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Group classification of nonlinear evolution equations: Semi-simple groups of contact transformations

Qing Huang [a,*], Renat Zhdanov [b]

[a] Department of Mathematics, Northwest University, Xi'an 710069, China
[b] BIO-key International, 55121 Eagan, MN, USA

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ABSTRACT

We generalize and modify the group classification approach of Zhdanov and Lahno (1999) to make it applicable beyond Lie point symmetries. This approach enables obtaining exhaustive classification of second-order nonlinear evolution equations in one spatial dimension invariant under semi-simple groups of contact transformations. What is more, all inequivalent second-order nonlinear evolution equations which admit semi-simple groups or groups having nontrivial Levi decompositions are constructed in explicit forms.

1. Introduction

In this paper, we study contact symmetries of general second-order nonlinear evolution equations of the form

$$u_t = F(t, x, u, u_1, u_2), \tag{1}$$

where $u = u(t,x), u_t = \partial u/\partial t, u_i = \partial^i u/\partial x^i$ $(i \in \mathbf{N})$, and F is an arbitrary sufficiently smooth real-valued function with $F_{u_2} \neq 0$. In fact, we intend to construct all possible forms of the function F such that the corresponding equation admits nontrivial contact transformation group which contains a semi-simple subgroup. The class (1) includes a number of important and fundamental equations of modern mathematical and theoretical physics, such as the heat, Fisher, Newell–Whitehead and Burgers equations, to mention just a few (see, e.g., [17,22,31]).

Lie group analysis is universally recognized as a versatile and convenient tool for analysis of partial differential equations (PDEs). However, there is a necessary prerequisite for efficient utilization of any group-theoretical method. Namely, the equation under study has to admit a nontrivial Lie group. By this very reason, the problem of group classification of nonlinear PDEs has attracted so much attention and resulted in numerous publications recently.

In the case when a transformation involves dependent and independent variables only, it is called point transformation. For the more general case of transformation including first derivatives of the dependent variables, the term contact transformation has been adopted in the literature. Nowadays, the point group classification of the class (1) has been extensively studied (see [32,3,33] and the references therein). In contrast, much less attention has been devoted to the contact symmetries of the class (1).

The notion of contact (tangential) transformation within the context of differential equations (DEs) was first presented in Sophus Lie's doctoral thesis [15]. He obtained a number of classical results on contact symmetries of ordinary differential

* Corresponding author.
E-mail address: hqing@nwu.edu.cn (Q. Huang).

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equations (ODEs). In addition, Lie described linear PDEs in two independent variables admitting contact symmetries [16]. Classification of contact symmetry groups of wave and related equations has been performed in [8]. Contact transformations of nonlinear hyperbolic type equations have been studied in [13,19–21]. The paper [27] tackles the contact symmetry reduction of second-order PDEs. The inter-relationship between contact symmetries and conservation laws has been investigated in [14]. Recently, local conservation laws of second-order evolution equations have been classified up to contact equivalence [25], and the action of contact equivalence transformation on low-order conservation laws of evolution equations of an arbitrary order has been considered in [26]. The concept of discrete transformation group has been modified to include contact transformations in [11].

Sokolov [28,29] dealt with the evolutionary symmetries of the evolution equations

$$u_t = F(x, u, u_1, u_2, \ldots, u_n), \quad n \geqslant 2, \tag{2}$$

and Magadeev [18] performed contact group classification of the equations of the form

$$u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \quad n \geqslant 2. \tag{3}$$

They obtained a number of nontrivial results on structure and dimension of contact symmetry algebras and described all possible realizations of symmetry algebras of the equations above by Lie vector fields over the field of complex numbers.

What is missing in the research of Sokolov and Magadeev is analysis which of the symmetries listed in [29,18] can be admitted by nonlinear evolution equations in the class (3) of a specific order n. What they have provided is the existence theorem, meaning that for any symmetry presented in [29,18] there exists an order n such that the corresponding PDE (3) can admit the symmetry in question. To utilize the results of [29,18] for complete description of second-order evolution equations admitting contact symmetries, one needs to solve the determining equations (if they are compatible) for each presented Lie algebra realization.

Consequently, description of all possible realizations of admissible symmetry algebras is only part of the group classification of nonlinear evolution equations. Another crucial element of the classification is an actual construction of inequivalent invariant equations of the form (3). Construction of invariant equations has not been considered in the papers [28,29,18] and therefore their solutions of group classification problem of the class (3) is incomplete, to some degree.

One more important point is that the method for computing algebras of contact symmetries developed by Sokolov relies heavily on the fact that an evolution equation by definition involves only the first-order derivative in the temporal variable t. As a result, the transformation law for t involves t only and the variable t enters contact symmetry almost like a parameter. This ensures that contact symmetries of an evolution PDE are closely related to contact symmetries of ordinary differential equations (ODEs) obtained by putting $u_t = 0$. So that efficiency of Sokolov's approach relies on the well-known classification of contact symmetries of ODEs. In fact one can search for finite-dimensional contact symmetries of (3) by considering linear combinations of the generators of contact symmetries of the corresponding ODEs and of the operator ∂_t with coefficients depending on t. In addition, the evolution equations that admit infinite-dimensional contact symmetry groups are known to be linearizable. Evidently, such approach has limited applicability beyond the class of parabolic type PDEs.

In this paper, we develop the alternative approach to classification of contact symmetries of PDEs, which is also a variation of the infinitesimal Lie method. As an application, we perform group classification of second-order evolution equations of the form (1) admitting contact symmetries. We solve completely the problem of constructing all inequivalent realizations of contact transformation groups containing semi-simple sub-groups. Utilizing these results, we construct all Eqs. (1) that are invariant with respect to semi-simple groups or groups containing semi-simple subgroup.

The approach of this paper is the direct generalization of the method used in our papers [3,33] to classify Lie point symmetries of second-order evolution equations. Action of the group of contact transformations imposes stronger equivalence relations than that of point transformations. By this very reason, we get fewer inequivalent realizations than we did for the case of point transformation groups in [3,33]. Any symmetry algebra containing a semi-simple subalgebra obtained in [3,33] is equivalent to one of the realizations derived in this paper with respect to a suitable contact transformation.

