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Symmetries and exact solutions of the time fractional Harry-Dym equation with Riemann–Liouville derivative

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HIGHLIGHTS

- Lie classical method is used to investigate the time fractional Harry-Dym equation.
- The maximal Lie symmetry group of the equation under study is derived.
- Optimal system of subgroups are determined and the optimization of the system is proved.
- Reduced fractional ODEs and some exact solutions in explicit forms are presented.
- The approach used here can also be applied to other nonlinear fractional PDEs.

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ABSTRACT

In this paper, group analysis of the time fractional Harry-Dym equation with Riemann–Liouville derivative is performed. Its maximal symmetry group in Lie's sense and the corresponding optimal system of subgroups are determined. Similarity reductions of the equation under study are performed. As a result, the reduced fractional ordinary differential equations are deduced, and some group invariant solutions in explicit form are obtained as well.

1. Introduction

The Harry-Dym equation (HD)

$$u_t = u^3 u_{xxx},$$

where $u = u(t, x)$, $u_x = \partial u/\partial x$, $u_{xx} = \partial^2 u/\partial x^2$, $u_{xxx} = \partial^3 u/\partial x^3$, is one of the most interesting and exotic soliton equations. It was discovered in an unpublished paper by Harry Dym in 1973, and appeared in a more general form in Ref. [1] within the classical string problem. Kruskal and Moster referred to it as Harry-Dym equation [2] and this name has been used ever since. HD equation is completely integrable and its properties are discussed in detail in Ref. [3]. We simply emphasize that it has many typical properties of the soliton equations, namely, it can be solved by the inverse scattering transform, it has a bi-Hamiltonian structure, infinitely many conservation laws and symmetries [4]. However, it does not possess Painlevé property. What is more, HD equation can be connected to KdV equation via the reciprocal transformation [3].

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Fractional calculus is a generalization of customary differentiation and integration to arbitrary non-integer order. It is as old as the classical calculus. And it can go back to the time when Leibnitz and Newton invented differential calculus. Fractional differential equations (FDEs) are generalizations of the previous differential equations (DEs) of integer order to non-integer one through the application of fractional calculus. It has been shown that new fractional-order models are more adequate than previously used integer-order models due to an exact description of nonlinear phenomena. Since FDEs appear more and more frequently in various research and engineering applications, such as rheology, viscoelasticity, biology, physics and electrochemistry (see Refs. [5–11] and references therein), they attract considerable interest and there has been a significant theoretical development recently (for example, see Refs. [7,8,10]). As far as the methods for solving such equations are concerned, there are no efficient general methods until now. Nevertheless, various methods such as Laplace transform [5], Fourier transform [8], variational iteration [12], finite difference [13], Adomian decomposition [14], homotopy perturbation [15,16], and generalized differential transform [17] methods, have been used to solve these equations and have achieved significant progress.

It is well-known that group theory is a universal and convenient tool for analysis of partial differential equations (PDEs) and symmetry properties of PDEs have been extensively studied [18,19]. Then can symmetry group theory be efficiently used in FDEs just as it has been done in PDEs? There are only few papers which are devoted to group analysis of FDEs and group properties of FDEs are much less understood. Scaling transformation of the time fractional linear wave-diffusion equation and its group invariant solutions have been described in Ref. [20]. Using groups of scaling transformation, self-similar solutions to a nonlinear fractional diffusion, Burgers and KdV equations are obtained [21]. Lie symmetries of the fractional nonlinear anomalous diffusion equations are studied in Refs. [22,23]. The problem of group analysis of fractional generalized Burgers and KdV equations is discussed in Refs. [24,25]. Complete group classifications and symmetry reductions of the fractional fifth-order KdV types of equations is performed in Ref. [26].

Time fractional PDEs are from classical PDEs by replacing its time derivative by fractional derivative. In this paper we perform Lie symmetry analysis, construct optimal system and determine group invariant solutions of the time fractional HD equation

$$\partial_t^\alpha u = u^3 u_{xxx}, \quad (0 < \alpha < 1), \tag{1}$$

where $\partial_t^\alpha u$ is the Riemann–Liouville fractional derivative of order α with respect to the variable t.

