

On the Sutherland Spin Model of B_N Type and its Associated Spin Chain

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Abstract

The B_N hyperbolic Sutherland spin model is expressed in terms of a suitable set of commuting Dunkl operators. This fact is exploited to derive a complete family of commuting integrals of motion of the model, thus establishing its integrability. The Dunkl operators are shown to possess a common flag of invariant finite-dimensional linear spaces of smooth scalar functions. This implies that the Hamiltonian of the model preserves a corresponding flag of smooth spin functions. The discrete spectrum of the restriction of the Hamiltonian to this spin flag is explicitly computed by triangularization. The integrability of the hyperbolic Sutherland spin chain of B_N type associated with the dynamical model is proved using Polychronakos's "freezing trick".

I. INTRODUCTION

Since the publication of the pioneering papers of Calogero [5] and Sutherland [32, 33], the study of solvable and integrable quantum many-body problems has become a fruitful field of research with multiple connections in many branches of contemporary mathematics and physics. From a mathematical standpoint, one of the key developments in the field was the discovery by Olshanetsky and Perelomov of an underlying A_N root system structure for both the Calogero and Sutherland models [25]. The integrability of these models follows by expressing the Hamiltonian as one of the radial parts of the Laplace–Beltrami operator in a symmetric space associated with the given root system. It was also shown in this paper that the original inverse square (Calogero) and trigonometric/hyperbolic (Sutherland) potentials arise as appropriate limits of the most general potential in this class, given by the Weierstrass \wp -function, and that integrable models associated to other root systems also exist. The rational and trigonometric Calogero–Sutherland (CS) models are also *exactly solvable*, in the sense that their eigenfunctions and eigenvalues can be computed algebraically. In fact, the study of the eigenfunctions of these models has led to significant advances in the theory of multivariate orthogonal polynomials [1, 11, 23]. Apart from their mathematical interest, CS models have found numerous applications in diverse areas of physics such as soliton theory [22, 29], fractional statistics and anyons [4, 6], random matrix theory [35], and Yang–Mills theories [9, 16], to name only a few.

During the last decade, CS models with internal degrees of freedom have been actively explored by a variety of methods, including the exchange operator formalism [24], the Dunkl operators approach [2, 8, 11], reduction by discrete symmetries [30], and construction of Lax pairs [19, 20, 21]. Historically, the first CS models with spin discussed in the literature were related to the original models of A_N type introduced by Calogero and Sutherland [2, 17, 19, 20, 24, 34]. The integrability of these CS spin models was established in some cases by relating the Hamiltonian to a quadratic combination of Dunkl operators of A_N type [7, 10, 26]. The B_N counterpart of the A_N CS spin models mentioned above were

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first considered by Yamamoto [38]. In this paper, the spectrum of the rational B_N spin model was explicitly determined, and its integrability was shown by means of the Lax pair approach. In Ref. [39], Yamamoto and Tsuchiya presented an alternative proof of the integrability of this model using Dunkl operators of B_N type. The same operators were later employed by Dunkl to construct a complete basis of eigenfunctions [11]. In contrast, the trigonometric/hyperbolic B_N spin model has received remarkably little attention. In our recent paper [13] we proved that this model is exactly solvable in the sense of Turbiner [36, 37], meaning that its Hamiltonian leaves invariant a known infinite increasing sequence (or *flag*) of finite-dimensional linear spaces of smooth spin functions. In fact, in [13] we developed a systematic method for constructing exactly (or in some cases partially) solvable B_N -type CS models with spin by combining several families of Dunkl operators. The key elements of this method—first introduced in the A_N case in [12]—are: i) the definition of a new family of Dunkl operators, and ii) the construction of a very wide class of quadratic combinations of these operators and those in the other two families considered by Dunkl in [11].

The interest on CS spin models has been further enhanced by their close connections with integrable spin chains of Haldane–Shastry type [18, 31]. Spin chains describe a fixed arrangement of particles that interact through their spins. A well-known example is the Heisenberg spin chain, whose spins are equally spaced and interact only with their nearest neighbors. The Haldane–Shastry model was actually the first one-dimensional spin chain with long range interactions whose spectrum could be computed exactly. In this model, the spin sites are equally spaced in a circle and interact with each other with strength decreasing as the inverse square of the chord distance between the sites. The integrability of the Haldane–Shastry spin chain was proved by Fowler and Minahan in [14]. Polychronakos later realized that the commuting conserved quantities of the Haldane–Shastry spin chain can be elegantly deduced from those of the (dynamical) Sutherland spin model of A_N type by applying what he called the “freezing trick” [27] (see also [34]). This corresponds to taking the strong coupling limit in the Sutherland spin model and restricting to states with no momentum excitations, so that the internal degrees of freedom remain the only relevant variables in the problem and the particles are “frozen” at their classical equilibrium positions. This observation is, in principle, valid for any integrable spin Calogero–Sutherland model. For instance, in Ref. [27] the freezing trick is applied to the spin Calogero model with rational interaction to construct a new integrable spin chain of rational type in which the sites are no longer equally spaced. The spectrum of this chain was later calculated by Frahm [15] and Polychronakos [28]. Bernard, Pasquier and Serban [3] studied the spin chain associated with the trigonometric Sutherland model, establishing its integrability for certain values of the parameters in the Hamiltonian.

The aim of this paper is twofold. In the first place, we prove the integrability of the hyperbolic Sutherland spin model of B_N type, from which we are also able to deduce the integrability of the spin chain associated with this model. Secondly, we give an explicit formula for the eigenvalues of the dynamical model whose corresponding (square-integrable) eigenfunctions lie in the invariant flag mentioned above. The paper is organized as follows. In Section II we introduce a family of commuting Dunkl operators of B_N type. We show that the sums of even powers of these Dunkl operators generate a complete set of commuting integrals of motion of the Hamiltonian of the model. The commutation relations satisfied by the Dunkl operators and the usual permutation and sign reversing operators, which possess a richer structure than in the rational case, play a key role in the proof of this result. In Section III we analyze the spectrum of the Hamiltonian for any value of the spin. Our analysis is based on the fact that the Dunkl operators leave invariant a flag of finite-dimensional linear spaces of smooth scalar functions. This flag yields a corresponding invariant flag of smooth spin functions for the hyperbolic B_N Sutherland Hamiltonian. We construct a partially ordered basis of this “spin” flag in which the Hamiltonian is represented by a triangular matrix. In this way we can explicitly compute the eigenvalues of the restriction of the Hamiltonian to the (finite-dimensional) intersection of the spin flag with

the Hilbert space of the system. We shall use the term *algebraic* in what follows to refer to these eigenvalues and its corresponding eigenfunctions. It remains an open problem to determine whether the algebraic sector of the spectrum actually coincides with the discrete spectrum. We also study in detail the (algebraic) ground state, determining its degeneracy for all values of the spin. In Section IV we define the spin chain associated with the hyperbolic Sutherland spin model of B_N type, and apply the freezing trick to derive a complete family of commuting integrals of motion of this chain.