If a finite-dimensional algebra L contains the radical N (the largest solvable ideal in L), then due to the Levi–Mal'cev theorem, there exists a semi-simple subalgebra S such that.

$$L = S \Subset N, \tag{4}$$

where S is the Levi factor. The relation (4) is called the Levi decomposition of L. Consequently, any Lie algebra falls into one of the following three categories: (i) semi-simple algebra, (ii) solvable algebra, and, (iii) semi-direct sum of solvable and semi-simple algebra.

Here we restrict our considerations to the cases when symmetry group is either semi-simple or have a nontrivial Levi factor. Note that we have obtained the complete group classification of the PDEs of the class (1) invariant under solvable groups of contact transformations of the dimension $n \leqslant 4$ in [10].

If one adopts the definition of potential symmetry of evolution PDEs suggested by Bluman in [5,6], the following assertion holds. Any potential symmetry of this type admitted by an evolution equation in one spatial dimension can be mapped into a contact symmetry of a related evolution equation of the same order [9,25,26,34]. Consequently, group classification of the class (1) admitting contact symmetries yields, as a by-product, description of PDEs possessing potential symmetries. In other words, it provides nontrivial insight into the largely unexplored world of non-local symmetries of PDEs, since a potential symmetry is a specific case of nonlocal symmetry.

To classify the class (1) admitting contact symmetries, we modify the classification approach of [32] based on point symmetries. Our group classification procedure is implemented in three major steps. First, we compute the most general group of contact symmetries that leaves the class (1) invariant. This yields the classifying equation for infinitesimal symmetry generator and unknown function F. And what is more, one gets the explicit form of the equivalence group of contact transformations for (1). Second, based on the well-known results on classification of low-dimensional abstract Lie algebras (see, e.g., [2,4,30]), we describe all inequivalent realizations of contact symmetry algebras by basis infinitesimal generators admitted by (1). Finally, we solve the classifying equations for the obtained realizations of Lie algebras, which gives the explicit forms of invariant equations.

The paper is organized as follows. In Section 2, we give a summary of the necessary facts and definitions which are used throughout the paper. There the classifying equation for the function F is also given. Section 3 is devoted to the classification of the class (1) admitting semi-simple groups. In Section 4, we construct all inequivalent equations of the form (1) which possess symmetry algebras with nontrivial Levi factors. The last section contains a brief discussion of the results obtained.

2. Admissible transformations of evolution equations

To keep exposition self-contained, we summarize below the well-known facts from the theory of contact symmetries of evolution equations. An interested reader can find further details in the monographs [1,7,12,23].

It is common knowledge that the most general contact transformation group admitted by (1) is generated by the infinitesimal operator

$$V_g = -g_{u_t}\partial_t - g_{u_1}\partial_x + (g - u_t g_{u_t} - u_1 g_{u_1})\partial_u + (g_x + u_1 g_u)\partial_{u_1} + (g_t + u_t g_u)\partial_{u_t}, \tag{5}$$

where the arbitrary real-valued smooth function $g = g(t,x,u,u_t,u_1)$ is called generating function or contact Hamiltonian for the vector field V_g. Since function g uniquely defines V_g, we usually use g as a short form of V_g throughout the paper. V_g can also be represented in the equivalent form of Lie-Bäcklund vector field (LBVF)

$$g\partial_u + (D_x g)\partial_{u_1} + (D_t g)\partial_{u_t} + (D_x^2 g)\partial_{u_2} + \cdots, \tag{6}$$

where $D_x^{j+1} = D_x(D_x^j)$, the symbols D_t and D_x stand for the total differentiation operators with respect to the variables t and x, respectively. Namely,

$$D_t = \partial_t + u_t\partial_u + u_{tt}\partial_{u_t} + u_{xt}\partial_{u_1} + \cdots,$$
$$D_x = \partial_x + u_1\partial_u + u_{tx}\partial_{u_t} + u_2\partial_{u_1} + \cdots.$$

Furthermore, provided the generating function g is linear in u_t and u_1

$$g = \eta(t,x,u) - \tau(t,x,u)u_t - \xi(t,x,u)u_1,$$

both the vector field (5) and LBVF (6) are equivalent to the vector field $\tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$ which generates Lie group of point transformations. If g do not explicitly depend on t and u_t, it is called evolutionary symmetry [29].

As the form of (6) is uniquely determined by the corresponding function g, it is convenient to use the shorthand notation $g\partial_u$ instead of its full version (6). The class (1) is invariant under the LBVF $g\partial_u$ or, equivalently, under the vector field (5) if and only if the condition

$$[gF_u + D_x g F_{u_1} + D_x^2 g F_{u_2} - D_t g]|_{u_t=F} = 0$$

holds.

Inserting $u_t = F$ and its differential consequences into the invariance criterion and splitting the obtained relation with respect to the functionally independent variables yield an over-determined system of linear PDEs for unknown functions g and F. Solving it, we arrive at the following assertion.

Lemma 1. *The most general contact symmetry admitted by (1) is of the form*

$$g(t,x,u,u_t,u_1) = \alpha(t)u_t + G(t,x,u,u_1), \tag{7}$$

where α and G are real-valued functions satisfying the classifying equation

$$\begin{aligned} &- G_{u_1}F_u u_1 + F_{u_2}G_u u_2 + 2F_{u_2}G_{u_1x}u_2 + 2F_{u_2}G_{ux}u_1 + F_{u_1}G_u u_1 \\ &+ F_{u_2}G_{uu}u_1^2 + F_{u_2}G_{u_1u_1}u_2^2 + 2F_{u_2}u_1G_{u_1u}u_2 + F_uG + F_{u_2}G_{xx} \\ &- G_{u_1}F_x + F_{u_1}G_x - \alpha F_t - FG_u - \alpha_t F - G_t = 0. \end{aligned} \tag{8}$$

Hence the problem of classifying the contact symmetries of the class (1) reduces to constructing all possible solutions of (8). The difficulty, however, is that (8) is an under-determined system of one PDE for three unknown functions α, G and F. To proceed with the classification, we need to get additional information on the structure of α, G and F. This information is provided by the classical results on the structure of abstract low dimensional Lie algebras.

