The structure of Lie algebras and the classification problem for partial differential equations.

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Abstract

The present paper solves completely the problem of the group classification of nonlinear heat-conductivity equations of the form $u_t = F(t, x, u, u_x)u_{xx} + G(t, x, u, u_x)$. We have proved, in particular, that the above class contains no nonlinear equations whose invariance algebra has dimension more than five. Furthermore, we have proved that there are two, thirty-four, thirty-five, and six inequivalent equations admitting one-, two-, three-, four- and five-dimensional Lie algebras, respectively. Since the procedure which we use, relies heavily upon the theory of abstract Lie algebras of low dimension, we give a detailed account of the necessary facts. This material is dispersed in the literature and is not fully available in English. After this algebraic part we give a detailed description of the method and then we derive the forms of inequivalent invariant evolution equations, and compute the corresponding maximal symmetry algebras. The list of invariant equations obtained in this way contains (up to a local change of variables) all the previously-known invariant evolution equations belonging to the class of partial differential equations under study.

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Introduction

Modeling phenomena in nature with partial differential equations is one of the central problems of mathematical physics and applied mathematics. One can even say that mathematical physics in its classical form was created in order to provide a rigorous mathematical foundation for describing different phenomena in physics, chemistry and biology by partial differential equations. However, when one has to decide which differential equation fits in the best way as a model for the process under study, one has to select from a broad class of possible partial differential equations. Even if one has taken into account all the peculiarities of the process under study (which is hardly possible!), there is still great freedom in choosing possible models. One of the principal criteria for choosing the partial differential equations modeling real processes is the *symmetry selection principle*. By this we mean that from the whole set of admissible models, those models which have the highest symmetry should be selected. This point of view is supported by the fact that the most successful mathematical models in theoretical and applied science have a rich symmetry structure. Indeed, the basic equations of modern physics, the wave, Schrödinger, Dirac and Maxwell equations are distinguished from the whole set of partial differential equations by their Lie and non-Lie (hidden) symmetry (see [1] for more details on symmetry properties of these equations).

The effectiveness of the symmetry (group-theoretical) approach to the classification of admissible partial differential equations relies heavily upon the availability of a constructive way of describing transformation groups leaving invariant the form of a given partial differential equation. This is done via the well-known infinitesimal method developed by Sophus Lie [2, 3] (see, e.g., [4]–[6]). Given a partial differential equation, the problem of investigating its maximal (in some sense) Lie invariance group reduces to solving an over-determined system of linear partial differential equations, called the determining equations. However, if the equation under study contains arbitrary elements (functions), then one has to solve an intermediate classification problem. Namely, it is necessary to describe all the possible forms of the functions involved such that this equation admits a non-trivial invariance group.

In principle, the classification problem is solved with the help of the Lie algebra approach. However, since the determining equations involve some arbitrary functions, there is an evident need for a modification of the basic Lie technique in order to obtain an efficient and systematic way of classifying these arbitrary elements. The idea of this modification was suggested by Sophus Lie himself. Indeed, his way of obtaining all ordinary differential equations in one variable admitting non-trivial symmetry algebras [2, 3] tells us what is to be done in the case at hand. We should first construct all the possible inequivalent realizations of symmetry algebras within some class of Lie vector fields. If we succeed in doing this, then the symmetry algebras will be specified, so that we can apply directly Lie's infinitesimal algorithm, thus getting inequivalent classes of invariant equations. In this way, Sophus Lie obtained his famous classification of realizations of all inequivalent complex Lie algebras in the plane [2, 3]. Recently, Lie's classification was exploited by Olver and Heredero [7] in order to classify nonlinear wave equations in two independent variables that are invariant with respect to transformation groups not changing the temporal variable.

A systematic implementation of these ideas for partial differential equations has been worked out by Ovsjannikov [4]. His approach is based on the concept of an equivalence group, which is a Lie transformation group acting in the extended space of independent variables, functions and their derivatives, and preserving the class of partial differential equations under study. It is possible to modify Lie's algorithm in order to make it applicable for the computation of this group [4]. Next, one constructs the optimal system of subgroups of the equivalence group. The last step uses Lie's algorithm for obtaining specific partial differential equations, that (a) belong to the class under study, and (b) are invariant with respect to the subgroups mentioned above. This approach has been applied to a number of equations of nonlinear gas dynamics and diffusion equations (Akhatov, Gazizov and Ibragimov [8, 9]). Ovsjannikov's ideas have also been exploited by Torrisi and co-workers in order to perform a preliminary group classification of some nonlinear diffusion and heat conductivity equations [10, 11]. Ibragimov and Torrisi have obtained a number of important results on the group classification of nonlinear detonation equations [12] and nonlinear hyperbolic type equations [13]. There is a number of papers (see, e.g., [14] and the references therein) devoted to a direct computation of equivalence groups of some PDEs. Since the transformations of the equivalence group are used in their finite form, this approach has the merit of giving the possibility of finding *discrete* equivalence groups or even *non-local* ones.

However, the possibility of implementing Ovsjannikov's approach in its full generality presupposes that we are able to construct the optimal system of subgroups of the equivalence group. So that, even for the case when the equivalence group has a finite number of parameters, there arise major algebraic difficulties, since for a number of known finite-parameter Lie groups the classification problem has not yet been solved (to say nothing about infinite-parameter Lie groups, where this problem is completely open). Consequently, there is an evident need for Ovsjannikov's approach to be modified so that it can be applied to the case of infinite-parameter equivalence groups.

In the paper [15] we have developed a new approach that enables us to solve efficiently the symmetry classification problem for partial differential equations even for the case of infinite-dimensional equivalence groups. It is mainly based on the following facts:

- If the partial differential equation possesses non-trivial symmetry, then it is invariant under some finite-dimensional Lie algebra of differential operators which is completely determined by its structural constants. In the event that the maximal algebra of invariance is infinite-dimensional, then it contains, as a rule, some finite-dimensional Lie algebra.

- If there are local non-singular changes of variables which transform a given differential equation into another, then the finite-dimensional Lie algebra of invariance of these equations are isomorphic, and in the group-theoretic analysis of differential equations such equations are considered to be equivalent.

What we have suggested in [15] is a preliminary classification of inequivalent realizations of low-dimensional Lie algebras within some specific class of first-order linear differential operators. This class is determined by the structure of the equation under study. Its elements form a representation space for realizations of Lie algebras of symmetry groups admitted by the equations belonging to the class of partial differential equations under study. A natural equivalence relation is introduced on the set of all possible realizations. Namely, two realizations are called equivalent if they are transformed into each other by the action of the equivalence group. In other words, solving the problem of symmetry classification of partial differential equations having some prescribed form, is equivalent to constructing a representation theory of Lie transformation groups (or Lie algebras of first-order differential operators) realized as symmetry groups (algebras) of the equations in question.

The first aim of the present paper is to give a detailed exposition of our approach. A full understanding of the techniques applied requires some basic facts from the general theory of Lie groups and algebras, some of which are dispersed in the literature and are not available in English (this is the case for the papers of Mubarakzyanov and Morozov). So, in addition to the exposition of the classification results, we give a survey of results on the structure of Lie algebras (with special emphasis on low-dimensional Lie algebras), which are of vital importance for the effective implementation of our approach.

The second aim is obtaining a complete description of the nonlinear heat conductivity equations of the form

$$u_t = F(t,x,u,u_x)u_{xx} + G(t,x,u,u_x) \tag{0.1}$$

that admit non-trivial symmetry group. Hereafter $u = u(t,x)$, F, G are sufficiently smooth functions of the corresponding arguments, $u_t = \dfrac{\partial u}{\partial t}$, $u_x = \dfrac{\partial u}{\partial x}$, $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $F \neq 0$.

Note that the above equation is, in some sense, the most general evolution equation in one dimension. Indeed, any equation of the most general form

$$u_t = H(t,x,u,u_x,u_{xx}) \tag{0.2}$$

which admits at least one-parameter symmetry group, not changing the temporal variable, can be reduced to the form (0.1) by a non-point transformation. So that our group classification of equations (0.1) will also cover invariant equations of the form (0.2) excepting for the small subclass of equations whose symmetry algebras are spanned by operators with non-vanishing coefficients by $\dfrac{\partial}{\partial t}$.

The principal scheme of the paper is as follows. Section I contains a general description of our approach. In the next section we give a brief overview of the necessary facts from the general theory of Lie algebras. Section III is devoted to group classification of PDEs (0.1). We consider subsequently, the cases of semi-simple, semi-direct sum of semi-simple and solvable and solvable symmetry algebras thus getting the full solution of the classification problem for nonlinear heat conductivity equations belonging to the class (0.1). The last section contains discussion of the results obtained and some conclusions.

I Description of the method

The approach to the classification of partial differential equations which we propound is, in fact, a synthesis of Lie's infinitesimal method, the use of equivalence transformations and the theory of classification of abstract finite-dimensional Lie algebras. It constitutes a constructive solution of the problem the group classification of partial differential equations possessing large classes of arbitrary elements and admitting *non-trivial finite-dimensional* invariance algebras.

The realization of group classification in the proposed approach consists in the implementation of the following algorithm:

I The first step involves finding the form of the infinitesimal operators which generate the symmetry group of the equation under consideration, and the construction of the equivalence group of this equation. To find the form of the infinitesimal operators one uses the usual Lie algorithm. As a result we obtain a system of linear partial differential equations of first order, which connect the coefficients of the infinitesimal operators with the arbitrary term of the equation. In what follows, we call this system the characterizing system of the equation. In order to construct the equivalence group $\mathcal{E}$ of the equation under consideration, one can use the infinitesimal as well as the direct method.

II In the second step, one carries out the group classification of those equations of the given form which admit finite-dimensional Lie algebras of invariance.

For this, one carries out a step-by-step classification of finite-dimensional Lie algebras within the specified class of infinitesimal operators, up to equivalence under transformations of the group $\mathcal{E}$. In this, one has to see if each algebra obtained in this way can be an invariance algebra of the equation at hand before proceeding from the realization of Lie algebras of lower dimension to the realization of Lie algebras of higher dimension. This eliminates superfluous realizations of Lie algebras. Also, those realizations of Lie algebras which are invariance algebras of the equation will, as their dimension increases, correspond to greater fixing of the arbitrary term.

This procedure is continued until the arbitrary term in the equation is completely determined or until it is no longer possible to extend the realization of Lie algebras beyond a given dimension within the specified class of infinitesimal operators.

III The third step is then to exploit the characterizing system or the infinitesimal method of Lie in order to find, for each of the particular choices of the arbitrary term, the maximal invariance algebra of the equation under consideration. Furthermore, the equivalence of the equations obtained in this manner is determined. We note that, in as much as equivalent equations have isomorphic invariance algebras, we may test the realizations of the invariance algebras for equivalence rather than the equations themselves.

Note that similar ideas have been used by Gangon and Winternitz [16] in order to classify symmetries of nonlinear Schroödinger equations having variable coefficients.

II Lie-algebraic structures involved in the classification algorithm

Let us take a more detailed look at the second step of the algorithm. As is clear from what has been said above, carrying out this step assumes that there is a classification of non-isomorphic finite-dimensional Lie algebras (in particular, we are interested in a classification of Lie algebras over the real numbers).

One of the central theorems which deals with the structure of Lie algebras is the Levi-Mal'cev theorem:

Theorem 2.1 *Let L be a finite-dimensional Lie algebra over* **R** *or* **C**, *and let N denote its radical (the largest solvable ideal in L). Then there exists a semi-simple Lie subalgebra S of L such that*

$$L = S \Subset N \tag{2.1}$$

Equation (2.1) is called the Levi decomposition of the Lie algebra L, and the semi-simple subalgebra S is called the Levi factor.

The Levi-Mal'cev decomposition gives us

$$[N, N] \subset N, \quad [S, S] \subset S, \quad [N, S] \subset N,$$

so that any Lie algebra L is the semi-direct sum $L = S \Subset N$ of its maximal solvable ideal N and the semi-simple subalgebra S. We see then that this result reduces the task of classifying all Lie algebras to the following problems:

1) the classification of all semi-simple Lie algebras;

2) the classification of all solvable Lie algebras;

3) the classification of all algebras which are semi-direct sums of semi-simple Lie algebras and solvable Lie algebras.

II.1 Semi-simple Lie algebras.

Of the problems listed above, only that of classifying all semi-simple Lie algebras is completely solved. We have the well-known theorem due to Cartan:

Theorem 2.2 *(**Cartan's theorem**) Any semi-simple complex or real semi-simple Lie algebra can be decomposed into a direct (Lie algebra) sum of ideals which are mutually orthogonal simple subalgebras. Here, orthogonality is with respect to the Cartan-Killing form $(X, Y) = Tr(ad\, X, ad\, Y)$.*

Let L be a semi-simple Lie algebra. Then, by Cartan's theorem, we have

$$L = S_1 \oplus S_2 \oplus \ldots \oplus S_m,$$

where $S_1, \ldots, S_m$ are simple Lie algebras. Thus, the problem of classifying semi-simple Lie algebras is equivalent to that of classifying all non-isomorphic simple Lie algebras. This classification is known (see, for instance, [17]).

There are four sequences of classical Lie algebras $A_n (n \geq 1)$, $B_n (n \geq 1)$, $C_n (n \geq 1)$, $D_n (n \geq 1)$ and five exceptional Lie algebras G_2, F_4, E_6, E_7, E_8 which together exhaust all the simple complex Lie algebras. There are some isomorphisms between some of these algebras. Indeed [18] there are the following isomorphisms:

$$A_1 \cong B_1 \cong C_1, \quad B_2 \cong C_2, \quad A_3 \cong D_3, \quad D_2 \cong A_1 \oplus A_1$$

and there are no other isomorphisms between the series.

The dimensions of the classical complex Lie algebras A_n, B_n, C_n and D_n are given in the following table:

Algebra	A_n	B_n	C_n	D_n
Dimension	$n(n+2)$	$n(2n+1)$	$n(2n+1)$	$n(2n-1)$

The dimensions of the exceptional Lie algebras are all even: $\dim G_2 = 14$, $\dim F_4 = 52$, $\dim E_6 = 78$, $\dim E_7 = 133$, $\dim E_8 = 248$.

To describe the real simple Lie algebras one uses the fact that every simple Lie algebra over the reals $\mathbf{R}$ is either a simple algebra over the complex field $\mathbf{C}$ (considered as an algebra over $\mathbf{R}$), or it is the real form of a simple Lie algebra over $\mathbf{C}$.

The real classical Lie algebras play an important role in the group analysis of differential equations. Below, we give a more detailed description of these Lie algebras. The symbol L_k denotes a compact simple Lie algebra.

I. Real forms of the algebras $sl(n, \mathbf{C})$ ($\cong A_{n-1}$, $n \geq 2$)

1) $L_k = su(n)$, the Lie algebra of all skew-symmetric matrices Z of order n with $Tr\, Z = 0$ of order n with $Tr\, Z = 0$.

2) $sl(n, \mathbf{R})$, the Lie algebra of all real matrices X of order n with $Tr\, X = 0$.

3) $su(p, q)$, $p + q = n$, $p \geq q$, the Lie algebra of all matrices of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_2^* & Z_3 \end{bmatrix}$$

where Z_1, Z_3 are skew-symmetric matrices of order p and q respectively, $Tr(Z_1 + Z_3) = 0$, and Z_2 is an arbitrary matrix of order q.

4) $su^*(2n)$, the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{bmatrix}$$

where Z_1, Z_2 are complex matrices of order n with $Tr(Z_1 + \overline{Z}_1) = 0$.

II. Real forms of the algebras $so(2n, \mathbf{C})$ ($\cong D_n$, $n \geq 1$)

1) $L_k = so(2n)$, the Lie algebra of all real skew-symmetric matrices of order $2n$.

2) $so(p, q)$, $p + q = 2n$, $p \geq q$, the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

where all the X_i are real matrices, and X_1, X_3 are skew-symmetric matrices of order p and q respectively, and X_2 is an arbitrary matrix of order q.

3) $so^*(2n)$, the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ -\overline{Z}_2 & \overline{Z}_1 \end{bmatrix}$$

Z_1 skew-symmetric and Z_2 Hermitian.

III. Real forms of the algebras $so(2n + 1, \mathbf{C})w$ ($\cong B_n$, $n \geq 1$)

1) $L_k = so(2n + 1)$, the Lie algebra of all real skew-symmetric matrices of order $2n + 1$.

2) $so(p, q)$, $p + q = 2n + 1$, $p \geq q$, the Lie algebra of all real matrices of order $2n + 1$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_2^T & X_3 \end{bmatrix}$$

where all the X_i are real matrices, and X_1, X_3 are skew-symmetric matrices of order p and q respectively, and X_2 is an arbitrary matrix of order q.

IV. Real forms of the algebras $sp(n, \mathbf{C})$ ($\cong C_n$, $n \geq 1$)

1) $L_k = sp(n)$, the Lie algebra of all matrices of order $2n$ of the form

$$\begin{bmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^T \end{bmatrix}$$

where all the Z_i are complex matrices of order n and Z_2, Z_3 are symmetric.

2) $sp(n, \mathbf{R})$, the Lie algebra of all real matrices of order $2n$ of the form

$$\begin{bmatrix} X_1 & X_2 \\ X_3 & -X_1^T \end{bmatrix}$$

where X_1, X_2, X_3 are all real matrices of order n, and X_2, X_3 are symmetric.

3) $sp(p,q)$, $p+q=n, p \geq q$, the Lie algebra of all complex matrices of order $2n$ of the form

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} \\ Z_{12}^* & Z_{22} & Z_{14}^T & Z_{24} \\ -\overline{Z}_{13} & \overline{Z}_{14} & \overline{Z}_{11} & -\overline{Z}_{12} \\ Z_{14}^* & -\overline{Z}_{24} & -\overline{Z}_{12}^T & \overline{Z}_{22} \end{bmatrix}$$

where the Z_{ij} are complex matrices, Z_{11} and Z_{13} are of order p, Z_{12} and Z_{14} are $p \times q$ matrices, Z_{11} and Z_{22} are skew-Hermitian, and Z_{13} and Z_{24} are symmetric.

The structure of the above real, simple classical Lie algebras is such that every algebra of a higher dimension contains, as a subalgebra, an algebra of the same class but of lower dimension. This allows us to proceed step-by-step when we study the realizations of these algebras as vector fields, at each stage extending the realizations of lower dimension to realizations of higher dimension. If at some stage in this procedure the chain stops, then this implies that there are no realizations within the given type of vector fields of Lie algebras of higher dimension.