II. INTEGRABILITY OF THE SUTHERLAND SPIN MODEL OF B_N TYPE

The Hamiltonian of the *hyperbolic B_N Sutherland spin model* is defined by

$$H^* = - \sum_i \partial_{x_i}^2 + 2a \sum_{i < j} \left[\sinh^{-2} x_{ij}^- (a + S_{ij}) + \sinh^{-2} x_{ij}^+ (a + \tilde{S}_{ij}) \right] + b \sum_i \sinh^{-2} x_i (b + S_i) - b' \sum_i \cosh^{-2} x_i (b' + S_i), \quad (1)$$

where $x_{ij}^\pm = x_i \pm x_j$ and a, b, b' are real parameters. Here and in what follows, any summation or product index without an explicit range will be understood to run from 1 to N , unless otherwise constrained. The operators S_{ij} and S_i in Eq. (1) act on the finite-dimensional Hilbert space

$$\mathcal{S} = \left\langle |s_1, \dots, s_N\rangle \mid s_i = -M, -M + 1, \dots, M; M \in \frac{1}{2}\mathbb{N} \right\rangle, \quad (2)$$

associated to the particles' internal degrees of freedom, as follows:

$$\begin{aligned} S_{ij} |s_1, \dots, s_i, \dots, s_j, \dots, s_N\rangle &= |s_1, \dots, s_j, \dots, s_i, \dots, s_N\rangle, \\ S_i |s_1, \dots, s_i, \dots, s_N\rangle &= |s_1, \dots, -s_i, \dots, s_N\rangle. \end{aligned} \quad (3)$$

We have also used the customary notation $\tilde{S}_{ij} = S_i S_j S_{ij}$.

The operators S_{ij} and S_i are represented in \mathcal{S} by $(2M + 1)^N$ -dimensional Hermitian matrices, and obey the following algebraic relations:

$$\begin{aligned} S_{ij}^2 &= 1, & S_{ij} S_{jk} &= S_{ik} S_{ij} = S_{jk} S_{ik}, & S_{ij} S_{kl} &= S_{kl} S_{ij}, \\ S_i^2 &= 1, & S_i S_j &= S_j S_i, & S_{ij} S_k &= S_k S_{ij}, & S_{ij} S_j &= S_i S_{ij}, \end{aligned} \quad (4)$$

where the indices i, j, k, l take distinct values in the range $1, \dots, N$. The algebra \mathfrak{S} generated by the operators S_{ij}, S_i is thus isomorphic to the group algebra of the Weyl group \mathcal{W}_N of type B_N , also known as the hyperoctahedral group.

We shall also make use of the permutation operators $K_{ij} = K_{ji}$ and the sign reversing operators K_i ($1 \leq i \neq j \leq N$), whose action on a function $f(\mathbf{x})$, with $\mathbf{x} = (x_1, \dots, x_N) \in \mathbb{R}^N$, is defined as follows:

$$\begin{aligned} (K_{ij} f)(x_1, \dots, x_i, \dots, x_j, \dots, x_N) &= f(x_1, \dots, x_j, \dots, x_i, \dots, x_N), \\ (K_i f)(x_1, \dots, x_i, \dots, x_N) &= f(x_1, \dots, -x_i, \dots, x_N). \end{aligned} \quad (5)$$

The operators K_{ij} and K_i obey algebraic identities analogous to (4). We shall denote by $\mathfrak{K} \simeq \mathfrak{S}$ the algebra generated by the coordinate permutation and sign changing operators K_{ij} and K_i . Note also that the operators $\Pi_{ij} = K_{ij} S_{ij}$ and $\Pi_i = K_i S_i$ generate an algebra \mathfrak{W} isomorphic to \mathfrak{K} and \mathfrak{S} . From now on we shall identify the abstract group \mathcal{W}_N with its realizations generated by the operators K_{ij}, K_i on $C^\infty(\mathbb{R}^N)$, S_{ij}, S_i on \mathcal{S} , or Π_{ij}, Π_i on $C^\infty(\mathbb{R}^N) \otimes \mathcal{S}$, depending on the context.

The Hamiltonian (1) describes a system of N identical particles, whose physical states are therefore either totally symmetric or totally antisymmetric under particle exchange. Moreover, since H^* clearly commutes with the family of commuting operators Π_i ($i = 1, \dots, N$), we can choose a basis of common eigenfunctions of H^* and all the operators Π_i . Given an element ψ_k of this basis, it follows from the commutation relations of the sign reversing operators Π_i with the permutation operators Π_{ij} that $\Pi_i \psi_k = \epsilon_k \psi_k$, independently of i . In principle, the parity ϵ_k could depend on k . However, we shall see in the following section that all the algebraic eigenfunctions have the same parity, and that this parity is determined by the sign of the parameter b . From now on we shall assume, for definiteness, that we are dealing with a system of fermions whose algebraic states are also antisymmetric under sign reversal of each particle's spatial and internal coordinates. This covers what is perhaps the physically most interesting case, namely that of a system of spin 1/2 particles, for which the internal degrees of freedom are naturally interpreted as the particles' spin. The results of this paper can be easily modified to treat any other choice of the particles' statistics and parity.

In the rest of this section we shall prove the integrability of the model (1) by expressing the Hamiltonian in terms of the following family of B_N -type Dunkl operators:

$$J_i = \partial_{x_i} - a \sum_{j \neq i} \left[(1 + \coth x_{ij}^-) K_{ij} + (1 + \coth x_{ij}^+) \tilde{K}_{ij} \right] - [b(1 + \coth x_i) + b'(1 + \tanh x_i)] K_i + 2a \sum_{j < i} K_{ij}, \quad (6)$$

where $\tilde{K}_{ij} = K_i K_j K_{ij}$. The operators J_i are related to the operators introduced by Yamamoto [38] in connection with the trigonometric BC_N spin Sutherland model.

A key property of the Dunkl operators (6) is their commutativity:

$$[J_i, J_j] = 0, \quad i, j = 1, \dots, N. \quad (7)$$

We shall also make use of the following commutation relations between the operators K_{ij}, K_i and the operators J_i :

$$[K_{ij}, J_k] = \begin{cases} 2a(K_{jk} - K_{ik})K_{ij}, & i < k < j \\ 0, & \text{otherwise} \end{cases} \quad (8)$$

$$K_{ij} J_i - J_j K_{ij} = -2a \left(1 + \sum_{i < l < j} K_{ij} K_{il} \right), \quad (9)$$

$$K_{ij} J_j - J_i K_{ij} = 2a \left(1 + \sum_{i < l < j} K_{ij} K_{jl} \right), \quad (10)$$

$$[K_i, J_j] = 2a K_{ij} (K_j - K_i), \quad [K_j, J_i] = 0, \quad (11)$$

$$\{K_i, J_i\} = -2(b + b') - 2a \sum_{l > i} K_{il} (K_i + K_l), \quad (12)$$

where $i < j$ and k are three distinct indices ranging from 1 to N . Note that the Dunkl operators of rational type considered in Ref. [39] satisfy the latter equations with the right-hand side replaced by zero. The nonvanishing of the right-hand side of Eqs. (8)–(12) gives rise to some nontrivial technical points in the proof of the integrability of the Hamiltonian (1), as we shall see below.