Following the approach of [32], we have succeeded in obtaining constraints on the admissible forms of the functions α and G. This enabled constructing all inequivalent contact symmetries of the form (7). Inserting the obtained functions α and G into classifying equations and solving the latter, we derive the explicit forms of F.

We now look for contact transformations which map the class (1) onto itself and form the contact equivalence group of (1). The direct computation shows that the contact equivalence group admitted by the class (1) read as

$$\tilde{t} = T(t), \quad \tilde{x} = X(t,x,u,u_1), \quad \tilde{u} = U(t,x,u,u_1), \tag{9}$$

where the functions T, X and U satisfy the regularity

$$T_t \neq 0, \qquad \operatorname{rank}\begin{pmatrix} X_x & X_u & X_{u_x} \\ U_x & U_u & U_{u_x} \end{pmatrix} = 2$$

and tangency conditions

$$X_{u_1}(u_1 U_u + U_x) = U_{u_1}(u_1 X_u + X_x). \tag{10}$$

Consider now the action of a contact transformation on the generating function and the corresponding equation. One can verify by straightforward computation that contact transformation (9) maps the generating function

$$\tilde{g}(\tilde{t},\tilde{x},\tilde{u},\tilde{u}_{\tilde{t}},\tilde{u}_{\tilde{x}}) = \tilde{\alpha}(\tilde{t})\tilde{u}_{\tilde{t}} + \widetilde{G}(\tilde{t},\tilde{x},\tilde{u},\tilde{u}_{\tilde{x}})$$

of contact symmetry of the equation

$$\tilde{u}_{\tilde{t}} = \widetilde{F}(\tilde{t},\tilde{x},\tilde{u},\tilde{u}_{\tilde{x}},\tilde{u}_{\tilde{x}\tilde{x}})$$

into the generating function

$$g(t,x,u,u_t,u_1) = \frac{\tilde{\alpha}(T)}{\dot{T}} u_t + \frac{D(X)}{J}\left[\frac{D(X)U_t - D(U)X_t}{D(X)\dot{T}}\tilde{\alpha}(T) + \widetilde{G}(T,X,U,\tilde{u}_{\tilde{x}})\right], \tag{11}$$

and the equation reduces to

$$u_t = \frac{D(X)}{J}\left[\dot{T}\widetilde{F}(T,X,U,\tilde{u}_{\tilde{x}},\tilde{u}_{\tilde{x}\tilde{x}}) + X_t\tilde{u}_{\tilde{x}} - U_t\right].$$

Here $D = \partial_x + \sum_{i=0}^{\infty} u_{i+1}\partial_{u_i}$, $D^2 = D \circ D$, $J = D(X)U_u - D(U)X_u$ and

$$\tilde{u}_{\tilde{x}} = \frac{D(U)}{D(X)}, \quad \tilde{u}_{\tilde{x}\tilde{x}} = \frac{D(X)D^2(U) - D(U)D^2(X)}{[D(X)]^3}.$$

The formulas above reflect the simple fact that there exist infinitely many equivalent realizations of a contact symmetry. Choosing appropriately the transformation functions T, X and U in (9), one can obtain g of the canonical form. If a contact symmetry can be mapped into a point symmetry by a suitable contact transformation, it is called reducible symmetry. Reducibility is an important concept in the theory of contact transformation groups and Lie algebras of contact vector fields [7]. Given a choice, inequivalent representatives of reducible contact realizations should be chosen among point realizations of the corresponding group or algebra. So, whenever possible, the generating function g should be transformed to become linear in u_t and u_1.

Lemma 2 ([28,29,18]). *Given an arbitrary generating function (7), there exists contact transformation (9) reducing it to either 1 or u_t.*

Solving the corresponding classifying equations yields the forms of all inequivalent equations belonging to (1) admitting one-dimensional Lie algebras of contact symmetries.

Theorem 1. *There exist only two inequivalent equations in the class (1) invariant under one-parameter contact symmetry groups*

$$A_1^1 = \langle u_t \rangle : \quad u_t = F(x,u,u_1,u_2),$$
$$A_1^2 = \langle 1 \rangle : \quad u_t = F(t,x,u_1,u_2).$$

Next, we give the definition of the Lie bracket for generating functions f and g, which corresponds to the commutation relation of the respective LBVFs. Denoting

$$f_*(g) = f_{u_t}D_t g + f_u g + f_{u_1}D_x g,$$

we define the Lie bracket as follows

$$[f,g] = g_*(f) - f_*(g).$$

Computing the right-hand side of the above formula gives

$$[f,g] = (g_{u_t}f_u - f_{u_t}g_u)u_t + (g_{u_1}f_u - f_{u_1}g_u)u_1 + g_{u_t}f_t - f_{u_t}g_t + g_{u_1}f_x - f_{u_1}g_x + fg_u - gf_u. \tag{12}$$

Hence we find that $[f,g]$ does not depend on u_2, u_{tx} or u_{tt}. Consequently, contact symmetries of the class (1) form a Lie algebra with respect to the so defined Lie bracket. It is straightforward to verify that $[V_f, V_g] = V_{[f,g]}$, where $[V_f, V_g]$ is the standard Lie bracket and $[f,g]$ is given by (12).

In what follows, we will concentrate on the PDEs of the form (1) invariant under semi-simple algebras and algebras having nontrivial Levi factors.

3. Equations invariant under semi-simple Lie algebras

In this section, we analyze equations from the class (1) whose contact symmetry algebras are semi-simple.

The lowest possible dimension of a semi-simple Lie algebra is three. There are two non-isomorphic three-dimensional semi-simple Lie algebra over the real field, namely,

$$\mathfrak{sl}(2,\mathbf{R}): \ [g_1,g_2] = g_1, \quad [g_1,g_3] = 2g_2, \quad [g_2,g_3] = g_3;$$

$$\mathfrak{so}(3): \ [g_1,g_2] = g_3, \quad [g_1,g_3] = -g_2, \quad [g_2,g_3] = g_1.$$

Note that only non-zero commutation relations are given. The following assertion holds.