The paper is organized as follows. In Section 2, we state the definition of Riemann–Liouville derivative and give an account of Lie symmetry analysis method for FDEs briefly. Section 3 is devoted to the description of Lie symmetries of the time fractional HD equation (1). In addition, we construct the optimal system of subgroups of the invariance group admitted by the equation under study. In Section 4 we perform similarity reductions of Eq. (1) and construct its group-invariant solutions. Section 5 contains discussion of the obtained results.

2. Preliminaries

For the fractional derivative operators, there exist various definitions which are not necessarily equivalent to each other. In this paper, we consider the most common definition named after Riemann and Liouville, which is the natural generalization of the Cauchy formula for the n-fold primitive of a function $f(t)$. The Riemann–Liouville fractional derivative is defined as

$$D^\alpha f(t) = \begin{cases} \dfrac{\mathrm{d}^n f}{\mathrm{d}t^n}, & \alpha = n, \\ \dfrac{\mathrm{d}^n}{\mathrm{d}t^n} I^{n-\alpha} f(t), & 0 \le n-1 < \alpha < n, \end{cases} \tag{2}$$

where $n \in \mathbf{N}$, $I^\mu f(t)$ is the Riemann–Liouville fractional integral of order μ, namely,

$$I^\mu f(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} f(s)\mathrm{d}s, \quad \mu > 0$$
$$I^0 f(t) = f(t)$$

and $\Gamma(z)$ is the standard Gamma function.

Similarly, the fractional partial derivative of order α for the function $u(t,x)$ with respect to the variable t can be defined as below.

Definition 1. The Riemann–Liouville fractional partial derivative is defined by

$$\partial_t^\alpha u(t,x) = \begin{cases} \dfrac{\partial^n u}{\partial t^n}, & \alpha = n, \\ \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t (t-s)^{n-\alpha-1} u(s,x)\mathrm{d}s, & 0 \le n-1 < \alpha < n. \end{cases} \tag{3}$$

If it exists, where ∂_t^n is the usual partial derivative of integer order n.

In literature, some alternative definitions of fractional derivative are considered, such as the Caputo, Weyl and Riesz fractional derivatives [6–10]. Each fractional derivative presents some advantages and disadvantages. For instance, the Riemann–Liouville derivative of a constant is not zero. The Caputo derivative of a constant is zero, but it is defined only for differentiable functions. While functions that have no first-order derivative might have fractional derivatives of all orders less than one in the Riemann–Liouville sense [27].

We now consider the symmetry analysis for a FDE of the form

$$\partial_t^\alpha u = F(t, x, u, u_x, u_{xx}, u_{xxx}), \quad (0 < \alpha < 1). \tag{4}$$

Given a one-parameter Lie group of infinitesimal transformations

$$t^* = t^*(t, x, u; \epsilon), \qquad x^* = x^*(t, x, u; \epsilon), \qquad u^* = u^*(t, x, u; \epsilon), \tag{5}$$

with the group parameter ϵ, the associated Lie algebra of (5) is spanned by vector fields

$$X = \tau\partial_t + \xi\partial_x + \eta\partial_u, \tag{6}$$

where

$$\tau = \left.\frac{dt^*}{d\epsilon}\right|_{\epsilon=0}, \qquad \xi = \left.\frac{dx^*}{d\epsilon}\right|_{\epsilon=0}, \qquad \eta = \left.\frac{du^*}{d\epsilon}\right|_{\epsilon=0}.$$

According to the infinitesimal invariance criterion, Eq. (4) admits transformation group (5) (or equivalently, vector fields (6)) iff the prolonged vector field $\mathrm{pr}^{(\alpha,3)}X$ annihilates (4) on its solution manifold, namely,

$$\mathrm{pr}^{(\alpha,3)}X(\Delta)|_{\Delta=0} = 0, \quad \Delta = \partial_t^\alpha u - F. \tag{7}$$