Another important property of the Dunkl operators J_i is that they preserve the linear space

$$\mathcal{R}_m = \left\langle \mu(\mathbf{x}) \exp \left(2 \sum_i n_i x_i \right) \mid n_i = -m, -m + 1, \dots, m, \quad i = 1, \dots, N \right\rangle, \quad (13)$$

where

$$\mu(\mathbf{x}) = \prod_{i < j} |\sinh x_{ij}^- \sinh x_{ij}^+|^a \cdot \prod_i |\sinh x_i|^b |\cosh x_i|^{b'}, \quad (14)$$

for any nonnegative integer m . This fact will prove crucial for the calculation of the spectrum of the hyperbolic B_N Sutherland spin model.

We define a mapping $*$: $\mathfrak{D} \otimes \mathfrak{K} \rightarrow \mathfrak{D} \otimes \mathfrak{S}$, where \mathfrak{D} denotes the algebra of linear differential operators on $C^\infty(\mathbb{R}^N)$, as follows. If $D \in \mathfrak{D}$, we set

$$(D K_{\alpha_1} \cdots K_{\alpha_r})^* = (-1)^r D S_{\alpha_r} \cdots S_{\alpha_1}, \quad (15)$$

where α_k stands for ij or i . The mapping $*$ is then linearly extended to $\mathfrak{D} \otimes \mathfrak{K}$. For instance, the Hamiltonian of the hyperbolic B_N Sutherland spin model (1) is obtained by applying the “star” mapping to the operator

$$\begin{aligned} H = & - \sum_i \partial_{x_i}^2 + 2a \sum_{i < j} \left[\sinh^{-2} x_{ij}^- (a - K_{ij}) + \sinh^{-2} x_{ij}^+ (a - \tilde{K}_{ij}) \right] \\ & + b \sum_i \sinh^{-2} x_i (b - K_i) - b' \sum_i \cosh^{-2} x_i (b' - K_i). \end{aligned} \quad (16)$$

Let Λ_0 be the antisymmetrisation operator, defined by the relations $\Lambda_0^2 = \Lambda_0$ and $\Pi_{ij} \Lambda_0 = -\Lambda_0$, $j > i = 1, \dots, N$. More explicitly,

$$\Lambda_0 = \frac{1}{N!} \sum_{l=1}^{N!} \epsilon_l P_l,$$

where P_l denotes an element of the realization of the symmetric group generated by the operators Π_{ij} , and ϵ_l is the signature of P_l . For instance, if $N = 2, 3$ the antisymmetriser Λ_0 is given by

$$\begin{aligned} N = 2: \quad \Lambda_0 &= \frac{1}{2} (1 - \Pi_{12}), \\ N = 3: \quad \Lambda_0 &= \frac{1}{6} (1 - \Pi_{12} - \Pi_{13} - \Pi_{23} + \Pi_{12}\Pi_{13} + \Pi_{12}\Pi_{23}). \end{aligned}$$

The total antisymmetriser Λ with respect to the action of the operators Π_{ij} and Π_i is determined by the relations $\Lambda^2 = \Lambda$ and

$$\Pi_{ij} \Lambda = -\Lambda, \quad \Pi_i \Lambda = -\Lambda, \quad j > i = 1, \dots, N. \quad (17)$$

It may be easily shown that

$$\Lambda = \frac{1}{2^N} \left(\prod_i (1 - \Pi_i) \right) \Lambda_0.$$

Since $K_{ij}^2 = K_i^2 = 1$, the relations (17) are equivalent to

$$K_{ij} \Lambda = -S_{ij} \Lambda, \quad K_i \Lambda = -S_i \Lambda, \quad j > i = 1, \dots, N. \quad (18)$$

From these relations and the definition of the star mapping it follows immediately that

$$A \Lambda = A^* \Lambda \quad (19)$$

for every operator $A \in \mathfrak{D} \otimes \mathfrak{K}$. The proof of the integrability of the hyperbolic B_N Sutherland spin model is based on the following lemmas.

Lemma 1. *If $B \in \mathfrak{D} \otimes \mathfrak{S}$ satisfies $B\Lambda = 0$, then $B = 0$.*

Proof. The operator $B \in \mathfrak{D} \otimes \mathfrak{S}$ is of the form

$$B = \sum_{i \in I} f_i(\mathbf{x}) B_i \partial^i, \quad (20)$$

where $B_i \in \mathfrak{S}$, $f_i \in C^\infty(\mathbb{R}^N)$, $i = (i_1, \dots, i_N)$ is a multiindex belonging to a finite subset $I \subset \mathbb{N}_0^N$ (with $\mathbb{N}_0 = \{0, 1, 2, \dots\}$), and $\partial^i = \partial_1^{i_1} \dots \partial_N^{i_N}$. Let us denote by W_l , with $l \in L = \{1, \dots, 2^N N!\}$, the elements of the realization of the Weyl group \mathcal{W}_N generated by the operators Π_{ij} and Π_i . The action of the total antisymmetrisation operator Λ over a factored state $\psi = \varphi(\mathbf{x}) |s\rangle$, with $\varphi \in C^\infty(\mathbb{R}^N)$ and $|s\rangle \in \mathcal{S}$, is given by

$$\Lambda \psi = \sum_{l \in L} \epsilon_l (W_l \varphi) W_l |s\rangle, \quad (21)$$

where $\epsilon_l = \pm 1$ is the parity of the total number of generators Π_{ij} , Π_i in any decomposition of W_l . By hypothesis,

$$B(\Lambda \psi) = \sum_{i \in I, l \in L} \epsilon_l f_i(\mathbf{x}) \partial^i (W_l \varphi) \cdot B_i (W_l |s\rangle) = 0. \quad (22)$$

Applying the latter equation to a family of functions $\{\varphi_j(\mathbf{x})\}_{j \in I \times L}$ satisfying the condition

$$\det \left[\partial^i (W_l \varphi_j) \right]_{j \in I \times L}^{i \in I, l \in L} \neq 0, \quad (23)$$

we obtain

$$B_i (W_l |s\rangle) = 0, \quad \text{for all } i \in I, l \in L.$$

In particular (taking $l \in L$ so that W_l is the identity) $B_i |s\rangle = 0$ for all $i \in I$. Since $|s\rangle \in \mathcal{S}$ is arbitrary, it follows that $B_i = 0$ for all $i \in I$, and hence B vanishes identically. \square

Lemma 2. *If $B \in \mathfrak{D} \otimes \mathfrak{K}$ commutes with Λ then $(AB)^* = A^* B^*$ for all $A \in \mathfrak{D} \otimes \mathfrak{K}$.*

Proof. Using Eq. (19) and the hypothesis repeatedly we obtain:

$$(AB)^* \Lambda = A B \Lambda = \Lambda A B = A^* \Lambda B = A^* B \Lambda = A^* B^* \Lambda.$$

The statement follows from the previous lemma. \square

We shall often make use of the following immediate consequence of Lemma 2:

Lemma 3. *If $A, B \in \mathfrak{D} \otimes \mathfrak{K}$ commute with Λ then $[A, B]^* = [A^*, B^*]$.*

