In this case the symmetry algebra is four-dimensional. In a convenient basis we have

$$\begin{aligned} X_1&=2\sqrt{t}\partial_x-\frac{1}{\sqrt{t}}\partial_u, \\ X_2&=4t^{3/2}\partial_t+2x\sqrt{t}\partial_x-\left(\frac{x}{\sqrt{t}}+4\sqrt{t}u\right)\partial_u, \\ X_3&=\partial_x, \quad X_4=3t\partial_t+x\partial_x-2u\partial_u, \end{aligned} \tag{7.6}$$

with nonzero commutation relations

$$[X_2,X_3]=-X_1, \quad [X_1,X_4]=-\tfrac{1}{2}X_1, \quad [X_2,X_4]=-\tfrac{3}{2}X_2, \quad [X_3,X_4]=X_3.$$

We see that the symmetry algebra of the cKdV equation is isomorphic to the algebra $A_{4,8}$ with $q=1$. The existence of such an isomorphism is a necessary, but not sufficient condition for a local point transformation to exist, transforming the two equations into each other. Comparing these generators with (2.10) and choosing (2.8) suitably, for example, first transforming the commuting elements $\{X_1,X_3\}$ into $\{\partial_{\tilde{x}},\tilde{t}\partial_{\tilde{x}}+\partial_{\tilde{u}}\}$ and then transforming the remaining ones with the aid of the freedom left in equivalence transformations we arrive at

$$\tilde{t}=2t^{-1/2}, \quad \tilde{x}=t^{-1/2}x, \quad \tilde{u}=tu+\frac{x}{2},$$

which establishes the equivalence of the Lie algebra with basis (7.6) and the cKdV equation to the KdV algebra ($A_{4,8}^1$) and KdV equation. This connection between the KdV and cKdV equations is well-known in the literature.[19]

As a further comparison of the results obtained in the article we consider

$$u_t+u_{xxx}+f(u)u_x^k=0, \quad k>0, \tag{7.7}$$

which is clearly a special case of (1.1). Group classification of this equation is given in a table (see Table II).[22] These results can immediately be derived from those obtained in this paper either directly or performing a change of independent or dependent variables.

Note that the equations that do not appear in the classification list can be recovered from those by suitable point transformations.

A number of integrable KdV type equations can be reproduced by restricting the arbitrary functions contained in invariant equations of this article. For example, the realization $A_{3,7}^2$ is equivalent to $\{\partial_t,\partial_x,t\partial_t+x/3\partial_x-u/3\partial_u\}$ under the transformation $u\to u^{-1/3}$. We have the invariant function

$$F=u^4\tilde{F}(\omega_1,\omega_2), \quad \omega_1=u^{-2}u_x, \quad \omega_2=u^{-3}u_{xx}.$$

Setting $\tilde{F}=\omega_1$ produces the modified KdV (mKdV) equation

TABLE II. Symmetry classification of (7.7).

N	k	$f(u)$	Symmetry generators	Symmetry algebra
1	arb.	arb.	∂_t,∂_x	$A^1_{2,1}$
2	k	u^n	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x+\frac{k-3}{k+n-1}u\partial_u,\quad k+n\neq 1$	$A^2_{3,7}$
3	k	e^u	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x+(k-3)\partial_u$	$A^2_{3,7}$
4	k	1	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x+\frac{k-3}{3(k-1)}u\partial_u,\partial_u,\quad k\neq 1$	$A^1_{4,3}$
5	3	arb.	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x$	$A^1_{3,7}$
6	3	u^{-2}	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x,u\partial_u$	$A^1_{3,7}\oplus A_1$
7	3	1	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x,\partial_u$	$A^1_{3,7}\oplus A_1$
8	1	u^n+c	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x-\frac{2}{n}u\partial_u,\quad n\neq 0$	$A^2_{3,7}$
9	1	u	$\partial_t,\partial_x,t\partial_t+\frac{x}{3}\partial_x-\frac{2}{3}u\partial_u,t\partial_x+\partial_u$	$A^1_{4,6}$
10	1	e^u+c	$\partial_t,\partial_x,t\partial_t+\frac{1}{3}(\partial_x+2ct)\partial_x-\frac{2}{3}\partial_u$	$A^2_{3,7}$
11	1	1	$\partial_t,\partial_x,t\partial_t+\frac{1}{3}(x+2t)\partial_x,u\partial_u,g(x,t)\partial_u$ $g_t+g_{xxx}+g_x=0$	Linear equation ($N=6$ in Table I)

$$u_t=u_{xxx}+u^2u_x. \tag{7.8}$$

Since the maximal symmetry algebra of the mKdV equation is three-dimensional, it is not isomorphic to the KdV algebra. This implies that there is no point transformation, transforming the mKdV equation into KdV equation. In this respect let us mention that there is the well-known nonlocal transformation (Miura transformation),

$$\tilde{u}=u^2\pm\sqrt{6}iu_x,$$

taking the mKdV (7.8) into the KdV equation $\tilde{u}_t=\tilde{u}_{xxx}+\tilde{u}\tilde{u}_x$. Another integrable equation which can be obtained from our classification is[23]

$$u_t=u_{xxx}+3(u_{xx}u^2+3uu_x^2)+3u^4u_x.$$

Its symmetry algebra is isomorphic to $A^2_{3,7}$. Note that this equation can be linearized by a change of dependent variable.

Let us mention that a classification based on higher order symmetries of third order integrable nonlinear equations of the form

$$u_t=u_{xxx}+F(u,u_x,u_{xx}) \tag{7.9}$$

is given in Ref. 24. We should also note that the question of finding PDEs admitting Lie point symmetries is different than finding integrable PDEs. In the latter case one requires the existence of a generalized one as opposed to Lie point symmetries. For the classification of integrable PDEs we refer the reader to Refs. 25–27.

Finally, let us point out that in a very recent work[28] a class of integrable (in the sense of existence of an infinite number of generalized symmetries) third order evolution equations of the form (7.9) for specific F admitting recursion operators have been analyzed. Among others, the special cases corresponding to $\tilde{F}=3\omega$ and $\tilde{F}=3/2-3\omega$ of (6.1) produce the following equations with 4-dimensional symmetry algebra $A^1_{3,7}\oplus A_1$:

$$u_t=u_{xxx}+3u^{-1}u_xu_{xx},$$

$$u_t=u_{xxx}-3u^{-1}u_xu_{xx}+\tfrac{3}{2}u^{-2}u_x^3,$$

both of which were shown to admit recursion operators. This fact indicates that many equations with relatively large symmetry groups in our classification are among the most probable candidates for being integrable.

We note that the maximal symmetry algebra of the first equation of the above list is infinite-dimensional with basis elements:

$$X_1=\partial_t,\quad X_2=\partial_x,\quad X_3=t\partial_t+\frac{x}{3}\partial_x+\frac{u}{2}\partial_u,$$

$$X(\rho)=\rho(x,t)u^{-1}\partial_u,\quad \rho_t=\rho_{xxx}.$$

The existence of an infinite-dimensional symmetry algebra suggests linearizability of the equation by point transformations and, indeed, it is linearized by the change of dependent variable $v(x,t)=u^2(x,t)/2$.

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