On quasi-exactly solvable matrix models

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Abstract

An efficient procedure for constructing quasi-exactly solvable matrix models is suggested. It is based on the fact that the representation spaces of representations of the algebra $sl(2,\mathbf{R})$ within the class of first-order matrix differential operators contain finite dimensional invariant subspaces.

The Lie-algebraic approach to constructing quasi-exactly solvable one-dimensional stationary Schrödinger equations as suggested by Shifman and Turbiner [1, 2] is based on the properties of the representations of the algebra $sl(2,\mathbf{R})$

$$[Q_0,\ Q_\pm] = \pm Q_\pm, \quad [Q_-,\ Q_+] = 2Q_0 \tag{1}$$

by first-order differential operators. Namely, the approach in question utilizes the fact that the representation space of the algebra $sl(2,\mathbf{R})$ having the basis elements

$$Q_- = \frac{d}{dx}, \quad Q_0 = x\frac{d}{dx} - \frac{n}{2}, \quad Q_+ = x^2\frac{d}{dx} - nx, \tag{2}$$

where n is an arbitrary natural number, has an $(n+1)$-dimensional invariant subspace. Its basis is formed by the polynomials in x of the order not higher than n (for further details see the monograph by Ushveridze [3] and references therein).

The aim of the present paper is to extend the above Lie-algebraic approach in order to make it applicable to analyzing eigenvalue problems for matrix differential operators.

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The key idea is that the basis elements of the algebra $sl(2,\mathbf{R})$ are searched for within the class of matrix differential operators

$$Q = \xi(x)\frac{d}{dx} + \eta(x), \tag{3}$$

where $\xi(x), \eta(x)$ are some matrix-valued functions of the corresponding dimension. Furthermore, the representation space of $sl(2,\mathbf{R})$ must contain a finite-dimensional subspace. Provided these requirements are met, a quasi-exactly solvable matrix model is obtained by composing a linear combination of the basis elements of the algebra $sl(2,\mathbf{R})$ with constant matrix coefficients.

Thus, to get a quasi-exactly solvable matrix model we need to solve two intermediate problems

1) to solve the relations (1) within the class (3),

2) to pick out from the set of thus obtained realizations of the algebra $sl(2,\mathrm{C})$ those ones whose representation space contains a finite-dimensional invariant subspace.

In a sequel we will restrict our considerations to the case when $\xi(x)$ is a scalar multiple of the unit matrix. Given this restriction, a simple computation yields that any representation of $sl(2,\mathbf{R})$ within the class of operators (3) is equivalent to the following one:

$$Q_- = \frac{d}{dx}, \quad Q_0 = x\frac{d}{dx} + A, \quad Q_+ = x^2\frac{d}{dx} + 2xA + B, \tag{4}$$

where A, B are constant $N \times N$ matrices satisfying the relation

$$[A,\ B] = B. \tag{5}$$

Next, we have to investigate under which circumstances the representation space of the algebra (4) has a finite-dimensional invariant subspace $\mathcal{I}_n$ with basis elements

$$\vec{f}_i(x) = \sum_{j=1}^{N} F_{ij}(x)\vec{e}_j, \quad i = 1, \ldots, n.$$

Here $\vec{e}_1, \ldots, \vec{e}_N$ is the orthonormal basis of the space $\mathbf{R}^N$. It occurs that the components of the functions $\vec{f}_i$ are necessarily polynomials in x of the order not higher than $n-1$.

Theorem 1 *Let the functions $\vec{f}_1(x), \ldots, \vec{f}_n(x)$ form the basis of the invariant subspace $\mathcal{I}_n$ of the representation space of the Lie algebra $\langle \frac{d}{dx}, x\frac{d}{dx} + A\rangle$, where A is a constant $N \times N$ matrix. Then*

$$\frac{d^n \vec{f}_i(x)}{dx^n} = \vec{0}, \quad i = 1, \ldots, n. \tag{6}$$

We will give a sketch of the proof for the case, when $N = 2$. The requirement that a representation space of a Lie algebra under study contains a finite-dimensional invariant subspace means that the following relations hold

$$\frac{d}{dx}\vec{f}_i = \sum_{j=1}^{n} \Lambda_{ij}\vec{f}_j, \tag{7}$$

$$\left(x\frac{d}{dx} + A\right)\vec{f}_i = \sum_{j=1}^{n} L_{ij}\vec{f}_j, \tag{8}$$

where Λ_{ij}, L_{ij} are arbitrary complex constants, $i, j = 1, \ldots, n$.

