The stationary KdV hierarchy and $so(2,1)$ as a spectrum generating algebra

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Abstract

The family $\mathcal{F}_L^\kappa$ of all potentials $V(x)$ for which the Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$ in one space dimension possesses a high order Lie symmetry is determined. A sub-family $\mathcal{F}_{SGA}^{(2)}$ of $\mathcal{F}_L^\kappa$, which contains a class of potentials allowing a realization of $so(2,1)$ as spectrum generating algebra of H through differential operators of finite order, is identified. Furthermore and surprisingly, the families $\mathcal{F}_{SGA}^{(2)}$ and $\mathcal{F}_L^\kappa$ are shown to be related to the stationary KdV hierarchy. Hence, the 'harmless' Hamiltonian H connects different mathematical objects: high order Lie symmetry, realization of $so(2,1)$-spectrum generating algebra and families of nonlinear differential equations. We describe in a physical context the interplay between these objects.

I. Motivation and background

In the two internal reports for the International Center of Theoretical Physics (ICTP) [1, 2] written in the early seventies, a complete classification of sym-

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metric Hamiltonians in one space dimension on $L^2(\mathbf{R}_x^1, dx)$

$$H = \gamma \frac{d^2}{dx^2} + V(x), \quad \gamma < 0 \tag{1}$$

having the Lie algebra $so(2,1)$ as a 'spectrum generating algebra' (SGA) has been obtained. This result has been published only recently in connection with an A.O. Barut-Memorial Lecture [3]. In [1, 2] the following definition of SGA is used: a differential operator A of the order n' has a spectrum generating (Lie) algebra L with generators g_i, $(i = 1, ..., m,\ m = \dim L)$ if there exist a realization R of L through differential operators of an order $n \geq n'$, such that

$$A = \sum_{i=1}^{m} \alpha_i R(g_i), \quad \alpha_i \in \mathbf{R}. \tag{2}$$

The Hamiltonian (1) is a differential operator on a suitable complex function space $\mathcal{G}$ over x with $n' = 2$. A realization R of $so(2,1)$ with standard basis $\mathcal{M}$ spanned by g_1, g_2, g_3 through nth order differential operators on $\mathcal{G}$ reads

$$R(g_i) = \sum_{j=0}^{n} a_{ij}(x) \frac{d^j}{dx^j}, \quad i = 1, 2, 3, \tag{3}$$

where $a_{ij}(x)$ are complex functions such that the commutation relations

$$\begin{aligned} &[R(g_1),\ R(g_2)] = -R(g_3), \quad [R(g_2),\ R(g_3)] = R(g_1), \\ &[R(g_3),\ R(g_1)] = -R(g_2) \end{aligned} \tag{4}$$

are fulfilled. So $R(so(2,1))$ is a SGA of (1) on $\mathcal{G}$ if there exist constants $\alpha_i \in \mathbf{R}$, $\quad (i = 1, 2, 3)$ such that the equality

$$H \equiv \gamma \frac{d^2}{dx^2} + V(x) = \sum_{j=1}^{3} \alpha_j R(g_j) \tag{5}$$

holds.

The relations (3)–(5) impose restrictions both on the coefficients $a_{ij}(x)$ and on the potential $V(x)$. Solving these restrictions we find two different families $\mathcal{F}_{SGA} = \left\{ \mathcal{F}_{SGA}^{(1)},\ \mathcal{F}_{SGA}^{(2)} \right\}$ of those potentials which allow $so(2,1)$ as SGA for (1) and we can use properties of $R(so(2,1))$ to calculate the spectrum of H, if H and $R(so(2,1))$ act on $L^2(\mathbf{R}_x^1, dx)$. If, furthermore, $R(g_i)$ are essentially self-adjoint in a common dense domain and if the representation is integrable, then one can use the known theory for unitary representations of $so(2,1)$. This is the background of the term 'spectrum-generating algebra' as suggested in [4, 5]. There were many results in this field for different Hamiltonians, Lie algebras and physical systems (see, e.g., the recent review [6]) but no general study in the sense of [1, 2].

The motivation of the present paper is to show (in Section III) that the family $\mathcal{F}^{(2)}_{SGA}$ can be read from the stationary KdV hierarchy. This surprising connection between the KdV hierarchy and $so(2,1)$ has its origin in a certain higher order Lie symmetry of the Hamiltonian (5) which has $so(2,1)$ as SGA. In Section II we sketch the results of [1, 2] on which our discussion is based.

