Initial-value problems for evolutionary partial differential equations and higher-order conditional symmetries

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Abstract

We suggest a new approach to the problem of dimensional reduction of initial/boundary value problems for evolution equations in one spatial variable. The approach is based on higher-order (generalized) conditional symmetries of the equations involved. It is shown that reducibility of an initial value problem for an evolution equation to a Cauchy problem for a system of ordinary differential equations can be fully characterized in terms of conditional symmetries which leave invariant the equation in question. We also give some examples of the solution of initial value problems for second- and third-order nonlinear differential equations by reduction by their conditional symmetries. We give a systematic classification of general second-order partial differential equations admitting second-order conditional symmetries, based on Lie's classification of invariant second-order ordinary differential equations. This yields five classes of principally new initial value problems for nonlinear evolution equations which admit no Lie symmetries and are reducible via second-order conditional symmetries.

1 Introduction

As is well-known, the basic idea of the classical Lie approach to solving partial differential equations (PDEs) is to use symmetries to reduce the number of independent variables in order to get an ordinary differential equation. If one can then solve the resulting ordinary differential equation, the whole procedure yields particular solutions of the initial PDE. This procedure of dimensional reduction is referred to in the literature as symmetry reduction, and it has proved to be a very efficient tool for constructing exact solutions of many linear

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and nonlinear differential equations arising in different areas of applied mathematics (see, e.g. [1]– [3] and the references therein).

On the other hand, the problem of the application of the symmetry approach to the analysis of boundary and initial value problems still remains a great challenge for mathematicians. The main difficulty is that the symmetry groups admitted by boundary and initial value problems arising in applications are not sufficiently rich to allow for the effective use of the technique of symmetry reduction. It would be natural to attempt to exploit conditional (non-classical) symmetries of PDEs in order to extend the range of the initial/boundary value problems that can be handled within the framework of the symmetry approach.

The aim of the present paper is to present our view of the problem of dimensional reduction of initial value problems for evolution equations in two independent variables t, x :

$$u_t = F(t, x, u, u_1, u_2, \dots, u_n), \tag{1}$$

where $u \in C^n(\mathbf{R^2}, \mathbf{R^1})$, $u_k = \partial^k u/\partial x^k$, $1 \leq k \leq n$. The approach we give below is based on higher-order conditional symmetries of PDEs of the form (1). The concept of higher-order conditional symmetry was introduced by Fushchych and Zhdanov [4], and independently by Fokas and Liu [5]. It was proved in [4] that it is higher-order conditional symmetries that are responsible for the "nonlinear separation of variables" introduced by Galaktionov [6](see, also the related papers addressing this problem [7]– [9]).

The structure of the paper is as follows. In Section 2 we give some necessary notations, definitions and theorems. In Section 3, we formulate the principal theorem on the reduction of the initial value problem for a PDE of the form (1) to a Cauchy problem for some system of ordinary differential equations. In addition, we give two illustrative examples of applications of the theorem to the reduction of initial value problems for the second- and third-order nonlinear evolution equations. Section 4 is devoted to the systematic classification of second-order evolutionary PDEs using their second-order conditional symmetries. The results of this classification are used to describe the initial value problems for second-order PDEs that admit dimensional reductions to Cauchy problems for some systems of ordinary differential equations.

2 Basic notations and definitions

It is known that reducibility of any PDE in two variables to a single ordinary differential equation is in one-to-one correspondence with its Q-conditional symmetry [10, 11] (see, also [12]–[17]). We have proved in [18] that reducibility of (1) to a system of several ordinary differential equations is in one-to-one correspondence with its invariance under a non-point group of infinitesimal transformations

$$\begin{aligned} u' &= u + \varepsilon\, \eta(t, x, u, u_1, \dots, u_N), \\ u_1' &= u_1 + \varepsilon\, D_x\, \eta(t, x, u, u_1, \dots, u_N), \dots \end{aligned} \tag{2}$$

provided some smoothness requirements are satisfied. The above group is generated by the Lie–Bäcklund vector field

$$Q = \sum_{k=0}^{\infty} \left(D_x^k \eta\right) \frac{\partial}{\partial u_k} \equiv \eta \frac{\partial}{\partial u} + (D_x \eta) \frac{\partial}{\partial u_1} + \left(D_x^2 \eta\right) \frac{\partial}{\partial u_2} + \cdots . \tag{3}$$

In formulas (1), (3) we use the following notation:

$$D_x = \frac{\partial}{\partial x} + \sum_{k=0}^{\infty} u_{k+1} \frac{\partial}{\partial u_k}, \quad D_x^{j+1} = D_x\,(D_x^j), \quad D_x^0 = 1.$$

Remark 1 *In general, the function η in (3) may also depend on derivatives of the function u with respect to t. However, on the solution manifold of PDE (1) we can express all the derivatives of u with respect to t in terms of $t, x, u, u_1, u_2, \ldots$ and thus eliminate derivatives with respect to the variable t.*

Remark 2 *If the function η has the structure*

$$\eta = \tilde{\eta}(t,x,u) - \xi_0(t,x,u)u_t - \xi_1(t,x,u)u_1, \tag{4}$$

then the Lie-Bäcklund vector field (3) is equivalent to a Lie vector field and can be represented in the standard form (see, e.g., [2, 19]):

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}.$$

As the form of a Lie-Bäcklund vector field is fully defined by the form of the coefficient η of $\dfrac{\partial}{\partial u}$ we will use the notation

$$Q = \eta \frac{\partial}{\partial u} + \ldots$$

instead of (3).