In searching for realizations of the classical simple Lie algebras over $\mathbf{R}$, it is important to take into account the isomorphisms for the lower-dimensional classical Lie algebras:

$$\begin{aligned}
&su(2) \cong so(3) \cong sp(1);\\
&sl(2,\mathbf{R}) \cong su(1,1) \cong so(2,1) \cong sp(1,\mathbf{R});\\
&so(5) \cong sp(2);\\
&so(3,2) \cong sp(2,\mathbf{R});\\
&so(4,1) \cong sp(1,1);\\
&so(4) \cong so(3) \oplus so(3) \cong sp(1) \oplus sp(1);\\
&so(5) \cong sp(2);\\
&so(2,2) \cong sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R});\\
&sl(2,C) \cong so(3,1);\\
&su(4) \cong so(6);\\
&sl(4,\mathbf{R}) \cong so(3,3);\\
&su(2,2) \cong so(4,2);\\
&su(3,1) \cong so^*(6);\\
&su^*(4) \cong so(5,1);\\
&so^*(8) \cong so(6,2);\\
&so^*(4) \cong su(2) \oplus sl(2,\mathbf{R}).
\end{aligned}$$

It is not difficult to see that the Lie algebras of the first two rows have the lowest dimension $n=3$. Thus, in constructing realizations of the classical simple Lie algebras over $\mathbf{R}$ one may begin with the algebras

$$so(3) = \langle e_1, e_2, e_3 \rangle,\ [e_1,e_2]=e_3, [e_2,e_3]=e_1, [e_3,e_1]=e_2;$$

$$sl(2,\mathbf{R}) = \langle e_1, e_2, e_3 \rangle, [e_1,e_3] = -2e_2,\ [e_1,e_2]=e_1,\ [e_2,e_3]=e_3.$$

Maximal compact subalgebras play an important role for the structure of the simple (and semi-simple) Lie algebras. We have:

Theorem 2.3 **(Cartan's Theorem)** *A semi-simple real Lie algebra has a decomposition of the form*

$$L = K \dot{+} P, \tag{2.2}$$

where

$$[K, K] \subset K, \quad [K, P] \subset P, \quad [P, P] \subset K, \tag{2.3}$$

and

$$\begin{array}{l} (X, X) < 0 \ \text{ for } \ X \neq 0 \ \text{ in } \ K, \\ (Y, Y) > 0 \ \text{ for } \ Y \neq 0 \ \text{ in } \ P. \end{array} \tag{2.4}$$

If the conditions (2.3), (2.4) are satisfied, then K is a maximal compact subalgebra of L.

The decomposition (2.2) for a real semi-simple Lie algebra is called the *Cartan decomposition.*
Consider as an example $so(3,1)$,which is the Lie algebra of the Lorentz group. Denoting by K_i $(i = 1, 2, 3)$ the generators of the compact algebra $so(3)$, and by N_i $(i = 1, 2, 3)$ the generators of the Lorentz boosts, we obtain the commutation relations

$$\begin{array}{rcl} [K_i, K_j] &=& \varepsilon_{ijl} K_l, \\ [K_i, N_j] &=& \varepsilon_{ijl} N_l, \\ [N_i, N_j] &=& -\varepsilon_{ijl} K_l. \end{array}$$

Thus, when looking for realizations of the Lie algebra $so(3,1)$, one may use the realizations obtained for the Lie algebra $so(3)$.

In the table below we give the maximal compact subalgebras of the real classical Lie algebras which are non-compact:

No/o	L	K	No/o	L	K
1	$sl(n, \mathbf{R})$	$so(n)$	5	$so^*(2n)$	$u(n)$
2	$su(p, q)$	$s(u(p) \oplus u(q))$	6	$sp(n, \mathbf{R})$	$u(n)$
3	$su^*(2n)$	$sp(n)$	7	$sp(p, q)$	$sp(p) \oplus sp(q)$
4	$so(p, q)$	$so(p) \oplus so(q)$			

Here, $u(n)$ is the Lie algebra of the unitary group $U(n)$, and $s(u(p) \oplus u(q))$ is the set of all elements $x \in u(p) \oplus u(q)$ such that $Tr\, x = 0$. Note that the matrix e_{ij}, defined as a matrix of order n with 1 in the (i, j) position and zeroes in all other entries, is an element of $u(n)$.

Because of their large dimension, the exceptional Lie algebras do not play as important a role as the classical simple Lie algebras do in the group analysis of differential equations. For this reason, we only mention briefly the real forms of the algebras of the type G_2, F_4, E_6, E_7, E_8, and we consider those subalgebras whose realizations one may use for the construction of realizations of the real exceptional simple Lie algebras. More details about these algebras can be found in [18].

The algebra G_2 has real compact form g_2 and one real non-compact form $g_2^{'}$. Moreover, $g_2 \cap g_2^{'} \cong su(2) \oplus su(2)$.

The algebra F_4 has real compact form f_4 and two real non-compact forms $f_4^{'}, f_4^{''}$. We also have $f_4^{'} \cap f_4 \cong sp(3) \oplus su(2)$, $f_4^{''} \cap f_4 \cong so(9)$.

The algebra E_6 has real compact form e_6 and four real non-compact forms $e_6^{'}$, $e_6^{''}$, $e_6^{'''}$, e_6^{IV}. Moreover, $e_6^{'} \cap e_6 \cong sp(4)$, $e_6^{''} \cap e_6 \cong su(6) \oplus su(2)$, $e_6^{'''} \cap e_6 \cong so(10) \oplus \mathbf{R}$, $e_6^{IV} \cap e_6 \cong f_4$.

The algebra E_7 has real compact form e_7 and four real non-compact forms $e_7^{'}$, $e_7^{''}$, $e_7^{'''}$. We also have $e_7^{'} \cap e_7 \cong su(8)$, $e_7^{''} \cap e_7 \cong so(12) \oplus su(2)$, $e_7^{'''} \cap e_7 \cong e_6 \oplus \mathbf{R}$.

The algebra E_8 has real compact form e_8 and two real non-compact forms $e_8^{'}, e_8^{''}$. Also, $e_8^{'} \cap e_8 \cong e_7 \oplus su(2)$, $e_8^{''} \cap e_8 \cong so(16)$.

II.2 Solvable Lie algebras.

The problem of classifying solvable Lie algebras up to isomorphism is, as far as we know, completely solved only for real Lie algebras of dimension up to and including six (see for example [19]–[24]). The difficulty in the classification of these algebras is, above all, connected with the fact that the number of non-isomorphic Lie algebras increases considerably with increasing dimension, beginning with dimension five. Thus, according to [21], there are 66 classes of non-isomorphic real, solvable Lie algebras of dimension five. Furthermore, for dimension six, there are 99 classes of non-isomorphic algebras just amongst the real solvable algebras containing a nilpotent element [22].

Let us consider in more detail at the structure of solvable Lie algebras over the field $\mathbf{R}$ with dimension no greater than five. We give a method of searching for their realizations in the class of differential operators.

Let L_n denote a solvable Lie algebra of dimension n, over a field of characteristic zero. It is known ([18]) that there exists a series of subalgebras

$$L_n \supset L_2 \supset \ldots \supset L_1 \supset L_0 = \{0\}$$

such that each subalgebra $L_i \, (i = 1, \ldots, n-1)$ is an ideal of the algebra L_{i+1}. This series is called the *composition series* of the algebra L_n.

The existence of the composition series for a real solvable Lie algebra allows us to make the following important conclusion: if, in the given class of differential operators, there is a realization of the solvable Lie algebras with $\dim L \leq m$, and there is no realization for algebras with $\dim L = m + 1$, then those realizations which appear give a complete description of the realizations of solvable algebras in the given class of vector fields.

Further, we shall use the following notation: $A_{k.i} = \langle e_1, \ldots, e_k \rangle$ denotes a Lie algebra of dimension k, $e_j \, (j = 1, \ldots, k)$ is its basis, and the index i denotes the number of the class to which the given Lie algebra belongs.

Fixing the type of the algebra $A_{k.i}$, we shall give only the non-zero commutation relations between the basis elements. Among the solvable Lie algebras over $\mathbf{R}$ of lowest dimension, we have only one algebra which is one-dimensional $A_1 = \langle e_1 \rangle$, and two algebras which have dimension two:

$$\begin{aligned} A_{2.1} &= \langle e_1, e_2 \rangle = A_1 \oplus A_1 = 2A_1; \\ A_{2.2} &= \langle e_1, e_2 \rangle, \quad [e_1, e_2] = e_2. \end{aligned}$$

Further, we shall call *decomposable* Lie algebras those algebras which can be decomposed as a direct sum of solvable algebras of lower dimension. We give a list of all solvable Lie algebras, up to and including dimension five, in Appendix 1.

It is clear that the search for realizations of solvable Lie algebras over **R** must be begun with the description of the inequivalent forms of the general infinitesimal operator, up to equivalence under the transformations of $\mathcal{E}$. Each of the operators obtained will be a basis for the inequivalent realizations of one-dimensional Lie algebras. Further, the completion of the basis operators of each of the one-dimensional Lie algebras, by an infinitesimal operator of the most general form, is done by extension of the realizations of the one-dimensional Lie algebras to realizations of two-dimensional Lie algebras. In doing this, in order to simplify the form of the second basis operator one uses those transformations from $\mathcal{E}$ which leave invariant the form of the first basis operator.

Analogously, the realizations of the two-dimensional Lie algebras which one obtains, are extended to realizations of three-dimensional solvable Lie algebras, and then the realizations of the three-dimensional Lie algebras are extended in the same way to realizations of the four-dimensional algebras, and so on.

In the extension of the realizations of Lie algebras of lower dimension to realizations of decomposable solvable Lie algebras is done simply by adding to each realization a basis operator which commutes with all the other basis elements.

For the construction of the realizations of non-decomposable solvable Lie algebras, as is shown by an analysis of their structure above, one may also carry out the extension of the realizations of Lie algebras of lower dimension to realizations of non-decomposable solvable Lie algebras of higher dimension.

II.3 Semi-direct sums of semi-simple and solvable Lie algebras.

Lie algebras which are semi-direct sums of semi-simple and solvable Lie algebras can be divided into two classes:

1) those Lie algebras which are direct sums of semi-simple and solvable Lie algebras (decomposable algebras);

2) algebras which cannot be written as a direct sum of semi-simple and solvable Lie algebras (indecomposable algebras).

Since decomposable algebras have the structure

$$L = S \oplus N$$

where S is the Levi factor and N is the radical (maximal solvable ideal of L), then a complete description of these algebras is easily obtained by combining the semi-simple and solvable Lie algebras. However, since the classification of solvable Lie algebras has only been done partially, there is a corresponding incompleteness in the classification of decomposable Lie algebras. The classification of indecomposable Lie algebras has been done only as far as for Lie algebras of dimension eight ([25]). These are Lie algebras whose Levi factor is $sl(2,\mathbf{R})$ or $so(3)$.

We give a complete list of these algebras in Appendix 2. We use the following notation:

$$sl(2,\mathbf{R}) = \langle e_1, e_2, e_3\rangle; \quad [e_1,e_2] = 2e_2,\ [e_1,e_3] = -2e_3,\ [e_2,e_3] = e_1$$
$$so(3) = \langle e_1, e_2, e_3\rangle; \quad [e_1,e_2] = e_3,\ [e_1,e_3] = -e_2,\ [e_2,e_3] = e_1.$$

We note that the basis of $sl(2,\mathbf{R})$ given here differs from that given previously, but it is not difficult to see that they are isomorphic. Indeed, if we make the transformations

$$e_1 \to 2e_2,\ \ e_2 \to e_3,\ \ e_3 \to e_1$$

then we have an isomorphism from the basis given here to the basis given previously.

In denoting the radicals $N = \langle e_4, \ldots, e_m \rangle$ ($m = \dim N - 3$), we keep to the notation used for the classification of solvable algebras given above. Moreover, the corresponding commutation relations for the basis operators of N can easily be obtained from those given by replacing the index i by the index $i+3$. Thus, in listing the algebras which are semi-direct sums of semi-simple and solvable Lie algebras, we give only those commutators $[e_i, e_j] = c_{ij}^k e_k$, $i = 1, 2, 3; j, k = 4, \ldots, m$ which are non-zero.

Taking into account the above classification of finite-dimensional real Lie algebras, we take a closer look at the second step of the algorithm for the group classification of differential equations.

After having completed the first step of the algorithm, we have the general form of the infinitesimal symmetry operator (together with a defining system) for the given equation, and we have group $\mathcal{E}$ of equivalence transformations of this equation.

At the beginning of the second step we have to bring the symmetry operator to the simplest form, using transformations from $\mathcal{E}$. We note that it is well-known ([4]) that linearization of the vector field is not possible since the group $\mathcal{E}$ is a subgroup of all the local transformations of the manifold V of dependent and independent variables which enter into the differential equation. Thus, we will obtain, up to equivalence, some finite set of simplest forms for the symmetry operator.

Further, using the determining equations, we find from the equation at hand, equations which admit the operators we have obtained as symmetry operators. With this, we will obtain the group classification of the differential equations of the given form which admit one-dimensional Lie algebras of invariance.

The list of simplest forms of the symmetry operator which we find, allows us to take one of the symmetry operators in its simplest form, when we consider the realizations of Lie algebras of higher dimension. Moreover, we may first consider realizations of semi-simple and solvable Lie algebras.

When we consider realizations of semi-simple Lie algebras, we must, as well as looking at those semi-simple Lie algebras which appear in our list, also take into account semi-simple Lie algebras which are direct sums of semi-simple Lie algebras which do not appear in the list given above.

We take a closer look at low-dimensional semi-simple Lie algebras. As we noted above, the semi-simple Lie algebras of lowest dimension are the algebras $sl(2, \mathbf{R})$ and $so(3)$, both having dimension 3. Then we have Lie algebras of dimension 6 (the algebras $so(4) \cong so(3) \oplus so(3), so(3,1), so(2,2) \cong sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R}), so^*(4) \cong so(3) \oplus sl(2,\mathbf{R})$); dimension 8 (the algebras $su(3), sl(3,\mathbf{R}), su(2,1)$).These algebras can be found in our list. However, the semi-simple Lie algebras of dimension 9 (the algebras $so(4) \oplus so(3), so(4) \oplus sl(2,\mathbf{R}), so(3,1) \oplus so(3), so(3,1) \oplus sl(3,\mathbf{R})$, $so(2,2) \oplus so(3), so(2,2) \oplus sl(2,\mathbf{R})$) are not to be found in our list.

For solvable Lie algebras, one may extend realizations of lower dimensional algebras to realizations of higher-dimensional algebras according to the scheme given above. Moreover, for solvable Lie algebras of higher dimension, the composition series play an important role: if there is a realization in terms of operators of a given class of solvable Lie algebras of dimension $m-1$, but no realization of a solvable Lie algebra of dimension m, the there will, *a priori*, be no realizations of solvable Lie algebras of dimension greater than m.

When looking at the realization of Lie algebras which are semi-direct sums of semi-simple and solvable Lie algebras, one must extend a given realization of semi-simple Lie algebras by operators which will be basis operators of the corresponding radicals. Moreover, it is only necessary to take into account those radicals which are isomorphic to solvable Lie algebras which have realizations in the given class of vector fields.

Finally, we note that since the classification of solvable Lie algebras, and those algebras which are semi-direct sums of semi-simple and solvable Lie algebras, is incomplete, then it is not possible to give a complete group classification of differential equations within the present framework. However, the problem of the complete group classification of, for instance, scalar equations in two-dimensional space-time which are invariant under finite-dimensional Lie algebras, is constructive in this approach.

III Classification results

First of all let us mention some papers in which group classification of particular equations of the form (0.1) has been carried out.

Ovsjannikov (1959) $F = F(u), \quad G = \frac{dF}{du}u_x^2$ [26];

Akhatov et al (1987) $F = F(u_x), \quad G = 0$ [8];

Dorodnitsyn (1982) $F = F(u), \quad G = \frac{dF}{du}u_x^2 + g(u)$ [27];

Oron & Rosenau (1986),

Edwards (1994) $F = F(u), \quad G = \frac{dF}{du}u_x^2 + f(u)u_x$ [28, 29];

Gandarias (1996) $F = u^n, \quad G = \frac{dF}{du}u_x^2 + g(x)u^m u_x + f(x)u^s$ [31];

Cherniha & Serov (1998) $F = F(u), \quad G = \frac{dF}{du}u_x^2 + f(u)u_x + g(u)$ [30];

Zhdanov & Lahno (1999) $F = 1, \quad G = G(t, x, u, u_x)$ [15].

We shall apply the algorithm described above in order to perform an exhaustive group classification of invariant equations of the general form (0.1). That is, we shall describe all inequivalent forms of functions F, G such that the corresponding equation admits a non-trivial symmetry group.

III.1 Computation of the equivalence group admitted by equation (0.1)

The first step of the algorithm is the determination of the most general form of the infinitesimal symmetry operator admitted by the PDE (0.1). To this end, we use Lie's method [2]–[5] and look for a symmetry generator in the form

$$Q = \tau(t, x, u)\,\partial_t + \xi(t, x, u)\,\partial_x + \eta(t, x, u)\,\partial_u, \tag{3.1}$$

where τ, ξ, η are arbitrary, real-valued smooth functions defined in some subspace of the space $V = X \otimes R^1$ of the independent variables $X = \langle t, x\rangle$ and the dependent variable $R^1 = \langle u\rangle$.

As a result, we find that the operator (3.1) generates a one-parameter symmetry group of equation (0.1) iff

$$\varphi^t - [\tau F_t + \xi F_x + \eta F_u + \varphi^x F_{u_x}]u_{xx} - \varphi^{xx}F - \tau G_t - \xi G_x - \eta G_u - \varphi^x G_{u_x}\Big|_{u_t = Fu_{xx}+G} = 0, \tag{3.2}$$

where

$$\begin{aligned}
\varphi^t &= D_t(\eta) - u_t D_t(\tau) - u_x D_t(\xi),\\
\varphi^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\
\varphi^{xx} &= D_x(\varphi^x) - u_{tx} D_x(\tau) - u_{xx} D_x(\xi)
\end{aligned}$$

and D_t, D_x are operators of total differentiation in t and x respectively.

After simplifying (3.2) we arrive at the following assertion.

Lemma 3.1 *The symmetry group of the nonlinear heat equation PDE (0.1) is generated by the infinitesimal operators of the form*

$$Q = a(t)\,\partial_t + b(t,x,u)\,\partial_x + c(t,x,u)\,\partial_u, \tag{3.3}$$

where $a,\ b,\ c$ are real-valued functions that satisfy the system of PDEs

$$\begin{aligned}&(2b_x + 2u_x b_u - \dot a)F = aF_t + bF_x + cF_u + (c_x + u_x c_u - u_x b_x - u_x^2 b_u)F_{u_x},\\ &c_t - u_x b_t + (c_u - \dot a - u_x b_u)G + (u_x b_{xx} - c_{xx} - 2u_x c_{ux} - u_x^2 c_{uu} +\\ &+2u_x^2 b_{xu} + u_x^3 b_{uu})F = aG_t + bG_x + cG_u + (c_x + u_x c_u - u_x b_x - u_x^2 b_u)G_{u_x}.\end{aligned} \tag{3.4}$$

In the rest of this paper we use the notation $\dot a = \dfrac{da}{dt},\ \ \ddot a = \dfrac{d^2a}{dt^2}$.