Solving (7) yields

$$\vec{f}_i(x) = \sum_{j=1}^{N}\sum_{k=1}^{n} \left(\mathrm{e}^{\Lambda x}\right)_{ik} C_{kj}\vec{e}_j, \tag{9}$$

where C_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, N$ are arbitrary complex constants and the symbol $(A)_{ij}$ stands for the (i, j)th entry of the matrix A.

Next, from the requirement (8) we get

$$x\Lambda C + CA = \mathrm{e}^{-\Lambda x}\, L\, \mathrm{e}^{\Lambda x} C. \tag{10}$$

Making use of the Cambell-Hausdorff formula and equating coefficients of the powers of x give the following infinite set of algebraic equations for unknown matrices L, Λ, C, A:

$$LC = CA, \tag{11}$$

$$[L,\ \Lambda]C = \Lambda C, \tag{12}$$

$$\{L,\ \Lambda\}^i\, C = 0, \quad i \geq 2, \tag{13}$$

where

$$\{L,\ \Lambda\}^0 = L, \quad \{L,\ \Lambda\}^i = [\{L,\ \Lambda\}^{i-1},\ \Lambda], \quad i \geq 1.$$

Choosing the basis vectors $\vec{e}_1, \vec{e}_2$ in an appropriate way, we can transform the constant matrix A to the Jordan form. There are two possibilities

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{14}$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}. \tag{15}$$

Here $\lambda, \lambda_1, \lambda_2$ are arbitrary constants.

Case 1. Let the matrix A be of the form (14). If we denote the first and the second columns of the 2×2 matrix C as $\vec{C}_1$ and $\vec{C}_2$, then equations (11) can be rewritten to become

$$L\vec{C}_1 = \lambda\vec{C}_1, \quad L\vec{C}_2 = \vec{C}_1 + \lambda\vec{C}_2.$$

Next, using equations (12), (13) we obtain the following relations

$$L\Lambda^j\vec{C}_1 = \lambda_j\Lambda^j\vec{C}_1, \tag{16}$$

$$L\Lambda^j\vec{C}_2 = \Lambda^j\vec{C}_1 + \lambda_j\Lambda^j\vec{C}_2. \tag{17}$$

Here $\lambda_0 = \lambda,\ \lambda_{j+1} = \lambda_j + 1,\ j = 0, 1, \ldots$.

It follows from (16) that $\vec{a}_{i+1} = \Lambda^i\vec{C}_1,\ i \geq 0$ are eigenvectors of the $n \times n$ matrix L corresponding to eigenvalues $\lambda_i = \lambda + i$ and, what is more, since these eigenvectors correspond to distinct eigenvalues, they are linearly independent. As there are at most n linearly independent eigenvectors of the matrix L, the relation

$$\Lambda^m\vec{C}_1 = \vec{0} \tag{18}$$

with some $m \leq n$ holds true.

Combining (17) and (18) yields

$$L\Lambda^{m+i}\vec{C}_1 = \lambda_{m+i}\Lambda^{m+i}\vec{C}_2, \quad i \geq 0. \tag{19}$$

Hence, we conclude that the vectors $\vec{a}_{m+i} = \Lambda^{m+i}\vec{C}_2,\ i \geq 0$ are eigenvectors of the matrix L forming together with the vectors $\vec{a}_1, \ldots, \vec{a}_m$ the system of its linearly independent eigenvectors. As an $n \times n$ matrix has at most n linearly independent eigenvectors, the relation

$$\Lambda^n\vec{C}_2 = \vec{0} \tag{20}$$

holds true.