Theorem 2. *There are at most three appropriate inequivalent contact realizations of the Lie algebra $\mathfrak{sl}(2,\mathbf{R})$. The realizations and the corresponding invariant equations are as follows*

$$\mathfrak{sl}^1(2,\mathbf{R}) = \langle 1, u + xu_1, u^2 + 2xuu_1 \rangle: \quad u_t = xu_1 F\left(t, \frac{3u_1 + 2xu_2}{x^4 u_1^3}\right),$$

$$\mathfrak{sl}^2(2,\mathbf{R}) = \langle 1, u + xu_1, u^2 + 2xuu_1 + u_1 \rangle: \quad u_t = (u_1 - x^2u_1^2)^{\frac{1}{2}} F\left(t, \frac{u_2 + 6xu_1^2 - 4x^3u_1^3}{(u_1 - x^2u_1^2)^{\frac{3}{2}}}\right),$$

$$\mathfrak{sl}^3(2,\mathbf{R}) = \langle u_t, -tu_t + xu_1, t^2u_t - 2txu_1 - u_1 \rangle: \quad u_t = -x^2u_1 + \frac{1}{u_1} F\left(u, \frac{u_2}{u_1^2}\right).$$

What is more, any contact realization of the algebra $\mathfrak{so}(3)$ is equivalent to

$$\mathfrak{so}^1(3) = \langle 1, \tan x \sin u - u_1 \cos u, \tan x \cos u + u_1 \sin u \rangle.$$

The most general $\mathfrak{so}^1(3)$ invariant equation in the class (1) has the form

$$u_t = (\sec^2 x + u_1^2)^{\frac{1}{2}} F\left(t, \frac{u_2 \cos x - (2 + u_1^2 \cos^2 x)u_1 \sin x}{(1 + u_1^2 \cos^2 x)^{\frac{3}{2}}}\right).$$

Proof. We begin by analyzing inequivalent realizations of the algebra $\mathfrak{sl}(2,\mathbf{R})$. Choosing the basis operators g_i $(i = 1,2,3)$ in the general form (7), inserting them into the commutation relations of $\mathfrak{sl}(2,\mathbf{R})$, and solving the equations obtained provide all possible realizations of the algebra under study.

In view of Lemma 2, we can assume, without any loss of generality, that one of the basis elements, say g_1, can be reduced to 1 or u_t. We consider the case $g_1 = 1$ in full detail.

Let $g_1 = 1$ and g_2, g_3 be of the form (7). Inserting g_2, g_3 into the first two commutation relations from $\mathfrak{sl}(2,\mathbf{R})$ gives

$$g_2 = u + \phi(t,x,u_1).$$

Hereafter ϕ is an arbitrary real-valued smooth function of indicated variables.

To get all possible forms of the function g_2, we need contact transformations (9) preserving the basis element $g_1 = 1$, namely,

$$\tilde{g_1} = 1 \to g_1 = \frac{D(X)}{J} = 1.$$

Hence, we conclude that

$$\frac{D(X)}{D(X)U_u - D(U)X_u} = 1 \tag{13}$$

and

$$X_{u_1} D(U) = U_{u_1} D(X). \tag{14}$$

Eq. (13) can be rewritten as follows

$$1 = U_u - \frac{D(U)}{D(X)} X_u. \tag{15}$$

Consider the cases $X_{u_1} \neq 0$ and $X_{u_1} = 0$ separately.

Case 1. If $X_{u_1} \neq 0$, then it follows from (14) that $D(U)/D(X) = U_{u_1}/X_{u_1}$. Inserting this expression into (15) gives the first order linear PDE

$$X_{u_1} = X_{u_1} U_u - X_u U_{u_1}$$

for U. Writing its associated system of ODEs in symmetric form as follows

$$\frac{\mathrm{d}t}{0} = \frac{\mathrm{d}x}{0} = \frac{\mathrm{d}u}{X_{u_1}} = \frac{\mathrm{d}u_1}{-X_u} = \frac{\mathrm{d}U}{X_{u_1}}$$

and solving the obtained equations, we get the functionally-independent first integrals t, x, X and $U - u$.

Consequently, we can choose the contact transformation in the form

$$\tilde{x} = X(t, x, u, u_1), \quad \tilde{u} = u + Y(t, x, X). \tag{16}$$

In view of (16), tangency condition (14) leads to

$$X_{u_1}(u_1 + Y_x) = 0,$$

whence

$$Y_x = -u_1. \tag{17}$$

Thus the most general contact transformation leaving $g_1 = 1$ invariant is given by (16), where Y satisfies the condition (17).

Case 2. If $X_{u_1} = 0$, we have $D(X) \neq 0$ (otherwise $X = X(t)$ and the non-degeneracy assumption does not hold). Consequently, (14) implies that $U_{u_1} = 0$ and contact transformation (9) turns out to be a point transformation.

It follows from (15) that the relation

$$X_x + u_1 X_u = X_x U_u - X_u U_x$$

holds. As functions X and U are independent of u_1, we have $X_u = 0$. Hence $X = X(t, x)$ and $U_u = 1$. Then the most general point transformation that preserves $g_1 = 1$ is of the form

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = u + Y(t, x), \tag{18}$$

where $X_x \neq 0$.

Consider now the action of contact transformation (16) on g_2,

$$\tilde{g}_2 = \tilde{u} + \phi(\tilde{t}, \tilde{x}, \tilde{u}_{\tilde{x}}) \to g_2 = u + Y(t, x, X) + \phi(T, X, Y_X).$$

By choosing Y which satisfy the compatible system of PDEs

$$Y(t, x, X) + \phi(T, X, Y_X) = 0$$

and (17), we arrive at $g_2 = u$. The commutation relations imply that $g_3 = u^2 + \psi(t, x) u_1^2$, where ψ is an arbitrary function. Performing transformation

$$\tilde{t} = T(t), \quad \tilde{x} = X(t, x), \quad \tilde{u} = u, \quad X_x \neq 0,$$

that preserves 1 and u, we get

$$\tilde{g_3} = \tilde{u}^2 + \psi(\tilde{t}, \tilde{x}) \tilde{u}_{\tilde{x}}^2 \to g_3 = u^2 + \frac{\psi(T, X)}{X_x^2} u_1^2.$$