The operator $\mathrm{pr}^{(\alpha,3)}X$ takes the form (note that here we only keep the essential terms)

$$\mathrm{pr}^{(\alpha,3)}X = X + \eta^{\alpha,t}\partial_{\partial_t^\alpha u} + \eta^x\partial_{u_x} + \eta^{xx}\partial_{u_{xx}} + \eta^{xxx}\partial_{u_{xxx}}, \tag{8}$$

where

$$\begin{aligned}
\eta^{\alpha,t} &= D_t^\alpha(\eta) + \xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) + D_t^\alpha(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u),\\
\eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\
\eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi),
\end{aligned}$$

the symbols D_t, D_x indicate the total derivatives with respect to t and x respectively,

$$\begin{aligned}
D_t &= \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_x} + \cdots,\\
D_x &= \partial_x + u_x\partial_u + u_{tx}\partial_{u_t} + u_{xx}\partial_{u_x} + \cdots,
\end{aligned}$$

and D_t^α is the total fractional derivative with respect to t.

In order to determine the vector fields (6) admitted by Eq. (4), it is necessary to obtain the concrete expression for $\mathrm{pr}^{(\alpha,3)}X$. Since the explicit expression for η^x, η^{xx} and η^{xxx} in (8) can be easily obtained in a standard fashion [19,28], here we concentrate on the expression for $\eta^{\alpha,t}$.

Using the generalized Leibnitz rule [8,10] given by

$$D_t^\alpha(f(t)g(t)) = \sum_{n=0}^{\infty}\binom{\alpha}{n} D_t^n f(t) D_t^{\alpha-n} g(t), \tag{9}$$

where

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{\Gamma(n+1)\Gamma(\alpha+1-n)}, \qquad D_t^0 f(t) = f(t), \qquad D_t^{n+1} f(t) = D_t(D_t^n f(t)),$$

we arrive at

$$\xi D_t^\alpha(u_x) - D_t^\alpha(\xi u_x) = -\sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x),$$

and

$$D_t^\alpha(uD_t(\tau)) - D_t^{\alpha+1}(\tau u) + \tau D_t^{\alpha+1}(u) = -\alpha D_t(\tau) D_t^\alpha u - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u).$$

Consequently, the relation

$$\eta^{\alpha,t} = D_t^\alpha(\eta) - \alpha D_t(\tau) D_t^\alpha u - \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi) D_t^{\alpha-n}(u_x) - \sum_{n=1}^{\infty}\binom{\alpha}{n+1} D_t^{n+1}(\tau) D_t^{\alpha-n}(u)$$

is obtained. Now the problem of finding the expression for $\eta^{\alpha,t}$ reduces to that of determining the first item $D_t^\alpha(\eta)$ of it. In view of the generalized chain rule for a composite function [9] of the form

$$\frac{\mathrm{d}^\alpha f(g(t))}{\mathrm{d}t^\alpha} = \sum_{n=0}^{\infty} \frac{U_n}{n!} \frac{\mathrm{d}^n f(z)}{\mathrm{d}z^n}|_{z=g(t)},$$

where

$$U_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^k(t) \partial_t^\alpha (g^{n-k}(t)),$$

and the generalized Leibnitz rule (9), we have

$$D_t^\alpha(\eta) = \partial_t^\alpha \eta + \eta_u \partial_t^\alpha u - u \partial_t^\alpha \eta_u + \sum_{n=1}^{\infty} \binom{\alpha}{n} \partial_t^n \eta_u \partial_t^{\alpha-n} u + \mu \tag{10}$$

with

$$\mu = \sum_{n=2}^{\infty} \sum_{m=2}^{n} \sum_{k=2}^{m} \binom{\alpha}{n} \binom{n}{m} \frac{t^{n-\alpha} U_k}{k! \Gamma(n+1-\alpha)} \frac{\partial^{n-m+k} \eta}{\partial t^{n-m} \partial u^k}.$$