We shall now construct a complete family of commuting integrals of motion for the hyperbolic B_N Sutherland spin Hamiltonian (1). The construction is based on the observation that the operator H in Eq. (16) can be expressed as

$$H = - \sum_i J_i^2. \quad (24)$$

By (7), the operators

$$I_p = \sum_i J_i^{2p}, \quad p \in \mathbb{N}, \quad (25)$$

commute with one another. In view of the previous lemma, it suffices to prove that I_p commutes with the total antisymmetriser Λ for all $p \in \mathbb{N}$ to conclude that the star operators I_p^* ($p \in \mathbb{N}$) form a commuting family of integrals of motion of $H^* = -I_1^*$. We shall in fact prove the following stronger result:

Lemma 4. *The operators I_p ($p \in \mathbb{N}$) commute with the permutation and sign changing operators K_{ij} and K_i .*

Proof. First of all, the elementary permutation $K_{i,i+1}$ commutes with I_p for $i = 1, \dots, N-1$. Indeed, $K_{i,i+1}$ commutes with J_j for $j \neq i, i+1$ by (8), while for $j = i, i+1$ we have

$$\begin{aligned} K_{i,i+1}J_i^{2p} &= J_{i+1}^{2p}K_{i,i+1} - 2a \sum_{r=0}^{2p-1} J_i^{2p-r-1} J_{i+1}^r, \\ K_{i,i+1}J_{i+1}^{2p} &= J_i^{2p}K_{i,i+1} + 2a \sum_{r=0}^{2p-1} J_i^{2p-r-1} J_{i+1}^r. \end{aligned}$$

Since an arbitrary permutation can be expressed as the product of elementary permutations, this shows that K_{ij} commutes with I_p for all $i \neq j$. Secondly, the sign reversing operator K_N commutes with I_p for all p , since

$$K_N J_i = J_i K_N, \quad \text{if } i < N,$$

while

$$K_N J_N^2 = -J_N K_N J_N - 2(b+b')J_N = J_N^2 K_N.$$

This implies that K_i commutes with I_p for an arbitrary $i = 1, \dots, N$, since

$$0 = K_{iN}[K_N K_{iN}, I_p] = K_{iN}[K_{iN} K_i, I_p] = [K_i, I_p].$$

□

The operators I_p^* ($p \in \mathbb{N}$) thus form an infinite commuting family. Moreover, by examining the terms of highest order in the partial derivatives one can easily conclude that the set $\{I_p^*\}_{p=1}^N$ is algebraically independent. We have thus proved the main result of this section:

Theorem 1. *The operators $\{I_p^*\}_{p=1}^N$ form a complete family of commuting integrals of motion for the hyperbolic B_N Sutherland spin Hamiltonian $H^* = -I_1^*$.*

We also note that the constants of motion I_p^* ($p \in \mathbb{N}$) commute with the total permutation and sign changing operators Π_{ij} and Π_i . This is a consequence of Lemma 4 and the following general fact:

Lemma 5. *If $A \in \mathfrak{D} \otimes \mathfrak{K}$ commutes with K_{ij} (resp. K_i) then A^* commutes with Π_{ij} (resp. Π_i).*

Proof. We can write

$$A = \sum_{\gamma \in \Gamma} D_\gamma K_\gamma, \tag{26}$$

where K_γ is a monomial in K_{kl} and K_l , $D_\gamma \in \mathfrak{D}$, and Γ is a finite set such that $\{K_\gamma \mid \gamma \in \Gamma\}$ is linearly independent. By hypothesis

$$A = K_{ij} A K_{ij} = \sum_{\gamma \in \Gamma} K_{ij}(D_\gamma) K_{ij} K_\gamma K_{ij} \tag{27}$$

where $K_{ij}(D_\gamma)$ is the image of D_γ under the natural action of K_{ij} in \mathfrak{D} . Comparing (27) with (26) we conclude that for each $\gamma \in \Gamma$ there exists $\gamma' \in \Gamma$ such that $K_{ij} K_\gamma K_{ij} = K_{\gamma'}$, and

$$K_{ij}(D_\gamma) = D_{\gamma'}.$$

On the other hand we have

$$\Pi_{ij} A^* \Pi_{ij} = \sum_{\gamma \in \Gamma} K_{ij}(D_\gamma) S_{ij} K_\gamma^* S_{ij} = \sum_{\gamma \in \Gamma} K_{ij}(D_\gamma) K_{\gamma'}^* = \sum_{\gamma \in \Gamma} D_{\gamma'} K_{\gamma'}^*.$$

Since $(\gamma')' = \gamma$, we have $\Gamma' = \Gamma$, and therefore the right-hand side of the previous formula equals A^* . The equality $\Pi_i A^* \Pi_i = A^*$ is established in a similar way. \square

III. SPECTRUM OF THE SUTHERLAND SPIN MODEL OF B_N TYPE

We shall now study the algebraic sector of the spectrum of the B_N -type Sutherland spin model (1). The starting point in our discussion is the invariance under H of the space \mathcal{R}_m for all $m = 0, 1, \dots$, which is an immediate consequence of Eq. (24) and the definition of the operators J_i . We shall construct a basis of the H -invariant space \mathcal{R}_m with respect to which the matrix of $H|_{\mathcal{R}_m}$ is upper triangular, thereby obtaining an exact formula for the spectrum of this operator.

To derive the spectrum of H^* from that of H , we shall make use of the identity

$$H^*[\Lambda(\varphi|s)] = \Lambda[(H\varphi)|s], \quad (28)$$

where $\varphi \in C^\infty(\mathbb{R}^N)$ and $|s\rangle \in \mathcal{S}$. The latter identity, which is an immediate consequence of Eq. (19) and Lemma 4, implies that the spaces

$$\mathcal{M}_m = \Lambda(\mathcal{R}_m \otimes \mathcal{S}), \quad m = 0, 1, \dots, \quad (29)$$

are invariant under H^* . From the basis of \mathcal{R}_m triangularizing $H|_{\mathcal{R}_m}$ we shall construct a basis of \mathcal{M}_m with respect to which $H^*|_{\mathcal{M}_m}$ is also represented by an upper triangular matrix. In this way we shall determine the spectrum of the Sutherland spin model of B_N type (1).

Let us start by computing the spectrum of the operator H . Following closely the approach of Ref. [2] for spin models of A_N type, we shall define a suitable partial ordering in the set of (scaled) exponential monomials

$$f_n(\mathbf{x}) = \mu(\mathbf{x}) \exp\left(2 \sum_i n_i x_i\right), \quad n = (n_1, \dots, n_N), \quad -m \leq n_i \leq m, \quad (30)$$

spanning the subspaces \mathcal{R}_m . We shall then show that the operator H is represented by a triangular matrix in any partially ordered basis of \mathcal{R}_m .

The partial ordering in the basis (30) is defined as follows. Given a multiindex $n = (n_1, \dots, n_N) \in \mathbb{Z}^N$, we define the nonnegative and nonincreasing multiindex $[n]$ by

$$[n] = (|n_{i_1}|, \dots, |n_{i_N}|), \quad \text{where } |n_{i_1}| \geq \dots \geq |n_{i_N}|. \quad (31)$$

If $n, n' \in [\mathbb{Z}^N]$ are nonnegative and nonincreasing multiindices, we shall say that $n \prec n'$ if $n_1 - n'_1 = \dots = n_{i-1} - n'_{i-1} = 0$ and $n_i < n'_i$. For two arbitrary multiindices $n, n' \in \mathbb{Z}^N$, by definition $n \prec n'$ if and only if $[n] \prec [n']$. The partial ordering \prec in \mathbb{Z}^N induces a partial ordering in the exponential monomial basis (30), namely $f_n \prec f_{n'}$ if and only if $n \prec n'$. The action of the Weyl group on the basis (30) preserves this partial ordering, i.e., if $f_n \prec f_{n'}$ then $Wf_n \prec Wf_{n'}$ for all $W \in \mathcal{W}_N$.