II. On some known results

To classify those $V(x)$ which are solutions of (3)–(5) we reduce the calculation to special choices of parameters $\alpha_1, \alpha_2, \alpha_3$ through basis transformations which leave $\mathcal{M}$ invariant. As result, we only have to treat the following two cases ($\lambda \in \mathbf{R}, \lambda \neq 0$):

$$\text{Case 1.}\quad \alpha_1^2+\alpha_2^2 \neq \alpha_3^2 \quad \text{with} \quad H = \lambda R(g_i),\ i=1,2, \tag{6}$$

$$\text{Case 2.}\quad \alpha_1^2+\alpha_2^2 = \alpha_3^2 \quad \text{with} \quad H = \lambda(R(g_1)+R(g_3)). \tag{7}$$

Case 2 is denoted as the 'light cone case' . Both cases (3)–(5) lead for a fixed n to set of coupled differential equations of the order n for $a_{ij}(x)$, $(i = 1,2,3,\ j=0,\ldots,n)$ and $V(x)$. We assume that $R(g_i)$ are symmetric operators and that $V(x)$ is a real function. A clumsy but straightforward calculation shows that in Case 1 a solution exists only for $n=2$ with a family $\mathcal{F}^{(1)}_{SGA}$ of corresponding potentials

$$\mathcal{F}^{(1)}_{SGA}(\lambda_1,\lambda_2,c) = \{V(x) \mid V(x) = \lambda_1(x-c)^2 + \lambda_2(x-c)^{-2},$$
$$\lambda_1,\lambda_2,c \in \mathbf{R}, \lambda_1 \neq 0\}. \tag{8}$$

In Case 2 a solution exists for all $n \geq 2$. The corresponding family $\mathcal{F}^{(2)}_{SGA}$ consists of potentials that are solutions of nonlinear differential equations

$$\mathcal{F}^{(2)}_{SGA} = \left\{ V(x) \mid -\left(\frac{x}{2}V' + V\right) + \sum_{j=0}^{N-1} c_j F_j + F_N = 0 \right\}, \tag{9}$$

where $N = \left[\frac{n-1}{2}\right]$, F_j are some polynomials of $V(x)$ and its derivatives up to the order $2j+1$ ($j = 0, 1, \ldots, N$) which will be given later in another context (see, below (12), (13)); $c_0, c_1, \ldots, c_{N-1}$ are arbitrary real constants.

The family $\mathcal{F}^{(2)}_{SGA}$ has a peculiar structure. The facts that the equations are equal for $n = 2N+1$ and $n = 2N+2$, $N = 1, 2, \ldots$ and that the equation for $n = n_1 > n_2$ contains all terms of the equation for $n = n_2$ are two of these peculiarities. This structure was not elucidated in [1, 2][1], also relations to other mathematical notions and objects were not found. The present paper fills this gap.

In order to simplify the following calculations we scale the variable x and thus get $\gamma = -1$. So the Schrödinger operator (1) takes the form

$$H = -\frac{d^2}{dx^2} + V(x). \tag{10}$$

III. SGA, Lie symmetries and KdV hierarchy

A. Aim and strategy

We present a view on the family $\mathcal{F}^{(2)}_{SGA}$ through a high order Lie symmetry Q of type C_κ, i.e. with $[Q, H] = \kappa H$ (see III. B) of the Schrödinger equation

$$\left(-\frac{d^2}{dx^2} + V(x)\right) f(x) = 0, \qquad f \in \mathcal{G} \tag{11}$$

and we show that because of this view the F_j, ($j = 0, 1, \ldots, N$) in (9) appear, surprisingly, in the stationary KdV hierarchy

$$\sum_{j=0}^{N-1} c_j F_j + F_N = 0.$$

We remind that the stationary KdV hierarchy is obtained successively by the repeated action of the integro-differential operator (second recursive operator for the KdV equation [7]–[9])

$$\mathcal{R} = -\frac{1}{4}\frac{d^2}{dx^2} - V(x) - \frac{1}{2}V'(x)\left(\frac{d}{dx}\right)^{-1} \tag{12}$$

[1]This was the reason why the authors of [1, 2] decided to present the results as internal reports.

generating $F_j, (j = 0, 1, \ldots)$ through

$$F_{j+1} = \mathcal{R}\, F_j, \quad j = 0, 1, \ldots \quad \text{with } F_0 = -\frac{1}{2}V'(x). \tag{13}$$

Our strategy is the following. We construct as a first step the family $\mathcal{F}_L^0$ of all potentials $V(x)$ for which $Hf = 0$ allows an nth order Lie symmetry $Q \in \mathcal{L}$ of type C_0 (Theorem 1). In a second step, we generalize this result for a Lie symmetry of type C_κ, κ is arbitrary, (Theorem 2) and get the family $\mathcal{F}_L^\kappa$. In the special case $\kappa = -1$ we find the family $\mathcal{F}_{SGA}^{(2)}$.