It should be noted that we have $\mu = 0$ when the infinitesimal η is linear of the variable u, because of the existence of the derivatives $\partial^k \eta / \partial u^k,\ k \geq 2$ in above expression. Summarizing the reasonings above, we obtain the explicit form of $\eta^{\alpha,t}$

$$\begin{aligned} \eta^{\alpha,t} = {} & \sum_{n=1}^{\infty} \left[\binom{\alpha}{n} \partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) \right] \partial_t^{\alpha-n} u - \sum_{n=1}^{\infty} \binom{\alpha}{n} D_t^n(\xi) \partial_t^{\alpha-n}(u_x) \\ & + \partial_t^\alpha \eta + (\eta_u - \alpha D_t(\tau)) \partial_t^\alpha u - u \partial_t^\alpha \eta_u + \mu. \end{aligned}$$

To determine the Lie symmetries (6) admitted by FDE (4), we act the differential operator $\mathrm{pr}^{(\alpha,3)}X$ on Eq. (4) according to the invariance criterion (7), split the obtained relation by independent variables, equate these coefficients to zero and solve the obtained over-determined system of linear PDEs and FDEs.

3. Lie symmetries and optimal system for the fractional HD equation

Following Lie group analysis approach for FDEs in Section 2, we devote to derive Lie symmetries admitted by Eq. (1) and construct its optimal system in this section.

For the fractional HD equation (1), the invariance criterion takes the form

$$\left[\eta^{\alpha,t} - 3u^2 u_{xxx} \eta - u^3 \eta^{xxx}\right]|_{\partial_t^\alpha u = u^3 u_{xxx}} = 0.$$

By equating to zero the coefficients of linearly independent derivatives $u_t, u_x, u_{xx}, \ldots$, and $\partial_t^{\alpha-n} u_x$, $\partial_t^{\alpha-n} u$ in above relation, we obtain the following over-determined system of linear PDEs and FDEs, which is called as determining equations:

$$\begin{aligned} & \binom{\alpha}{n} \partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau) = 0 \qquad n = 1, 2, 3, \ldots, \\ & \tau_x = \tau_u = \xi_t = \xi_u = \eta_{uu} = 0, \\ & 3u\xi_x - \alpha \tau_t u - 3\eta = 0, \\ & \partial_t^\alpha \eta - u \partial_t^\alpha \eta_u - u^3 \eta_{xxx} = 0. \end{aligned}$$

Determining equations above are easily integrated to yield the following general solution:

$$\tau = c_3 t + c_5, \qquad \xi = c_4 x^2 + c_2 x + c_1, \qquad \eta = \left(2c_4 x + c_2 - \frac{c_3 \alpha}{3}\right) u,$$

where c_i $(i = 1, 2, \ldots, 5)$ are arbitrary constants. Further, imposing the requirement that transformation (5) should preserve the structure of the Riemann–Liouville fractional derivative operator, we have

$$\tau(t, x, u)|_{t=0} = 0.$$

This is because that the lower limit of the integral in (3) is fixed, and therefore the equation $t = 0$ should be invariant with respect to such transformation. Thus the following assertion holds.

Theorem 1. *The symmetry group of the fractional HD equation is spanned by the vector fields*

$$X_1 = \partial_x, \qquad X_2 = x\partial_x + u\partial_u, \qquad X_3 = t\partial_t - \frac{\alpha}{3} u \partial_u, \qquad X_4 = x^2 \partial_x + 2xu\partial_u. \tag{11}$$

Table 1
The adjoint representation of (11).