Definition 1 *PDE (1) is invariant under the Lie-Bäcklund vector field (3) if the condition*

$$Q(u_t - F)\Big|_M = 0 \tag{5}$$

holds, where M is the set of all differential consequences of the equation $u_t - F = 0$, that is $D_x^j D_t^k (u_t - F) = 0$, $j, k = 0, 1, 2, \ldots$.

Definition 2 *PDE (1) is conditionally-invariant under the Lie-Bäcklund vector field (3) if the following condition holds*

$$Q(u_t - F)\Big|_{M \cap L_x} = 0 \tag{6}$$

Here the symbol L_x stands for the set of all differential consequences of the equation $\eta = 0$ with respect to x, that is $D_x^j \eta = 0$, $j = 0, 1, 2, \ldots$.

The procedure for constructing Lie-Bäcklund vector fields is, in fact, encoded in the above definitions. If one considers the problem of finding Lie-Bäcklund symmetries in the sense of Definition 1, then the first thing to do is to act with the operator Q on the expression $u_t - F$, considered as a function of the independent variables $t, x, u, u_t, u_1, \ldots, u_n$. The next step is to eliminate the derivatives $u_{tj}, j = 0, 1, 2, \ldots$ with the use of the equation $u_t - F = 0$ and its differential consequences $D_x^j(u_t - F) = 0, j = 1, 2, \ldots$. Equating the resulting expression to zero yields a system of linear PDEs called the system of determining equations. Solving this system yields the most general form of the Lie-Bäcklund vector field (3) admitted by equation (1). Further details about the procedure for calculating higher symmetries of PDEs and numerous examples of equations possessing these symmetries can be found in [2, 19, 20]. The procedure for calculating higher conditional symmetries is essentially the same. The only difference is the necessity of taking into account not only the differential consequences of equation (1) but also the differential consequences $D_x^j \eta = 0, \quad j = 1, 2, \ldots$ of the invariance condition $\eta = 0$. This additional restriction, on the one hand, extends considerably the scope of invariant equations (as the number of variables to be split, decreases) but, on the other hand, yields a nonlinear system of determining equations.

Clearly, if PDE (1) is invariant under the Lie-Bäcklund vector field (3), then it is conditionally invariant under it; however, the converse is not true. Consequently, this means that Definition 2 is a generalization of the standard definition of invariance of partial differential equation with respect to Lie-Bäcklund vector field.

If we consider the nonlinear PDE

$$\eta(t, x, u, u_1, \ldots, u_N) = 0 \tag{7}$$

as an N-th order ordinary differential equation with respect to variable x, then its general integral can be given (locally) in the form

$$u(t, x) = U\left(t, x, \varphi_1(t), \varphi_2(t), \ldots, \varphi_N(t)\right), \tag{8}$$

where $\varphi_j(t)$, $(j = 1, \ldots N)$ are arbitrary smooth functions. In the following, we call expression (8) the *ansatz* invariant under the Lie-Bäcklund vector field (3). The following result (see [18]) establishes the connection between reducibility of a given PDE (1) to ordinary differential equations and its higher-order conditional symmetry.

Theorem 1 *Let equation (1) with $F \in C^{N+1}(\mathcal{D})$, where $\mathcal{D}$ is an open domain in $\mathbf{R}^{n+3}$, be conditionally invariant under Lie-Bäcklund vector field (3) with $\eta \in C^2(\mathcal{D}')$, where $\mathcal{D}'$ is an open domain in $\mathbf{R}^{N+3}$ and $\partial\eta/\partial u_N \neq 0$ on $\mathcal{D}'$. Then ansatz (8) invariant under the Lie-Bäcklund vector field (3) reduces PDE (1) to a system of N ordinary differential equations for some functions $\varphi_j(t)$, $(j = 1, \ldots, N)$*

$$\dot{\varphi}_j = F_j(t, \varphi_1, \ldots, \varphi_N), \qquad j = 1, \ldots, N. \tag{9}$$

Now suppose the converse. Namely, that ansatz (8), where the function U and its derivatives $\partial U^{k+1}/\partial\varphi_j \partial x^k$, $(j = 1, \ldots, N,\ k = 0, \ldots, N)$ exist and are continuous on an open domain $\mathcal{D}_1$ in $\mathbf{R}^{N+2}$, reduces (1) to a system of ordinary differential equations (9) with $F_i \in C^1(\mathcal{D}_1')$, where $\mathcal{D}_1'$ is an open domain in $\mathbf{R}^{N+2}$. Then there exists a Lie-Bäcklund vector field (3) such that equation (1) is conditionally-invariant with respect to it.

The principal reason for the existence of the phenomenon of nonlinear separation of variables in non-integrable evolution equations is their higher-order conditional symmetry. Of course, the usual Lie and higher-order Lie symmetries can be also used to reduce nonlinear PDEs that have the necessary algebraic properties, but there exist equations that admit no Lie symmetries but are nonetheless reducible to one or several ordinary differential equations due to their (first or higher-order) conditional symmetry. As an example we give the following nonlinear PDE:

$$u_t = u_{xx} + f_1(t, r(u_x) - x)u_x + f_2(t, r(u_x) - x) - \frac{1}{r'(u_x)}, \tag{10}$$

where f_1, f_2, r are arbitrary smooth functions of their arguments with $r'(u_x) \neq 0$. This equation admits neither first nor higher-order Lie symmetries (except for the trivial one-parameter translation group by u). However, it can be reduced to a system of two ordinary differential equations due to the fact that equation (10) is conditionally invariant with respect to the Lie-Bäcklund field

$$Q = (r'(u_x)u_{xx} - 1)\frac{\partial}{\partial u} + \ldots.$$

3 Reduction of initial values problems for evolution PDEs

Consider the following initial value problem for the evolution equation (1)

$$\begin{aligned} &u_t = F(t, x, u, u_1, u_2, \ldots, u_n), \\ &(\alpha(x)u_1 + \beta(x)u)|_{t=t_0} = \gamma(x), \end{aligned} \tag{11}$$

where $\alpha(x)$, $\beta(x)$, $\gamma(x)$ are smooth functions.