If $n = (n_1, \dots, n_N) \in [\mathbb{Z}^N]$ and $s \in \mathbb{Z}$, we shall use the following notation:

$$\begin{aligned} \#(s) &= \text{card}\{i : n_i = s\}, \\ \ell(s) &= \min\{i : n_i = s\}, \end{aligned}$$

with $\ell(s) = +\infty$ if $n_i \neq s$ for all $i = 1, \dots, N$. For instance, if $n = (5, 2, 2, 1, 1, 1, 0)$ then $\#(1) = 3$ and $\ell(1) = 4$. The computation of the spectrum of H is based on the following result:

Proposition 1. *If $n \in [\mathbb{Z}^N]$ is a nonnegative and nonincreasing multiindex, the following identity holds:*

$$J_i f_n = \lambda_{n,i} f_n + \sum_{\substack{n' \in \mathbb{Z}^N \\ n' \prec n}} c_{n,i}^{n'} f_{n'}, \quad (32)$$

where $c_{n,i}^{n'} \in \mathbb{R}$ and

$$\lambda_{n,i} = \begin{cases} 2n_i + b + b' + 2a(N + i + 1 - \#(n_i) - 2\ell(n_i)), & n_i > 0 \\ -b - b' + 2a(i - N), & n_i = 0. \end{cases} \quad (33)$$

Proof. After some algebra one readily obtains the following expression:

$$\begin{aligned} J_i f_n = f_n & \left[2n_i + b + b' + 2a(N - 1) - 2a \sum_{j < i} \left(\frac{\alpha_{ij}^{n_j - n_i} - 1}{\alpha_{ij} - 1} + \frac{\beta_{ij}^{1 - n_i - n_j} - 1}{\beta_{ij} - 1} \right) \right. \\ & - 2a \sum_{j > i} \left(\frac{\alpha_{ij}^{1 + n_j - n_i} - 1}{\alpha_{ij} - 1} + \frac{\beta_{ij}^{1 - n_i - n_j} - 1}{\beta_{ij} - 1} \right) \\ & \left. - 2b \frac{z_i^{1 - 2n_i} - 1}{z_i - 1} - 2b' \frac{z_i^{1 - 2n_i} + 1}{z_i + 1} \right], \end{aligned} \quad (34)$$

where

$$z_i = e^{2x_i}, \quad \alpha_{ij} = z_i z_j^{-1}, \quad \beta_{ij} = z_i z_j.$$

Consider, for instance, the first term in α_{ij} in Eq. (34). Since $j < i$, $n_j \geq n_i$. If $n_j = n_i$ this term vanishes. If $n_j > n_i$ we have

$$f_n \frac{\alpha_{ij}^{n_j - n_i} - 1}{\alpha_{ij} - 1} = f_n \left(1 + \sum_{r=1}^{n_j - n_i - 1} z_j^{-r} z_i^r \right), \quad (35)$$

where the last sum only appears if $n_j - n_i > 1$. In this case we have $0 < \max\{n_j - r, n_i + r\} < n_j$ for all $r = 1, \dots, n_j - n_i - 1$, so the multiindices of the monomials in the summation symbol in Eq. (35) satisfy

$$(n_1, \dots, n_j - r, \dots, n_i + r, \dots, n_N) \prec n.$$

It may be likewise verified that the multiindices n' of the monomials arising from the remaining terms in Eq. (34) either coincide with n or satisfy $n' \prec n$. The value of $\lambda_{n,i}$ given in Eq. (33) can be computed by evaluating the constant part of the expression in square brackets in the right-hand side of Eq. (34). For instance, the first term in α_{ij} in Eq. (34) contributes the quantity $-2a(\ell(n_i) - 1)$ to $\lambda_{n,i}$. \square

Note that Eq. (32) does not hold if n does not belong to $[\mathbb{Z}^N]$, so that Proposition 1 in general does not determine the spectrum of the restriction of J_i to \mathcal{R}_m . On the other hand, for an arbitrary multiindex $n \in \mathbb{Z}^N$ we shall only need the following weaker result:

Corollary 1. *If $n \in \mathbb{Z}^N$ then*

$$J_i f_n = \sum_{\substack{n' \in \mathbb{Z}^N \\ [n'] \preceq [n]}} \gamma_{n,i}^{n'} f_{n'}, \quad (36)$$

for some real constants $\gamma_{n,i}^{n'}$.

Proof. We have

$$J_i f_n = J_i W f_{[n]},$$

where W is any element of \mathcal{W}_N such that $f_n = W f_{[n]}$. Eq. (36) then follows from the previous proposition, the commutation relations (8)–(12) and the invariance of the partial ordering \prec under the action of the Weyl group. \square

The algebraic spectrum of H can be computed in closed form using the previous results, which imply the following proposition:

Proposition 2. *For all $n \in \mathbb{Z}^N$ the following identity holds:*

$$H f_n = - \sum_i \lambda_{[n],i}^2 f_n + \sum_{\substack{n' \in \mathbb{Z}^N \\ n' \prec n}} c_n^{n'} f_{n'}, \quad \text{with } c_n^{n'} \in \mathbb{R}. \quad (37)$$

Proof. Let W be any element of \mathcal{W}_N such that $f_n = W f_{[n]}$. Since $H = -I_1$ commutes with W by Lemma 4, from (32) we obtain

$$H f_n = W H f_{[n]} = - \sum_i \lambda_{[n],i}^2 f_n - \sum_{\substack{n' \in \mathbb{Z}^N, i \\ n' \prec [n]}} \lambda_{[n],i} c_{[n],i}^{n'} W f_{n'} - \sum_{\substack{n' \in \mathbb{Z}^N, i \\ n' \prec [n]}} c_{[n],i}^{n'} W J_i f_{n'}.$$

Eq. (37) follows immediately from the latter equation, Corollary 1 and the invariance of the partial ordering \prec under the action of the Weyl group. \square

Let $\mathcal{B}_m = \{f_{n(j)} \mid j = 1, \dots, (2m+1)^N\}$ be any exponential monomial basis of the linear space \mathcal{R}_m partially ordered according to \prec , i.e., such that if $n(j) \prec n(k)$ then $j < k$. The previous proposition implies that the matrix of the restriction of H to \mathcal{R}_m with respect to \mathcal{B}_m is upper triangular. The eigenvalues of this matrix are its diagonal elements

$$E_n = - \sum_i \lambda_{[n],i}^2; \quad -m \leq n_j \leq m, \quad j = 1, \dots, N. \quad (38)$$