With properly chosen X and U, the function ψ/X_x^2 reduces to 0, 1 or -1. As a result, we obtain the following three inequivalent contact realizations of $\mathfrak{sl}(2, \mathbf{R})$:

$$\langle 1, u, u^2 \rangle, \quad \langle 1, u, u^2 - u_1^2 \rangle, \quad \langle 1, u, u^2 + u_1^2 \rangle.$$

In view of the reducibility requirement, the inequivalent representatives of reducible contact realizations should be chosen among point realizations, whenever possible. Let us analyze reducibility of the last two realizations. Applying the contact transformation

$$\tilde{t} = t, \quad \tilde{x} = \ln |x^2 u_1|, \quad \tilde{u} = u + x u_1,$$

to the realization $\langle 1, u, u^2 - u_1^2 \rangle$, we get

$$\langle 1, u + x u_1, u^2 + 2 x u u_1 \rangle.$$

Analogously, the realization $\langle 1, u, u^2 + u_1^2 \rangle$ can be reduced to $\langle 1, u + xu_1, u^2 + 2xuu_1 + u_1 \rangle$ by the contact transformation

$$\tilde{t} = t, \quad \tilde{x} = \arcsin(2x^2u_1 - 1), \quad \tilde{u} = u + xu_1.$$

Turn now to the case $g_1 = u_t$. Analysis of this case yields two more inequivalent realizations of the algebra $\mathfrak{sl}(2, \mathbf{R})$, namely, $\langle u_t, -tu_t + xu_1, t^2u_t - 2txu_1 - u_1 \rangle$ and $\langle u_t, -tu_t, t^2u_t \rangle$.

The algebra $\mathfrak{so}(3)$ is handled in a similar fashion. We get the realization $\mathfrak{so}^1(3)$ listed in the theorem.

To complete the proof, we need to construct the corresponding invariant equations. Inserting the realizations obtained above into the classifying Eq. (8), we verify that the algebras $\langle 1, u, u^2 \rangle$ and $\langle u_t, -tu_t, t^2u_t \rangle$ cannot be symmetry algebras of (1). The remaining realizations give rise to the invariant equations given above. □

Remark 1. The realizations $\mathfrak{sl}^1(2, \mathbf{R})$ and $\mathfrak{sl}^2(2, \mathbf{R})$ are equivalent when considered over the field of complex numbers. Replacing the underlying field of real numbers by complex numbers, we have stronger equivalence relation and fewer inequivalent cases. Consequently, there are two inequivalent realizations of $\mathfrak{sl}(2, \mathbf{C})$ evolutionary symmetries

$$\langle 1, u, u^2 \rangle, \qquad \langle 1, u, u^2 - u_1^2 \rangle,$$

and another realizations of $\mathfrak{sl}(2, \mathbf{C})$ algebra of type III [18]. These results coincide with the realizations obtained in [18,28,29].

Remark 2. The algebra $\mathfrak{so}(3)$ over the complex field is isomorphic to $\mathfrak{sl}(2, \mathbf{C})$. So that the realization $\mathfrak{so}^1(3)$ is equivalent to the realization $\mathfrak{sl}^1(2, \mathbf{R})$.

Theorem 3. *Any equation of the form (1) invariant under a semi-simple Lie algebra of contact symmetries is equivalent to one of the PDEs listed in Theorem 2.*

Proof is carried out in the same way as it has been done in our paper [33], where we studied the less general case of Lie point transformations.

4. Equations invariant under the algebras having nontrivial Levi factor

In what follows, we utilize the results of the previous section to classify equations of the form (1) admitting the contact symmetry algebras having nontrivial Levi decomposition. Namely, we consider contact symmetry algebras that are represented as either direct or truly semi-direct sums of semi-simple and solvable Lie algebras. These two cases are to be considered separately.

A. Direct sums of semi-simple and solvable Lie algebras

It suffices to consider only the direct sums of each realization given in Theorem 2 and solvable algebras. Take $\mathfrak{sl}^1(2, \mathbf{R})$, as an example. We look for all possible extensions of the realization $\mathfrak{sl}^1(2, \mathbf{R})$ by functions (7) which commute with its basis functions. Analysis of the commutation conditions gives the most general form of the generating function

$$g = \alpha(t)u_t + \phi(t)xu_1. \tag{19}$$

Now we need to construct all possible solvable Lie algebras having the generating function (19). Skipping the intermediate computations, we formulate the final result: any solvable Lie algebra having generators (19) is isomorphic to one-dimensional algebras $\langle u_t \rangle, \langle \phi(t)xu_1 \rangle$ with $\dot{\phi} \neq 0$ or the two-dimensional non-Abelian algebra $\langle u_t, -tu_t + \lambda xu_1 \rangle$. Consequently, the list of all possible extensions of $\mathfrak{sl}^1(2, \mathbf{R})$ consists of three inequivalent algebras

$$\mathfrak{sl}^1(2, \mathbf{R}) \oplus \langle u_t \rangle : u_t = xu_1 F\left(\frac{3u_1 + 2xu_2}{x^4u_1^3}\right),$$

$$\mathfrak{sl}^1(2, \mathbf{R}) \oplus \langle \phi(t)xu_1 \rangle,\ \dot{\phi} \neq 0 : u_t = \frac{x\dot{\phi}u_1}{2\phi} \ln\left|\frac{3u_1 + 2xu_2}{x^4u_1^3}\right| + xu_1F(t),$$

$$\mathfrak{sl}^1(2, \mathbf{R}) \oplus \langle u_t, -tu_t + \lambda xu_1 \rangle,\ \lambda \neq 0 : u_t = Cx^{\frac{2+\lambda}{\lambda}} u_1^{\frac{3+2\lambda}{2\lambda}} (3u_1 + 2xu_2)^{-\frac{1}{2\lambda}}.$$