In order to construct the equivalence group $\mathcal{E}$ of the class of PDEs (0.1) one has to select from the set of invertible changes of variables of the space V

$$\bar t = \alpha(t,x,u),\quad \bar x = \beta(t,x,u),\quad v = \gamma(t,x,u),\quad \frac{D(\alpha,\beta,\gamma)}{D(t,x,u)} \neq 0, \tag{3.5}$$

those changes of variable which do not alter the form of the class of PDEs (0.1).

Lemma 3.2 *The maximal equivalence group $\mathcal{E}$ of the class of PDEs (0.1) reads as*

$$\bar t = T(t),\quad \bar x = X(t,x,u),\quad v = U(t,x,u), \tag{3.6}$$

where $\dot T \neq 0,\ \ \dfrac{D(X,U)}{D(x,u)} \neq 0$.

Proof. Let (3.5) be an invertible change of variables that transforms equation (0.1) into another equation of the same form (0.1), namely,

$$v_{\bar t} = \tilde F(\bar t, \bar x, v, v_{\bar x})v_{\bar x\bar x} + \tilde G(\bar t, \bar x, \bar v, \bar v_x). \tag{3.7}$$

Computing u_x according to (3.5) we get

$$u_x = \frac{v_{\bar t}\alpha_x + v_{\bar x}\beta_x - \gamma_x}{\gamma_u - v_{\bar t}\alpha_u - v_{\bar x}\beta_u}. \tag{3.8}$$

As the functions $F,\ G$ in (0.1) and $\tilde F,\ \tilde G$ in (3.7) are arbitrary functions of the corresponding arguments, we must have that

$$u_x \to g(\bar t, \bar x, v, v_{\bar x})$$

for some function g. This implies that $\alpha_x = \alpha_u = 0$ in (3.8). Consequently, $\alpha = T(t),\ \dot T \neq 0$.

Next, making the change of variables (3.5), where $\alpha = T(t)$, we arrive at the relations

$$\begin{aligned}&u_t \to v_{\bar t}\dot T(\gamma_u - v_{\bar x}\beta_u)^{-1} + \theta_1(\bar t, \bar x, v, v_{\bar x}),\\ &u_{xx} \to v_{\bar x\bar x}\theta_2(\bar t, \bar x, v, v_{\bar x}) + \theta_3(\bar t, \bar x, v, v_{\bar x}),\end{aligned} \tag{3.9}$$

where $\theta_1,\ \theta_2 \neq 0,\ \theta_3$ are some functions of $\alpha,\ \beta,\ \gamma$ and of their derivatives. Then, inserting $u_t,\ u_{xx}$ from (3.9) into (0.1), we arrive at a PDE of the form (3.7). The lemma is proved.

III.2 Classification of equations (0.1) invariant under semi-simple Lie algebras

Now we proceed to solving the classification problem for the nonlinear heat-conductivity equation (0.1). Our first step is to construct realizations of finite-dimensional real Lie algebras whose representation space is spanned by operators of the form (3.3). It should be noted that the realizations are constructed up to equivalence as determined by the transformations (3.6). In the second step, we choose those realizations which are invariance algebras of PDE (0.1) and thus specify the form of the functions F, G. Finally, in the third step, we find the maximal symmetry groups of the equations we obtain, and thus complete the group classification of PDE (0.1).

By the Levi-Mal'cev theorem, we need only consider the cases of semi-simple, solvable and semi-direct sums of semi-simple and solvable symmetry algebras. This will yield an exhaustive description of invariant equations of the form (0.1). In this subsection, we analyze the case of semi-simple symmetry algebras. As semi-simple Lie algebras can always be decomposed into a direct sum of simple Lie algebras (which is the content of Cartan's theorem), we begin with the lowest dimensional simple Lie algebras $sl(2,\mathbf{R})$ and $so(3)$.

We begin by proving the following useful lemma:

Lemma 3.3 *There are changes of variables (3.6), that reduce an operator (3.3) to one of the operators below:*

$$Q = \partial_t, \tag{3.10}$$
$$Q = \partial_x. \tag{3.11}$$

Proof. Making the change of variables (3.6) transforms operator (3.3) to the following one:

$$Q \to Q' = a\dot{T}\,\partial_{\bar{t}} + (aX_t + bX_x + cX_u)\,\partial_{\bar{x}} + (aU_t + bU_x + cU_u)\,\partial_v. \tag{3.12}$$

Suppose $a \neq 0$. Then, choosing in (3.6) the function T to be a solution of the equation $\dot{T} = a^{-1}$ and the functions X and U to be independent fundamental solutions of the first-order PDE

$$aY_t + bY_x + cY_u = 0, \quad Y = Y(t,x,u),$$

we find that the operator (3.12) takes the form $Q' = \partial_{\bar{t}}$.

Now suppose $a = 0$. Then $b^2 + c^2 \neq 0$. If $b \neq 0$, then choosing in (3.6) a particular solution of PDE $bX_x + cX_u = 1$ as the function X and a fundamental solution of PDE $bU_x + cU_u = 0$ as the function U, we transform (3.12) to become $Q' = \partial_{\bar{x}}$.

If $b = 0$, $c \neq 0$, then making the change of variables (3.6) with $\bar{t} = t,\ \bar{x} = u,\ v = x$, we again get the case $b \neq 0$.

By the direct calculation we can verify that there is no transformation from $\mathcal{E}$, that reduce operator (3.10) to the form (3.11).

The lemma is proved.

Theorem 3.1 *Within the equivalence relation $\mathcal{E}$, there exists only one realization of the algebra $so(3)$ by operators of the form (3.3):*

$$\langle \partial_x,\ \tan u\,\sin x\,\partial_x + \cos x\,\partial_u,\ \tan u\,\cos x\,\partial_x - \sin x\,\partial_u \rangle, \tag{3.13}$$

It is the invariance algebra of an equation from the class (0.1). Furthermore, the most general form of the functions F, G allowing for PDE (0.1) to be invariant under the above realization is given by

$$F = \frac{\sec^2 u}{1+\omega^2}, \quad G = \frac{2\omega^2+1}{1+\omega^2}\tan u + \sqrt{1+\omega^2}\,\tilde{G}(t), \quad \omega = u_x \sec u. \tag{3.14}$$

Provided the function $\tilde{G}$ is arbitrary, the realization (3.13) is the maximal symmetry algebra of the corresponding equation (0.1).

Proof. The Lie algebra $so(3) = \langle Q_1, Q_2, Q_3\rangle$ is defined by the following commutation relations:

$$[Q_1, Q_2] = Q_3, \quad [Q_1, Q_3] = -Q_2, \quad [Q_2, Q_3] = Q_1. \tag{3.15}$$

To describe all inequivalent realizations of the algebra $so(3)$ we take operators of the form (3.3) as the basis elements Q_i $(i = 1, 2, 3)$ of $so(3)$ and then study the restrictions imposed on their coefficients by relations (3.15). We also use transformations (3.6) in order to simplify the final forms of the basis elements.

In view of Lemma 3.3, we can take one of the basis elements of the algebra $so(3)$ (say, Q_1) either in the form ∂_t or ∂_x.

Let $Q_1 = \partial_t$. Using the first two commutation relations from (3.15) yields

$$\begin{aligned} Q_2 &= \lambda \cos t\, \partial_t + [b \cos t + \beta \sin t]\, \partial_x + [c \cos t + \gamma \sin t]\, \partial_u, \\ Q_3 &= -\lambda \sin t\, \partial_t + [-b \sin t + \beta \cos t]\, \partial_x + [-c \sin t + \gamma \cos t]\, \partial_u, \end{aligned}$$

where $\lambda = \text{const} \ \in \mathbf{R},\ b = b(x, u),\ c = c(x, u),\ \beta = \beta(x, u),\ \gamma = \gamma(x, u)$ are arbitrary smooth functions.Then, using the third commutation relation, we arrive at the equation $\lambda^2 = -1$ which has no real solutions λ. Consequently, in the case when the operator Q_1 is equivalent to the operator ∂_t, there are no realizations of the algebra $so(3)$.

Turn now to the case $Q_1 = \partial_x$. As a straightforward calculation shows, the most extensive subgroup of the equivalence group $\mathcal{E}$ not altering the form of Q_1 is of the form

$$\bar{t} = T(t),\ \bar{x} = x + X(t, u),\ v = U(t, u),\ \dot{T} \neq 0,\ U_u \neq 0. \tag{3.16}$$

Using the first two commutation relations from (3.15) we get

$$\begin{aligned} Q_2 &= \alpha \cos(x + \gamma)\, \partial_x + \beta \cos(x + \theta)\, \partial_u, \\ Q_3 &= -\alpha \sin(x + \gamma)\, \partial_x - \beta \sin(x + \theta)\, \partial_u, \end{aligned} \tag{3.17}$$

where $\alpha = \alpha(t, u),\ \gamma = \gamma(t, u),\ \beta = \beta(t, u),\ \theta = \theta(t, u)$ are arbitrary smooth functions. Now, either $\beta = 0$ or $\beta \neq 0$. If $\beta = 0$, the third commutation relation givews $\alpha^2 = -1$, which has no real solutions. Consequently, $\beta \neq 0$.

Choosing in (3.16) $X = \theta$ and furthermore, taking an arbitrary solution of the equation $U_u = \beta^{-1}$, as U, we simplify the forms of the operators $Q_2,\ Q_3$ to obtain

$$\begin{aligned} Q_2 &= \alpha \cos(x + \gamma)\, \partial_x + \cos x\, \partial_u, \\ Q_3 &= -\alpha \sin(x + \gamma)\, \partial_x - \sin x\, \partial_u, \end{aligned}$$

where $\alpha = \alpha(t, u),\ \gamma = \gamma(t, u)$ are arbitrary smooth functions (here and in the following, we keep the initial designations for the transformed operators to simplify the notation).

The third commutation relation for the operators $Q_1,\ Q_2$ which we have obtained, yields the equations $\cos\gamma = 0,\ \alpha^2 + \alpha_u \sin\gamma = -1$, whence

$$\begin{aligned} Q_2 &= \tan[u \pm \tilde{\alpha}(t)] \sin x\, \partial_x + \cos x\, \partial_u, \\ Q_3 &= \tan[u \pm \tilde{\alpha}(t)] \cos x\, \partial_x - \sin x\, \partial_u, \end{aligned}$$

where $\tilde{\alpha}(t)$ is an arbitrary smooth function.

Finally, putting $T = t,\ X = 0,\ U = u \pm \tilde{\alpha}(t)$ in (3.16) we find that the above realization is equivalent to (3.13).

To complete the proof we have to verify whether there exists an equation of the form (0.1), whose symmetry algebra contains subalgebra (3.13). Invariance of (0.1) with respect to the one-parameter group having the generator Q_1 means that $F = F(t, u, u_x)$, $G = G(t, u, u_x)$. Writing down condition (3.4) for the operators Q_2, Q_3 we get the following system of PDEs:

$$F_u - u_x \tan u F_{u_x} = 2 \tan u F,$$
$$(1 + u_x^2 \sec^2 u) F_{u_x} = -2u_x \sec^2 u F,$$
$$u_x \sec^2 u G + u_x \tan u (1 - 2u_x^2 \sec^2 u) F = (1 + u_x^2 \sec^2 u) G_{u_x},$$
$$(1 + 2u_x^2 \sec^2 u) F = G_u - u_x \tan u G_{u_x}.$$

Solving the first two equations of the above system gives

$$F = \frac{\sec^2 u}{1 + \omega^2} \tilde{F}(t),$$

where $\omega = u_x \sec u$. Integrating the fourth equation yields $G = \dfrac{2\omega^2 + 1}{1 + \omega^2} \tan u \tilde{F}(t) + \bar{G}(t, \omega)$. Finally, solving the third equation we get the form of $\bar{G}(t, \omega)$

$$\bar{G}(t, \omega) = \sqrt{1 + \omega^2} \tilde{G}(t).$$

Thus PDE (0.1) is invariant with respect to the algebra (3.13) iff

$$F = \frac{\sec^2 u}{1 + \omega^2} \tilde{F}(t), \;\; G = \frac{2\omega^2 + 1}{1 + \omega^2} \tan u \tilde{F}(t) + \sqrt{1 + \omega^2} \tilde{G}(t), \;\; \omega = u_x \sec u. \tag{3.18}$$

with arbitrary smooth functions $\tilde{F}(t)$, $\tilde{G}(t)$ provided that $\tilde{F}(t) \neq 0$.

Evidently, the change of variables (3.16) with $X = 0$, $U = u$ does not alter the forms of the operators of the realization (3.13). Choosing a solution of the equation $\dot{T} = \tilde{F}$ as T, we get $\tilde{F}(t) = 1$. By direct computation one shows that if the function $\tilde{G}(t)$ is arbitrary, then the algebra (3.13) is the maximal invariance algebra admitted by the equation obtained. The theorem is proved.

Theorem 3.2 *There exist five inequivalent realizations of the algebra $sl(2, \mathbf{R})$ by operators (3.3), which are admitted by PDEs of the form (0.1)*

$$\langle 2t\,\partial_t + x\,\partial_x, \;\; -t^2\,\partial_t - tx\,\partial_x + x^2\,\partial_u, \;\; \partial_t \rangle, \tag{3.19}$$
$$\langle 2t\,\partial_t + x\,\partial_x, \;\; -t^2\,\partial_t + x(x^2 - t)\,\partial_x, \;\; \partial_t \rangle, \tag{3.20}$$
$$\langle 2x\,\partial_x - u\,\partial_u, -x^2\,\partial_x + xu\,\partial_u, \;\; \partial_x \rangle, \tag{3.21}$$
$$\langle 2x\,\partial_x - u\,\partial_u, \;\; (u^{-4} - x^2)\,\partial_x + xu\,\partial_u, \;\; \partial_x \rangle, \tag{3.22}$$
$$\langle 2x\,\partial_x - u\,\partial_u, \;\; -(u^{-4} + x^2)\,\partial_x + xu\,\partial_u, \;\; \partial_x \rangle. \tag{3.23}$$

The forms of the functions F, G determining the corresponding invariant equations are given as follows:

$sl(2,\mathbf{R})$	F	G
(3.19)	$\tilde{F}(\omega)$	$x^{-2}\left[\tilde{G}(\omega) - 2u\tilde{F}(\omega) + u^2 - u\omega\right], \ \omega = 2u - xu_x$
(3.20)	ω^{-3}	$x^{-2}\left[-\frac{1}{4}\omega + 3\omega^{-2} + \omega^{-1}\tilde{G}(u)\right], \ \omega = xu_x$
(3.21)	u^{-4}	$-2u^{-5}u_x^2$
(3.22)	$u^{-4}\left(1+4\omega^2\right)^{-1}$	$u\left[\sqrt{1+4\omega^2}\tilde{G}(t) - \frac{10\omega^2+1}{8\omega^2+2}\right], \ \omega = u^{-3}u_x$
(3.23)	$u^{-4}\left(1-4\omega^2\right)^{-1}$	$u\left[\sqrt{\lvert 1-4\omega^2\rvert}\tilde{G}(t) + \frac{10\omega^2-1}{8\omega^2-2}\right], \ \omega = u^{-3}u_x$

If the functions $\tilde{F}$, $\tilde{G}$ are arbitrary, then the corresponding realizations of the algebra $sl(2,\mathbf{R})$ are maximal invariance algebras of the respective PDEs (0.1). Furthermore, the maximal symmetry group admitted by the third PDE from the above list

$$u_t = u^{-4}u_{xx} - 2u^{-5}u_x^2$$

is the five-dimensional Lie algebra $sl(2,\mathbf{R}) \oplus L_{2.1}$, where $sl(2,\mathbf{R})$ is given in (3.21) and $L_{2.1} = \langle 4t\,\partial_t + u\,\partial_u,\ \partial_t\rangle$.

Proof. The Lie algebra $sl(2,\mathbf{R}) = \langle Q_1,\ Q_2,\ Q_3\rangle$ is defined by the following commutation relations:

$$[Q_1,\ Q_2] = 2Q_2, \quad [Q_1,\ Q_3] = -2Q_3, \quad [Q_2,\ Q_3] = Q_1. \tag{3.24}$$

In view of Lemma 3.3 we can choose the operator Q_3 either in the form ∂_t or ∂_x.

Let $Q_3 = \partial_t$. Imposing the second commutation relation from (3.24) gives (up to equivalence under $\mathcal{E}$) the operator Q_1 either equals to $2t\,\partial_t$ or $2t\,\partial_t + x\,\partial_x$.

If $Q_1 = 2t\,\partial_t$, then it follows from the remaining commutation relation that $Q_2 = -t^2\,\partial_t$, so that we obtain the realization $\langle 2t\,\partial_t,\ -t^2\,\partial_t,\ \partial_t\rangle$. However, PDE (0.1) can admit this algebra only when the condition $F = 0$ holds. This contradicts the assumption $F \neq 0$ and, consequently, there are no corresponding invariant equations within the class (0.1).

If now $Q_1 = 2t\,\partial_t + x\,\partial_x$, we get (up to equivalence under $\mathcal{E}$) the realization $\langle 2t\,\partial_t + x\,\partial_x,\ -t^2\,\partial_t - tx\,\partial_x,\ \partial_t\rangle$ and the realizations (3.19), (3.20) of the algebra $sl(2,\mathbf{R})$. Substituting into the invariance conditions (3.4) shows that the first realization cannot be admitted by PDE (0.1). This leaves us with the realizations (3.19), (3.20).

The most general equations (0.1) invariant with respect to realizations (3.19), (3.20) are given by:

$$F = \tilde{F}(\omega), \quad G = x^{-2}\left[\tilde{G}(\omega) - 2u\tilde{F}(\omega) + u^2 - u\omega\right], \quad \omega = 2u - xu_x,$$
$$F = \omega^{-3}\tilde{F}(u), \quad G = x^{-2}\left[-\frac{1}{4}\omega + 3\omega^{-2}\tilde{F}(u) + \omega^{-1}\tilde{G}(u)\right], \quad \omega = xu_x.$$

It is not difficult to show that the change of variables

$$t = t, \quad x = x, \quad u = U(v), \quad U' \neq 0, \quad v = v(t,x)$$

does not alter the form of the basis operators of the realization (3.20). So, choosing the function U to be a solution of the equation $(U')^3 = \tilde{F}(U)$, we can transform PDE (0.1) invariant under (3.20) in such a way that $\tilde{F} \equiv 1$.