If the problem (11) is invariant under a one-parameter transformation group, then it is possible to reduce it by symmetry reduction. However, this approach is too restrictive, since a choice of initial conditions that are invariant with respect to a Lie group can be very limited. Indeed, the transformation group in question must leave invariant the initial surface at $t = t_0$, and this imposes rather strong limitations on the exploitation of symmetry reduction in its classical setting. For instance, even such a simple symmetry as invariance under $t-$translations violates the symmetry of problem (11).

We will show that using higher-order conditional symmetries enables us to overcome this difficulty. The principal idea is to rewrite the generator of a one-parameter transformation group

$$Q = \xi_0(t,x,u)\frac{\partial}{\partial t} + \xi_1(t,x,u)\frac{\partial}{\partial x} + \tilde{\eta}(t,x,u)\frac{\partial}{\partial u}$$

as a canonical generator of a Lie-Bäcklund field of the form (3), with $\eta = \tilde{\eta}(t,x,u) - \xi_0(t,x,u)u_t - \xi_1(t,x,u)u_1$, and then to eliminate all t derivatives with the use of equation (1). Since the vector field (3) generates an infinitesimal transformation group of the form (2), the initial surface at $t = t_0$, as well as any boundary surface $X(x) = 0$, is invariant with respect to it, and an *arbitrary* Lie symmetry can, in principle, be used for dimensional reduction of the initial value problem (11)! However, the price for this is that the order of the symmetry operator is no longer equal to one, and the standard procedure of symmetry reduction has to be modified in order to be applicable to the case under investigation. To this end we will use Theorem 1.

One of the immediate conclusions is that higher-order conditional symmetry is not just one more exotic concept, but the natural and efficient tool for handling initial value problems (11) within the framework of the symmetry approach.

It is natural to expect that, provided PDE (1) admits higher-order conditional symmetry (and some reasonable restrictions are satisfied), there exist functions $\alpha(x)$, $\beta(x)$, $\gamma(x)$ such that the initial value problem (11) reduces (by virtue of the ansatz (8)) to the Cauchy problem for the functions $\varphi_j(t)$, $(j = 1,\ldots,N)$. This means that PDE (1) should reduce to a system of ordinary differential equations (9), and the initial condition given in (11) should reduce to a set of algebraic relations prescribing the values of the functions $\varphi_j(t)$, $(j = 1,\ldots,N)$ at $t = t_0$.

To make the above procedure meaningful one has to give a formal definition of what is meant by a reduction of the initial-value problem

$$(\alpha(x)u_1 + \beta(x)u)|_{t=t_0} = \gamma(x). \tag{12}$$

Definition 3 *We say that ansatz (8) reduces the initial-value condition (12) if substituting it into (12) yields an expression that vanishes identically when*

$$\varphi_i(t_0) = C_i, \quad i = 1,2,\ldots,N$$

where $C_1, \ldots, C_N$ are some constants.

We prove the following assertion:

Theorem 2 *Ansatz (8), invariant with respect to the Lie-Bäcklund field (3), reduces the initial-value condition (12) if and only if*

a) the system of two PDEs

$$\eta(t, x, u, u_1, \ldots, u_N) = 0, \quad a(t, x)u_1 + b(t, x)u - c(t, x) = 0 \tag{13}$$

is compatible, and

b) its solution is obtained from the general integral (8) of equation $\eta = 0$ considered as an ordinary differential equation with respect to x.

Proof. The implication *reduction* ⇒ *compatibility* is evident. We prove the converse assertion.

Suppose that conditions a and b of the theorem hold. Then, substituting ansatz (8) into the left-hand side of the second equation of (13) yields an expression of the form $R(t,\ x,\ \varphi_1(t),\ \varphi_1(t),\ \ldots,\ \varphi_N(t))$.

As the solution of system (13) exists (by assumption) and belongs to the family (8), there are constants $C_1, C_2, \ldots, C_N$ such that the relation

$$R(t, x, C_1, C_2, \ldots, C_N) = 0$$

holds. This means that ansatz (8) reduces the initial value condition. This establishes the theorem.

Definition 4 *We say that the equation*

$$\eta(t, x, u, u_1, \ldots, u_N) = 0, \tag{14}$$

considered as an ordinary differential equation with respect to x, is conditionally invariant under the operator

$$X = \xi(t, x, u)\frac{\partial}{\partial x} + \zeta(t, x, u)\frac{\partial}{\partial u} \tag{15}$$

with $\xi \neq 0$ if the following relation hold:

$$\widetilde{X}\eta|_{\mathcal{M}} = 0, \tag{16}$$

Here, $\tilde{X}$ is the Nth prolongation of the operator X. The symbol $\mathcal{M}$ stands for the intersection of the surface defined in the space $\mathbf{R}^{N+3}$ of the variables t, x, u, u_t, u_x, …, u_N by the equation

$\eta = 0$ and differential consequences (with respect to x) of the equation $\xi(t,x,u)u_1 - \xi(t,x,u) = 0$ up to the order $N-1$. When $\xi = 0$, we say that (14) is conditionally invariant under the operator (15), if solving the relation $\zeta(t,x,u) = 0$ with respect to u yields an exact solution of (14).