It should be noted, however, that the algebraic eigenfunctions of H must satisfy appropriate boundary conditions that we shall now discuss. In the first place, since the potential of the B_N -type spin Sutherland Hamiltonian (1) diverges on the hyperplanes $x_i \pm x_j = 0$, $1 \leq i \leq j \leq N$, as $(x_i \pm x_j)^{-2}$, we must require that the eigenfunctions of H vanish faster than $(x_i \pm x_j)^{1/2}$ near these hyperplanes. This yields the conditions (cf. Eq. (14))

$$a, b > \frac{1}{2}. \quad (39)$$

Secondly, the eigenfunctions must be square-integrable on their domain, which (without loss of generality) shall be taken as the open set $X \subset \mathbb{R}^N$ given by

$$0 < x_N < \dots < x_1. \quad (40)$$

The algebraic eigenfunctions lying in \mathcal{R}_m will satisfy this condition if and only if

$$\frac{1}{2}(b + b') + a(N - 1) + m < 0. \quad (41)$$

The latter inequality implies that the number of algebraic levels of H is finite, since m cannot exceed the integer m_1 defined by

$$m_1 = \max \left\{ m \in \mathbb{N}_0 \mid \frac{1}{2}(b + b') + a(N - 1) + m < 0 \right\}. \quad (42)$$

Note, in particular, that there are no algebraic eigenfunctions unless the parameters in the potential verify the inequality

$$\frac{1}{2}(b+b') + a(N-1) < 0. \quad (43)$$

From now on, we shall work on the maximal H -invariant subspace \mathcal{R}_{m_1} . *Remark 1.* We could also have considered algebraic eigenfunctions of the Hamiltonian (1) antisymmetric under permutations but even under sign reversals. On these eigenfunctions, the action of the operators S_{ij} and S_i coincides with that of the operators $-K_{ij}$ and K_i , respectively. Therefore Eq. (15) in the definition of the star mapping should be replaced by

$$(DK_{\alpha_1} \cdots K_{\alpha_r})^* = (-1)^{r'} DS_{\alpha_r} \cdots S_{\alpha_1},$$

where r' is the number of permutation operators in the monomial $K_{\alpha_1} \cdots K_{\alpha_r}$. As a consequence, the Hamiltonian (1) is the image under the new star mapping of the operator $H(-b, -b')$, with $H(b, b')$ given by Eq. (16). It follows from Eqs. (39) and (43) that H^* possesses algebraic eigenfunctions of even parity if and only if

$$a > \frac{1}{2}, \quad b < -\frac{1}{2}, \quad -\frac{1}{2}(b+b') + a(N-1) < 0. \quad (44)$$

In particular, these inequalities and Eqs. (39) and (43) imply that H^* cannot have both odd and even parity algebraic eigenfunctions for any values of the parameters.

Let us turn now to the algebraic spectrum of the B_N -type Sutherland spin model (1). First of all, if Eqs. (39) and (43) are satisfied the wavefunctions in \mathcal{M}_{m_1} are normalizable and well behaved near the singular hyperplanes. Secondly, if $\varphi \in \mathcal{R}_{m_1}$ is an eigenfunction of H with eigenvalue E and $|s\rangle \in \mathcal{S}$ is an arbitrary spin state, it follows from Eq. (28) that $\Lambda(\varphi|s) \in \mathcal{M}_{m_1}$ is either zero or an eigenfunction of H^* with the same energy E . The algebraic spectrum of H^* is thus a subset of the algebraic spectrum of H . Instead of studying the conditions under which $\Lambda(\varphi|s)$ does not vanish, in the next proposition we shall directly construct from \mathcal{B}_{m_1} a basis of \mathcal{M}_{m_1} with respect to which the matrix of $H^*|_{\mathcal{M}_{m_1}}$ is upper triangular. We shall use the following notation:

$$m_0 = \frac{\overline{N - \overline{M}}}{2M + 1}, \quad (45)$$

where \overline{x} denotes the smallest integer greater than or equal to $x \in \mathbb{R}$.

Proposition 3. *The H^* -invariant space \mathcal{M}_{m_1} is nonzero if and only if $m_0 \leq m_1$. If this condition holds, a basis of \mathcal{M}_{m_1} consists of states of the form*

$$\Lambda(f_n|s_1, \dots, s_N), \quad (46)$$

where $n \in [\mathbb{Z}^N]$ and $|s_1, \dots, s_N\rangle$ satisfy:

$$i) \quad \#(n_i) \leq \begin{cases} 2M + 1, & \text{if } 0 < n_i \leq m_1 \\ \overline{M}, & \text{if } n_i = 0; \end{cases} \quad (47)$$

$$ii) \quad s_i > s_j, \quad \text{if } n_i = n_j \quad \text{and } i < j; \quad (48)$$

$$iii) \quad s_i > 0, \quad \text{if } n_i = 0. \quad (49)$$

Proof. In the first place, since $\Lambda W = \epsilon(W)\Lambda$ for any element W of the realization of \mathcal{W}_N generated by Π_{ij} , Π_i , the space \mathcal{M}_{m_1} is spanned by states of the form $\Lambda(f_n|s)$, where $n \in [\mathbb{Z}^N]$ and $|s\rangle \in \mathcal{S}$. Moreover, from the definition of the total antisymmetriser Λ it follows that a state of the form (46) with $n \in [\mathbb{Z}^N]$ vanishes if and only if either $s_i = s_j$

when $n_i = n_j > 0$ and $i \neq j$, or $s_i = \pm s_j$ when $n_i = n_j = 0$ and $i \neq j$, or $s_i = 0$ when $n_i = 0$. In particular, the condition (47) is necessary to ensure that the state (46) does not vanish. Since this condition cannot hold if $n_1 < m_0$, and $n_1 \leq m_1$ for all states in \mathcal{M}_{m_1} , it follows that \mathcal{M}_{m_1} is trivial if $m_1 < m_0$.

On the other hand, if $m_0 \leq m_1$ all the states (46)–(49) are nonzero, and it is immediate to show that they are also linearly independent. Moreover, any nonzero state of the form (46) with $n \in [\mathbb{Z}^N]$ can be written as

$$\Lambda[W(f_n|s_1, \dots, s_N)] = \epsilon(W) \Lambda(f_n|s_1, \dots, s_N),$$

where n and $|s_1, \dots, s_N\rangle$ satisfy (47)–(49), and $W \in \mathcal{W}_N$ is an element of the stabilizer of f_n . \square

Corollary 2. *If $m_0 \leq m_1$, the dimension of the H^* -invariant space \mathcal{M}_{m_1} is given by*

$$\dim(\mathcal{M}_{m_1}) = \binom{m_1(2M+1) + \overline{M}}{N}. \quad (50)$$

Proof. Indeed, from Eqs. (48)–(49) it easily follows that

$$\dim(\mathcal{M}_{m_1}) = \sum_{N_0 + \dots + N_{m_1} = N} \binom{\overline{M}}{N_0} \binom{2M+1}{N_1} \dots \binom{2M+1}{N_{m_1}} = \binom{m_1(2M+1) + \overline{M}}{N}. \quad \square$$

The algebraic spectrum of the hyperbolic B_N Sutherland spin model (1) follows directly from Proposition 3:

Theorem 2. *If $m_0 \leq m_1$, the algebraic energies of H^* are given by*

$$E_n^* = - \sum_i \lambda_{n,i}^2, \quad (51)$$

where $n \in [\mathbb{Z}^N]$ satisfies the condition (47) and $\lambda_{n,i}$ is given by Eq. (33).