We now turn to the case $Q_3 = \partial_x$. Using the commutation relations (3.24) we find that the inequivalent realizations of $sl(2, \mathbf{R})$ within the class of operators (3.3) are exhausted by the realization $\langle 2x\,\partial_x,\ -x^2\,\partial_x,\ \partial_x\rangle$ and by the realizations (3.21), (3.22), (3.23). The invariance conditions (3.4) show that the first realization cannot be an invariance algebra of PDE of the form (0.1). The remaining realizations are invariance algebras of PDEs (0.1) under proper specification of the functions F, G :

$$F = u^{-4}\tilde{F}(t),\quad G = -2u^{-5}u_x^2\tilde{F}(t) + u\tilde{G}(t),\ (\text{ for the realization (3.21)}); \qquad (3.25)$$

$$F = \frac{1}{u^4\,(1+4\omega^2)}\tilde{F}(t),\quad G = u\left[\sqrt{1+4\omega^2}\tilde{G}(t) - \frac{10\omega^2+1}{8\omega^2+2}\tilde{F}(t)\right],$$
$$\omega = u^{-3}u_x,\ (\text{ for the realization (3.22)}; \qquad (3.26)$$

$$F = \frac{1}{u^4\,(1-4\omega^2)}\tilde{F}(t),\quad G = u\left[\sqrt{|1-4\omega^2|}\tilde{G}(t) + \frac{10\omega^2-1}{8\omega^2-2}\tilde{F}(t)\right],$$
$$\omega = u^{-3}u_x,\ (\text{ for the realization (3.23)}). \qquad (3.27)$$

As the change of variables

$$\bar{t} = T,\quad \bar{x} = x,\quad v = U(t)u,\quad T \neq 0,\quad U \neq 0$$

does not alter the form of the basis operators of the realization (3.21), we can use it in order to simplify the forms of F, G. Choosing the functions T and U to be solutions of the equations $\dot{U} = U\tilde{G}(t)$, $U \neq 0$, $\dot{T} = \tilde{F}U^4$, we obtain $\tilde{F} \equiv 1$, $\tilde{G} \equiv 0$ in (3.25).

Similarly, using the change of variables

$$\bar{t} = T(t),\quad \bar{x} = x,\quad v = u$$

which preserve the form of the basis operators of the realizations (3.22), (3.23) we can choose $\tilde{F} \equiv 1$ in (3.26), (3.27).

Computing the maximal invariance algebra of the PDE which admits the realization (3.21) we get the five-dimensional Lie algebra which is the direct sum of $sl(2, \mathbf{R})$ having the basis elements (3.21) and the two-dimensional solvable Lie algebra $L_{2.1} = \langle 4t\,\partial_t + u\,\partial_u,\ \partial_t\rangle$.

The remaining invariant equations contain an arbitrary function. If there are no additional constraints on this function, then the realizations (3.19), (3.20), (3.22), (3.23) of $sl(2, \mathbf{R})$ are easily shown to be the maximal invariance algebras of the corresponding invariant equations.

The theorem is proved.

Theorem 3.3 *The realizations of the algebras $so(3)$ and $sl(2, \mathbf{R})$, given in Theorems 3.1, 3.2, exhaust the set of all possible realizations of semi-simple Lie algebras by operators (3.3) which are admitted by PDEs of the form (0.1).*

Proof. The simple Lie algebras of the lowest dimension admit the following isomorphisms:

$$su(2) \sim so(3) \sim sp(1),\ sl(2, \mathbf{R}) \sim su(1,1) \sim so(2,1) \sim sp(1, \mathbf{R}).$$

¿From this it follows that the realizations given in Theorems 3.1, 3.2 exhaust the set of all possible realizations of three-dimensional simple Lie algebras which are symmetry algebras of (0.1).

The next admissible dimension for simple Lie algebras is six. There are four distinct six-dimensional simple Lie algebras over the field of real numbers, namely, $so(4)$, $so(3,1)$, $so(2,2)$, and $so^*(4)$.

As $so(4) = so(3) \oplus so(3)$, we have $so(4) = \langle Q_i, K_i | \ i = 1,2,3\rangle$, where $\langle Q_1, Q_2, Q_3\rangle = so(3)$, $\langle K_1, K_2, K_3\rangle = so(3)$, and we have the commutation relations, $[Q_i, K_j] = 0$, $i,j = 1,2,3$. Making use of Theorem 3.1 we put the basis operators Q_i $(i = 1,2,3)$ to be equal to the corresponding basis operators of realization (3.13). Next, the commutation relations $[Q_i, K_j] = 0$, $(i,j = 1,2,3)$ imply the following form of the operators K_j:

$$K_j = a_j(t)\,\partial_t, \quad a_j \neq 0, \quad j = 1,2,3. \tag{3.28}$$

Using the change of variables

$$\bar{t} = T(t), \quad \bar{x} = x, \quad v = u,$$

(which does not change the form of operators (3.13)), we can transform the operator K_1 to become $K_1 = \partial_t$. Checking the commutation relations for the algebra $so(3)$ yields that

$$K_2 = \lambda \cos(t + \lambda_1)\,\partial_t, \quad K_3 = -\lambda \sin(t + \lambda_1),$$

where $\{\lambda_1, \lambda\} \subset \mathbf{R}$ with $\lambda^2 = -1$. Consequently, there are no realizations of the algebra $so(4)$ within the class of operators (3.3), which are symmetry algebras of (0.1).

We have the relation $so^*(4) \sim so(3) \oplus sl(2,\mathbf{R})$. So, in order to construct realizations of $so^*(4)$ we have to describe realizations of the algebra $sl(2,\mathbf{R})$ by operators of the form (3.28). Now, in proving Theorem 3.2, we established, in particular, that there is a unique realization of $sl(2,\mathbf{R})$ by operators (3.28), $\langle 2t\,\partial_t, \ -t^2\,\partial_t, \ \partial_t\rangle$, which, however, cannot be admitted by a PDE of the form (0.1). This eliminates $so^*(4)$.

The algebra $so(3,1)$ admits the Cartan decomposition $\langle Q_1, Q_2, Q_3\rangle \dotplus \langle N_1, N_2, N_3\rangle$, where $\langle Q_1, Q_2, Q_3\rangle = so(3)$, $[Q_i, \ N_j] = N_k$, $[N_i, \ N_j] = -Q_k$, $i,j,k = \mathrm{cycle}(1,2,3)$. Thus, taking as Q_i $(i = 1,2,3)$ the corresponding basis operators of realization (3.13) and computing the forms of the operators N_1, N_2, N_3, we get within the equivalence relation $\mathcal{E}$ the following relations:

$$N_1 = \cos u\,\partial_u, \quad N_2 = -\sec u \ \cos x\,\partial_x + \sin u \ \sin x\,\partial_u, \quad N_3 = \sec u \ \sin x\,\partial_x + \sin u \ \cos x\,\partial_u.$$

Imposing the invariance conditions (3.4) for the operator N_1 gives that $F = 0$, contradicting our initial assumption $F \neq 0$.

In studying realizations of the algebra $so(2,2)$ we use the fact that $so(2,2) \sim sl(2,\mathbf{R}) \oplus sl(2,\mathbf{R})$. In view of this, we can choose the basis operators of this algebra so that $so(2,2) = \langle Q_i, K_i \mid i = 1,2,3\rangle$, where $\langle Q_1, Q_2, Q_3\rangle = sl(2,\mathbf{R})$, $\langle K_1, K_2, K_3\rangle = sl(2,\mathbf{R})$, and , $[Q_i \ K_j] = 0$, $i,j = 1,2,3$. Now we can take as Q_1, Q_2, Q_3 the corresponding basis operators of the realizations of $sl(2,\mathbf{R})$ given by (3.19)–(3.23). However, as further analysis shows, these realizations cannot be extended to a realization of $so(2,2)$ which could be a symmetry algebra of PDE of the form (0.1).

Thus there no are realizations of six-dimensional simple Lie algebras by operators (3.3), which are symmetry algebras of (0.1).

The same assertion holds true for the simple Lie algebras of dimension eight ($sl(3,\mathbf{R})$, $su(3)$, $su(2,1)$), which is the next admissible dimension for real simple Lie algebras.

As $su^*(4) \sim so(5,1)$ and since the algebra $so(5,1)$ contains $so(4)$, we conclude that the algebras A_{n-1} $(n > 1)$ have no realizations by operators of the form (3.3), which generate symmetry algebras of (0.1), except for those given in Theorems 3.1 and 3.2.

There are also no realizations of the desired form for simple Lie algebras of the type D_n ($n > 1$), since the lowest dimensional algebras of this type ($so(4)$, $so(2,2)$, $so^*(4)$) have no realizations within the class (3.3) which could be symmetry algebras of (0.1).

By the same reasoning, we conclude that the realizations (3.13), (3.19)–(3.23) exhaust the set of all possible realizations of the simple Lie algebras B_n ($n > 1$) and C_n ($n \geq 1$). Indeed, taking the least possible value of n and putting $n = 2$ we see that the algebras of the type B_n contain subalgebras that are isomorphic to $so(4)$, $so(1,3)$. The same assertion for the simple Lie algebras of the type C_n ($n \geq 1$) follows from the relations:

$$sp(2,\mathbf{R}) \sim so(3,2), \quad sp(1,1) \sim so(4,1), \quad sp(2) \sim so(5),$$

if we take into account that the algebras $so(3,2)$, $so(4,1)$, contain $so(3,1)$, and that the algebra $so(5)$ contains $so(4)$.

To complete the proof we have to consider the exceptional simple Lie algebras G_2, F_4, E_6, E_7, E_8.

We consider in detail the first two algebras, the remaining algebras being treated in the same way.

A Lie algebra of the type G_2 contains a compact real form g_2 and a non-compact real form g_2'. We also have $g_2 \cap g_2' \sim su(2) \oplus su(2) \sim so(4)$, from which we conclude that G_2 has no realizations by operators of the form (3.3) which are symmetry operators of (0.1).

A Lie algebra of the type F_4 contains a compact real form f_4 and two non-compact real forms f_4', f_4''. We also have $f_4' \cap f_4 \sim sp(3) \oplus su(2)$, $f_4'' \cap f_4 \sim so(9)$. Hence, it follows that the algebra F_4 has no realizations within the class of operators of the form (3.3) which are admitted by PDE of the form (0.1).

The theorem is proved.

III.3 Equations invariant under semi-direct sums of simple and solvable Lie algebras

In order to describe equations of the form (0.1) which are invariant with respect to the Lie algebras that are semi-direct sums of simple and solvable Lie algebras, we could follow the same strategy as in the previous section. However, with Theorems 3.1–3.3 in hand, the most effective way is a direct application of the Lie infinitesimal method in order to specify the arbitrary functions of one variable, given in Theorems 3.1, 3.2, with the aim of obtaining all the possible extensions of the algebras $so(3)$, $sl(2,\mathbf{R})$ admitted by PDEs (0.1). In this way we will get all the possible equations of the form (0.1) admitting Lie algebras which are semi-direct sums of simple and solvable Lie algebras.

So we insert the corresponding forms of the functions F, G into invariance conditions (3.4) and then investigate the consistency of the system of determining equations which are obtained in this way. Substituting formulas (3.14) into the first equation of (3.4) yields the following system of PDEs:

$$\begin{aligned} &(a) \quad 2b_x - \dot{a} - 2c\ \tan u = 0,\\ &(b) \quad b_u + c_x \sec^2 u = 0,\\ &(c) \quad 2c_u - \dot{a} = 0. \end{aligned}$$

It follows from (c) that $c = \dfrac{1}{2}\dot{a}u + \tilde{c}(t,x)$. Then the compatibility requirement for equations (a) and (b) gives $\dot{a} = 0$, $\tilde{c}_{xx} + \tilde{c} = 0$, whence

$$\dot{a} = 0, \quad b = [f(t)\sin x - g(t)\cos x]\tan u + h(t), \quad c = f(t)\cos x + g(t)\sin x,$$

where $f,\ g,\ h$ are arbitrary smooth functions of t.

Next, substituting the expressions obtained for F and G into the second equation from (3.14) we see that $a\dot{\tilde{G}} = 0,\ \dot{f} = \dot{g} = \dot{h} = 0$. Hence it follows that extension of the symmetry algebra is only possible if $\tilde{G} = \lambda,\ \lambda = const$. In this case, the maximal symmetry algebra of the corresponding PDE is the four-dimensional Lie algebra $so(3) \oplus L_1$, where $so(3)$ is given in (3.13), and $L_1 = \langle \partial_t \rangle$.

We get similar results for PDEs invariant under the realizations (3.20), (3.22) and (3.23). Namely, extension of the symmetry algebra is only possible when $\tilde{G} = \lambda,\ \lambda = \mathrm{const}$. Moreover, the maximal invariance algebras are the four-dimensional Lie algebras of the form $sl(2, \mathbf{R}) \oplus L_1$, where $L_1 = \langle \partial_u \rangle$ when $sl(2, \mathbf{R})$ given by (3.20), and $L_1 = \langle \partial_t \rangle$ when $sl(2, \mathbf{R})$ is given by (3.22) or (3.23).

We turn now to the remaining case of PDE (0.1) invariant with respect to realization (3.19) of the algebra $sl(2, \mathbf{R})$. Inserting the corresponding expressions for F and G into (3.4), we find the following equations:

$$(A - 2B\omega)\tilde{F} = (C + D\omega + B\omega^2)\dot{\tilde{F}}, \tag{3.29}$$

$$\begin{aligned}(E + B\omega)\tilde{G} - (C + D\omega + B\omega^2)\dot{\tilde{G}} &= \\ = K + L\omega + (M + N\omega + P\omega^2 + S\omega^3)\tilde{F} &- 2u(C + D\omega + B\omega^2)\dot{\tilde{F}},\end{aligned} \tag{3.30}$$

where

$$\begin{aligned}
A &= 2xb_x - x\dot{a} + 4ub_u, \quad B = b_u,\\
C &= 2xc - x^2c_x - 2u(b + xc_u - xb_x) + 4u^2b_u,\\
D &= b + xc_u - xb_x - 4ub_u,\\
K &= -x^3c_t + 2x^2ub_t + x^2uc_x + xu^2(c_u - 2b_x + \dot{a}) - 2u^3b_u,\\
L &= -x^2b_t - xc + ub + ux(b_x - \dot{a}) + u^2b_u,\\
E &= 2b + x(c_u - \dot{a}) - 2ub_u,\\
M &= 2uE - 2xc + x^3c_{xx} - 2x^2u(b_{xx} - 2c_{xu}) - 4xu^2(2b_{xu} - c_{uu}) - 8u^3b_{uu},\\
N &= 2ub_u + x^2(b_{xx} - 2c_{xu}) + 4xu(2b_{xu} - c_{uu}) + 12u^2b_{uu},\\
P &= -x(2b_{xu} - c_{uu}) - 6ub_{uu}, \quad S = b_{uu},\\
\dot{\tilde{F}} &= \frac{d\tilde{F}}{d\omega}, \quad \dot{\tilde{G}} = \frac{d\tilde{G}}{d\omega}, \quad \omega = 2u - xu_x.
\end{aligned}$$

If $\tilde{F}$ is an arbitrary function of ω, then we have $A = B = C = D = 0\dot{\text{I}}$t follows that $b = \frac{1}{2}x\dot{a}$, $c = x^2\tilde{c}(t)$. Equation (3.30) now takes the form

$$K + L\omega = 0,$$

where

$$K = -x^5\dot{\tilde{c}} + x^3u(\ddot{a} + 2\tilde{c}), \quad L = -\frac{1}{2}x^3(\ddot{a} + 2\tilde{c}),$$

so that, $\dot{\tilde{c}} = 0,\ \ddot{a} + 2\tilde{c} = 0$. Thus realization (3.19) of the algebra $sl(2, \mathbf{R})$ is the maximal invariance algebra of the corresponding PDE (0.1).

Thus, extension of realization (3.19) is only possible when not all of the coefficients A, B, C, D in (3.29) vanish as a result of (3.29), (3.30). In order to classify all these cases we note that (3.29) is equivalent to the following relation:

$$(k - 2m\omega)\tilde{F} = (n + p\omega + m\omega^2)\dot{\tilde{F}}, \tag{3.31}$$

where the coefficients k, m, n, p are constant. Indeed, since $\tilde{F}$ is a function of ω only, relation (3.29) can be valid if and only if all its coefficients have the form const. $\times\, R(t, x, u)$ with some non-vanishing function R. If all the coefficients in (3.29) are equal to zero, then we get the case of an arbitrary function $\tilde{F}$. Consequently, extension of the invariance algebra is only possible when the function $\tilde{F}(\omega)$ satisfies an equation of the form (3.31), where k, m, n, p, are constants not vanishing simultaneously.

Summing up, we conclude that the problem of the group classification of PDEs (0.1) invariant under realization (3.19) of the algebra $sl(2,\mathbf{R})$, reduces to classifying all admissible forms of the function $\tilde{F}$. Solving this problem requires simple but very tedious computations and so we give only the result, omitting the intermediate calculations. The admissible forms of the functions $\tilde{F}(\omega)$ are:

$$\begin{aligned}
\tilde{F} &= 1;\\
\tilde{F} &= \lambda\omega^{\alpha};\\
\tilde{F} &= \lambda\exp\omega;\\
\tilde{F} &= \lambda(\omega^2+\alpha)^{-1};\\
\tilde{F} &= \frac{\lambda}{(\omega+\alpha)^2}\exp\left(-\frac{2\alpha}{\omega+\alpha}\right);\\
\tilde{F} &= \frac{\lambda}{(\omega+\alpha)^2+\beta^2}\exp\left(\frac{2\alpha}{\beta}\arctan\frac{\omega+\alpha}{\beta}\right);\\
\tilde{F} &= \frac{\lambda}{(\omega+\alpha)^2-\beta^2}\exp\left|\frac{\omega+\alpha-\beta}{\omega+\alpha+\beta}\right|^{\frac{\alpha}{\beta}},
\end{aligned}\tag{3.32}$$

where $\{\alpha, \lambda, \beta\} \subset \mathbf{R}$, $\alpha\lambda\beta \neq 0$.