B. Lie symmetries

An n-th order differential operator

$$Q = \sum_{j=0}^{n} q_j(x) \frac{d^j}{dx^j}, \qquad \frac{d^0}{dx^0} \stackrel{\text{def}}{=} 1 \tag{14}$$

is called an n-th order (Lie) symmetry operator of the Schrödinger equation (11) if it transforms the set of its solutions γ_H into itself. An equivalent (and more algorithmic) definition is the following one [10]. The operator Q is a symmetry operator of equation (11) if there is such an m-th order differential operator P that on γ_H

$$[Q,\ H] = PH. \tag{15}$$

In the following we use $P = \kappa\mathbf{1}$, κ is a real constant, i.e.

$$[Q,\ H] = \kappa H \qquad \text{on } \gamma_H \tag{16}$$

and denote this type of Q as C_κ. If Q is an nth order symmetry operator of $Hf = 0$, $f \in \gamma_H$, then

$$\tilde{Q} = Q + \sum_{j=0}^{n'} \gamma_j H^j$$

with $n' < n$ and with arbitrary constants γ_j is also a symmetry operator. Hence, $\tilde{Q}$ gives no new information on the structure of the set of solutions and is excluded from further considerations. To formalize this we introduce the equivalence relation

$$Q_1 \sim Q_2 \qquad \text{if} \qquad Q_1 - Q_2 = \sum_{j=0}^{n'} \gamma_j H^j$$

and consider representatives of these equivalence classes in the linear space of differential operators (14) satisfying (16) We denote this quotient space as $\mathcal{L}$.

A higher order Lie symmetry Q of H depends on $V(x)$ and on κ. We denote the family of such potentials (for all finite κ) as $\mathcal{F}_L$. There is a connection between $\mathcal{F}_{SGA}$ and $\mathcal{F}_L$. Consider $\mathcal{F}^{(2)}_{SGA}$, i.e. the singular case (16), the realization $R(g_2)$ of $g_2 \in so(2,1)$ is a Lie symmetry (16) of H for $\kappa = -1$

$$[R(g_2), H] = -H. \tag{17}$$

Hence, we have

$$\mathcal{F}^{(2)}_{SGA} \subset \mathcal{F}_L.$$

It is just the higher order Lie symmetry of H which is “responsible” for the family $\mathcal{F}^{(1)}_{SGA}$. We add that $R(g_j),\ j = 1,3$ can be constructed if $R(g_2)$ is known.

C. The reduction lemma

Because a first order Lie symmetry Q of H is easier to analyze than an nth order one, it would be convenient to reduce the order of Q under the condition that no information is lost. This is possible if we use in addition to H the operator $H + \epsilon\mathbf{1}$. The following result holds:

Lemma 1 *The Schrödinger equation $Hf = 0$, $H = -\frac{d^2}{dx^2} + V(x)$ allows an nth order Lie symmetry $Q \in \mathcal{L}$ of type C_0 if and only if*

$$\left(-\frac{d^2}{dx^2} + V(x) + \varepsilon\right) f = 0, \quad \varepsilon \neq 0, \quad \textit{real} \tag{18}$$

admits a first order Lie symmetry $\hat{Q}$ of the form

$$\hat{Q} = a(x,\varepsilon)\frac{d}{dx} + b(x,\varepsilon) \equiv \left(\sum_{j=0}^{N} a_j(x)\varepsilon^j\right)\frac{d}{dx} + \sum_{j=0}^{N} b_j(x)\varepsilon^j, \tag{19}$$

where $N = \left[\frac{n-1}{2}\right]$, $a_N = 1$.