Theorem 3 *Ansatz (8), invariant with respect to the Lie-Bäcklund field (3), reduces the initial value condition (12) if and only if*

a) equation (9) is conditionally invariant under the operator (15) with $\xi = \alpha(x)$, $\zeta = -\beta(x)u + \gamma(x)$, and

b) the solution of the corresponding system (13) exists and is obtained from the general integral (8) of the equation $\eta = 0$, considered as ordinary differential equation with respect to x.

Proof. To prove the theorem it is sufficient to show that, given condition b, then condition a of Theorem 3 is equivalent to condition a of Theorem 2.

Since the assertion to be proved is evident for the case $\alpha = 0$, we will concentrate on the case when $\alpha \neq 0$.

When $\alpha \neq 0$, we can make a change of variables

$$\tilde{x} = X(t,x), \quad \tilde{u} = U_1(t,x)u + U_2(t,x),$$

so that the operator (15) takes the form $\tilde{X} = \dfrac{\partial}{\partial \tilde{x}}$ (to simplify notations, we omit the tilde in the sequel). Now, system (13) reads as follows:

$$\tilde{\eta}(t,x,u,u_1,\ldots,u_N) = 0, \quad u_1 = 0. \tag{17}$$

Compatibility of this system means that the function $\tilde{\eta}(t,x,u,0,\ldots,0)$ is independent of x, whence we find that there exist smooth functions

$$a_0(t,u,u_1, \ \ldots,u_N), \ \ a_i(t,x,u,u_1,\ldots,u_N), \ \ i = 1,2,\ldots,N$$

such that

$$\tilde{\eta}(t,x,u,u_1,\ldots,u_N) = a_0(t,u,u_1,\ldots,u_N) + \sum_{i=1}^{N} a_i(t,x,u,u_1,\ldots,u_N)u_i.$$

Hence, we conclude that the equation $\tilde{\eta} = 0$ is conditionally invariant with respect to the operator $\dfrac{\partial}{\partial x}$. Reversing this argument completes the proof of the equivalence of conditions a of Theorems 2 and 3. This proves Theorem 3.

As an immediate consequence of Theorem 3, we see that a necessary condition for the reducibility of a Cauchy problem by solutions invariant under some Lie-Bäcklund field (3) is the conditional invariance of the equation $\eta = 0$ with respect to the operator (15) with $\xi = \alpha(x), \zeta = -\beta(x)u + \gamma(x)$. In view of this fact, we can formulate the following symmetry approach to the reduction of initial value problems (11) for the case when the corresponding evolution equation admits first or higher-order conditional symmetry:

- rewrite the symmetry operator in the canonical form (3),
- calculate the classical or conditional symmetry of the equation $\eta = 0$ within the class of operators
$$X = \alpha(x)\frac{\partial}{\partial x} + (-\beta(x)u + \gamma(x))\frac{\partial}{\partial u}, \tag{18}$$
- carry out the reduction of the initial value problem (11) to a Cauchy problem for a system of ordinary differential equations using ansatz (8), invariant with respect to the Lie-Bäcklund vector field (3).

Note that condition b of Theorem 3 cannot be neglected, since conditional invariance alone cannot guarantee reducibility of the corresponding initial value problem. As an example, consider the Lie-Bäcklund vector field

$$Q = (u_x + u^2 - 1)\frac{\partial}{\partial u} + \dots .$$

The equation $u_x + u^2 - 1 = 0$ is evidently invariant under translations by x. Furthermore, system

$$u_x + u^2 - 1 = 0, \quad u_1 = 0$$

is compatible (it has the solution $u(t,x) = 1$). However, the ansatz

$$u(t,x) = \tanh(x + \varphi(t)) \tag{19}$$

which is invariant under Q, does not reduce the initial condition $u_1|_{t=t_0} = 0$. The reason for this is that the function $u(t,x) = 1$ does not belong to the family (19), and thus the condition b of Theorem 3 fails to hold.

Let us now proceed to giving examples which illustrate the main features of the whole procedure. As the first example, we consider the nonlinear heat-conduction equation

$$u_t = u_x^{-1} u_{xx}, \quad u_x \neq 0. \tag{20}$$

It is evidently invariant with respect to the one-parameter transformation group having the generator $Q = \dfrac{\partial}{\partial t} + a\dfrac{\partial}{\partial u}$ with a being an arbitrary constant. Our aim is to describe the initial value problems

$$
\begin{aligned}
&u_t = u_x^{-1} u_{xx},\\
&(\alpha(x)u_x + \beta(x)u)|_{t=t_0} = \gamma(x).
\end{aligned}
\tag{21}
$$

that are reducible to Cauchy problems for some first-order systems of ordinary differential equations. First we note that, since the initial surface at $t = t_0$ is not invariant with respect to the group generated by the operator Q, we cannot apply the symmetry reduction method in its standard form. However, if we rewrite Q in the canonical form (3) and eliminate the t derivative on the solution manifold of PDE (20), then the resulting Lie-Bäcklund vector field

$$
\tilde{Q} = \left(u_x^{-1} u_{xx} - a\right) \frac{\partial}{\partial u} + \ldots
$$

generates an infinitesimal transformation group leaving invariant the initial surface $t = t_0$. Consequently, we can apply the procedure developed above and based on Theorem 3. Solving the invariance condition $u_x^{-1} u_{xx} - a = 0$ gives the ansatz for $u = u(t, x)$

$$
u(t, x) = \varphi_1(t)\exp(ax) + \varphi_2(t) \tag{22}
$$

which reduces the initial PDE (20) to a system of two ordinary differential equations for φ_1, φ_2 (Theorem 1).