In a similar fashion, the extensions of $\mathfrak{sl}^2(2, \mathbf{R})$ and $\mathfrak{sl}^3(2, \mathbf{R})$ are obtained

$$\mathfrak{sl}^2(2, \mathbf{R}) \oplus \langle u_t \rangle : u_t = (u_1 - x^2u_1^2)^{\frac{1}{2}} F\left(\frac{u_2 + 6xu_1^2 - 4x^3u_1^3}{(u_1 - x^2u_1^2)^{\frac{3}{2}}}\right),$$

$$\mathfrak{sl}^2(2,\mathbf{R}) \oplus \left\langle \frac{\phi(t)}{x}(x^2u_1 - x^4u_1^2)^{\frac{1}{2}} \right\rangle,\ \dot{\phi} \neq 0 : u_t = \frac{u_1^{\frac{1}{2}}(x^2u_1-1)^{\frac{1}{2}}(-u_1^3+3u_1^4x^2-3u_1^5x^4+u_1^6x^6)^{\frac{1}{2}}}{u_1^{\frac{3}{2}}(x^2u_1-1)^{\frac{3}{2}}}F(t) + \frac{\dot{\phi}u_1^2(x^2u_1-1)^2 \operatorname{arctanh}\frac{u_2+6u_1^2x-4x^3u_1^3}{4u_1^{\frac{3}{2}}(x^2u_1-1)^{\frac{3}{2}}}}{\phi u_1^{\frac{3}{2}}(x^2u_1-1)^{\frac{3}{2}}},$$

$$\mathfrak{sl}^2(2,\mathbf{R}) \oplus \langle u_t, -tu_t + \frac{\lambda}{x}(x^2u_1 - x^4u_1^2)^{\frac{1}{2}} \rangle,\ \lambda \neq 0 : u_t = Cu_1^{\frac{1}{2}}(x^2u_1-1)^{\frac{1}{2}} \mathrm{e}^{\frac{(x^2u_1-x^4u_1^2)^{\frac{3}{2}}\operatorname{arctanh}\frac{-u_2-6u_1^2x+4x^3u_1^3}{4u_1^{\frac{3}{2}}(x^2u_1-1)^{\frac{3}{2}}}}{\lambda x^3u_1^{\frac{3}{2}}(x^2u_1-1)^{\frac{3}{2}}}},$$

$$\mathfrak{sl}^3(2,\mathbf{R}) \oplus \langle 1 \rangle : u_t = -x^2u_1 + \frac{1}{u_1}F\left(\frac{u_2}{u_1^2}\right), \mathfrak{sl}^3(2,\mathbf{R}) \oplus \langle 1, u \rangle : u_t = -x^2u_1 + C\frac{u_1^3}{u_2^2}.$$

Analogously, the extension $\mathfrak{so}^1(3)$ yields one more invariant equation

$$\mathfrak{so}^1(3) \oplus \langle u_t \rangle : u_t = (\sec^2 x + u_1^2)^{\frac{1}{2}} F\left(\frac{u_2\cos x - (2+u_1^2\cos^2 x)u_1\sin x}{(1+u_1^2\cos^2 x)^{\frac{3}{2}}}\right).$$

Note that above symmetry algebras are maximal invariance algebras, in Lie's sense, of the corresponding equations provided the function F and constant C are arbitrary.

When considering the direct sums of semi-simple and solvable Lie algebras over the complex field, we only need to take into account the extensions of the realizations $\mathfrak{sl}^1(2,\mathbf{R})$ and $\mathfrak{sl}^3(2,\mathbf{R})$, since $\mathfrak{sl}^1(2,\mathbf{R})$, $\mathfrak{sl}^2(2,\mathbf{R})$ and $\mathfrak{so}^1(3)$ are equivalent there. In this way we obtain all the above realizations except for those associated with $\mathfrak{sl}^2(2,\mathbf{R})$ and $\mathfrak{so}^1(3)$.

B. Semi-direct sums of semi-simple and solvable Lie algebras

To classify PDEs (1), whose invariance algebras are isomorphic to semi-direct sums of semi-simple and solvable Lie algebras, we apply the following two-step approach.

Firstly, we utilize the results of classification of low dimensional Lie algebras, which are semi-direct sum of Levi factor and solvable radical obtained in [30], and describe all inequivalent equations within the class (1) admitting those algebras. Without loss of generality, we can restrict our considerations to the Lie algebras having Levi decomposition $\mathfrak{sl}(2,\mathbf{N}) \Subset A_{2,1}$ and $\mathfrak{so}(3) \Subset A_{3,1}$, where $A_{2.1}$ and $A_{3.1}$ are two- and three-dimensional Abelian algebras respectively [30].

Secondly, applying the Ovsiannikov's approach [24], we complete the group classification of PDEs (1) containing arbitrary functions of one variable or arbitrary constants.

Consider first the algebra $\mathfrak{sl}(2,\mathbf{N} \Subset A_{2,1}$. Taking $\mathfrak{sl}(2,\mathbf{R}) = \langle g_1, g_2, g_3 \rangle$ and $A_{2.1} = \langle g_4, g_5 \rangle$, the nonzero commutation relations

$$[g_1,g_2] = g_1, \quad [g_1,g_3] = 2g_2, \quad [g_2,g_3] = g_3, \quad [g_1,g_5] = g_4, \quad [g_2,g_4] = -\frac{1}{2}g_4, \quad [g_2,g_5] = \frac{1}{2}g_5, \quad [g_3,g_4] = -g_5,$$

hold for the algebra in question.

Here we provide full details of the analysis of the algebra $\mathfrak{sl}^1(2,\mathbf{R}) \Subset A_{2,1}$. Other cases are treated in a similar way, and we only present the final results. Let $g_1 = 1,\ g_2 = u + xu_1,\ g_3 = u^2 + 2xuu_1$, and $g_4,\ g_5$ be of the general form (7). Inserting these expressions into the commutation relations above and solving the obtained equations yield

$$g_4 = \phi(t)|x|^{-\frac{1}{2}}, \quad g_5 = \phi(t)|x|^{-\frac{1}{2}}(u + 2xu_1).$$

Performing the contact transformation

$$\tilde{t} = T(t), \quad \tilde{x} = xY(t), \quad \tilde{u} = u,$$

which preserves $\mathfrak{sl}^1(2,\mathbf{R})$, we have

$$g_4 = \phi(T)|Y(t)|^{-\frac{1}{2}}|x|^{-\frac{1}{2}}, \quad g_5 = \phi(T)|Y(t)|^{-\frac{1}{2}}|x|^{-\frac{1}{2}}(u + 2xu_1).$$