Proof. Computing the commutator $[Q,\ H]=0$ on $\mathcal{G}$ yields the following operator relation

$$\sum_{j=0}^{n} q_j \left[\frac{d^j}{dx^j},\ V(x)\right] + \sum_{j=0}^{n} \left(q_j'' + 2q_j'\frac{d}{dx}\right)\frac{d^j}{dx^j} = 0.$$

Equating the coefficients of the operator $\frac{d^{n+1}}{dx^{n+1}}$ we get $q_n = \text{const}$. Without loss of generality we may choose $q_n = 1$ thus getting

$$Q = \frac{d^n}{dx^n} + \sum_{j=0}^{n-1} q_j(x)\frac{d^j}{dx^j}. \tag{20}$$

For even n the term $\frac{d^n}{dx^n}$ can be canceled as Q belongs to $\mathcal{L}$; it is only necessary to consider odd n, i.e., $n = 2N+1$, $N \in \mathbf{N}$.

As Q commutes with H, it commutes also with $H+\varepsilon\mathbf{1}$ and is a symmetry operator of (18). To reduce Q to a first order symmetry we use a well-known fact in the theory of high-order Lie symmetries of linear differential equations (c.f., [10]): Let Q be an nth order symmetry operator of (18) and R be a differential operator of order $n-2$, then $\tilde{Q} = Q + R(H+\varepsilon\mathbf{1})$ is also a symmetry operator of (18). Choosing $R = \frac{d^{2N-1}}{dx^{2N-1}}$ we reduce operator Q to the symmetry operator of the order $2N$

$$\tilde{Q} = q_{2N}\frac{d^{2N}}{dx^{2N}} + (q_{2N-1} + V + \varepsilon)\frac{d^{2N-1}}{dx^{2N-1}} + \sum_{j=0}^{2N-2} \tilde{q}_j \frac{d^j}{dx^j}.$$

Repeating this procedure $2N-1$ times we cancel all powers of the operator $\frac{d}{dx}$ of degree $k > 1$ and obtain a first-order operator $\hat{Q}$ which is again a symmetry operator of (18). What is more, each time when we are cancelling an odd power of the operator $\frac{d}{dx}$ the highest power of ε is increased by 1, which means that at the very end of the procedure the highest power of ε will be N.

Consequently, if the Schrödinger equation (11) admits a high order symmetry operator of the type C_0, then $H+\varepsilon\mathbf{1}$ necessarily admits a first-order Lie symmetry of the form (19); the a_j and b_j depend on $V(x)$ and its derivatives.

Suppose now that (18) admits a first-order Lie symmetry (16) of the form (19). Cancelling all the powers of ε by adding an appropriately chosen polynomial of $H+\varepsilon$ with variable coefficients yields a $(2N+1)$th order differential operator of the form (20) which is still a symmetry of the (18) (see the remark made at the beginning of the proof). To complete the proof we have to verify this operator commutes with H on γ_H. Indeed, inserting (14) into the invariance condition $[Q,\ H+\varepsilon] = R(H+\varepsilon)$, where P is some $(n-2)$th order differential operator on γ_H, and collecting the coefficients of ε we see that the operator R necessarily vanishes. Consequently, the relation $[Q,\ H+\varepsilon] = 0$ holds, whence it follows that $[Q,\ H] = 0$. $\triangleright$

D. The family $\mathcal{F}_L^0$ of potentials which allow Lie symmetries Q of type C_0

The reduction lemma restricts the potentials $V(x)$ which allow an nth-order Lie symmetry of type C_0. We show that this set is the solution set of a family of non-linear differential equations. Integro-differential operators are applied in order to characterize the structure of this family.

Our main result is:

Theorem 1 *The Schrödinger equation $Hf = 0$, $H = -\frac{d^2}{dx^2} + V(x)$ admits an n-th order Lie symmetry of type C_0 if and only if the potential $V(x)$ satisfies a nonlinear differential equation contained in the family*

$$\mathcal{F}_L^0 = \left\{ V(x) \mid G(V) \equiv \sum_{j=0}^{N-1} c_j F_j + F_N = 0, \right\}, \tag{21}$$

where $N = \left[\frac{n-1}{2}\right]$; $F_j = \mathcal{R}^j F_0$, $F_0 = -\frac{1}{2}V'$, i.e. the F_j represent the stationary KdV hierarchy (13), c_j are real constants.

Proof. The proof is simplified substantially if we use Lemma 1 and well-known techniques of soliton theory.