In view of (18), (20) the matrix $C = (\vec{C}_1, \vec{C}_2)$ satisfy the following matrix equation

$$\Lambda^n\, C = 0.$$

Due to this fact, (9) reads as

$$\vec{f}_i(x) = \sum_{j=1}^{N} \sum_{k=1}^{n} \left(1 + x\Lambda + \frac{x^2}{2!}\Lambda^2 + \cdots + \frac{x^{n-1}}{(n-1)!}\Lambda^{n-1}\right)_{ik} C_{kj}\vec{e}_j. \tag{21}$$

Thus, the components of the vectors $\vec{f}_j$ are polynomials of the order not higher than $n-1$, which is the same as what was to be proved.

Case 2. We turn now to the case when the matrix A is given by (15). Denoting the first and the second columns of the 2×2 matrix C as $\vec{C}_1$ and $\vec{C}_2$, we rewrite equations (11) as follows

$$L\vec{C}_1 = \lambda_1\vec{C}_1, \quad L\vec{C}_2 = \lambda_2\vec{C}_2.$$

Next, using equations (12), (13) we obtain the relations:

$$L\Lambda^j\vec{C}_1 = \alpha_j\Lambda^j\vec{C}_1, \tag{22}$$

$$L\Lambda^j\vec{C}_2 = \beta_j\Lambda^j\vec{C}_2, \tag{23}$$

where

$$\alpha_0 = \lambda_1, \quad \alpha_{j+1} = \alpha_j + 1, \quad j = 0, 1, \ldots,$$
$$\beta_0 = \lambda_2, \quad \beta_{j+1} = \beta_j + 1, \quad j = 0, 1, \ldots.$$

Thus, the vectors $\vec{e}_{i+1} = \Lambda^i\vec{C}_1$, $i = 0, 1, \ldots$ are eigenvectors of the $n\times n$ matrix L with eigenvalues $\alpha_i = \lambda_1 + i$, $i = 0, 1, \ldots$. As these eigenvalues are distinct, the vectors $\vec{e}_i$ are linearly independent. Taking into account that an $n\times n$ matrix can have at most n linearly independent eigenvectors we conclude that $\Lambda^n\vec{C}_1 = \vec{0}$. Similarly, we get the relation $\Lambda^n\vec{C}_2 = \vec{0}$. Hence it follows that the expression (9) takes the form (21), which is the same as what was to be proved. •

Consequently, the most general finite-dimensional invariant subspace of the representation space of the algebra (4) is spanned by vectors whose components are finite-order polynomials in x. We postpone a detailed study of these representations with arbitrary N for future publications and concentrate on the case $N = 2$. There are two families of inequivalent finite dimensional representations of the algebra $sl(2, \mathbf{R})$ having the basis elements (4)

$$\text{I.} \quad A = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & -\frac{m}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

$$\text{II.} \quad A = \begin{pmatrix} -\frac{n}{2} & 0 \\ 0 & \frac{2-n}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}. \tag{25}$$

Here n, m are abritrary natural numbers with $n \geq m$.

Representations of the form (4), (24) are the direct sums of two irreducible representations realized on the representation spaces

$$\mathcal{R}_1 = \langle \vec{e}_1, x\vec{e}_1, \ldots, x^n\vec{e}_1 \rangle, \quad \mathcal{R}_2 = \langle \vec{e}_2, x\vec{e}_2, \ldots, x^m\vec{e}_2 \rangle,$$

where $\vec{e}_1 = (1,0)^{\mathrm{T}}$, $\vec{e}_2 = (0,1)^{\mathrm{T}}$.

Next, representations (4), (25) are also the direct sums of two irreducible representations realized on the representation spaces

$$\begin{aligned} \mathcal{R}_1 &= \langle n\vec{e}_1, \ldots, nx^j\vec{e}_1 + jx^{j-1}\vec{e}_2, \ldots, nx^n\vec{e}_1 + nx^{n-1}\vec{e}_2 \rangle, \\ \mathcal{R}_2 &= \langle \vec{e}_2, x\vec{e}_2, \ldots, x^{n-2}\vec{e}_2 \rangle. \end{aligned}$$