$\mathrm{Ad}(\epsilon\cdot)$	X_1	X_2	X_3	X_4
X_1	X_1	$X_2-\epsilon X_1$	X_3	$X_4-2\epsilon X_2+\epsilon^2 X_1$
X_2	$e^{\epsilon}X_1$	X_2	X_3	$e^{-\epsilon}X_4$
X_3	X_1	X_2	X_3	X_4
X_4	$X_1+2\epsilon X_2+\epsilon^2 X_4$	$X_2+\epsilon X_4$	X_3	X_4

Note that X_1 is the space translation generator, X_2 and X_3 are scaling symmetries, and X_4 is the conformal symmetry. The nonzero commutators of the algebra (11) read as

$$[X_1, X_2] = X_1, \qquad [X_1, X_4] = 2X_2, \qquad [X_2, X_4] = X_4.$$

Here we should mention that HD equation admits a five dimensional Lie symmetry group [29]

$$\left\langle \partial_x, x\partial_x + u\partial_u, t\partial_t - \frac{u}{3}\partial_u, x^2\partial_x + 2xu\partial_u, \partial_t \right\rangle.$$

The transformation group corresponding to the infinitesimal generator $x^2\partial_x + 2xu\partial_u$ is also given in Ref. [29].

Before performing similarity reduction and finding exact solutions of (1), we construct the optimal system of (11). Any transformation from the full symmetry group converts a solution into another one, so we only need to find invariant solutions which are not related by transformations. There are infinitely many subgroups of the symmetry group of a given DE, for any linear combination of infinitesimal generators is also an infinitesimal generator. In order to achieve a complete description of invariant solutions for a DE, we need the subgroups which give essentially different solutions, namely, the optimal system [18,19,28]. Constructing an optimal system of subgroups is evidently equivalent to finding an optimal system of subalgebras. For one dimensional subalgebras, the construction of optimal system reduces to the classification of the orbits for the adjoint representation. The simplest method to this problem is taking a general element of the Lie algebra and simplifying it by a properly chosen adjoint transformation [28].

The action of the adjoint operator is given by the Lie series of the form

$$\mathrm{Ad}(\exp(\epsilon X_i))X_j = X_j - \epsilon[X_i, X_j] + \frac{\epsilon^2}{2}[X_i, [X_i, X_j]] - \cdots,$$

where $[X_i, X_j]$ is the usual commutator and ϵ is a parameter. In Table 1, we list the adjoint actions of the generators of algebra (11), where the (i, j)th entry indicates $\mathrm{Ad}(\exp(\epsilon X_i))X_j$.

We construct the one-dimensional optimal system of the algebra (11) and get the following result.

Theorem 2. *The one-dimensional optimal system of (11) is given by*

$$\begin{aligned} &V_1 = X_1, \quad V_2 = X_2, \quad V_3 = X_3, \quad V_4 = X_1 + X_3, \quad V_5 = X_1 - X_3, \\ &V_6 = X_1 + X_4, \quad V_7 = X_2 + \beta X_3 \quad (\beta \neq 0), \quad V_8 = X_1 + X_4 + \gamma X_3 \quad (\gamma \neq 0). \end{aligned} \tag{12}$$

Proof. Let V be the most general element, namely, $V = \sum_{i=1}^4 a_i X_i$, and then simplify it by utilizing suitable adjoint maps. Applying $\mathrm{Ad}(\exp(\epsilon_1 X_1))$ and $\mathrm{Ad}(\exp(\epsilon_2 X_4))$ to it yields

$$\tilde{V} = \sum_{i=1}^4 \tilde{a}_i X_i = \mathrm{Ad}(\exp(\epsilon_2 X_4)) \circ \mathrm{Ad}(\exp(\epsilon_1 X_1))V,$$

where

$$\tilde{a}_1 = a_1 - \epsilon_1 a_2 + \epsilon_1^2 a_4,$$

$$\tilde{a}_2 = a_3 - 2\epsilon_1 a_4 + 2\epsilon_2(a_1 - \epsilon_1 a_2 + \epsilon_1^2 a_4),$$

$$\tilde{a}_4 = a_4 + \epsilon_2(a_2 - 2\epsilon_1 a_4) + \epsilon_2^2(a_1 - \epsilon_1 a_2 + \epsilon_1^2 a_4).$$

There are three possible inequivalent cases for the value of $a_2^2 - 4a_1a_4$. Now we consider these three cases separately.