On analyzing the above cases, we conclude that the only forms of the function $\tilde{F}$ from the list (3.32) that provide an extension of invariance algebra of the equation under study are:

$$\tilde{F} = 1, \quad \tilde{F} = \omega.$$

We finally find that there exist five nonlinear equations of the form (0.1) invariant under four-dimensional algebras and two nonlinear PDEs admitting five-dimensional Lie algebras. Below we the give these equations together with their maximal symmetry algebras $L_{\max}$.

$$u_t = \frac{\sec^2 u}{1+u_x^2\sec^2 u}u_{xx} + \frac{1+2u_x^2\sec^2 u}{1+u_x^2\sec^2 u}\tan u + \lambda\sqrt{1+u_x^2\sec^2 u},\ \lambda\in\mathbf{R},$$

$L_{\max} = so(3)\oplus L_1$, $so(3)$ has the form (3.13), $L_1 = \langle\partial_t\rangle$;

$$u_t = x^{-3}u_x^{-3}u_{xx} - \frac{1}{4}x^{-1}u_x + 3x^{-4}u_x^{-2} + \lambda x^{-3}u_x^{-1},\ \lambda\in\mathbf{R},$$

$L_{\max} = sl(2,\mathbf{R})\oplus L_1$, $sl(2,\mathbf{R})$ has the form (3.20), $L_1 = \langle\partial_u\rangle$;

$$u_t = \frac{u^2}{u^6+4u_x^2}u_{xx} - \frac{10uu_x^2+u^7}{8u_x^2+2u^6} + \lambda u^{-2}\sqrt{u^6+4u_x^2},\ \lambda\in\mathbf{R},$$

$L_{\max} = sl(2,\mathbf{R})\oplus L_1$, $sl(2,\mathbf{R})$ has the form (3.22), $L_1 = \langle\partial_t\rangle$;

$$u_t = \frac{u^2}{u^6-4u_x^2}u_{xx} + \frac{10uu_x^2-u^7}{8u_x^2-2u^6} + \lambda u^{-2}\sqrt{|u^6-4u_x^2|},\ \lambda\in\mathbf{R},$$

$L_{\max} = sl(2,\mathbf{R})\oplus L_1$, $sl(2,\mathbf{R})$ has the form (3.23), $L_1 = \langle\partial_t\rangle$;

$$u_t = \lambda[2u - xu_x]u_{xx} + [4\gamma - 4\lambda - 1]x^{-2}u^2 + [1 + 2\lambda - 4\gamma]x^{-1}uu_x + \gamma u_x^2,\ \lambda \neq 0,\ \gamma \in \mathbf{R},$$

$L_{\max} = sl(2,\mathbf{R}) \oplus L_1$, $sl(2,\mathbf{R})$ has the form (3.19), $L_1 = \langle x\,\partial_x + 2u\,\partial_u\rangle$;

$$u_t = u^{-4}u_{xx} - 2u^{-5}u_x^2,$$

$L_{\max} = sl(2,\mathbf{R}) \oplus L_{2.1}$, $sl(2,\mathbf{R})$ has the form (3.21), $L_{2.1} = \langle 4t\,\partial_t + u\,\partial_u, \partial_t\rangle$;

$$u_t = u_{xx} + x^{-1}uu_x - x^{-2}u^2 - 2x^{-2}u,$$

$L_{\max} = sl(2,\mathbf{R}) \subset\!\!\!\!+ L_{2.2}$, $sl(2,\mathbf{R})$ has the form (3.19), $L_{2.2} = \langle t\,\partial_x + [tx^{-1}(u+2) - x]\,\partial_u, \partial_x + x^{-1}(u+2)\,\partial_u\rangle$.

The above formulas provide the full solution of the problem of describing all PDEs of the type (0.1) admitting symmetry Lie algebras which are semi-direct sums of semi-simple and solvable Lie algebras.

III.4 Classification of equations (0.1) invariant with respect to solvable Lie algebras

To complete the classification of invariant PDEs of the form (0.1) we have to construct all possible inequivalent realizations of solvable Lie algebras within the class of operators (3.3) which are invariance algebras of (0.1). First, we shall perform a preliminary classification: we shall describe inequivalent PDEs (0.1) admitting one-, two- and three-dimensional solvable invariance algebras and then proceed to classifying equations invariant with respect to higher dimensional solvable Lie algebras.

III.4.1 Preliminary classification

Equations (0.1) invariant with respect to one-dimensional algebras have already been constructed, so that we can start by considering two-dimensional solvable Lie algebras. As mentioned in Section II.2 there are two inequivalent solvable Lie algebras

$$\begin{aligned} A_{2.1} &: \ [e_1, e_2] = 0; \\ A_{2.2} &: \ [e_1, e_2] = e_2. \end{aligned} \tag{3.33}$$

As each of the above algebras contains the algebra A_1, when studying realizations of two-dimensional Lie algebras we can take as one of the basis operators either $\frac{\partial}{\partial t}$ or $\frac{\partial}{\partial x}$. Consider in more detail the case of the algebra $A_{2.1}$.

Let $e_1 = \partial_t$ and e_2 be an operator of the form (3.3). Then it follows from (3.33) that within a choice of a basis of the algebra $A_{2.1}$ we can put

$$e_2 = b(x,u)\partial_x + c(x,u)\partial_u. \tag{3.34}$$

Since operator (3.34) can be treated as the non-zero vector field acting on smooth functions of x, u, we can choose $e_2 = \partial_u$ thus getting the realization $\langle\partial_t, \partial_u\rangle$.

Now take the case $e_1 = \partial_x$ and e_2 is an operator of the form (3.3). Using the commutation relation (3.33) yields

$$e_2 = a(t)\partial_t + b(t,u)\partial_x + c(t,u)\partial_u. \tag{3.35}$$

If $a \neq 0$, we make the change of variables

$$\bar{t} = T(t),\ \bar{x} = x + X(t,u),\ v = U(t,u),\ \dot{T} \neq 0,\ U_u \neq 0, \tag{3.36}$$

where $\dot{T} = a^{-1}$, $aX_t + xX_u + b = 0$; $aU_t + cU_u = 0,\ U_u \neq 0$. This reduces operator (3.35) to the form $e_2 = \partial_{\bar{t}}$.

If in (3.35) $a = 0,\ c \neq 0$, then we put $T = t$ in (3.36) and take as X and U solutions of the equations

$$cX_u + b = 0, \quad cU_u = 1,$$

which reduces operator (3.35) to $e_2 = \partial_v$.

Finally, turning to the remaining case, when $a = c = 0$ in (3.35). There are transformations of the form (3.36) which transform operator (3.35) to the form $e_2 = \bar{t}\partial_{\bar{x}}$ (if $b_u = 0$) or to $e_2 = v\partial_{\bar{x}}$ (if $b_u \neq 0$).

Summing up we conclude that, up to equivalence defined by transformations of the group $\mathcal{E}$, there are four inequivalent realizations of the algebra $A_{2.1}:\ \langle\partial_t, \partial_u\rangle$, $\langle\partial_x, \partial_u\rangle$, $\langle\partial_x, t\partial_x\rangle$, $\langle\partial_x, u\partial_x\rangle$.

The conditions (3.4) imply that the third realization, $\langle\partial_x, t\partial_x\rangle$, cannot be an invariance algebra of PDEs of the form (0.1). The equation invariant under the fourth realization is

$$u_t = u_x^{-2}F(t,x)u_{xx} + u_xG(t,u). \tag{3.37}$$

and is linearizable by the change of variables

$$\bar{t} = t, \quad \bar{x} = u, \quad v = x.$$

Thus we see that there exist two inequivalent realizations of the algebra $A_{2.1}$ which are invariance algebras of nonlinear PDEs of the form (0.1):

$$\begin{aligned} A^1_{2.1} &= \langle\partial_t, \partial_u\rangle; \\ A^2_{2.1} &= \langle\partial_x, \partial_u\rangle. \end{aligned}$$

The corresponding forms of the functions F, G are given in Table 1.

The same reasoning gives all inequivalent realizations of the abstract Lie algebra $A_{2.2}$. The full list of these contains three realizations which are admitted by PDEs of the form (0.1):

$$\begin{aligned} A^1_{2.2} &= \langle -t\partial_t - x\partial_x, \partial_t\rangle; \\ A^2_{2.2} &= \langle -t\partial_t - x\partial_x, \partial_x\rangle; \\ A^3_{2.2} &= \langle -x\partial_x - u\partial_u, \partial_x\rangle. \end{aligned}$$

The corresponding forms of the functions F and G are given in Table 1.

Let us note that provided the functions $\tilde{F}, \tilde{G}$ are arbitrary, the corresponding realizations of the two-dimensional Lie algebras are maximal invariance algebras of these equations.

Table 1. *Invariance of (0.1) under two-dimensional solvable Lie algebras*

Algebra	F	G
$A^1_{2.1}$	$\tilde{F}(x, u_x)$	$\tilde{G}(x, u_x)$
$A^2_{2.1}$	$\tilde{F}(t, u_x)$	$\tilde{G}(t, u_x)$
$A^1_{2.2}$	$x\tilde{F}(u, \omega)$	$x^{-1}\tilde{G}(u, \omega),\ \omega = xu_x$
$A^2_{2.2}$	$t\tilde{F}(u, \omega)$	$t^{-1}\tilde{G}(u, \omega),\ \omega = tu_x$
$A^3_{2.2}$	$u^2\tilde{F}(t, u_x)$	$u\tilde{G}(t, u_x)$

We begin the search for realizations of three-dimensional solvable Lie algebras $A_3 = \langle e_1, e_2, e_3\rangle$ by considering decomposable algebras $A_{3.1}, A_{3.2}$.

Evidently, in order to get all the possible realizations of these algebras within the class of operators (3.3), it suffices to extend the realizations already known for the two-dimensional algebras $A^i_{2.1} = \langle e_1, e_2\rangle$ $(i = 1, 2)$ (for the algebra $A_{3.1}$) and $A^i_{2.2} = \langle e_1, e_2\rangle$ $(i = 1, 2, 3)$ (for the algebra $A_{3.2}$). This follows from the definition of decomposable solvable Lie algebras. As a result we get one realization of the algebra $A_{3.1}$ and six inequivalent realizations of the algebra $A_{3.2}$, which are admissible as invariance algebras of PDEs (0.1):

$$
\begin{aligned}
A^1_{3.1} &= \langle \partial_t, \partial_u, \partial_x\rangle; \\
A^1_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_t, \partial_u\rangle; \\
A^2_{3.2} &= \langle -t\partial_t - u\partial_u, \partial_t, xu\partial_u\rangle; \\
A^3_{3.2} &= \langle -t\partial_t - u\partial_u, \partial_u, t\partial_t + x\partial_x\rangle; \\
A^4_{3.2} &= \langle -t\partial_t - x\partial_x, \partial_x, \partial_u\rangle; \\
A^5_{3.2} &= \langle -x\partial_x - u\partial_u, \partial_u, \partial_t\rangle; \\
A^6_{3.2} &= \langle -x\partial_x - u\partial_u, \partial_u, tx\partial_x\rangle.
\end{aligned}
$$

The explicit forms of the invariant equations (0.1) are determined by the forms of the functions F, G which are given in Table 2, where $\tilde{F}$, $\tilde{G}$ are arbitrary smooth functions.

One can verify by direct computation that the realizations given above are the maximal invariance algebras of the corresponding equations, provided the functions $\tilde{F}$ and $\tilde{G}$ are arbitrary smooth functions.

As mentioned in Section II.2, there are seven abstract non-isomorphic non-decomposable Lie algebras. All of them contain the two-dimensional commutative ideal $A_{2.1} = \langle e_1, e_2\rangle$. Thus, to construct their realizations within the class of operators under consideration, it suffices to describe all the possible extensions of the realizations of $A_{2.1}$ with the operator e_3 of the form (3.3). Moreover, we have to consider both the realizations $A^i_{2.1} = \langle e_1, e_2\rangle$ $(i = 1, 2)$, and $\tilde{A}^i_{2.1} = \langle \tilde{e}_1, \tilde{e}_2\rangle$ $(i = 1, 2)$, where $\tilde{e}_1 = e_2, \tilde{e}_2 = e_1$. We will consider in more detail the procedure for constructing realizations of the Weyl algebra $A_{3.3}$, which is a nilpotent Lie algebra.

Table 2. *Invariance of (0.1) under three-dimensional decomposable solvable Lie algebras*

Algebra	F	G
$A^1_{3.1}$	$\tilde{F}(u_x)$	$\tilde{G}(u_x)$
$A^1_{3.2}$	$x\tilde{F}(\omega)$	$x^{-1}\tilde{G}(\omega),\ \omega = xu_x$
$A^2_{3.2}$	$u^{-1}e^{x\omega}\tilde{F}(x)$	$e^{x\omega}[\tilde{G}(x) - \omega^2\tilde{F}(x)],\ \omega = u^{-1}u_x$
$A^3_{3.2}$	$t^{-1}x^2\tilde{F}(\omega)$	$x^{-1}\tilde{G}(\omega),\ \omega = t^{-1}x^2u_x$
$A^4_{3.2}$	$t\tilde{F}(\omega)$	$t^{-1}\tilde{G}(\omega),\ \omega = tu_x$
$A^5_{3.2}$	$x^2\tilde{F}(u_x)$	$x\tilde{G}(u_x)$
$A^6_{3.2}$	$x^2\tilde{F}(t)$	$xt^{-1}u_x \ln\|u_x\| + xu_x\tilde{G}(t)$

We begin with the realization $A^1_{2.1}$. If $e_1 = \partial_t, e_2 = \partial_u$, then it follows from the commutation relation $[e_2, e_3] = e_1$, where e_3 is of the form (3.3), that the equation $b_u\partial_x + c_u\partial_u = \partial_t$ holds true. Since this equation cannot be satisfied for any choice of the functions a, b, c contained in e_3, this realization cannot be extended to that of a three-dimensional solvable Lie algebra. Next, if $e_1 = \partial_u, e_2 = \partial_t$, then $e_3 = \tilde{b}(x)\partial_x + [t + \tilde{c}(x)]\partial_u$, and we obtain, up to equivalence under $\mathcal{E}$, the following three realizations of the algebra $A_{3.3}$:

$$\langle \partial_u, \partial_t, \partial_x + t\partial_u\rangle,$$
$$\langle \partial_u, \partial_t, t\partial_u\rangle,$$
$$\langle \partial_u, \partial_t, (t + x)\partial_u\rangle.$$

Checking conditions (3.4) we find that the second realization from the above list cannot be an invariance algebra of PDE (0.1). Moreover, the equation admitting the third realization is necessarily linear.

In studying the realization $A^2_{2.1}$ we have to take into account the existence of two possibilities. The first possibility is $e_1 = \partial_x, e_2 = \partial_u$ and the second one is $e_1 = \partial_u, e_2 = \partial_x$. However, the above realizations are transformed one into another by the change of variables

$$\bar{t} = t, \quad \bar{x} = u, \quad v = x. \tag{3.38}$$

So we may consider without loss of generality the second realization only. Performing the necessary computations yields a realization of the algebra $A_{3.3}$ which can be admitted by a nonlinear equation of the form (0.1): $\langle \partial_u, \partial_x, \partial_t + x\partial_u\rangle$.

We conclude that there are two inequivalent realizations of the algebra $A_{3.3}$, which are invariance algebras of nonlinear PDEs from the class (0.1),

$$A^1_{3.3} = \langle \partial_u, \partial_t, t\partial_u + \partial_x\rangle;$$
$$A^2_{3.3} = \langle \partial_u, \partial_x, t\partial_x + x\partial_u\rangle.$$

The forms of the functions F and G defining the corresponding nonlinear heat conductivity equations are given in Table 3.

Table 3. *Invariance of (0.1) with respect to the Weyl algebra*

Algebra	F	G
$A^1_{3.3}$	$\tilde{F}(u_x)$	$x + \tilde{G}(u_x)$
$A^2_{3.3}$	$\tilde{F}(t)$	$-\frac{1}{2}u_x^2 + \tilde{G}(t)$

The remaining non-decomposable solvable Lie algebras are treated in an analogous way. We present below those of their inequivalent realizations which are admitted by nonlinear PDEs of the form (0.1). In Table 4 we give the various forms of the functions F, G defining the forms of the invariant equations.

$$
\begin{aligned}
A^1_{3.4} &= \langle \partial_u, \partial_t, t\partial_t + x\partial_x + [t+u]\partial_u \rangle; \\
A^2_{3.4} &= \langle \partial_u, \partial_t, t\partial_t + [t+u]\partial_u \rangle; \\
A^3_{3.4} &= \langle \partial_x, \partial_u, 2t\partial_t + (x+u)\partial_x + u\partial_u \rangle; \\
A^4_{3.4} &= \langle \partial_x, \partial_u, (x+u)\partial_x + u\partial_u \rangle; \\
A^1_{3.5} &= \langle \partial_t, \partial_u, t\partial_t + x\partial_x + u\partial_u \rangle; \\
A^2_{3.5} &= \langle \partial_t, \partial_u, t\partial_t + u\partial_u \rangle; \\
A^3_{3.5} &= \langle \partial_x, \partial_u, 2t\partial_t + x\partial_x + u\partial_u \rangle; \\
A^1_{3.6} &= \langle \partial_t, \partial_u, t\partial_t + x\partial_x - u\partial_u \rangle; \\
A^2_{3.6} &= \langle \partial_t, \partial_u, t\partial_t - u\partial_u \rangle; \\
A^3_{3.6} &= \langle \partial_x, \partial_u, t\partial_t + x\partial_x - u\partial_u \rangle; \\
A^4_{3.6} &= \langle \partial_x, \partial_u, x\partial_x - u\partial_u \rangle; \\
A^1_{3.7} &= \langle \partial_u, \partial_t, qt\partial_t + x\partial_x + u\partial_u \rangle \ (q \neq 0, \pm 1); \\
A^2_{3.7} &= \langle \partial_u, \partial_t, qt\partial_t + u\partial_u \rangle \ (q \neq 0, \pm 1); \\
A^3_{3.7} &= \langle \partial_x, \partial_u, t\partial_t + x\partial_x + qu\partial_u \rangle \ (0 < |q| < 1); \\
A^4_{3.7} &= \langle \partial_x, \partial_u, x\partial_x + qu\partial_u \rangle \ (0 < |q| < 1); \\
A^1_{3.8} &= \langle \partial_x, \partial_u, \partial_t + u\partial_x - x\partial_u \rangle; \\
A^2_{3.8} &= \langle \partial_x, \partial_u, u\partial_x - x\partial_u \rangle; \\
A^1_{3.9} &= \langle \partial_x, \partial_u, \partial_t + (u+qx)\partial_x + (qu-x)\partial_u \rangle \ (q > 0); \\
A^2_{3.9} &= \langle \partial_x, \partial_u, (u+qx)\partial_x + (qu-x)\partial_u \rangle \ (q > 0).
\end{aligned}
$$

Let us note that if the functions $\tilde{F}$, $\tilde{G}$ from Tables 3, 4 are arbitrary, then the corresponding realizations are the maximal invariance algebras of the equations obtained.

III.4.2 Complete classification of nonlinear PDEs (0.1) invariant with respect to solvable Lie algebras

The next step of our approach to the group classification of nonlinear PDEs of the form (0.1) is to describe equations which are invariant under four-dimensional solvable Lie algebras.