In order to exploit Theorem 3, we calculate the Lie symmetry of equation $u_{xx} - au_x = 0$ and obtain the five-dimensional invariance algebra with basis elements

$$
\begin{aligned}
&e_1 = \frac{\partial}{\partial x}, \quad e_2 = \exp(ax)\left(\frac{\partial}{\partial x} + au\frac{\partial}{\partial u}\right),\\
&e_3 = u\frac{\partial}{\partial u} \quad e_4 = \exp(ax)\frac{\partial}{\partial u}, \quad \frac{\partial}{\partial u}.
\end{aligned}
$$

In other words, the most general form of the Lie vector field admitted by PDE $u_{xx} - au_x = 0$ reads as

$$
C_1\frac{\partial}{\partial x} + C_2\exp(ax)\left(\frac{\partial}{\partial x} + au\frac{\partial}{\partial u}\right) + C_3 u\frac{\partial}{\partial u} + C_4\exp(ax)\frac{\partial}{\partial u} + C_5\frac{\partial}{\partial u},
$$

where $C_1, C_2, \ldots, C_5$ are arbitrary real constants. Hence, using Theorem 3, we find the form of the unknown functions $\alpha(x),\ \beta(x),\ \gamma(x)$ to be

$$
\begin{aligned}
&\alpha(x) = (C_1 + C_2\exp(ax)), \quad \beta(x) = -C_3 - aC_2\exp(ax),\\
&\gamma(x) = C_4 + C_5\exp(ax).
\end{aligned}
$$

Direct verification shows that the initial value problem

$$
\begin{aligned}
&u_t = u_x^{-1} u_{xx},\\
&\Big((C_1 + C_2\exp(ax))u_x - (C_3 + aC_2\exp(ax))u\Big)\Big|_{t=t_0}\\
&\quad = C_4 + C_5\exp(ax)
\end{aligned}
\tag{23}
$$

under $C_2(aC_1-C_3)\neq 0$ fulfills the conditions of Theorem 3 and, consequently, can be reduced to a Cauchy problem for a system of ordinary differential equations. Indeed, inserting ansatz (22) into (23) yields

$$\begin{aligned}
&\varphi_1'(t)=0, \quad \varphi_2'(t)=a,\\
&\varphi_1(t_0)=\frac{aC_5+C_4}{aC_1-C_3}, \quad \varphi_2(t_0)=\frac{C_5}{C_2}, \quad C_2(aC_1-C_3)\neq 0.
\end{aligned}$$

Finally, solving the above Cauchy problem gives the following unique solution of the initial value problem (23):

$$u(t,x)=\frac{aC_5+C_4}{aC_1-C_3}\exp(ax)+a(t-t_0)+\frac{C_5}{C_2}.$$

Next, we apply the above scheme for reduction of the initial value problem for the third-order nonlinear PDE

$$u_t=u_{xxx}+u_x^2-4auu_x+4a^2u^2, \tag{24}$$

where a is an arbitrary real parameter. Note that when $a\neq 0$, equation (24) cannot be solved by the methods of soliton theory, and in this sense it is non-integrable. When $a=0$, (24) is reduced to the KdV equation by the (non-point) change of dependent variable $v(t,x)=u_x(t,x)$. In what follows we suppose that $a\neq 0$.

Direct calculation shows that equation (24) is conditionally invariant with respect to the Lie-Bäcklund vector field (3) with $\eta=u_{xxx}-3au_{xx}+2a^2u_x$. On calculating the Lie symmetry of the equation $\eta=u_{xxx}-3au_{xx}+2a^2u_x=0$, and taking into account Theorem 3, we get the following initial value problem for PDE (24):

$$\begin{aligned}
&u_t=u_{xxx}+u_x^2-4auu_x+4a^2u^2, \qquad (25)\\
&\Big((C_1+C_2\exp(ax)+C_3\exp(-ax))u_x+(C_4-2aC_2\exp(ax))u\Big)\Big|_{t=t_0}\\
&\quad=C_5+C_6\exp(ax)+C_7\exp(2ax).
\end{aligned}$$

Integrating the equation $\eta=0$ and substituting its general solution

$$u(t,x)=\varphi_(t)+\varphi_2(t)\exp(ax)+\varphi_3(t)\exp(2ax) \tag{26}$$

into (25) give the following Cauchy problem:

$$\begin{aligned}
&\vec{\varphi}'(t)=\mathcal{F}(\vec{\varphi}(t)),\\
&\mathcal{C}\vec{\varphi}(t_0)=\vec{C},
\end{aligned}$$

where

$$\vec{\varphi}(t)=\begin{pmatrix}\varphi_1(t)\\ \varphi_2(t)\\ \varphi_3(t)\end{pmatrix},\quad \mathcal{F}(\vec{\varphi})=\begin{pmatrix}4a^2\varphi_1^2\\ a^3\varphi_2+4a^2\varphi_1\varphi_2\\ 8a^3\varphi_3+a^2\varphi_2^2\end{pmatrix},$$
$$\mathcal{C}=\begin{pmatrix}C_4 & aC_3 & 0\\ -2aC_2 & aC_1+C_4 & 2aC_3\\ 0 & -aC_2 & C_4+2aC_1\end{pmatrix},\quad \vec{C}=\begin{pmatrix}C_5\\ C_6\\ C_7\end{pmatrix}.$$