Proof. Let $\psi_{n,s} = \Lambda(f_n|s_1, \dots, s_N)$ be an element of the basis (46)–(49) of \mathcal{M}_{m_1} . Using Eqs. (37) and (28) we easily obtain

$$H^* \psi_{n,s} = - \sum_i \lambda_{n,i}^2 \psi_{n,s} + \sum_{\substack{n' \in \mathbb{Z}^N \\ n' \prec n}} c_n^{n'} \Lambda(f_{n'}|s_1, \dots, s_N). \quad (52)$$

The state $\Lambda(f_{n'}|s_1, \dots, s_N)$ is proportional to a basis element of the form $\psi_{n'',s'}$, with $f_{n''} = W f_{n'}$ for some element W of \mathcal{W}_N and $n'' \prec n$. Therefore the matrix of $H^*|_{\mathcal{M}_{m_1}}$ in the basis (46)–(49) is also upper triangular, with diagonal elements given by (51). \square

The algebraic ground state of the Hamiltonian H^* and its degeneracy d can be determined using Proposition 3 and the previous theorem:

Proposition 4. *The multiindex n yielding the algebraic ground state and the degeneracy of this state are given by*

$$\begin{aligned} i) \quad N \leq \overline{M}: \quad n = 0, \quad d = \binom{\overline{M}}{N}; \\ ii) \quad N > \overline{M}: \quad n = (\overbrace{m_0, \dots, m_0}^r, \overbrace{m_0 - 1, \dots, m_0 - 1}^{2M+1}, \dots, \overbrace{1, \dots, 1}^{2M+1}, \overbrace{0, \dots, 0}^{\overline{M}}), \\ d = \binom{2M+1}{r}, \end{aligned} \quad (53)$$

where $r = N - (m_0 - 1)(2M + 1) - \overline{M}$.

Proof. Note, first of all, that from the definition (45) of m_0 it follows that $1 \leq r \leq 2M + 1$. Let $n \in [\mathbb{Z}^N]$ be a nonnegative and nonincreasing multiindex satisfying $n_1 \leq m_1$ and Eq. (47). The contribution to the algebraic energy (51) of all the terms $\lambda_{n,i}^2$ such that n_i is equal to a fixed value $k \in \{n_j\}_{j=1}^N$ can be easily evaluated in closed form. Indeed, denoting for brevity

$$\begin{aligned} i_0 &= \ell(k), & i_1 &= \ell(k) + \#(k) - 1, \\ \alpha &= -\frac{1}{2a}(b + b' + 2a(N - 1)), & \alpha_k &= \alpha - \frac{k}{a}, \end{aligned}$$

from (33) we have, for $k > 0$,

$$\begin{aligned} -\sum_{i=i_0}^{i_1} \lambda_{n,i}^2 &= -4a^2 \sum_{i=i_0}^{i_1} (i - i_0 - i_1 - \alpha_k + 1)^2 \\ &= -4a^2 \#(k) \left[\alpha_k^2 + (i_0 + i_1 - 2)\alpha_k + \frac{1}{3}(i_0^2 + i_0 i_1 + i_1^2) - \frac{1}{6}(7i_0 + 5i_1) + 1 \right]. \end{aligned} \quad (54)$$

Similarly, the contribution to the algebraic energy of all the terms in Eq. (51) with $k = 0$

$$-\sum_{i=i_0}^{i_1} \lambda_{n,i}^2 = -4a^2 \sum_{i=i_0}^{i_1} (i + \alpha - 1)^2 \quad (55)$$

is easily seen to equal the right-hand side of Eq. (54), since $\alpha_0 = \alpha$. The derivative of the right-hand side of Eq. (54) with respect to k , with i_0 and i_1 fixed, is given by

$$4a\#(k)(2\alpha_k + i_0 + i_1 - 2). \quad (56)$$

This is clearly positive, since $i_0 + i_1 - 2 \geq 0$ and $\alpha_k \geq \alpha - \frac{m_1}{a} > 0$ on account of (42). Hence the energy decreases if k decreases, i_0 and i_1 being fixed. It follows that any multiindex n corresponding to the minimum value of the algebraic energy must be of the form

$$n = (m, \dots, m, m - 1, \dots, m - 1, \dots, \epsilon, \dots, \epsilon), \quad (57)$$

where $\epsilon = 0$ or $\epsilon = 1$ and $m_0 \leq m \leq m_1$.

Let k be an integer in the range 1 to m . We shall consider next the change in the algebraic energy associated to the multiindex (57) when $\#(k)$ decreases by 1, while $\#(k - 1)$ increases by 1 (including the case in which $k = \epsilon = 1$ and therefore $\#(k - 1) = \#(0) = 0$). Suppose, for instance, that $k \geq 2$. Denoting $i_2 = \ell(k - 1) + \#(k - 1) - 1$, the change in the algebraic energy is given by

$$\begin{aligned} &4a^2 \left(-\sum_{i=i_0}^{i_1-1} (i - i_0 - i_1 - \alpha_k + 2)^2 - \sum_{i=i_1}^{i_2} (i - i_1 - i_2 - \alpha_{k-1} + 1)^2 \right. \\ &\quad \left. + \sum_{i=i_0}^{i_1} (i - i_0 - i_1 - \alpha_k + 1)^2 + \sum_{i=i_1+1}^{i_2} (i - i_1 - i_2 - \alpha_{k-1})^2 \right) \\ &= -4(1 + 2a(\alpha_k + i_1 - 1)) \leq -4(1 + 2a\alpha_k) < 0. \end{aligned} \quad (58)$$

It may be similarly verified that when $k = 1$ and either $\epsilon = 0$ or $\epsilon = 1$ the change in the algebraic energy is negative. This implies that the multiindex n yielding the algebraic ground state is of the form (53) if $N > \overline{M}$, and zero otherwise. The degeneracy of the algebraic ground state then follows immediately from Proposition 3. \square

The algebraic ground energy can be easily obtained from the previous proposition and Eqs. (33) and (38). Indeed, denoting

$$\nu = 2M + 1, \quad c = b + b' + 2m_0 - a,$$

the algebraic ground energy for even ν (that is, half-integer M) is given by

$$E_{0,e}^* = \frac{1}{3} \nu m_0 (4m_0^2 (a\nu - 1) - 6cm_0 - a\nu - 2) + \frac{1}{3} (a^2 - 3c^2)N + 2acN^2 - \frac{4}{3} a^2 N^3,$$

while for odd ν (integer M), the algebraic ground energy reads

$$E_{0,o}^* = E_{0,e}^* + m_0 (a + 2c + 2m_0(1 - a\nu)).$$

IV. THE B_N -TYPE SUTHERLAND SPIN CHAIN

In this section we shall introduce a quantum spin chain closely related to the hyperbolic B_N Sutherland spin model (1). We shall establish the integrability of this chain by explicitly constructing a complete family of commuting integrals of motion associated with the integrals I_p^* of the Hamiltonian (1).