We will finish the paper with an example of utilizing the above results for obtaining an exactly solvable two-component Dirac-type equation, which is one of the two differential equations composing the Lax pair for the cubic Schrödinger equation (see, e.g. [4, 5]). Consider the following two-component matrix model:

$$\mathcal{H}\vec{w} \equiv i(ax\sigma_2 + b\sigma_1)\frac{d\vec{w}}{dx} + (c_1\sigma_1 + (c_2 + ia)\,\sigma_2)\,\vec{w} = \lambda\vec{w}, \tag{26}$$

where a, b, c_1, c_2 are arbitrary real parameters with $ab \neq 0$ and σ_1, σ_2 are 2×2 Pauli matrices. As

$$\mathcal{H} = ib\sigma_1 Q_- + ia\sigma_2 Q_0 + c_1\sigma_1 + c_2\sigma_2,$$

where Q_-, Q_+ are given by (4) with $A = 1$, the operator $\mathcal{H}$ transforms the $(2n+2)$-dimensional vector space

$$\vec{f}_j(x) = x^{j-1}\vec{e}_1, \quad \vec{f}_{n+j+1}(x) = x^{j-1}\vec{e}_2, \quad j = 1, \ldots, n+1$$

into itself. This means that there exists the constant $(2n+2) \times (2n+2)$ matrix H such that

$$\mathcal{H}\vec{f}_i = \sum_{j=1}^{2n+2} H_{ij}\vec{f}_j, \quad i = 1, \ldots, 2n+2.$$

Hence it immediately follows that the vector-function

$$\vec{\psi}(x) = \sum_{j=1}^{2n+2} \alpha_j \vec{f}_j(x)$$

is the solution of the system of ordinary differential equations (26) with $\lambda = \lambda_0$, provided $(\alpha_1, \ldots, \alpha_{2n+2})$ is an eigenvector of the matrix H with the eigenvalue λ_0.

Making a transformation

$$\begin{aligned} x &= \frac{b}{a}\sinh(ay), \\ \vec{w}(x) &= (\cosh(ay))^{1/2}\exp\left\{-\frac{i}{a}\left(c_1 \arctan\sinh(ay) + c_2 \ln\cosh(ay)\right)\right\} \\ &\quad \times \exp\{-i\sigma_3 \arctan\sinh(ay)\}\,\vec{\psi}(y) \end{aligned}$$

we reduce (26) to the Dirac-type equation

$$i\sigma_1 \frac{d\vec{\psi}}{dy} + \sigma_2 V(y)\vec{\psi}(y) = \lambda\vec{\psi}, \tag{27}$$

where

$$V(y) = \frac{a^2c_2 - b^2c_1\sinh(ay)}{ab\cosh(ay)}$$

is the well-known hyperbolic Pöschel-Teller potential. It has exact solutions of the form

$$\begin{aligned} \psi(y) &= (\cosh(ay))^{-1/2}\exp\left\{\frac{i}{a}\left(c_1 \arctan\sinh(ay) + c_2 \ln\cosh(ay)\right)\right\} \\ &\quad \times \exp\{i \arctan\sinh(ay)\sigma_3\} \sum_{j=1}^{2n+2} \alpha_j \vec{f}_j\left(\frac{b}{a}\sinh(ay)\right). \end{aligned}$$

As the potential V does not depend explicitly on n, the order of the polynomials P_n, Q_n may be arbitrarily large. This means that the Dirac equation (27) with the hyperbolic Pöschel-Teller potential is *exactly-solvable.*

In [6] we suggest an alternative approach to construction of quasi-exactly solvable stationary Schrödinger equations based on their conditional symmetry. We believe that a similar idea should work for matrix models as well. It is intended to devote one of the future publications to a comparison of the conditional symmetry and Lie-algebraic approaches to constructing quasi-exactly solvable Dirac-type equations (27).

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References

[1] A.V. Turbiner, Comm. Math. Phys., **118**, 467 (1988).

[2] M.A. Shifman and A.V. Turbiner, Comm. Math. Phys., **126**, 347 (1989).

[3] A.G. Ushveridze, *Quasi-exactly solvable models in quantum mechanics* (IOP Publ., Bristol, 1993).

[4] Ablovitz M.J., Segur H. *Solitons and the Inverse Scattering Transform*, SIAM, Philadelphia (1981).

[5] Zakharov V.E., Manakov S.V., Novikov S.P. and Pitaevski L.P. *Theory of Solitons: the Inverse Scattering Method*, Consultants Bureau, New York (1980).

[6] Zhdanov R.Z., *J. Math. Phys.*, **37**, 3198 (1996).