Case 1. If $a_2^2 - 4a_1a_4 > 0$, by choosing ϵ_1 be the real root of the quadratic equation $a_4\epsilon_1^2 - a_2\epsilon_1 + a_1 = 0$ and putting $\epsilon_2 = a_4/(2\epsilon_1 a_4 - a_2)$, we obtain $\tilde{a}_1 = \tilde{a}_4 = 0$ and $\tilde{a}_2 \neq 0$. Consequently, V is equivalent to a multiple of

$$\tilde{V} = X_2 + \tilde{a}_3 X_3$$

and thus we have two vector fields X_2 and $X_2 + \beta X_3$ ($\beta \neq 0$).

Case 2. Assuming $a_2^2 - 4a_1a_4 < 0$, we set $\epsilon_1 = a_3/(2a_4)$ and $\epsilon_2 = 0$. Hence $\tilde{a}_2 = 0$. Acting on this V by the suitable group generated by X_2, we can make the coefficients of X_1 and X_4 equal. So V is equivalent to a scalar multiple of

$$\tilde{V} = X_1 + X_4 + \tilde{a}_3 X_3.$$

Table 2
The invariants of the algebra (12).

	V_1	V_2	V_3	V_4	V_5	V_6	V_7	V_8
Q	0	1	0	0	0	−4	1	−4
B	0	0	1	1	−1	0	β	γ
C	1	0	0	1	1	0	0	0

Therefore V is equivalent to $X_1 + X_4$ or $X_1 + X_4 + \gamma X_3$ ($\gamma \neq 0$).

Case 3. If $a_2^2 - 4a_1a_4 = 0$, there exist two subcases.

If $a_1^2 + a_2^2 + a_4^2 \neq 0$, we can choose ϵ_1 and ϵ_2 so that $\tilde{a}_2 = \tilde{a}_4 = 0$ and what is more $\tilde{a}_1 \neq 0$. Thus V is equivalent to

$$\tilde{V} = X_1 + \tilde{a}_3 X_3.$$

Provided $\tilde{a}_3 \neq 0$, we use the group generated by X_2 to scale the coefficients of X_1 and X_3 and get that $\tilde{V}$ is equivalent to $X_1 + X_3$ or $X_1 - X_3$. If $\tilde{a}_3 = 0$, $\tilde{V}$ reduces to X_1.

Given $a_1 = a_2 = a_4 = 0$, we must have $a_3 \neq 0$ (otherwise, V is trivial). Now the only remaining vectors are the multiples of X_3 and the adjoint representation acts trivially on it. Then V is equivalent to X_3. □

Now we have shown that any one-dimensional subspace of (11) is equivalent to one of the subspaces spanned by $V_1, \ldots, V_8$ in Theorem 2. To complete the proof of Theorem 2, we still need to deal with the mutual inequivalence of any two algebras in (12). Following Ref. [30], we introduce the adjoint invariant, which is a real valued function ϕ on a Lie algebra $\mathfrak{g}$ satisfying $\phi(\mathrm{Ad}(\mathfrak{g})X) = \phi(X)$ for all $X \in \mathfrak{g}$. If vectors X and Y generate conjugate one-dimensional subalgebras, the relation $\phi(X) = \phi(Y)$ holds for any invariant ϕ. Given the general vector $V = \sum_{i=1}^{4} a_i X_i$ of (11), ϕ can be expressed as a function of the coefficients $a_1, \ldots, a_4$.

Lemma 1. *$Q = a_2^2 - 4a_1a_4$ is an invariant.*

Proof. It is a common knowledge that the Killing form is invariant under the adjoint action. Through straightforward calculation, we have the killing form

$$K(V, V) = 2(a_2^2 - 4a_1a_4)$$

of the Lie algebra (11). Consequently, $Q = K(V, V)/2$ is an invariant. □

We should like to point out that the invariants of the full adjoint action are often used to simplify the general vector field when constructing the optimal system. Actually, the deduction of (12) is based on the fact that Q is an invariant.

Lemma 2. *$B = a_3$ and*

$$C = \begin{cases} \mathrm{sign}\, a_1, & a_2 = a_4 = 0, \\ 0, & otherwise \end{cases}$$

are invariants.