The starting point in this construction is the following expansion of the hyperbolic B_N Sutherland spin Hamiltonian (1) in terms of the parameter a :

$$H^* = - \sum_i \partial_{x_i}^2 + a \mathcal{H}^* + a^2 U(\mathbf{x}), \quad (59)$$

where

$$\mathcal{H}^* = \sum_{i \neq j} \left[\sinh^{-2} x_{ij}^- S_{ij} + \sinh^{-2} x_{ij}^+ \tilde{S}_{ij} \right] + \sum_i (\beta \sinh^{-2} x_i - \beta' \cosh^{-2} x_i) S_i \quad (60)$$

$$U(\mathbf{x}) = \sum_{i \neq j} (\sinh^{-2} x_{ij}^- + \sinh^{-2} x_{ij}^+) + \sum_i (\beta^2 \sinh^{-2} x_i - \beta'^2 \cosh^{-2} x_i) \quad (61)$$

and

$$\beta = \frac{b}{a}, \quad \beta' = \frac{b'}{a}.$$

The Hamiltonian of the *hyperbolic Sutherland spin chain of B_N type* is by definition the operator \mathcal{H}^{*0} , where the superscript 0 means that the coordinates x_i are replaced by the equilibrium points x_i^0 of the potential U , which satisfy the system

$$\frac{\partial U}{\partial x_i}(x_1^0, \dots, x_N^0) = 0, \quad i = 1, \dots, N. \quad (62)$$

A necessary condition for the system (62) to have a solution in the region $x_i > 0$, $i = 1, \dots, N$, is that $\beta'^2 > \beta^2 + 2(N-1)$. In fact, there is strong numerical evidence that a solution exists if and only if $|\beta'| > |\beta| + 2(N-1)$. Note that this inequality corresponds to the condition (43) (when $b > 0$) or (44) (when $b < 0$) necessary for the existence of square-integrable algebraic eigenfunctions of the dynamical model, a fact certainly deserving further study.

Let us define the operator $\mathcal{J}_i \in C^\infty(\mathbb{R}^N) \otimes \mathfrak{K}$ by

$$J_i = \partial_{x_i} - a \mathcal{J}_i, \quad i = 1, \dots, N. \quad (63)$$

We shall also denote

$$\mathcal{I}_p = \sum_i \mathcal{J}_i^{2p}, \quad p \in \mathbb{N}. \quad (64)$$

Note that \mathcal{I}_1 is the coefficient of a^2 in $I_1 = -H$, and thus equals $-U(\mathbf{x})$ by Eq. (59). We shall prove below that the operators $\{\mathcal{I}_p^{*0}\}_{p=1}^N$ form a complete family of commuting integrals of motion for the Sutherland B_N spin chain Hamiltonian \mathcal{H}^{*0} . Let us begin by establishing the commutativity of the operators \mathcal{I}_p^{*0} for all $p \in \mathbb{N}$. In fact, the following stronger result holds:

Proposition 5. *The operators \mathcal{I}_p^* ($p \in \mathbb{N}$) form a commuting family.*

Proof. The proposition follows directly from the commutativity of the operators I_p^* , taking into account that \mathcal{I}_p^* is the coefficient of a^{2p} in the expansion of I_p^* in powers of a . \square

We show next that \mathcal{I}_p^{*0} commutes with \mathcal{H}^{*0} for all $p \in \mathbb{N}$. Note that this is not a consequence of the previous proposition, since \mathcal{H}^{*0} is not proportional to $\mathcal{I}_1^{*0} = -U(\mathbf{x}^0)$.

Proposition 6. *The operators \mathcal{I}_p^{*0} ($p \in \mathbb{N}$) commute with the B_N Sutherland spin chain Hamiltonian \mathcal{H}^{*0} .*

Proof. From (24) it follows that

$$[H, J_i] = 0, \quad i = 1, \dots, N.$$

Using Eqs. (6) and (59) in the previous identity and equating to zero the coefficient of a^2 in the resulting expression we obtain

$$[\mathcal{H}, \mathcal{J}_i] = -\frac{\partial U}{\partial x_i}, \quad i = 1, \dots, N.$$

It follows that

$$[\mathcal{H}, \mathcal{I}_p] = -\sum_{i=1}^N \sum_{r=0}^{2p-1} \binom{2p-1}{r} \mathcal{J}_i^r \frac{\partial U}{\partial x_i} \mathcal{J}_i^{2p-r-1} \equiv \mathcal{C}_p.$$

The operator $\mathcal{C}_1 = -[\mathcal{H}, U(\mathbf{x})]$ vanishes identically, since \mathcal{H} does not contain partial derivatives and $U(\mathbf{x})$ is a symmetric even function of \mathbf{x} . Note, however, that \mathcal{C}_p need not vanish for $p > 1$. Expanding in powers of a the identities $[H, \Lambda] = [I_p, \Lambda] = 0$ we obtain

$$[\mathcal{H}, \Lambda] = [\mathcal{I}_p, \Lambda] = 0, \quad p \in \mathbb{N}.$$

By Lemma 3 we have

$$[\mathcal{H}^*, \mathcal{I}_p^*] = \mathcal{C}_p^*, \quad p \in \mathbb{N}. \quad (65)$$

From the symmetry of the function $U(\mathbf{x})$ with respect to permutations and sign changes it follows that $\partial U / \partial x_i$ commutes with K_{jk} and K_j for $j, k \neq i$, while

$$K_{ij} \frac{\partial U}{\partial x_i} = \frac{\partial U}{\partial x_j} K_{ij}, \quad K_i \frac{\partial U}{\partial x_i} = -\frac{\partial U}{\partial x_i} K_i.$$

From these identities one can easily show that \mathcal{C}_p^* is of the form

$$\mathcal{C}_p^* = \sum_i \frac{\partial U}{\partial x_i} \mathcal{C}_{p,i}^*, \quad p \in \mathbb{N}.$$

The commutativity of \mathcal{H}^{*0} with \mathcal{I}_p^{*0} follows from the latter equation, Eq. (65) and the definition of the equilibrium points (62). \square

Finally, we prove the algebraic independence of the set $\{\mathcal{I}_p^{*0}\}_{p=1}^N$, thus establishing the integrability of the hyperbolic Sutherland spin chain of B_N type:

Theorem 3. *The operators $\{\mathcal{I}_p^{*0}\}_{p=1}^N$ form a complete family of commuting integrals of motion for the B_N Sutherland spin chain Hamiltonian \mathcal{H}^{*0} .*

Proof. The set $\{\mathcal{I}_p^0\}_{p=1}^N$ is algebraically independent, since the operators \mathcal{I}_i^0 ($i = 1, \dots, N$) are linearly independent and commute with each other. The counterpart of Lemma 2 for the inverse of the star operator implies that the family $\{\mathcal{I}_p^{0*}\}_{p=1}^N$ is also algebraically independent. The lemma now follows from the identity $\mathcal{I}_p^{0*} = \mathcal{I}_p^{*0}$. \square

Note also that the constants of motion \mathcal{I}_p^{*0} ($p \in \mathbb{N}$) commute with the total permutation and sign reversing operators Π_{ij} and Π_i . This follows from the identities $[\mathcal{I}_p^*, \Pi_{ij}] = [\mathcal{I}_p^*, \Pi_i] = 0$ by taking the coefficient of a^{2p} .

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