Table 4. *Invariance of (0.1) under non-decomposable three-dimensional solvable Lie algebras*

Algebra	F	G
$A^1_{3.4}$	$x\tilde{F}(u_x)$	$\tilde{G}(u_x) + \ln\lvert x\rvert$
$A^2_{3.4}$	$u_x^{-1}\tilde{F}(x)$	$\tilde{G}(x) + \ln\lvert u_x\rvert$
$A^3_{3.4}$	$u_x^{-2}\tilde{F}(\omega)$	$u_x e^{-\frac{1}{u_x}}\tilde{G}(\omega),\ \omega = 2u_x^{-1} - \ln\lvert t\rvert$
$A^4_{3.4}$	$u_x^{-2}\tilde{F}(t)\exp(2u_x^{-1})$	$u_x\tilde{G}(t)\exp(u_x^{-1})$
$A^1_{3.5}$	$x\tilde{F}(u_x)$	$\tilde{G}(u_x)$
$A^2_{3.5}$	$u_x^{-1}\tilde{F}(x)$	$\tilde{G}(x)$
$A^3_{3.5}$	$\tilde{F}(u_x)$	$\lvert t\rvert^{-\frac{1}{2}}\tilde{G}(u_x)$
$A^1_{3.6}$	$x\tilde{F}(\omega)$	$x^{-2}\tilde{G}(\omega),\ \omega = x^2u_x$
$A^2_{3.6}$	$u_x\tilde{F}(x)$	$u_x^2\tilde{G}(x)$
$A^3_{3.6}$	$t\tilde{F}(\omega)$	$t^{-2}\tilde{G}(\omega),\ \omega = t^2u_x$
$A^4_{3.6}$	$u_x^{-1}\tilde{F}(t)$	$\sqrt{\lvert u_x\rvert}\tilde{G}(t)$
$A^1_{3.7}$	$\lvert x\rvert^{2-q}\tilde{F}(u_x)$	$\lvert x\rvert^{1-q}\tilde{G}(u_x)\ (q =\neq 0, \pm 1)$
$A^2_{3.7}$	$\lvert u_x\rvert^{-q}\tilde{F}(x)$	$\lvert u_x\rvert^{1-q}\tilde{G}(x), q \neq 0, \pm 1$
$A^3_{3.7}$	$t\tilde{F}(\omega)$	$\lvert t\rvert^{q-1}\tilde{G}(\omega),\ \omega = \lvert t\rvert^{1-q}u_x(0 < \lvert q\rvert < 1)$
$A^4_{3.7}$	$\lvert u_x\rvert^{\frac{2}{q-1}}\tilde{F}(t)$	$\lvert u_x\rvert^{\frac{q}{q-1}}\tilde{G}(t)\ (0 < \lvert q\rvert < 1)$
$A^1_{3.8}$	$(1+u_x^2)^{-1}\tilde{F}(\omega)$	$\sqrt{1+u_x^2}\tilde{G}(\omega),\ \omega = t + \arctan u_x$
$A^2_{3.8}$	$(1+u_x^2)^{-1}\tilde{F}(t)$	$\sqrt{1+u_x^2}\tilde{G}(t)$
$A^1_{3.9}$	$\frac{\exp(-2q\arctan u_x)\tilde{F}(\omega)}{1+u_x^2}$	$\sqrt{1+u_x^2}\exp(-q\arctan u_x)\tilde{G}(\omega),$ $\omega = t + \arctan u_x\ (q > 0)$
$A^2_{3.9}$	$\frac{\exp(-2q\arctan u_x)\tilde{F}(t)}{1+u_x^2}$	$\sqrt{1+u_x^2}\exp(-q\arctan u_x)\tilde{G}(t),\ (q > 0)$

As we mentioned in Section II.2, there are ten decomposable and ten non-decomposable, non-isomorphic, solvable four-dimensional Lie algebras. Since nonlinear equations of the form (0.1) which admit three-dimensional solvable algebras contain arbitrary functions of one argument, it is only natural to expect that PDEs admitting four-dimensional algebras will depend on arbitrary parameters at most. In other words, the arbitrary functions in question will take specific forms dictated by the extension of symmetry group. This is, indeed, the case for all the invariant PDEs except for the equation

$$u_t = F(u_x)u_{xx}. \tag{3.39}$$

Group classification of PDEs of the form (3.39) has been carried out in [8] and we give below the results obtained in the form of theorem.

Theorem 3.4 **([8])** *Provided F is an arbitrary smooth function, the maximal invariance algebra admitted by (3.39) is the four-dimensional Lie algebra*

$$A^1_{3.1} \niplus \langle 2t\partial_t + x\partial_x + u\partial_u \rangle.$$

An extension of the symmetry algebra of PDE (3.39) is only possible for the three cases given below:

$$\begin{aligned} F = \exp u_x &\quad : \quad e_5 = t\partial_t - x\partial_u; \\ F = u_x^n &\quad : \quad e_5 = nt\partial_t - u\partial_u, \quad n \geq -1, \quad n \neq 0; \\ F = \frac{\exp(n \arctan u_x)}{1 + u_x^2} &\quad : \quad e_5 = nt\partial_t - u\partial_x - x\partial_u, \quad n \geq 0. \end{aligned}$$

In view of this result, we will exclude from further consideration equations which are equivalent to an equation of the form (3.39).

Consider first the decomposable solvable four-dimensional Lie algebras

$$\begin{aligned} &4A_1 = A_{3.1} \oplus A_1, && A_{3.2} \oplus A_1, \\ &2A_{2.2} = A_{2.2} \oplus A_{2.2}, && A_{3.i} \oplus A_1 \quad (i = 3, 4, \ldots, 9). \end{aligned}$$

On analyzing extensions of the realization $A^1_{3.1}$ for the algebra $4A_1$ and of the realizations $A^i_{3.2}$ $(i = 1, \ldots, 6)$ by an operator e_4 of the form (3.3), we conclude that there are no realizations of the algebras $4A_1$ and $A_{3.2} \oplus A_1$ which could be invariance algebras of PDEs of the form (0.1).

Studying realizations of the algebra $2A_{2.2}$ yields four inequivalent realizations admitted by PDEs from the class (0.1). We give these realizations below, as well as the corresponding forms of the functions F, G.

$$\begin{aligned} 2A^1_{2.2} &= A^1_{3.2} \niplus \langle -u\partial_u + kx\partial_x \rangle \ (k \neq 0): \\ &\quad F = \lambda x|\omega|^{-k}, \quad G = \beta x^{-1}|\omega|^{1-k}, \quad \lambda \neq 0, \beta \in R, \omega = xu_x; \\ 2A^2_{2.2} &= A^2_{3.2} \niplus \langle x\partial_x \rangle: \\ &\quad F = \lambda x^2 u^{-1} \exp \omega, \quad G = (\beta - \lambda\omega^2) \exp \omega, \quad \lambda \neq 0, \beta \in R, \omega = xu^{-1}u_x; \\ 2A^3_{2.2} &= A^4_{3.2} \niplus \langle -u\partial_u + kt\partial_t \rangle \ (k \neq 0, 1): \\ &\quad F = \lambda t|\omega|^{\frac{2k}{1-k}}, \quad G = \beta t^{-1}|\omega|^{\frac{1}{1-k}}, \quad \omega = tu_x, \quad \lambda \neq 0, \beta \in R; \\ 2A^4_{2.2} &= A^4_{3.2} \niplus \langle -u\partial_u + t\partial_x \rangle: \\ &\quad F = \lambda t, \quad G = u_x \ln|tu_x| + \beta u_x, \quad \lambda \neq 0, \quad \beta \in R. \end{aligned}$$

Now, in order to complete group classification of the PDEs given above, one has to compute their maximal invariance algebras. To this end, it is necessary to solve the determining equations (3.4) for each choice of the functions F, G. Note, that we have simplified the forms of the functions F, G with the use of transformations from the corresponding equivalence groups which are subgroups of $\mathcal{E}$. As a result, we get the following simplified forms of the above invariant equations:

$$2A^1_{2.2}: u_t = |x|^{1-k}|u_x|^{-k}u_{xx} + \beta|x|^{-k}|u_x|^{1-k}, \quad \beta \in R, \quad k \neq 0; \tag{3.40}$$
$$2A^2_{2.2}: u_t = x^2u^{-1}\exp(\omega)u_{xx} + (\beta - \omega^2)\exp\omega, \quad \omega = xu^{-1}u_x, \quad \beta \in R; \tag{3.41}$$
$$2A^3_{2.2}: u_t = \pm|t|^{\frac{k+1}{1-k}}|u_x|^{\frac{2k}{1-k}}u_{xx} + \epsilon|t|^{\frac{k}{1-k}}|u_x|^{\frac{1}{1-k}}, \quad \epsilon = 0, 1, \quad k \neq 0, 1; \tag{3.42}$$
$$2A^4_{2.2}: u_t = \lambda t u_{xx} + u_x \ln|tu_x|, \quad \lambda \neq 0. \tag{3.43}$$

Inserting the functions F, G defining the above PDEs (3.40)–(3.43) into the determining equations (3.4) and analyzing the equations obtained, we arrive at the following conclusions.

1. If $k \neq 0, 2$, $\beta \neq \frac{k-1}{k-2}$ or $k = 2, \beta \neq \frac{5}{4}$ in (3.40), the realization $2A^1_{2.2}$ is the maximal invariance algebra of the nonlinear heat conductivity equation (3.40). If $k = 2, \beta = \frac{5}{4}$, then the maximal invariance algebra of the equation in question is five-dimensional. Its basis if formed by the operators of the realization $2A^1_{2.2}$ $(k = 2)$ and the operator $4xu\partial_x - u^2\partial_u$. However, this algebra is isomorphic to the Lie algebra $sl(2, R) \oplus A_{2.2}$ and the change of variables

   $$\bar{t} = t, \qquad \bar{x} = u, \qquad v = \alpha|x|^{\frac{1}{4}}, \quad \alpha \neq 0$$

   transforms its basis operators to become basis operators of the realization of $sl(2, R) \oplus L_{2.1}$, where $sl(2, R)$ is the realization (3.21) and $L_{2.1} = \langle 4t\partial_t + u\partial_u, \partial_t\rangle$. Thus equation (3.40) with $k = 2$, $\beta = \frac{5}{4}$ is equivalent to an invariant PDE obtained in the previous section.

   Finally, if $k \neq 0, 2$ and $\beta = \frac{k-1}{k-2}$ in equation (3.40), then the latter is transformed by transformations from the group $\mathcal{E}$ to the form (3.39).

2. If $\beta \neq -2$ in equation (3.40), the realization $2A^2_{2.2}$ is the maximal invariance algebra of this equation. Given the condition $\beta = -2$, the maximal invariance algebra is the five-dimensional Lie algebra spanned by the operators

   $$\langle \partial_t, -xu\partial_u, x^2\partial_x + \ln|x^2u|xu\partial_u, 2t\partial_t + 2u\partial_u - x\partial_x, t\partial_t + u\partial_u\rangle.$$

   The change of variables

   $$\bar{t} = t, \qquad \bar{x} = -x^{-1}, \qquad v = x^{-1}\ln|u| + 2x^{-1}(1 + \ln|x|)$$

   reduce equation (3.41) with $\beta = -2$ to the equation

   $$v_{\bar{t}} = \exp(v_{\bar{x}})v_{\bar{x}\bar{x}},$$

   which is contained in the class of PDEs (3.39).

3. The realization $2A^3_{2.2}$ $(k \neq 0, 1)$ is the maximal invariance algebra of PDE (3.42), provided $\epsilon = 1$. If $\epsilon = 0$, then its maximal invariance algebra is the five-dimensional Lie algebra spanned by the operators

   $$2A^3_{2.2} \ \ (k \neq 0, 1) \Subset \langle |t|^{\frac{1+k}{k-1}}\partial_t\rangle.$$

   However, with this choice of ε, equation (3.42) is reduced through the change of variables

   $$\bar{t} = \frac{1}{2}(1-k)|t|^{\frac{2}{1-k}}, \qquad \bar{x} = x, \qquad v = u,$$

   to equation

   $$v_{\bar{t}} = \pm|v_{\bar{x}}|^{\frac{2k}{1-k}}v_{\bar{x}\bar{x}},$$

   which belongs to the class of PDEs (3.39).

4. The realization $2A^4_{2.2}$ is the maximal invariance algebra of PDE (3.43). Analyzing the algebra $A_{3.3} \oplus A_1$ we find that it has no realizations which are admissible for PDEs (0.1). Next, we get a realization $A^2_{3.5} \oplus \langle \partial_x \rangle$ of the algebra $A_{3.5} \oplus A_1$ but the corresponding invariant equation
$$u_t = u_x^{-1} u_{xx}$$
belong to the class of PDEs (3.39). Studying realizations of the algebra $A_{3.7} \oplus A_1$ we get the nonlinear heat conductivity equation
$$u_t = u_x^{-2} u_{xx} + u_x^{-1},$$
whose maximal invariance algebra is the five-dimensional algebra having the following basis elements:
$$A^1_5 = A^2_{3.7}\ (q = 2) \oplus \langle \partial_x, e^x \partial_x \rangle.$$
A similar analysis of the remaining decomposable solvable four-dimensional algebras yields eight inequivalent realizations that are maximal invariance algebras of nonlinear PDEs of the form (0.1). We list the nonlinear PDEs (0.1) whose maximal invariance algebras are four-dimensional decomposable solvable Lie algebras in Table 5.

We turn now to non-decomposable algebras. There are ten non-isomorphic non-decomposable solvable four-dimensional Lie algebras $A_4 = \langle e_i | i = 1, 2, 3, 4 \rangle$ (the full list is given in Section II.2). Their structure implies that studying realizations of these algebras can be carried out by extension of the (already known) realizations of three-dimensional solvable algebras $A_3 = \langle e_1, e_2, e_3 \rangle$ by the operator e_4 of the form (3.3). Moreover, one must use the following extension scheme: $A_{4.i} = A_{3.1} \Subset \langle e_4 \rangle \quad (i = 1, \ldots, 6),\ A_{4.i} = A_{3.3} \Subset \langle e_4 \rangle \quad (i = 7, 8, 9), \quad A_{4.10} = A_{3.5} \Subset \langle e_4 \rangle$.

There exists only one realization of the algebra $A_{3.1}$, and it is the maximal invariance algebra of the PDE
$$u_t = F(u_x) u_{xx} + G(u_x), \tag{3.44}$$
so that nonlinear PDEs invariant under realizations of the algebras $A_{4.i}$ $(i = 1, \ldots, 6)$ must belong to the class of equations (3.44).

Direct computation shows that the algebra $A_{4.1}$ has no realizations that are admitted by PDEs (0.1). For the remaining abstract Lie algebras from the class under study we get seven realizations which are invariance algebras of nonlinear PDEs of the form (0.1).

$$\begin{aligned}
&A^1_{4.2} = A^1_{3.1} \Subset \langle qt\partial_t + x\partial_x + (u + x)\partial_u \rangle \quad (q \neq 0, 1);\\
&A^2_{4.2} = A^1_{3.1} \Subset \langle t\partial_t + (t + x)\partial_x + qu\partial_u \rangle \quad (q \neq 0, 1);\\
&A^1_{4.3} = A^1_{3.1} \Subset \langle t\partial_t + x\partial_u \rangle;\\
&A^2_{4.3} = A^1_{3.1} \Subset \langle t\partial_x + u\partial_u \rangle;\\
&A^1_{4.4} = A^1_{3.1} \Subset \langle t\partial_t + (t + x)\partial_x + (x + u)\partial_u \rangle;\\
&A^1_{4.5} = A^1_{3.1} \Subset \langle t\partial_t + px\partial_x + qu\partial_u \rangle \quad (p < q,\ p \cdot q \neq 0;\ p, q, \neq 1);\\
&A^1_{4.6} = A^1_{3.1} \Subset \langle qt\partial_t + (px + u)\partial_x + (pu - x)\partial_u \rangle \quad (q \neq 0;\ p \geq 0).
\end{aligned}$$

The corresponding forms of the functions F, G defining invariant equations (0.1) are given in Table 6. Note that, for the sake of completeness, we give in Table 6 equation (3.39), whose maximal invariance algebra for arbitrary F is
$$A^2_{4.5} = A^1_{3.1} \Subset \langle t\partial_t + \frac{1}{2} x\partial_x + \frac{1}{2} u\partial_u \rangle.$$

Table 5. *Invariance of (0.1) under decomposable four-dimensional solvable algebras*

Algebra	F	G
$2A^1_{2.2},\quad (k\neq 0,2)$	$\lvert x\rvert^{1-k}\lvert u_x\rvert^{-k}$	$\beta\lvert x\rvert^{-k}\lvert u_x\rvert^{1-k},\beta\neq\frac{k-1}{k-2}$
$2A^1_{2.2}\ (k=2)$	$x^{-1}u_x^{-2}$	$\beta x^{-2}u_x^{-1},\beta\neq\frac{5}{4}$
$2A^2_{2.2}$	$x^2u^{-1}\exp\omega$	$(\beta-\omega^2)\exp\omega,\ \omega=xu^{-1}u_x,\beta\neq -2$
$2A^3_{2.2}\ (k\neq 0,1)$	$\pm\lvert t\rvert^{\frac{k+1}{1-k}}\lvert u_x\rvert^{\frac{2k}{1-k}}$	$\lvert t\rvert^{\frac{k}{1-k}}\lvert u_x\rvert^{\frac{1}{1-k}}$
$2A^4_{2.2}$	$\lambda t,\quad \lambda\neq 0$	$u_x\ln\lvert tu_x\rvert$
$A^2_{3.4}\oplus\langle\partial_x\rangle$	u_x^{-1}	$\ln\lvert u_x\rvert$
$A^4_{3.4}\oplus\langle\partial_t\rangle$	$u_x^{-2}\exp(2u_x^{-1})$	$u_x\exp(u_x^{-1})$
$A^2_{3.6}\oplus\langle\partial_x\rangle$	u_x	u_x^2
$A^4_{3.6}\oplus\langle\partial_t\rangle$	u_x^{-1}	$\sqrt{\lvert u_x\rvert}$
$A^2_{3.7}\oplus\langle\partial_x\rangle\ (q\neq 0,\pm1,2)$	$\lvert u_x\rvert^{-q}$	$\lvert u_x\rvert^{1-q}$
$A^4_{3.7}\oplus\langle\partial_t\rangle\ (0<\lvert q\rvert<1)$	$\lvert u_x\rvert^{\frac{2}{q-1}}$	$\lvert u_x\rvert^{\frac{q}{q-1}}$
$A^2_{3.8}\oplus\langle\partial_t\rangle$	$(1+u_x^2)^{-1}$	$\sqrt{1+u_x^2}$
$A^2_{3.9}\oplus\langle\partial_t\rangle\ (q>0)$	$\dfrac{\exp(-2q\arctan u_x)}{1+u_x^2}$	$\sqrt{1+u_x^2}\exp(-q\arctan u_x)$

As we have already mentioned, the realizations of the algebras $A_{4.i}$ $(i=7,8,9)$ are constructed by extension of the realizations of the algebra $A_{3.3}$ by an operator e_4 of the type (3.3). Also, while considering the realizations of $A_{3.3}=\langle e_1,e_2,e_3\rangle$, we have taken into account the isomorphism of this algebra given by $e_1\rightarrow e_1,\ e_2\rightarrow -e_3,\ e_3\rightarrow e_2$.