Proof. Using Table 1, we can prove this lemma directly. □

Now we determine the invariants Q, B and C for each algebra V_i ($i = 1, 2, \ldots, 8$) in (12) and give the results in Table 2. From Table 2, we can easily verify the inequivalence of V_i. Thus Theorem 2 holds.

4. Similarity reductions of the fractional HD equation

Similar to the group invariant solution of PDEs [18,19], the group invariant solution of FDEs is defined as follows. The definition of group invariant solution of FDEs is the same as that of PDEs.

Definition 2. $u = \theta(t, x)$ is an invariant solution of Eq. (4) corresponding to the infinitesimal generator (6) iff

- $u = \theta(t, x)$ satisfies Eq. (4);
- $u = \theta(t, x)$ is an invariant surface of (6), namely, it fulfils the invariant surface condition

$$\tau(t, x, \theta)\theta_t + \xi(t, x, \theta)\theta_x = \eta(t, x, \theta).$$

In what follows, we perform similarity reductions, present the reduced nonlinear fractional ordinary differential equations (ODEs) and classify the corresponding group-invariant solutions of the fractional HD equation (1) for each inequivalent subalgebra (12).

Case 1. $V_1 = \partial_x$. Integration of the invariant surface condition

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{0}$$

gives the similarity variables $t,\ u$. Thus we have the ansatz $u = g(t)$. Inserting it into Eq. (1) yields the reduced fractional ODE

$$\partial_t^\alpha g(t) = 0, \tag{13}$$

which implies that $g(t) = a_1 t^{\alpha-1}$. Here and hereafter a_i are arbitrary real constants. Consequently, Eq. (1) has group-invariant solutions of the form

$$u = a_1 t^{\alpha-1}.$$

Case 2. $V_2 = x\partial_x + u\partial_u$. According to the invariant surface condition, we get two independent similarity variables t and u/x. Substitute the ansatz $u = xg(t)$ into Eq. (1), and the reduced equation (13) is obtained. Then we have the group-invariant solutions

$$u = a_2 x t^{\alpha-1}$$

of Eq. (1) corresponding to V_2.

Case 3. $V_3 = t\partial_t - \frac{\alpha}{3}u\partial_u$. Solving the invariant surface condition

$$\frac{\mathrm{d}t}{t} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{-\frac{\alpha}{3}u}$$

yields the similarity variables x and $t^{\frac{\alpha}{3}}u$. Inserting the ansatz $u = t^{-\frac{\alpha}{3}}g(x)$ into Eq. (1), we arrive at the reduced equation

$$g^2(x)g'''(x) = \frac{\Gamma(1-\frac{\alpha}{3})}{\Gamma(1-\frac{4}{3}\alpha)}. \tag{14}$$

Consequently, Eq. (1) has group invariant solution of the form $u = t^{-\frac{\alpha}{3}}g(x)$, where $g(x)$ satisfies Eq. (14).

Case 4. $V_4 = t\partial_t + \partial_x - \frac{\alpha}{3}u\partial_u$. In this case, the similarity variables are $\mathrm{e}^{-x}t$ and $\mathrm{e}^{\frac{\alpha}{3}x}u$. In view of the reduction ansatz $u = \mathrm{e}^{-\frac{\alpha}{3}x}g(\mathrm{e}^{-x}t)$, we arrive at

$$\partial_z^\alpha g(z) = -g^3(z)\left[\frac{\alpha^3}{27}g(z) + \left(\frac{\alpha^2}{3}+\alpha+1\right)zg'(z) + (\alpha+3)z^2g''(z) + z^3g'''(z)\right],$$

where $z = \mathrm{e}^{-x}t$.