Choosing $T = t$ and $Y = \phi^2(t)$, this $A_{2.1}$ algebra reduces to $\langle |x|^{-\frac{1}{2}}, |x|^{-\frac{1}{2}}(u + 2xu_1) \rangle$. Consequently, any extension of the algebra $\mathfrak{sl}^1(2,\mathbf{R})$ is equivalent to

$$\mathfrak{sl}^1(2,\mathbf{R}) \Subset \left\langle |x|^{-\frac{1}{2}}, |x|^{-\frac{1}{2}}(u + 2xu_1) \right\rangle.$$

The corresponding invariant equation takes the form

$$u_t = \frac{(3u_1 + 2xu_2)^{\frac{1}{3}}}{x^{\frac{1}{3}}}F(t). \tag{20}$$

However, the five-dimensional Lie algebra is not the maximal one admitted by (20). In order to determine its maximal symmetry algebra, we simplify the equation under study. By making the change of variables

$$\tilde{t} = \int F(t)\mathrm{d}t, \quad \tilde{x} = x, \quad \tilde{u} = u,$$

Eq. (20) is transformed to

$$u_t = \frac{(3u_1 + 2xu_2)^{\frac{1}{3}}}{x^{\frac{1}{3}}}.$$

Applying the Lie infinitesimal algorithm, we prove that the maximal invariance algebra of the transformed equation is the seven-dimensional Lie algebra

$$\mathfrak{sl}^1(2,\mathbf{R}) \Subset \left\langle |x|^{-\frac{1}{2}}, |x|^{-\frac{1}{2}}(u + 2xu_1), u_t, -\frac{4}{3}tu_t - 2xu_1 \right\rangle,$$

which is isomorphic to $\mathfrak{sl}^1(2,\mathbf{R}) \Subset A_{4.5}$. Here $A_{4.5}$ is the four-dimensional decomposable Lie algebra with $q = 1$ and $p = 4/3$ given in [3].

Analogous analysis shows that $\mathfrak{sl}^2(2,\mathbf{R})$ cannot be extended up to an invariance algebra of (1) isomorphic to $\mathfrak{sl}(2,\mathbf{R}) \Subset A_{2.1}$ and algebra $\mathfrak{sl}^3(2,\mathbf{R})$ admits an extensional realization,

$$\mathfrak{sl}^3(2,\mathbf{R}) \Subset \left\langle |u_x|^{-\frac{1}{2}}(xu_1 + u), -|u_x|^{-\frac{1}{2}}((tx+1)u_1 + tu) \right\rangle$$

together with the invariant PDE

$$u_t = -2\lambda\frac{u_1}{u_2} + \frac{u^2 + \lambda u}{u_1} - x^2u_1, \quad \lambda \neq 0,$$

whose maximal invariance algebra is five-dimensional.

Turn now to the algebra $\mathfrak{so} \Subset A_{3.1}$. Let $\mathfrak{so}(3) = \langle g_1, g_2, g_3 \rangle$ and $A_{3.1} = \langle g_4, g_5, g_6 \rangle$, where g_i $(i = 1, 2, \ldots, 5)$, satisfy the nonzero commutation relations

$$[g_1, g_5] = g_6, \quad [g_1, g_6] = -g_5, \quad [g_2, g_4] = -g_6, \quad [g_2, g_6] = g_4, \quad [g_3, g_4] = g_5, \quad [g_3, g_5] = -g_4.$$

Letting $g_1 = 1,\ g_2 = \tan x \sin u - u_1 \cos u,\ g_3 = \tan x \cos u + u_1 \sin u, g_4,\ g_5,\ g_6$ be of the general form (7), and inserting the expressions for g_i into the commutation relations above, lead to the trivial generators $g_4 = g_5 = g_6 = 0$. By this reason PDEs of the form (1) cannot admit a symmetry algebra having Levi decomposition $\mathfrak{so}(3) \Subset A_{3.1}$.

Note that considering realizations of the semi-direct sums of semi-simple and solvable Lie algebras over the complex field yields the same realizations as those obtained above.

5. Concluding remarks

In this paper, we generalize the Lie group classification method of [32] and make it applicable to contact symmetries. As a result, we develop an efficient algebraic approach to classify contact symmetry groups of nonlinear evolution equations. It enables us to obtain exhaustive description of all second-order evolution equations that possess contact symmetry algebras containing semi-simple subalgebras.

The study of symmetry properties of equations of the form (1) in the present paper can be briefly summarized up as follows.

- There are four inequivalent subclasses of equations of the form (1) which are invariant under semi-simple Lie algebras;
- There exist eleven inequivalent subclasses of equations of the form (1) admitting symmetry algebras having nontrivial Levi factors.

The number of inequivalent realizations obtained here is lower than that of the realizations based on point symmetries [3,33]. The reason for this is that contact transformations provide stronger equivalence relations, i.e., some of realizations being inequivalent with respect to point transformations may be equivalent with respect to contact ones [23]. Take semi-simple algebras for instance. There are five realizations of the $\mathfrak{sl}(2,\mathbf{R})$ algebra in [33]. Among them, two realizations in Theorem 3.1 are equivalent to $\mathfrak{sl}^3(2,\mathbf{R})$, and three realizations from Lemma 4.2 can be transformed to $\mathfrak{sl}^1(2,\mathbf{R})$ or $\mathfrak{sl}^2(2,\mathbf{R})$. More specifically, the first realizations $\langle \partial_t, 2t\partial_t + x\partial_x, -t^2\partial_t - tx\partial_x + x^2\partial_u \rangle$, in Theorem 3.1 of [33], can be mapped into $\langle u_t, -2tu_t + 2xu_1, t^2u_t - 2txu_1 - u_1 \rangle$ by the contact transformation

$$\tilde{t} = t, \quad \tilde{x} = u_1^{\frac{1}{2}}, \quad \tilde{u} = -u + xu_1.$$

Evidently, any algebra containing semi-simple subalgebra in [3,33] can be reduced to one of our realizations by a suitable contact transformation.

It is important to emphasize that our approach can be efficiently applied to hyperbolic PDEs as well. We have classified the broad classes of nonlinear wave equations admitting nontrivial point symmetries [35]. We intend to apply this approach to classify nonlinear hyperbolic PDEs admitting nontrivial contact symmetries.