Thus, using higher-order conditional symmetries of the nonlinear PDE (24), we reduce the initial value problem (25) to the Cauchy problem for a system of three first-order ordinary differential equations for the functions $\varphi_1(t),\varphi_2(t),\varphi_3(t)$. The general solution of the system in question reads

$$\begin{aligned}\varphi_1(t) &= (\kappa_1-4a^2t)^{-1},\\ \varphi_2(t) &= \kappa_2\exp(a^3t)(\kappa_1-4a^2t)^{-1},\\ \varphi_3(t) &= \exp(8a^3t)\left(\kappa_3+a^2\kappa_2^2\int_{t_0}^{t}\exp(-6a^3z)(\kappa_1-4a^2z)^{-2}dz\right),\end{aligned}$$

where $\kappa_1,\kappa_2,\kappa_3$ are arbitrary real parameters (integration constants).

We emphasize that the exact analytic solution of the nonlinear PDE (24) that is obtained via substitution of the corresponding expressions for the functions $\varphi_1(t),\varphi_2(t),\varphi_3(t)$ into (26) cannot be constructed within the standard symmetry reduction approach. The reason for this is that the solution is not invariant with respect to the two-parameter displacement group defined by $\frac{\partial}{\partial t},\frac{\partial}{\partial x}$, which is the maximal Lie symmetry algebra of (24).

Specifying $\kappa_1,\kappa_2,\kappa_3$ appropriately, we can solve any representative of the seven-parameter family of initial value problems given in (25). Consider, for example, the case $C_1=C_2=C_3=C_7=0$ and $C_4=1, C_5=k_1, C_6=k_2$. Then the unique solution of the corresponding initial value problem

$$\begin{aligned}&u_t=u_{xxx}+u_x^2-4auu_x+4a^2u^2,\\ &u(t_0,x)=k_1+k_2\exp(ax)\end{aligned}$$

is given by

$$\begin{aligned}u(t,x) = &\left(k_1+k_2\exp(ax+a^3(t-t_0))\right)(1-4k_1a^2(t-t_0))^{-1}\\ &+a^2k_2^2\exp(2ax+8a^3t-2a^3t_0)\int_{t_0}^{t}\exp(-6a^3z)(1-4k_1a^2z)^{-2}dz.\end{aligned}$$

4 Conditionally-invariant second-order PDEs

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An analysis of the examples given in the previous section reveals an evident restriction on the scope of applicability of the reduction techniques based on higher conditional symmetries. It comes from a necessity to integrate nonlinear high order ordinary differential equation (7). Moreover, in order to use Theorem 3, we need to ensure that the ordinary differential equation $\eta = 0$ has a non-trivial Lie symmetry. These observations suggest the idea of using Lie's classification of invariant second-order ordinary differential equations as the source of the functions η. With this choice of the conditional symmetry we are guaranteed that differential equation $\eta = 0$ admits two-, three- or eight-parameter transformation group and, consequently, is integrable by quadratures. Surprisingly, it is possible to implement this approach to classifying second-order PDEs (1) by their second-order conditional symmetries in full generality.

In Table 1 we present the complete list of invariant real second-order ordinary differential equations together with their maximal invariance algebras, obtained by Lie ([21, 22]). Note that a, k are arbitrary real parameters and f is an arbitrary function. As classification has been done to within an arbitrary reversible transformation of the variables x, y, the equations given in Table 1 are representatives of the conjugacy classes of invariant ordinary differential equations.

Table 1. *Lie's classification of invariant second-order ordinary differential equations*

No.	Equation	Symmetry algebra
1	$y'' = f(y,\ y')$	$\frac{\partial}{\partial x}$
2	$y'' = f(y')$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$
3	$y'' = x^{-1} f(y')$	$\frac{\partial}{\partial x}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$
4	$y'' = 2(y-x)^{-1}((y')^2 + ay'\sqrt{y'} + y')$	$\frac{\partial}{\partial x} + \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$ $x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$
5	$y'' = ay^{-3}$	$\frac{\partial}{\partial x}, 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$ $x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$
6	$y'' = a\exp(-y')$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + (x+y)\frac{\partial}{\partial y}$
7	$y'' = a(y')^{\frac{k-2}{k-1}},\ k \neq 1, 2$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + ky\frac{\partial}{\partial y}$
8	$y'' = a(1+(y')^2)^{\frac{3}{2}}\exp(k\arctan y')$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y},$ $(kx+y)\frac{\partial}{\partial x} + (ky-x)\frac{\partial}{\partial y}$
9	$y'' = 0$	$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, x\frac{\partial}{\partial y}, x\frac{\partial}{\partial x}, y\frac{\partial}{\partial x},$ $y\frac{\partial}{\partial y}, x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},$ $xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}$

We exclude from further consideration case 1 of Table 1, since the corresponding ordinary differential equation is not integrable by quadratures. Next, since our final aim is to exploit conditional symmetries for the description and reduction of initial value problems, it make no sense to consider case 4. This is because the symmetry group admitted by the corresponding ordinary differential equation within the class (18) is the same as that of the more general equation given in case 3 of Table 1. The same argument applies to case 8. Consequently, we will deal only with the remaining cases 2, 3, 5–7, 9.