Using the reduction lemma and inserting (18), (19) into (15) we get a system of determining equations for the coefficients a_j, b_j in (19)

$$a''(x,\varepsilon) + 2b'(x,\varepsilon) = 0,$$
$$b''(x,\varepsilon) + a(x,\varepsilon)V'(x) + 2a'(x,\varepsilon)(V(x)+\varepsilon) = 0,$$

where primes denote differentiation with respect to x. Using the polynomial ε-dependence we find after integration for $b_j(x)$ (B_j are integration constants)

$$b_j(x) = -\frac{1}{2}a_j'(x) + B_j, \quad j = 0, 1, \ldots, N \tag{22}$$

and $N+2$ recurrence relations for $a_j(x)$ depending on $V(x)$ and on its derivatives

$$\begin{aligned} a_N(x) &= 1, \\ a'_{j-1}(x) &= -\frac{1}{4}a'''_j(x) - V(x)a'_j(x) - \frac{1}{2}V'(x)a_j(x), \\ a_{-1}(x) &\stackrel{\text{def}}{=} 0, \end{aligned} \tag{23}$$

where $j = 0, 1, \ldots, N$.

The first $N+1$ relations of (23) are solved by subsequent integrations yielding expressions for the functions $a_0(x), \ldots, a_{N-1}(x)$ through $V(x)$ and its derivatives. Substituting these results into the last equation for $j = 0$, i.e. $a_{-1} = 0$ we arrive at a nonlinear differential equation for $V(x)$ of order $2N+1$. Its solutions generate all solutions of (22) and (23).

To reveal the structure of (23), especially of $a_{-1} = 0$, we introduce new functions $\mathcal{U}_0(x)$, $\mathcal{U}_1(x)$,... by the following recurrence relation (D_x denotes $\frac{d}{dx}$):

$$\begin{aligned} &\mathcal{U}_j(x) = \mathcal{P}\,\mathcal{U}_{j-1}, \quad \mathcal{U}_{-1} \equiv 1, \\ &\mathcal{P} = -\frac{1}{4}D_x^2 - V(x) + \frac{1}{2}D_x^{-1}V'(x), \end{aligned} \tag{24}$$

where $j = 0, 1, \ldots$ and $\mathcal{U}_{-1}(x) \stackrel{\text{def}}{=} 1$. Note that $\mathcal{P}$ is the first recursive (integro-differential) operator for the KdV equation (see, e.g. [7]–[9]). The action of $\mathcal{P}$ on some (initial conserved density) $\mathcal{U}_0 = -\frac{1}{2}V(x)$ yields the whole hierarchy of the conserved densities $\mathcal{U}_1, \mathcal{U}_2, \ldots$. The second recursive operator $\mathcal{R}$ in (12) is related to $\mathcal{P}$ through

$$\mathcal{R} = D_x \circ \mathcal{P} \circ D_x^{-1}.$$

With this $\mathcal{U}_j(x)$ the $a_j(x)$ in (23) can be written as

$$a_{j-1}(x) = \mathcal{U}_{N-j} + \sum_{k=1}^{N-j+1} c_{N-k}\,\mathcal{U}_{N-j-k}, \quad j = 1, \ldots, N, \tag{25}$$

where $c_0, \ldots, c_{N-1}$ are integration constants (independent of the B_j).

The equation $a_{-1} = 0$ can be rewritten in a more transparent form as

$$D_x \circ \left(\sum_{j=0}^{N-1} c_j\,\mathcal{P}^j + \mathcal{P}^N \right) \mathcal{U}_0 = 0, \tag{26}$$

which shows a direct relation to the stationary higher KdV equation. Using the operator identity

$$D_x \circ \mathcal{P}^j \equiv \left(D_x \circ \mathcal{P} \circ D_x^{-1}\right)^j \circ D_x = \mathcal{R} \circ D_x$$

(26) takes the following form:

$$\left(\sum_{j=0}^{N-1} c_j \mathcal{R}^j + \mathcal{R}^N\right) \circ D_x \mathcal{U}_0 = 0, \quad \mathcal{U}_0 = -\frac{1}{2} V'(x).$$

Because of $D_x \mathcal{U}_0 = -\frac{1}{2} V' = F_0$ and $F_j = \mathcal{R}^j F_0$ we get finally (21). $\triangleright$

E. The family $\mathcal{F}_L^\kappa$ of potentials which allow Lie-symmetries of type C_κ

The results in Section D for nth order Lie-symmetries of type C_0 can be utilized to construct also nth order Lie symmetries of type C_κ for $\kappa \neq 0$. We have:

Theorem 2 *The Schrödinger equation $Hf = 0$, $H = -\frac{d^2}{dx^2} + V(x)$ admits an nth order Lie-symmetry $Q \in \mathcal{L}$ of type C_κ, i.e., $[Q, H] = \kappa H$, if and only if the potential $V(x)$ satisfies a nonlinear differential equation contained in the family*

$$\mathcal{F}_L^\kappa = \left\{ V(x) \mid \kappa \left(\frac{x}{2} V' + V\right) + \sum_{j=0}^{N-1} c_j F_j + F_N = 0 \right\} \tag{27}$$

where $N = [\frac{n-1}{2}]$, $F_j = \mathcal{R}^j F_0$, $F_0 = -\frac{1}{2} V'$; c_j are real constants.

Proof. With the same arguments as in the proof of the reduction lemma we get for Q (see (14)) $q_n = 1$. It is only necessary to consider odd n. We insert this Q and H in (16):

$$\begin{aligned} &\sum_{j=0}^{n-1} \begin{pmatrix} j \\ n \end{pmatrix} V^{(n-j)} \frac{d^j}{dx^j} + \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} q_i \begin{pmatrix} j \\ i \end{pmatrix} V^{(i-j)} \frac{d^j}{dx^j} \\ &\qquad + \sum_{j=0}^{n-1} (2q_j' \frac{d}{dx} + q_j'') \frac{d^j}{dx^j} = \kappa \left(-\frac{d^2}{dx^2} + V(x)\right). \end{aligned} \tag{28}$$

Comparing coefficients in front of the linearly-independent operators $\frac{d^j}{dx^j}$, $(j = 1, \dots n)$ yields n recurrence integro-differential relations for the coefficients $q_i(x) = q_i(x, V(x), \kappa)$, $(i = 0, 1, \dots, n-1)$ in (14)

$$\begin{aligned} q_{n-1}(x) &= \hat{c}_{n-1}, \\ q_{j-1}(x) &= -\frac{1}{2}\Bigg(q_j'(x) + \left(\begin{array}{c} j \\ n \end{array} \right) V^{(n-j-1)}(x) + \sum_{i=j+1}^{n-1} \left(\begin{array}{c} j \\ i \end{array} \right) \\ &\quad \times \int^x q_i(y) V^{(i-j)}(y) dy + \kappa \delta_{j2} x \Bigg) + \hat{c}_{j-1}, \end{aligned} \tag{29}$$

where $\hat{c}_j$ are integration constants and

$$\hat{c}_j = 0, \quad \text{for } j \text{ even.}$$

Note that only q_1 depends on κ through an additive part $\frac{1}{2}\kappa x$, i.e.

$$q_1(x, V(x), \kappa) = \hat{q}_1(x, V(x)) + \frac{1}{2}\kappa x.$$

Collecting the terms with $\left(\frac{d}{dx}\right)^0$ in (28) we get a differential equation $G(V, \kappa)$ for $V(x)$ of the type

$$G(V, \kappa) \equiv q_0'' + V^{(n)} + \hat{q}_1 V' + \kappa \frac{x}{2} V' + \sum_{j=2}^{n-1} q_j V^{(j)} + \kappa V = 0. \tag{30}$$

¿From Theorem 1 we know the differential equation for $V(x)$ if $\kappa = 0$. Hence

$$G(V, 0) = G(V). \tag{31}$$

Because the κ-dependence in (30) is explicitly known we have

$$G(V, \kappa) \equiv G(V) + \kappa \left(\frac{x}{2} V' + V \right) = 0\ ; \tag{32}$$

so we arrive at (27). $\triangleright$

This concludes the discussion of high order Lie symmetry of the Hamiltonian $H = -\frac{d^2}{dx^2} + V(x)$.

IV. Relation to spectrum generating algebras

The Lie symmetry $[Q, H] = \kappa H$ (Theorem 2) is related to the spectrum generating algebra $so(2,1)$ of H through (17), where $\kappa = -1$. The realization $R(g_2)$ of $g_2 \in so(2,1)$ is given by Q of the form (14), (29) through solutions of (27) with $\kappa = -1$ and

$$\mathcal{F}^{(2)}_{SGA} = \mathcal{F}^{\kappa}_{L}, \qquad \kappa = -1 \tag{33}$$

holds. As $R(g_2)$ is explicitly known, we can insert $R(g_2)$ and the differential operators $R(g_1)$, $R(g_3)$ into the commutation relations of the algebra $so(2,1)$ and thus find the latter (for further details, see [1, 2]).