We have briefly analyzed reducibility of the obtained realizations. Whenever possible, we choose the inequivalent representatives of reducible contact realizations among their point realizations. Note that some reducible contact realizations obtained in [18,29] have not been reduced to point realizations.

As we have already mentioned, all inequivalent equations of the form (1) that admit solvable Lie algebras of the dimension $n \leq 4$ have been constructed in [10].

Note that relation (7) holds for evolution equations of an arbitrary order [18] and our approach can also be applied to higher dimensional evolution equations as well.

We intend to utilize the obtained results to study potential symmetries of the class (1), based on the relation between contact and potential symmetries [9,25,26,34]. We proved in [9] the following assertion:

Theorem 4. *Let L be the algebra of contact symmetries of evolution Eq. (1) such that, (i) dim(L)⩾2, (ii) $[L, L] \neq 0$. Then Eq. (1) can be mapped to another evolution equation, belonging to the class (1), which admits potential symmetries.*

Since any semi-simple algebra is non-commutative, every realization obtained in this paper leads to a potential symmetry. To construct second-order evolution PDEs possessing potential symmetries, one needs to classify all inequivalent non-commutative subalgebras of the symmetry algebras constructed above and then follow the procedure suggested in [9,34] to obtain a wide range of Eqs. (1) admitting potential symmetries. This research is in progress and will be reported in one of our future publications.

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References

[1] Anderson RL, Ibragimov NH. Lie-Bäcklund Transformations in Applications. Philadelphia, PA: SIAM; 1979.
[2] Barut AO, Rączka R. Theory of Group Representations and Applications. Singapore: World Scientific; 1986.
[3] Basarab-Horwath P, Lahno V, Zhdanov R. The structure of Lie algebras and the classification problem for partial differential equations. Acta Appl. Math. 2001;69:43–94.
[4] Bianchi L. Lezioni sulla teoria dei Gruppi Continui Finiti di Trasformazioni. Pisa: Enrico Spoerri; 1918.
[5] Bluman GW, Kumei S. Symmetries and Differential Equations. New York: Springer-Verlag; 1989.
[6] Bluman GW, Cheviakov AF, Anco SC. Applications of Symmetry Methods to Partial Differential Equations. New York: Springer; 2010.
[7] Campbell JE. Introductory Treatise on Lie's Theory of Finite Continuous Transformation Groups. Oxford: The Clarendon Press; 1903.
[8] Heredero RH, Olver PJ. Classification of invariant wave equations. J. Math. Phys. 1996;37:6414–38.
[9] Huang Q, Qu CZ, Zhdanov R. Group-theoretical framework for potential symmetries of evolution equations. J. Math. Phys. 2011;52:023514.
[10] Q. Huang, R. Zhdanov, C. Z. Qu, Contact symmetries of evolution equations, arxiv:1301.2097, 2013.
[11] Hydon PE. How to find discrete contact symmetries. J. Nonlinear Math. Phys. 1998;5:405–16.
[12] Ibragimov NH. Transformation Groups Applied to Mathematical Physics. Dordrecht: D. Reidel; 1985.
[13] Ibragimov NH, Khabirov SV. Contact transformation group classification of nonlinear wave equations. Nonlinear Dynam. 2000;22:61–71.
[14] Kara AH, Mahomed FM. Relationship between symmetries and conservation laws. Int. J. Theor. Phys. 2000;39:23–40.
[15] S. Lie, Ph. D. thesis, University of Christiana, 1871.
[16] Lie S. Über die Integration durch bestimmte Integrable von einer Klasse linear partialler Differentialgleichungen. Arch. Math. 1881;6:328.
[17] Loskutov AYu, Mikhalov AS. Introduction to Synergetics. Moscow: Nauka; 1990.
[18] Magadeev BA. On group classification of nonlinear evolution equations. Algebra i Analiz 1993;5:141–56.
[19] Momoniat E. True contact transformations admitted by a class of non-linear wave equations. Int. J. Nonlinear Mech. 2001;36:705–8.
[20] Momoniat E. An equation admitting infinite true contact transformations. J. Math. Anal. Appl. 2003;282:668–72.
[21] Morozov OI. Contact-equivalence problem for linear hyperbolic equations. J. Math. Sci. (NY) 2006;135:2680–94.
[22] Newell AC, Whitehead JA. Finite bandwidth, finite amplitude convection. J. Fluid Mech. 1969;38:279–303.
[23] Olver PJ. Equivalence, Invariants, and Symmetry. Cambridge: Cambridge University Press; 1995.
[24] Ovsiannikov LV. Group Analysis of Differential Equations. New York: Academic; 1982.
[25] Popovych RO, Samoilenko AM. Local conservation laws of second-order evolution equations. J. Phys. A 2008;41:362002.
[26] Popovych RO, Sergyeyev A. Conservation laws and normal forms of evolution equations. Phys. Lett. A 2010;374:2210–7.
[27] Pucci E, Saccomandi G. Contact symmetries and solutions by reduction of partial differential equations. J. Phys. A: Math. Gen. 1994;27:177–84.
[28] Sokolov VV. On the structure of the symmetry algebra for one-field evolution equation. Sov. Math. Dokl. 1987;35:635–8.
[29] Sokolov VV. On the symmetries of evolution equations. Russ. Math. Surv. 1988;43:165–204.
[30] Turkowski P. Low-dimensional real Lie algebras. J. Math. Phys. 1988;29:2139–44.
[31] Yefimova OYu, Kudryashov NA. Exact solutions of the Burgers–Huxley equation. J. Appl. Math. Mech. 2004;68:413–20.
[32] Zhdanov RZ, Lahno VI. Group classification of heat conductivity equations with a nonlinear source. J. Phys. A Math. Gen. 1999;32:7405–18.

[33] Zhdanov R, Lahno V. Group classification of the general second-order evolution equation: semi-simple invariance groups. J. Phys. A Math. Theor. 2007;40:5083–103.
[34] Zhdanov R. On relation between potential and contact symmetries of evolution equations. J. Math. Phys. 2009;50:053522.
[35] Lahno V, Zhdanov R, Magda O. Group classification and exact solutions of nonlinear wave equations. Acta Appl. Math. 2006;91:253–313.