In this way, we get three inequivalent realizations of the algebras $A_{4.7}$ and $A_{4.9}$

$$
\begin{aligned}
&A^1_{4.7}=A^1_{3.3}\Supset\langle t\partial_t+(x-t)\partial_x+(2u-\frac{1}{2}t^2)\partial_u\rangle,\\
&A^2_{4.7}=A^2_{3.3}\Supset\langle -\partial_t+x\partial_x+2u\partial_u\rangle,\\
&A^1_{4.9}=A^2_{3.3}\Supset\langle -(1+t^2)\partial_t+(q-t)x\partial_x+(2qu-\frac{1}{2}x^2)\partial_u\rangle\quad (q>0),
\end{aligned}
$$

which are maximal invariance algebras of nonlinear PDEs (0.1). The corresponding forms of the functions F, G are given in Table 6.

Next, we have constructed four inequivalent realizations of the algebra $A_{4.8}$ that are admitted by nonlinear PDEs from the class (0.1):

$$
A^1_{4.8}=A^1_{3.3}\Supset\langle t\partial_t+qx\partial_x+(1+q)u\partial_u\rangle\quad (q\in R),
$$

$$A^2_{4.8} = A^1_{3.3} \supset\!\!\!\!+ \langle t\partial_t + k\partial_x + u\partial_u\rangle \quad (k \neq 0),$$
$$A^3_{4.8} = A^1_{3.3} \supset\!\!\!\!+ \langle x\partial_x + u\partial_u + k^{-1}(\partial_t + x\partial_u)\rangle \quad (k \neq 0),$$
$$A^4_{4.8} = A^2_{3.3} \supset\!\!\!\!+ \langle (1-q)t\partial_t + x\partial_x + (1+q)u\partial_u\rangle \quad (|q| \neq 1).$$

The realizations $A^2_{4.8}, A^4_{4.8}$ are the maximal invariance algebras of nonlinear heat conductivity equations belonging to the class of PDEs (0.1), and the corresponding forms of the functions F, G are given in Table 6.

The PDE invariant with respect to the realization $A^1_{4.8}$ reduces to the form

$$u_t = \lambda |u_x|^{2q-1} u_{xx} + x + \epsilon |u_x|^q, \tag{3.45}$$

where $\epsilon = 0,\ \ \lambda = \pm 1$, provided $q = 0, 1$ and $\epsilon = 0, \lambda = \pm 1$ or $\epsilon = 1, \lambda \neq 0$ if $q \neq 0, 1$.

Investigating the maximal symmetry admitted by (3.45)we find that for $q \neq -\frac{1}{2}$ the realization $A^1_{4.8}$ is its maximal invariance algebra. If $q = -\frac{1}{2}$, the change of variables (3.38) reduces PDE (3.45) to the Burgers equation

$$v_{\overline{t}} = \lambda v_{\overline{x}\overline{x}} - v v_{\overline{x}}.$$

The maximal invariance algebra of the Burgers equation is the semi-direct sum of the algebra $sl(2, R)$ and a two-dimensional solvable radical.

The equation invariant under the realization $A^3_{4.8}$ is

$$u_t = \pm \exp(2ku_x) u_{xx} + x + \epsilon \exp(ku_x), \quad k \neq 0, \epsilon = 0, 1.$$

If $\epsilon = 1$, then the realization $A^3_{4.8}$ is the maximal invariance algebra of this equation. For $\epsilon = 0$, the change of variables

$$\overline{t} = \frac{1}{2k} e^{2kt}, \quad \overline{x} = -x, \quad v = -2ku + 2ktx, \quad k \neq 0,$$

reduces the equation in question to the PDE

$$v_{\overline{t}} = \pm \exp(v_{\overline{x}}) v_{\overline{x}\overline{x}},$$

which belongs to the class of equations (3.39).

Finally, after extending the realizations of the Lie algebra $A_{3.5} = \langle e_1, e_2, e_3\rangle$ by an operator e_4 of the form (3.3), we obtain a realization of the algebra $A_{4.10}$ of the form

$$A^1_{4.10} = A^3_{3.5} \supset\!\!\!\!+ \langle 2kt\partial_t + u\partial_x - x\partial_u\rangle, \quad k \geq 0,$$

which is the maximal invariance algebra of the equation

$$u_t = \frac{\exp(2k \arctan u_x)}{1 + u_x^2} u_{xx} + \beta |t|^{-\frac{1}{2}} \sqrt{1 + u_x^2} \exp(k \arctan u_x), \quad k \geq 0, \ \beta \neq 0.$$

We give in Table 6 a complete list of inequivalent PDEs of the form (0.1), whose maximal invariance algebras are non-decomposable four-dimensional solvable Lie algebras.

Table 6. *Invariance of (0.1) under non-decomposable four-dimensional solvable Lie algebras*

Algebra	F	G
$A^1_{4.2}$	$\exp(2-q)u_x$	$\exp(1-q)u_x,\quad q\neq 0,1$
$A^2_{4.2}$	$\lvert u_x\rvert^{\frac{1}{q-1}}$	$(1-q)^{-1}u_x\ln\lvert u_x\rvert,\quad q\neq 0,1$
$A^1_{4.3}$	$\exp(-u_x)$	$\exp(-u_x)$
$A^2_{4.3}$	1	$-u_x\ln\lvert u_x\rvert$
$A^1_{4.4}$	$\exp u_x$	$-\frac{1}{2}u_x^2$
$A^1_{4.5}$	$\lvert u_x\rvert^{\frac{2p-1}{q-p}}$	$\lvert u_x\rvert^{\frac{q-1}{q-p}}, p<q, p\cdot q\neq 0,\quad p,q\neq 1$
$A^2_{4.5}$	$\tilde{F}(u_x)$	0
$A^1_{4.6}$	$\dfrac{\exp[(q-2p)\arctan u_x]}{1+u_x^2}$	$\sqrt{1+u_x^2}\exp[(q-p)\arctan u_x], q\neq p,\ p\geq 0$
$A^1_{4.7}$	$\lambda u_x,\ \lambda\neq 0$	$x+u_x\ln\lvert u_x\rvert$
$A^2_{4.7}$	$\pm\exp(-2t)$	$-\frac{1}{2}u_x^2$
$A^1_{4.8}\ (q\neq-\frac{1}{2})$	$\pm\lvert u_x\rvert^{2q-1}$	x
$A^1_{4.8}\ (q\neq 0,1)$	$\lambda\lvert u_x\rvert^{2q-1},\ \lambda\neq 0$	$x+\lvert u_x\rvert^q$
$A^2_{4.8}$	$\lambda\lvert u_x\rvert^{-1},\ \lambda\neq 0$	$x-k\ln\lvert u_x\rvert,\ k\neq 0$
$A^3_{4.8}$	$\pm\exp(2ku_x)$	$x+\exp(ku_x),\ k\neq 0$
$A^4_{4.8}\ (\lvert q\rvert\neq 1)$	$\lvert t\rvert^{\frac{1+q}{1-q}}$	$-\frac{1}{2}u_x^2$
$A^1_{4.9}\ (q>0)$	$\pm\exp(-2q\arctan t)$	$\mp\dfrac{t\exp(-2q\arctan t)}{1+t^2}-\frac{1}{2}u_x^2$
$A^1_{4.10}$	$\dfrac{\exp(2k\arctan u_x]}{1+u_x^2}$	$\beta\lvert t\rvert^{-\frac{1}{2}}\sqrt{1+u_x^2}\exp(k\arctan u_x), k\geq 0,\ \beta\neq 0$

In order to complete the group classification, we have to analyze nonlinear equations of the form (0.1) which admit five-dimensional invariance algebras. In Section III.3 we have constructed two nonlinear heat conductivity equations whose invariance algebras are five-dimensional semi-direct products of semi-simple and solvable Lie algebras. According to the results of [26] there are three more PDEs belonging to the class (3.39), admitting the five-dimensional algebras

$$
\begin{aligned}
A^2_5 &= A^2_{4.5}\Subset\langle t\partial_t-x\partial_u\rangle;\\
A^3_5 &= A^2_{4.5}\Subset\langle nt\partial_t-u\partial_u\rangle,\ (n\geq -1, n\neq 0);
\end{aligned}
$$

$$A_5^4 \; = \; A_{4.5}^2 \niplus \langle nt\partial_t + u\partial_x - x\partial_u \rangle, \;\; (n \geq 0),$$

and there is one PDE admitting the realization $A^1_{4.5}$. It is not difficult to verify that all the algebras A_5^i $(i = 1, \ldots, 4)$ are solvable five-dimensional Lie algebras. Moreover, the algebra A_5^1 is decomposable $A_5^1 \sim A_{3.7} \oplus A_{2.2}$ and the algebras A_5^i $(i = 2, 3, 4)$ are non-isomorphic non-decomposable five-dimensional solvable Lie algebras (see, e.g., [21])

$$A_5^2 \sim A_{5.34}(p = 2), A_5^3 \sim A_{5.33}(p = 2 + n, q = -n)$$
$$A_5^4 \sim A_{5.35}(p = 2, q = n).$$

Consequently, the PDEs invariant with respect to the above algebras are inequivalent. We give in Table 7 a complete list of nonlinear PDEs of the form (0.1) whose maximal invariance algebras are five-dimensional.

IV Some conclusions

Surprisingly, the number of inequivalent nonlinear PDEs of the general form under consideration, and which admit non-trivial symmetry groups is reasonably small. Summarizing the results of our group classification of nonlinear heat conductivity equations of the form (0.1) we conclude that

1. There are two inequivalent nonlinear PDEs (0.1), that admit a one-dimensional invariance algebra.
2. There are five inequivalent PDEs (0.1) given in Table 1, which are invariant with respect to two-dimensional Lie algebras. Note that all two-dimensional Lie algebras are solvable.
3. Nonlinear heat conductivity equations (0.1) invariant under three-dimensional Lie algebras (note that a three-dimensional Lie algebra is either semi-simple or solvable).
   (a) There are six PDEs (0.1) admitting three-dimensional semi-simple invariance algebras (see Theorems 3.1 and 3.2).
   (b) There are twenty eight equations (0.1), given in Tables 2–4, which are invariant with respect to three-dimensional solvable Lie algebras.
4. Nonlinear heat conductivity equations (0.1) invariant under four-dimensional Lie algebras (note that there are no semi-simple four-dimensional Lie algebras).
   (a) There are five PDEs (0.1) admitting four-dimensional invariance algebras, that are semi-direct sums of semi-simple and solvable Lie algebras (see PDEs given at the end of Section III.3).
   (b) There are thirty equations (0.1) given in Tables 5,6, which are invariant with respect to four-dimensional solvable Lie algebras.
5. Nonlinear heat conductivity equations (0.1) invariant under five-dimensional Lie algebras (note that there are no semi-simple five-dimensional Lie algebras).
   (a) There are two PDEs (0.1) admitting five-dimensional invariance algebras, that are semi-direct sums of semi-simple and solvable Lie algebras (see PDEs given at the end of Section III.3).

(b) There are four equations (0.1) given in Table 7, which are invariant with respect to five-dimensional solvable Lie algebras.

Table 7. *Invariance of (0.1) under four-dimensional solvable Lie algebras*

Algebra	F	G
A_5^1	u_x^{-2}	u_x^{-1}
A_5^2	$\exp u_x$	0
A_5^3	$u_x^n,\ n \geq -1,\ n \neq 0$	0
A_5^4	$\dfrac{\exp(n \arctan u_x)}{1+u_x^2},\ n \geq 0$	0

We have shown that there are no nonlinear PDEs of the form (0.1) admitting invariance algebras of the dimension higher than five. Consequently, the classification of invariant nonlinear heat conductivity equations (0.1) presented above is complete in the sense that **any** PDEs of the form (0.1), which possess non-trivial Lie symmetry, can be reduced to one of the canonical forms given above.

Furthermore, we have shown that the results on group classification of particular equations from the class (0.1) obtained in [8, 15], [26]–[31] can be derived from our considerations. That is to say, for each invariant PDE (0.1) obtained in the papers enumerated above we can give an invariant equation from our list which is equivalent to it. The procedure of looking for a corresponding change of variables is purely algebraic. We start with identifying the invariance algebra by determining whether it is semi-simple, or solvable, or a semi-direct sum of semi-simple and solvable algebras, and then we find the corresponding realizations of the Lie algebras of the same dimension as the algebra under study. Comparing the two realizations, it is not difficult to find the explicit form of the change of variables connecting these realizations (and, consequently transforming the corresponding invariant equations into one another).

One more point is that our classification is in full accordance with the results of Sokolov [32] and Magadeev [33]. These papers discuss, in particular, estimates for the dimension of the symmetry algebras of evolution PDEs with one or more spatial variables.

Another important point is the so called quasi-local or non-local symmetries of nonlinear heat conductivity equations. One can construct a number of these kind of symmetries as indicated in [9] and combine them with non-local transformations like the Legendre, Laplace and Bäcklund transformations [6, 34]. As mentioned in the introduction, there is an intriguing possibility of reducing the problem of group classification of the general second order evolution equation (0.2) to that of PDE (0.1). Moreover, the "singular points" of this reduction are the quasi-local symmetries of (0.1) which might correspond to usual Lie symmetries of (0.2). However, these very important questions go beyond the scope of the present paper and, in fact, can be a basis of a separate paper.

Let us stress again that it is our belief that the models most adequately describing real processes should possess the highest symmetry. That is why the most probable candidates for the roles of such models are PDEs admitting four and five-dimensional invariance algebras. The corresponding list of PDEs contain the well-known equations (like the Burgers equation) and several principally new equations which certainly deserve further investigation.

The questions mentioned above are under now study and will be reported on in future publications.

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Appendix 1: solvable Lie algebras.

Three-dimensional solvable Lie algebras ($L = \langle e_1, e_2, e_3\rangle$) over $\mathbf{R}$

The set of three-dimensional solvable Lie algebras consists of the following two decomposable Lie algebras:

$$\begin{aligned} A_{3.1} &= A_1 \oplus A_1 \oplus A_1 = 3A_1; \\ A_{3.2} &= A_{2.2} \oplus A_1, \quad [e_1, e_2] = e_2, \end{aligned}$$

and the following eight classes of non-decomposable Lie algebras:

$$\begin{aligned} A_{3.3} &: \ [e_2, e_3] = e_1; \\ A_{3.4} &: \ [e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2; \\ A_{3.5} &: \ [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2; \\ A_{3.6} &: \ [e_1, e_3] = e_1, \quad [e_2, e_3] = -e_2; \\ A_{3.7} &: \ [e_1, e_3] = e_1, \quad [e_2, e_3] = qe_2 \ \ (0 < |q| < 1); \\ A_{3.8} &: \ [e_1, e_3] = -e_2, \quad [e_2, e_3] = e_1; \\ A_{3.9} &: \ [e_1, e_3] = qe_1 - e_2, \quad [e_2, e_3] = e_1 + qe_2, \ (q > 0). \end{aligned}$$

We note that the algebra $A_{3.3}$ is nilpotent. Note also that we have $A_{3.i}$ $(i = 3, 4, \ldots, 9)$ such that $\langle e_1, e_2\rangle = A_{2.1} = 2A_1$.

Four-dimensional solvable Lie algebras ($L = \langle e_1, e_2, e_3, e_4\rangle$) over $\mathbf{R}$

Amongst the four-dimensional Lie algebras there are 10 decomposable algebras: $4A_1 = A_{3.1} \oplus A_1$, $A_{2.2} \oplus 2A_1 = A_{2.2} \oplus A_{2.1}$, $A_{2.2} \oplus A_{2.2} = 2A_{2.2}, A_{3.i} \oplus A_1$ $(i = 3, 4, \ldots, 9)$; and 10 non-decomposable solvable Lie algebras:

$$\begin{aligned} A_{4.1} &: \ [e_2, e_4] = e_1, \quad [e_3, e_4] = e_2; \\ A_{4.2} &: \ [e_1, e_4] = qe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3, \quad q \neq 0; \\ A_{4.3} &: \ [e_1, e_4] = e_1, \quad [e_3, e_4] = e_2; \\ A_{4.4} &: \ [e_1, e_4] = e_1, \quad [e_2, e_4] = e_1 + e_2, \quad [e_3, e_4] = e_2 + e_3; \\ A_{4.5} &: \ [e_1, e_4] = e_1, \quad [e_2, e_4] = qe_2, \quad [e_3, e_4] = pe_3, \quad -1 \le p \le q \le 1, \quad p \cdot q \neq 0; \\ A_{4.6} &: \ [e_1, e_4] = qe_1, \quad [e_2, e_4] = pe_2 - e_3, \quad [e_3, e_4] = e_2 + pe_3, \quad q \neq 0, \quad p \ge 0; \\ A_{4.7} &: \ [e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_2 + e_3; \\ A_{4.8} &: \ [e_2, e_3] = e_1, \quad [e_1, e_4] = (1+q)e_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = qe_3, \quad |q| \le 1; \\ A_{4.9} &: \ [e_2, e_3] = e_1, \quad [e_1, e_4] = 2qe_1, \quad [e_2, e_4] = qe_2 - e_3, \quad [e_3, e_4] = e_2 + qe_3, \quad q \ge 0; \\ A_{4.10} &: \ [e_1, e_3] = e_1, \quad [e_2, e_3] = e_2, \quad [e_1, e_4] = -e_2, \quad [e_2, e_4] = e_1. \end{aligned}$$

Five-dimensional solvable Lie algebras ($L = \langle e_1, e_2, \ldots, e_5 \rangle$) over $\mathbf{R}$

The set of non-isomorphic five-dimensional Lie algebras is exhausted by 27 types of decomposable algebras: $5A_1, A_{2.2} \oplus 3A_1, 2A_{2.2} \oplus A_1, A_{3.i} \oplus 2A_1$ $(i = 3, 4, \ldots, 8), A_{3.i} \oplus A_{2.2} (i = 3, 4, \ldots 8), A_{4.i} \oplus A_1$ $(i = 1, \ldots, 10)$; and 39 non-decomposable solvable algebras:

$A_{5.1}$: $[e_3, e_5] = e_1, \quad [e_4, e_5] = e_2;$

$A_{5.2}$: $[e_2, e_5] = e_1, \quad [e_3, e_5] = e_2, \quad [e_4, e_5] = e_3;$

$A_{5.3}$: $[e_2, e_4] = e_3, \quad [e_2, e_5] = e_1, \quad [e_4, e_5] = e_2;$

$A_{5.4}$: $[e_2, e_4] = e_1, \quad [e_3, e_5] = e_1;$

$A_{5.5}$: $[e_3, e_4] = e_1, \quad [e_2, e_5] = e_1, \quad [e_3, e_5] = e_2;$

$A_{5.6}$: $[e_3, e_4] = e_1, \quad [e_2, e_5] = e_1, \quad [e_3, e_5] = e_2, \quad [e_4, e_5] = e_3;$

$A_{5.7}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = pe_2, \quad [e_3, e_5] = qe_3,$
$[e_4, e_5] = re_4, \quad -1 \leq r \leq q \leq p \leq 1, \; rpq \neq 0;$

$A_{5.8}$: $[e_2, e_5] = e_1, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = pe_4, \quad 0 < |p| \leq 1;$