We take as the function η in operator (3) the expressions $y'' - f(x, y, y')$, where f is one of the right-hand sides of equations listed in the second column of Table 1 and make the replacements $y \to u$, $y' \to u_x$ and $y'' \to u_{xx}$. We classify PDEs of the form

$$u_t = u_{xx} + F(t, x, u, u_x) \tag{27}$$

admitting the corresponding Lie-Bäcklund vector fields. As we have already mentioned, this programme can be realised in full generality. Indeed, given the above choice of higher conditional symmetry operators, we are able to describe the most general PDEs (27) admitting these symmetries. The form of the function F is the same for all the cases, namely:

$$F(t, x, u, u_x) = rF_1(t, \omega_1, \omega_2) + pF_2(t, \omega_1, \omega_2) + q.$$

Here F_1, F_2 are arbitrary smooth functions of the indicated variables and the forms of the coefficients r, p, q and of the "invariants" ω_1, ω_2 are presented in Table 2.

As an example, we consider the case of the Lie-Bäcklund field $Q = (u_{xx} - a\exp(-u_x))\frac{\partial}{\partial u} + \ldots$. Inserting this operator into the invariance condition (6) yields the second-order PDE for the function $F = F(t, x, u, u_x)$ (we have generated it using *MATHEMATICA*)

$$\begin{aligned}
&a^3\exp(-3u_x) + a^2\exp(-2u_x)F_{u_x u_x} + a\exp(-u_x)(u_x + 1)F_u \\
&\quad + 2au_x\exp(-u_x)F_{uu_x} + u_x^2 F_{uu} + a\exp(-u_x)F_x + 2a\exp(-u_x)F_{xu_x} \\
&\quad + 2u_x F_{xu} + F_{xx} = 0.
\end{aligned}$$

The equation obtained looks rather fiersome. However, it simplifies radically with the change of variables $F = \tilde{F}(t, x, \omega_1, \omega_2)$, where ω_1, ω_2 are the integrals of the ordinary differential equation $u_{xx} - a\exp(-u_x) = 0$. As a result, we find that the function $\tilde{F}$ satisfies the following PDE:

$$a^3\exp(-3u_x) + a\exp(-u_x)\tilde{F}_x + \tilde{F}_{xx} = 0.$$

Now, we eliminate u_x, using the second invariant $\omega_2 = -ax + \exp(u_x)$, and thus get an ordinary differential equation with respect to x. Solving it yields formulae 6 from Table 2. The other cases are dealt with in an analogous way.

Now we proceed to calculating the initial conditions (12) such that the initial value problems for conditionally invariant PDEs (27) given in Table 2 are reducible to Cauchy problems for systems of two ordinary differential equations. To this end we exploit Theorem 3. First of all we note that for equations 2, 3 of Table 2, the condition b of Theorem 3 fails to hold. For this reason, the corresponding initial value problem is not reduced to a Cauchy problem within the framework of our approach. For the remaining cases 5, 6, 7.1, 7.2 and 9 we get the following initial conditions:

Table 2. *Conditionally-invariant PDEs (27)*

No.	Coefficients r, p, q	Functions ω_1, ω_2
2	$r = 1$ $p = u_x$ $q = -(f'(u_x))^{-1}$	$\omega_1 = u - \int^{f(u_x)} g(z)dz$ $\omega_2 = f(u_x) - x$
3	$r = 1$ $p = xu_x - u$ $q = -(xf'(u_x))^{-1}$	$\omega_1 = u - x\exp(-f(u_x))\times$ $\times \int^{f(u_x)} g(z)\exp(z)dz$ $\omega_2 = f(u_x) - \ln x$
5	$r = u^{-1}u_x(a + u^2u_x^2)^{\frac{1}{2}}$ $p = u^3(2xu_x - u)(a + u^2u_x^2)^{-\frac{3}{2}}$ $q = -au^{-3}$	$\omega_1 = 2u^3u_x(a + u^2u_x^2)^{-1} - 2x$ $\omega_2 = u^{-2}(u_x^2u^2 - a)$
6	$r = 1$ $p = u_x$ $q = -a\exp(-u_x)$	$\omega_1 = au + (1 - u_x)\exp(u_x)$ $\omega_2 = -ax + \exp(u_x)$
7.1	$r = 1$ $p = u_x$ $q = -au_x^{\frac{k-2}{k-1}}, \; k \neq 0, 1, 2$	$\omega_1 = u + \frac{1-k}{ak}u_x^{\frac{k}{k-1}}$ $\omega_2 = \frac{ax}{1-k} + u_x^{\frac{1}{k-1}}$
7.2	$r = 1$ $p = u_x$ $q = -au_x^2$	$\omega_1 = u - a^{-1}\ln(u_x)$ $\omega_2 = ax + (u_x)^{-1}$
9	$r = 1$ $p = x$ $q = 0$	$\omega_1 = xu_x - u$ $\omega_2 = u_x$

Case 5. $\quad (C_1 + 2C_2x + C_3x^2)u_x(t_0, x) - (C_2 + C_3x)u(t_0, x) = 0,$
$\quad C_3 \neq 0, \quad C_1C_3 \neq C_2^2;$

Case 6. $\quad (C_1 + C_3x)u_x(t_0, x) - C_3u(t_0, x) = C_2 + C_3x, \quad C_3 \neq 0;$

Case 7.1. $\quad (C_1 + C_3x)u_x(t_0, x) - kC_3u(t_0, x) = C_2, \quad C_3 \neq 0;$

Case 7.2. $\quad (C_1 + C_3x)u_x(t_0, x) = C_2, \quad C_2 \neq 0;$

Case 9. $(C_1 + C_4x + C_6x^2)u_x(t_0, x) - (C_5 + C_6x)u(t_0, x) = C_2 + C_3x,$
$\Delta = C_1C_6 + C_5(C_5 - C_4) \neq 0, \quad \Delta_1 = C_2(C_4 - C_5) - C_1C_3,$
$\Delta_2 = C_2C_6 - C_3C_5.$

In the above formulae $C_1, C_2, \ldots, C_6$ are arbitrary real constants satisfying the given constraints. These constraints ensure that the corresponding initial value problem fulfill both the conditions of Theorem 3.