With the results obtained in III we can elucidate the peculiar features of the nonlinear differential equation (9) mentioned in Section II. The fact that the potentials $V(x)$ are identical for $n = 2N+1$ and $n = 2N+2$ is explained as follows. In a symmetry operator Q of order $2N+2$ the term with $\left(\frac{d}{dx}\right)^{2N+2}$ can be canceled because of $Q \in \mathcal{L}$ thus getting a symmetry operator of order $2N+1$ which is again a Lie symmetry of the Schrödinger equation. The stronger statement that the equations for $V(x)$ in $\mathcal{F}^{(2)}_{SGA}$ are identical for $n = 4k, 4k+1, 4k+2, 4k+3,\ k = 1, 2, \ldots$ is valid because a SGA-symmetry is stronger than a Lie symmetry. The way of constructing the equations for $V(x)$ used while proving Lemma 1, makes it also evident, why this equation with some fixed $n = n_1$ contains all the terms of an equation for $V(x)$ under $n = n_2 < n_1$. Indeed, the equation for $n = n_1$ is obtained from one for $n_1 - 1$ by the action of the recursive operator $\mathcal{R}$ which transforms a term F_j into F_{j+1}.

V. Integrability

Hamiltonians (10) admitting $so(2,1)$-spectrum generating algebra have a further useful property: they are *integrable* in the sense that the corresponding Schrödinger equation (11) can be integrated by quadratures. This is so because (11) admits a first-order Lie symmetry of the form $\hat{Q} = \xi(x)\frac{d}{dx} + \eta(x)$ with

$$-\eta'' + 2V\xi' + V'\xi = 0, \quad 2\eta' + \xi'' = 0. \tag{34}$$

and because one can apply for an integration of (11) the classical method (see, e.g., [11, 12]) based on its Lie symmetry. The first integral for system

(34) is:

$$-\eta'\xi + V\xi^2 - \frac{1}{4} = \alpha \equiv \text{const}.$$

Depending on the sign of α the general solution of the equation (11) reads

$$f(x) = \sqrt{\xi(x)} \begin{cases} C_1\rho(x) + C_2, & \alpha = 0 \\ C_1 \cos a\rho(x) + C_2 \sin a\rho(x), & \alpha = a^2 > 0, \\ C_1 \cosh a\rho(x) + C_2 \sinh a\rho(x), & \alpha = -a^2 < 0, \end{cases}$$

where

$$\rho(x) = \int \frac{dx}{\xi(x)}.$$

Now inserting the explicit expressions for $\xi(x), \eta(x)$ from (19) into the above formulae yields the general solution of (11) provided the function $V(x)$ fulfills an equation of the form (9).

VI. Concluding remarks

Given a physical observable quantized through a linear differential operator A in $L^2(\mathbf{R}^d, dx^d)$, e.g., a Hamiltonian H (1) in one space dimension, a spectrum generating algebra for A is specified through a Lie algebra L with $\dim L = m$ with generators g_i and a realization through differential operators of the order n such that $A = \sum_{k=1}^{m} \alpha_k R(g_k)$, e.g. $L = so(2)$, $m = 3$. A high order Lie symmetry for the linear operator A is defined through finite order differential operators Q, P with $[Q, A] = PA$, e.g., $A = H$ and $P = \kappa$. Both SGA and high order Lie symmetry are different methods to model a physical symmetry with different mathematical structures.

We have shown that for the Hamiltonian (1) an $so(2,1)$-SGA and an nth order Lie symmetry with $P = -1$ are directly related via (17). However, a Lie symmetry is more general than an $so(2,1)$-SGA symmetry. The interesting result is the connection between an nth order C_κ Lie-symmetry of the Hamiltonian (1) and the stationary KdV hierarchy. It is understandable that for the singular case of $so(2,1)$, which reflects the light cone structure in $so(2,1)$, a family of non-linear differential equations for $V(x)$ appears. But it is, as we already mentioned, surprising that this family is related to the stationary KdV hierarchy, a mathematical object not connected directly to a symmetry concept of observables. We suspect that this connection to the

KdV hierarchy is somehow encoded in the geometry of $so(2,1)$ and its realizations. An investigation of Hamiltonians of the type (1) in higher space dimensions, their Lie symmetry in the above sense and L-SGA with non-compact L, $m > 3$, seems to be appropriate.

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