$A_{5.9}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_5, \quad [e_3, e_5] = pe_3, \quad [e_4, e_5] = qe_4, \quad 0 \neq q \leq p;$

$A_{5.10}$: $[e_2, e_5] = e_1, \quad [e_3, e_5] = e_2, \quad [e_4, e_5] = e_4;$

$A_{5.11}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_2, \quad [e_3, e_5] = e_2 + e_3, \quad [e_4, e_5] = pe_4, \quad p \neq 0;$

$A_{5.12}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_2, \quad [e_3, e_5] = e_2 + e_3, \quad [e_4, e_5] = e_3 + e_4;$

$A_{5.13}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = pe_2, \quad [e_3, e_5] = qe_3 - re_4,$
$[e_4, e_5] = qe_4 + re_3, \quad |p| \leq 1, \quad p \cdot r \neq 0, \quad q \geq 0;$

$A_{5.14}$: $[e_2, e_5] = e_1, \quad [e_3, e_5] = pe_3 - e_4, \quad [e_4, e_5] = e_3 + pe_4, \quad p \geq 0;$

$A_{5.15}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_2, \quad [e_3, e_5] = pe_3,$
$[e_4, e_5] = e_3 + pe_4, \; -1 \leq p \leq 1;$

$A_{5.16}$: $[e_1, e_5] = e_1, \quad [e_2, e_5] = e_1 + e_2, \quad [e_3, e_5] = pe_3 - qe_4,$
$[e_4, e_5] = qe_3 + pe_4, \quad p \geq 0, \quad q \neq 0;$

$A_{5.17}$: $[e_1, e_5] = pe_1 - e_2, \quad [e_2, e_5] = e_1 + pe_2, \quad [e_3, e_5] = qe_3 - re_4,$
$[e_4, e_5] = re_3 + qe_4, \quad r \neq 0, \quad p, q \in \mathbf{R};$

$A_{5.18}$: $[e_1, e_5] = pe_1 - e_2, \quad [e_2, e_5] = e_1 + pe_2, \quad [e_3, e_5] = e_1 + pe_3 - e_4,$
$[e_4, e_5] = e_2 + e_3 - pe_4, \quad p \in \mathbf{R};$

$A_{5.19}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = (1 + p)e_1, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = pe_3,$
$[e_4, e_5] = qe_4, \; p \in \mathbf{R}, \; q \neq 0;$

$A_{5.20}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = (1 + p)e_2, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = pe_3,$
$[e_4, e_5] = e_1 + (1 + p)e_4, \quad p, q \in \mathbf{R};$

$A_{5.21}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = 2e_1, \quad [e_2, e_5] = e_2 + e_3, \quad [e_3, e_5] = e_3 + e_4, \quad [e_4, e_5] = e_4;$

$A_{5.22}$: $[e_2, e_3] = e_1, \quad [e_2, e_5] = e_3, \quad [e_4, e_5] = e_4;$

$A_{5.23}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = 2e_1, \quad [e_2, e_5] = e_2 + e_3,$
$[e_3, e_5] = e_3, \quad [e_4, e_5] = pe_4, \quad p \neq 0;$

$A_{5.24}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = 2e_1, \quad [e_2, e_5] = e_2 + e_3,$
$[e_3, e_5] = e_3, \quad [e_4, e_5] = \epsilon e_1 + 2e_4, \quad \epsilon = \pm 1;$

$A_{5.25}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = 2pe_1, \quad [e_2, e_5] = pe_2 + e_3, \quad [e_3, e_5] = -e_2 + pe_3,$
$[e_4, e_5] = qe_4, \quad p \in \mathbf{R}, \quad q \neq 0;$

$A_{5.26}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = 2pe_1, \quad [e_2, e_5] = pe_2 + e_3, \quad [e_3, e_5] = -e_2 + pe_3,$

$[e_4, e_5] = \epsilon e_1 + 2pe_4, \quad \epsilon = \pm 1, \quad p \in \mathbf{R};$

$A_{5.27}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = e_1, \quad [e_3, e_5] = e_3 + e_4, \quad [e_4, e_5] = e_1 + e_4;$

$A_{5.28}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = (1+p)e_1, \quad [e_2, e_5] = pe_2,$
$[e_3, e_5] = e_3 + e_4, \quad [e_4, e_5] = e_4, \quad p \in \mathbf{R};$

$A_{5.29}$: $[e_2, e_3] = e_1, \quad [e_1, e_5] = e_1, \quad [e_2, e_5] = e_2, \quad [e_3, e_5] = e_4;$

$A_{5.30}$: $[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2, \quad [e_1, e_5] = (2+p)e_1, \quad [e_2, e_5] = (1+p)e_2,$
$[e_3, e_5] = pe_3, \quad [e_4, e_5] = e_4, p \in \mathbf{R};$

$A_{5.31}$: $[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2, \quad [e_1, e_5] = 3e_1,$
$[e_2, e_5] = 2e_2, \quad [e_3, e_5] = e_3, \quad [e_4, e_5] = e_3 + e_4;$

$A_{5.32}$: $[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2, \quad [e_1, e_5] = e_1, \quad [e_2, e_5] = e_2,$
$[e_3, e_5] = pe_1 + e_3, \quad p \in \mathbf{R};$

$A_{5.33}$: $[e_1, e_4] = e_1, \quad [e_3, e_4] = pe_3, \quad [e_2, e_5] = e_2,$
$[e_3, e_5] = qe_3, \quad p, q \in \mathbf{R}, \quad p^2 + q^2 \neq 0;$

$A_{5.34}$: $[e_1, e_4] = pe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \quad [e_1, e_5] = e_1, \quad [e_3, e_5] = e_2, \quad p \in \mathbf{R};$

$A_{5.35}$: $[e_1, e_4] = pe_1, \quad [e_2, e_4] = e_2, \quad [e_3, e_4] = e_3, \quad [e_1, e_5] = qe_1,$
$[e_2, e_5] = -e_3, \quad [e_3, e_5] = e_2, p, q \in \mathbf{R}, \quad p^2 + q^2 \neq 0;$

$A_{5.36}$: $[e_2, e_3] = e_1, \quad [e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_2, e_5] = -e_2, \quad [e_3, e_5] = e_3;$

$A_{5.37}$: $[e_2, e_3] = e_1, \quad [e_1, e_4] = 2e_1, \quad [e_2, e_4] = e_2,$
$[e_3, e_4] = e_3, \quad [e_2, e_5] = -e_3, \quad [e_3, e_5] = e_2;$

$A_{5.38}$: $[e_1, e_4] = e_1, \quad [e_2, e_5] = e_2, \quad [e_4, e_5] = e_3;$

$A_{5.39}$: $[e_1, e_4] = e_1, \quad [e_2, e_4] = e_2, \quad [e_1, e_5] = -e_2, \quad [e_2, e_5] = e_1, \quad [e_4, e_5] = e_3.$

Appendix 2: Lie algebras which are semi-direct sums of semi-simple and solvable algebras.

1. Lie algebras of dimensions 5 and 6.

$sl(2,\mathbf{R}) \Subset A_{2.1}$: $[e_1, e_4] = e_4, \quad [e_2, e_5] = e_4, \; [e_3, e_4] = e_5, \; [e_1, e_5] = -e_5;$

$so(3) \Subset A_{3.1}$: $[e_1, e_5] = e_6, \quad [e_2, e_4] = -e_6, \; [e_3, e_4] = e_5, \; [e_1, e_6] = -e_5,$
$[e_2, e_6] = e_4, \quad [e_3, e_5] = -e_4;$

$sl(2,\mathbf{R}) \Subset A_{3.i}$: $[e_1, e_4] = e_4, \quad [e_2, e_5] = e_4,$
$i = 3, A_{3.3} = \langle e_6, e_4, e_5 \rangle$ $\quad [e_3, e_4] = e_5, \; [e_1, e_5] = -e_5,$
$i = 5, \; A_{3.5} = \langle e_4, e_5, e_6 \rangle;$

$sl(2,\mathbf{R}) \Subset A_{3.1}$: $[e_1, e_4] = 2e_4, \; [e_2, e_5] = 2e_4, \; [e_3, e_4] = e_5,$
$[e_1, e_6] = -2e_6, \; [e_2, e_6] = e_5, \; [e_3, e_5] = 2e_6.$

2. Lie algebras of dimension 7.

$so(3) \Subset A_{4.5}(p=q=1)$: $[e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,\ [e_3,e_4]=e_5,$
$[e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4;$

$so(3) \Subset 4A_1$: $[e_1,e_4]=\frac{1}{2}e_7,\ [e_2,e_4]=\frac{1}{2}e_5,\ [e_3,e_4]=\frac{1}{2}e_6,$
$[e_1,e_5]=\frac{1}{2}e_6,\ [e_2,e_5]=-\frac{1}{2}e_4,\ [e_3,e_5]=-\frac{1}{2}e_7,$
$[e_1,e_6]=-\frac{1}{2}e_5,\ [e_2,e_6]=\frac{1}{2}e_7,\ [e_3,e_6]=-\frac{1}{2}e_4,$
$[e_1,e_7]=-\frac{1}{2}e_4,\ [e_2,e_7]=-\frac{1}{2}e_6,\ [e_3,e_7]=\frac{1}{2}e_5;$

$sl(2,\mathbf{R}) \Subset A_{4.i}$: $[e_1,e_4]=e_4,\ [e_2,e_5]=e_4,$
$i=5: A_{4.5}\ (q=1),$ $[e_3,e_4]=e_5,\ [e_1,e_5]=-e_5,$
$i=8: A_{4.8}(q=1),$
$A_{4.8}=\langle e_6,e_4,e_5,e_7\rangle;$

$sl(2,\mathbf{R}) \Subset A_{4.5}$: $[e_1,e_4]=2e_4,\ [e_2,e_5]=2e_4,\ [e_3,e_4]=e_5,$
$A_{4.5}(p=q=1)$ $[e_1,e_6]=-2e_6,\ [e_2,e_6]=e_5,\ [e_3,e_5]=2e_6;$

$sl(2,\mathbf{R}) \Subset 4A_1$: $[e_1,e_4]=3e_4,\ [e_2,e_5]=3e_4,\ [e_3,e_4]=e_5,$
$[e_1,e_5]=e_5,\ [e_2,e_6]=2e_5,\ [e_3,e_5]=2e_6,$
$[e_1,e_6]=-e_6,\ [e_2,e_7]=e_6,\ [e_3,e_6]=3e_7,\ [e_1,e_7]=-3e_7;$

$sl(2,\mathbf{R}) \Subset 4A_1$: $[e_1,e_4]=e_4,\ [e_2,e_5]=e_4,\ [e_3,e_4]=e_5,\ [e_1,e_5]=-e_5,$
$[e_1,e_6]=e_6,\ [e_2,e_7]=e_6,\ [e_3,e_6]=e_7,\ [e_1,e_7]=-e_7.$

3. Lie algebras of dimension 8.

$so(3) \Subset A_{5.7}$: $[e_1,e_5]=e_6,\ [e_2,e_4]=-e_6,\ [e_3,e_4]=e_5,$
$A_{5.7}(p=q=1)$ $[e_1,e_6]=-e_5,\ [e_2,e_6]=e_4,\ [e_3,e_5]=-e_4;$

$so(3) \Subset A_{5.i}$: $[e_1,e_4]=\frac{1}{2}e_7,\ [e_2,e_4]=\frac{1}{2}e_5,$
$i=4: A_{5.4}=\langle e_8,e_4,e_7,e_5,e_6\rangle,$ $[e_3,e_4]=\frac{1}{2}e_6,\ [e_1,e_5]=\frac{1}{2}e_6,$
$i=7: A_{5.7}(p=q=r=1),$ $[e_2,e_5]=-\frac{1}{2}e_4,\ [e_3,e_5]=-\frac{1}{2}e_7,$
$i=17: A_{5.17}\ (p=q,\ r=1),$ $[e_1,e_6]=-\frac{1}{2}e_5,\ [e_2,e_6]=\frac{1}{2}e_7,$
$A_{5.17}=\langle e_4,e_6,e_5,e_7,e_8\rangle,$ $[e_3,e_6]=-\frac{1}{2}e_4,\ [e_1,e_7]=-\frac{1}{2}e_4,$
$[e_2,e_7]=-\frac{1}{2}e_6,\ [e_3,e_7]=\frac{1}{2}e_5;$

$$
\begin{aligned}
so(3) \Subset 5A_1 \quad &: \quad [e_1,e_4]=\frac{1}{2}e_7,\ [e_1,e_5]=-\frac{1}{2}e_6,\\
&\quad [e_1,e_6]=2e_5-e_8,\ [e_1,e_7]=-2e_4,\\
&\quad [e_1,e_8]=3e_6,\ [e_2,e_4]=\frac{1}{2}e_6,\\
&\quad [e_2,e_5]=\frac{1}{2}e_7,\ [e_2,e_6]=-2e_4,\\
&\quad [e_2,e_7]=-2e_5-e_8,\ [e_2,e_8]=3e_7,\\
&\quad [e_3,e_4]=2e_5,\ [e_3,e_5]=-2e_4,\\
&\quad [e_3,e_6]=e_7,\ [e_3,e_7]=-e_6;
\end{aligned}
$$

$$
\begin{aligned}
sl(2,\mathbf{R}) \Subset A_{5.i} \quad &: \quad [e_1,e_4]=e_4,\ [e_2,e_5]=e_4,\\
&\quad [e_3,e_4]=e_5,\ [e_1,e_5]=-e_5;\\
i=4: A_{5.4}=\langle e_8,e_4,e_6,e_5,e_7\rangle,&\\
i=7,8: A_i(p=1),&\\
A_{5.8}=\langle e_6,e_7,e_4,e_5,e_8\rangle,&\\
i=9:\cong A_{5.9},&\\
i=13,19,20: A_{5.i}\ (p=1),&\\
A_{5.i}\ (i=19,20)=\langle e_6,e_4,e_5,e_7,e_8\rangle,&
\end{aligned}
$$

$$
\begin{aligned}
sl(2,\mathbf{R}) \Subset A_{5.7} \quad &: \quad [e_1,e_4]=2e_4,\ [e_2,e_5]=2e_4,\ [e_3,e_4]=e_5,\\
&\quad [e_1,e_5]=-2e_6;\ \ [e_2,e_6]=e_5,\ \ [e_3,e_5]=2e_6;
\end{aligned}
$$

$$
\begin{aligned}
sl(2,\mathbf{R}) \Subset A_{5.i} \quad &: \quad [e_1,e_4]=e_4,\ [e_2,e_5]=e_4,\\
i=4: A^{\epsilon}_{5.4}, &\quad\quad [e_3,e_4]=e_5,\ [e_1,e_5]=-e_5,\\
i=1: A_{5.1}, &\quad\quad [e_1,e_6]=e_6,\ [e_2,e_7]=e_6,\\
i=3:\cong A_{5.3}, &\quad\quad [e_3,e_6]=e_7,\ [e_1,e_7]=-e_7,\\
i=15: A_{5.15}\ (p=1)&\\
i=7: A_{5.7}\ (p=q=1,\ -1\le r\le 1)&\\
i=17: A_{5.17}\ (p=q,\ r=1, p\ge 0)&\\
A_{5.i}\ (i=7,17)=\langle e_4,e_6,e_5,e_7,e_8\rangle;&
\end{aligned}
$$

$$
\begin{aligned}
sl(2,\mathbf{R}) \Subset A_{5.i} \quad &: \quad [e_1,e_4]=3e_4,\ [e_2,e_5]=3e_4,\\
&\quad [e_3,e_4]=e_5,\ [e_1,e_5]=e_5;\\
i=4:\cong A_{5.4} &\quad\quad [e_2,e_6]=2e_5,\ [e_3,e_5]=2e_6,\ [e_1,e_6]=-e_6,\\
i=7: A_{5.7}(p=q=r=1) &\quad\quad [e_2,e_7]=e_6,\ [e_3,e_6]=3e_7,\ [e_1,e_7]=-3e_7;
\end{aligned}
$$

$$
\begin{aligned}
sl(2,\mathbf{R}) \Subset 5A_1 \quad &: \quad [e_1,e_4]=4e_4,\ [e_2,e_5]=4e_4,\\
&\quad [e_3,e_4]=e_5,\ [e_1,e_5]=2e_5;\\
&\quad [e_2,e_6]=3e_5,\ [e_3,e_5]=2e_6,\\
&\quad [e_1,e_7]=-2e_7,\ [e_2,e_7]=2e_6,\\
&\quad [e_3,e_6]=3e_7,\ [e_1,e_8]=-4e_8,
\end{aligned}
$$

$$[e_2, e_8] = e_7,\ [e_3, e_7] = 4e_8;$$

$$\begin{aligned} sl(2,\mathbf{R}) \inplus 5A_1 \quad : \quad & [e_1, e_4] = 2e_4,\ [e_2, e_5] = 2e_4, \\ & [e_3, e_4] = e_5,\ [e_1, e_6] = -2e_6; \\ & [e_2, e_6] = e_5,\ [e_3, e_5] = 2e_6, \\ & [e_1, e_7] = e_7,\ [e_2, e_8] = e_7, \\ & [e_3, e_7] = e_8,\ [e_1, e_8] = -e_8. \end{aligned}$$

In giving the type of the radicals, we have followed the rule that the bases of the radicals consist of the operators $e_4, \ldots, e_m$, whenever the basis is not given explicitly; where it is given explicitly, then the basis operators are ordered as in the corresponding solvable algebra. For instance, in the algebra $sl(2,\mathbf{R}) \inplus A_{5.17}$ we have written $A_{5.17} = \langle e_4, e_6, e_5, e_7, e_8 \rangle$. This means that the basis operators satisfy the commutation relations which define the algebra $A_{5.17}$ given in the list of solvable algebras. To obtain the commutation relations for the algebra $A_{5.17}$, we replace the operators e_4, e_5, e_6, e_7, e_8 as follows:

$$e_4 \to e_1, \quad e_6 \to e_2, \quad e_5 \to e_3, \quad e_7 \to e_4, \quad e_8 \to e_5.$$

Furthermore, for the five-dimensional radicals $N = \langle e_4, e_5, e_6, e_7, e_8 \rangle$ we use the notation

$$\begin{aligned} \cong A_{5.9} \quad : \quad & [e_4, e_8] = e_4, \quad [e_5, e_8] = e_5, \quad [e_6, e_8] = pe_6, \\ & [e_7, e_8] = e_6 + pe_7,\ p \neq 0; \\ A^{\epsilon}_{5.4} \quad : \quad & [e_4, e_8] = e_8, \quad [e_6, e_7] = \epsilon e_8, \quad \epsilon = \pm 1; \\ \cong A_{5.3} \quad : \quad & [e_6, e_8] = e_4, \quad [e_7, e_8] = e_5, \quad [e_6, e_7] = e_8; \\ \cong A_{5.4} \quad : \quad & [e_4, e_7] = e_8, \quad [e_5, e_6] = -3e_8. \end{aligned}$$

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