The last step of the reduction algorithm is the construction of ansatzes for the function $u(t, x)$ and reduction of both invariant equation and initial conditions to systems of ordinary differential equations and algebraic equations, correspondingly. We present the list of obtained results in Table 3, where $F_i = F_i(t, \varphi_1(t), \varphi_2(t)), i = 1, 2$, the functions $\varphi_1(t), \varphi_2(t)$ are new dependent variables and the symbol g stands for the converse of the function $f(u_x)$, i.e., $g(f(u_x)) \equiv u_x$.

Table 3. *Ansatzes and reduced equations*

No.	Ansatz for $u(t,x)$	ODEs	Cauchy data
2	$\varphi_1(t) + g(x + \varphi_2(t))$	$\varphi_1' = F_1$ $\varphi_1' = F_2$	No reduction
3	$\varphi_1(t) + \exp(-\varphi_2(t))\times$ $\times \int^{\ln(x)+\varphi_2(t)} \exp(y)g(y)dy$	$\varphi_1' = F_1-$ $-\varphi_1 F_2$ $\varphi_2' = F_2$	No reduction
5	$\frac{1}{2\sqrt{\varphi_2(t)}}\Big(4a + \varphi_2^2(t)\times$ $\times(2x + \varphi_1(t))^2\Big)^{\frac{1}{2}}$	$\varphi_1' = 2\varphi_1^{\frac{1}{2}}F_1-$ $-2\varphi_1\varphi_2^{\frac{1}{2}}F_2$ $\varphi_2' = 2\varphi_2^{\frac{1}{2}}F_2$	$\varphi_1(t_0) = \frac{2C_2}{C_3}$ $\varphi_2(t_0) = \frac{C_3\sqrt{a}}{\sqrt{C_1C_3 - C_2^2}}$
6	$\varphi_1(t) + a^{-1}(ax + \varphi_2(t))\times$ $\times(\ln(ax + \varphi_2(t)) - 1)$	$\varphi_1' = F_1$ $\varphi_2' = aF_2$	$\varphi_1(t_0) = \frac{C_2 - C_1}{C_3}$ $\varphi_2(t_0) = \frac{aC_1}{C_3}$
7.1	$\varphi_1(t) + \frac{k-1}{ak}\left(\frac{ax}{k-1} + \varphi_2(t)\right)^k$	$\varphi_1' = F_1$ $\varphi_2' = \frac{a}{k-1}F_2$	$\varphi_1(t_0) = \frac{C_2}{kC_3}$ $\varphi_2(t_0) = \frac{aC_1}{(k-1)C_3}$
7.2	$\varphi_1(t) - a^{-1}\ln(\varphi_2(t) - ax)$	$\varphi_1' = F_1$ $\varphi_2' = -aF_2$	$C_3 = -aC_2$ $\varphi_2(t_0) = \frac{C_1}{C_2}$
9	$\varphi_1(t) + \varphi_2(t)x$	$\varphi_1' = F_1$ $\varphi_2' = F_2$	$\varphi = \frac{\Delta_1}{\Delta}$ $\varphi_2(t_0) = \frac{\Delta_2}{\Delta}$

Let us emphasize that the nonlinear evolution equations constructed above, for arbitrary functions F_1, F_2, admit no Lie symmetry. Consequently, the symmetry reduction algorithm cannot be applied to perform dimensional reductions of these equations. This means that the reductions obtained in this section (both for PDEs and initial value problems) are purely non-Lie reductions and cannot be obtained within the framework of the standard Lie's approach.

5 Conclusions

Our work shows that higher-order conditional symmetry is a useful and efficient concept in handling initial value problems. In fact, in view of Theorem 3, one may claim that the method of conditional symmetries is the most appropriate tool for reductions of initial value problems for evolution type equations. It extends the scope of applicability of symmetry methods to the analysis of initial value problems via dimensional reductions.

Furthermore, the technique developed in this paper can be applied to boundary value problems as well, provided the PDE under study admits some higher-order Lie-Bäcklund vector field. Indeed, inserting ansatz (8) into the boundary condition

$$U(t, u, u_1, \ldots, u_M)|_{x=x_0} = 0, \quad M < N$$

yields an algebraic relation for the functions $\varphi_i, i = 1, 2, \ldots, N$. This means that, if we are given a reducible initial value problem, then we can add *any* boundary condition, so that the resulting problem remains reducible. However, the problem of compatibility of initial and boundary conditions must be investigated separately (see, also the paper [23] and the references therein).

We consider in the present paper the case of one dependent variable. However, all our arguments can be directly applied to the analysis of systems of evolution equations in one spatial dimension. One final point is that it seems to be possible to apply the same technique for reduction of multi-dimensional evolution equations admitting classical or non-classical symmetries. These and related problems are under study now and will be reported on in future publications.

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