Case 5. $V_5 = -t\partial_t + \partial_x + \frac{\alpha}{3}u\partial_u$. According to the invariant surface condition

$$\frac{\mathrm{d}t}{-t} = \frac{\mathrm{d}x}{1} = \frac{\mathrm{d}u}{\frac{\alpha}{3}u},$$

the similarity variables $\mathrm{e}^x t,\ \mathrm{e}^{-\frac{\alpha}{3}x}u$ are obtained. The group invariant solution has the form

$$u = \mathrm{e}^{\frac{\alpha}{3}x}g(\mathrm{e}^x t),$$

where $g(z)$ satisfies the equation

$$\partial_z^\alpha g(z) = g^3(z)\left[\frac{\alpha^3}{27}g(z) + \left(\frac{\alpha^2}{3}+\alpha+1\right)zg'(z) + (\alpha+3)z^2g''(z) + z^3g'''(z)\right].$$

Case 6. $V_6 = (1+x^2)\partial_x + 2xu\partial_u$. Considering the invariant surface condition

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{1+x^2} = \frac{\mathrm{d}u}{2xu},$$

we arrive at the similarity variables t and $u/(1+x^2)$. Inserting the ansatz $u = (1+x^2)g(t)$ into Eq. (1) yields Eq. (13). Thus we have the group-invariant solution

$$u = a_3(1+x^2)t^{\alpha-1}$$

of (1) that corresponds to V_6. Since the linear combination of the invariant solutions is still invariant solution. We have the following exact solutions

$$u = (a_3x^2 + a_2x + a_1)t^{\alpha-1}$$

of Eq. (1).

Case 7. $V_7 = \beta t\partial_t + x\partial_x + (1-\frac{\alpha\beta}{3})u\partial_u$. The similarity variables for this generator are $x^{-\beta}t$ and $x^{\frac{\alpha\beta}{3}-1}u$. We now look for a similarity reduction for (1) in consideration of the reduction ansatz

$$u = x^{1-\frac{\alpha\beta}{3}}g(x^{-\beta}t).$$

Substituting the above expression into Eq. (1) leads to the reduced equation

$$\partial_z^\alpha g(z) = -g^3(z)\left[\frac{\alpha\beta(\alpha^2\beta^2-9)}{27}g(z) + \frac{\beta(\alpha^2\beta^2+3\beta^2\alpha+3\beta^2-3)}{3}zg'(z) + (\alpha+3)\beta^3z^2g''(z) + \beta^3z^3g'''(z)\right],$$

where $z = x^{-\beta}t$.

Case 8. $V_8 = \gamma t\partial_t + (1+x^2)\partial_x + (2x - \frac{\alpha\gamma}{3})u\partial_u$. Due to the invariant surface condition

$$\frac{dt}{\gamma t} = \frac{dx}{1+x^2} = \frac{du}{(2x-\frac{\alpha\gamma}{3})u},$$

we obtain the following similarity variables

$$e^{-\gamma \arctan x}t, \qquad \frac{e^{\frac{\alpha\gamma}{3}\arctan x}}{1+x^2}u.$$

Thus Eq. (1) has group invariant solution of the form

$$u = (1+x^2)e^{-\frac{\alpha\gamma}{3}\arctan x}g(e^{-\gamma\arctan x}t)$$

where $g(z)$ satisfies the equation

$$\partial_z^\alpha g(z) = -g^3(z)\left[\frac{\alpha\gamma(\alpha^2\gamma^2+36)}{27}g(z) + \frac{\gamma(\alpha^2\gamma^2+3\alpha\gamma^2+3\gamma^2+12)}{3}zg'(z) + (\alpha+3)\gamma^3z^2g''(z) + \gamma^3z^3g'''(z)\right].$$

5. Concluding remarks

In this paper we demonstrate the efficiency of the classical Lie group approach to analysis of FDEs. We perform exhaustive analysis of subgroup structure of the invariance group admitted by the time-fractional Harry-Dym equation with Riemann–Liouville derivative. As an application of the so obtained classification of subgroups, we construct all inequivalent reduced fractional ODEs and also obtain some explicit exact solutions of the equation in consideration.

Integration of reduced fractional ODEs is a very difficult and challenging problem. In order to complete the construction of group-invariant solution of Eq. (1), one needs to develop symmetry approach to the integration of the obtained fractional ODEs. This work is in progress now and will be reported elsewhere.

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