New conditional symmetries and exact solutions of the nonlinear wave equation

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Abstract

We have constructed the new class of conditional symmetries of the nonlinear wave equation in 1+3 dimensions. These symmetries enable us to obtain broad families of new exact solutions of the nonlinear wave equation containing up to the four arbitrary functions.

1 Introduction

One of the principal motivations for developing efficient procedures for constructing exact solutions of nonlinear multi-dimensional partial differential equations (PDEs) was not to get some particular solutions as such but to obtain broad families of these in order to be able to solve some initial value or boundary problems for PDEs. It is generally recognized that for the multi-dimensional case the most powerful and systematic method for constructing exact solutions of PDEs having non-trivial symmetry is the symmetry reduction routine [1, 2]. The principal idea of this approach is a reduction of PDE under study to PDEs in two or one independent variables using special substitutions (Ansätze, invariant solutions). Given a symmetry group, the procedure of constructing a complete (in some sense) set of inequivalent Ansätze is fairly algorithmic. For a number of principal nonlinear equations of the mathematical and theoretical physics (the wave, Dirac, Maxwell, $SU(2)$ Yang-Mills, Navier-Stokes equations) the problem of symmetry reduction has

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already been solved in a full generality [3]–[7]. This progress was possible due to a number of strong results on subgroup classification of principal symmetry groups of modern physics, of the Poincaré and Galilei groups and their extensions and different generalizations [8]–[11]. Using so constructed Ansätze broad families of exact solutions of the above mentioned PDEs have been constructed. However, with all the importance of these results they, in fact, cannot be applied to solving initial value or boundary problems since these solutions, as a rule, contain no arbitrary functions. Only for the case when the symmetry group of the equation under study is infinite dimensional there is a regular procedure (generating solutions by final transformations from the symmetry group see, e.g. [1, 2])) enabling to get exact solutions containing arbitrary functions.

By these very reasons there is a lot of activity aimed at exploring non-classical (conditional) symmetries of multi-dimensional PDEs in order to get exact solutions involving arbitrary functions. Though non-classical (conditional) symmetries of differential equations were introduced long ago (see [12] and also [13]–[17]) a problem of constructing conditional symmetries of multi-dimensional nonlinear partial differential equations still remains a challenging one. The reason is quite obvious, since finding a conditional symmetry of PDE for a function u of n variables requires solving an over-determined system of PDEs in $n+1$ dimensions. That is why, a progress in investigating conditional symmetries of multi-dimensional ($n>2$) PDEs relies heavily on existence of efficient methods for handling over-determined systems of PDEs in four and higher dimensions.

In papers [15], [18]–[21] we have developed a technique enabling us to integrate some (1+3)-dimensional over-determined Poincaré-invariant over-determined systems of PDEs. Using these results we have constructed broad classes of Ansätze corresponding to conditional symmetries of the nonlinear wave, Dirac and Yang-Mills equations [7]. An underlying idea of our approach is quite simple and is based on the following observation. We have noted that quite a few of invariant solutions (i.e., solutions obtained via symmetry reduction routine) of the above enumerated equations are particular cases of more general solutions which correspond to conditional symmetry of the equations under study and can be efficiently constructed in explicit form. Putting this idea in a more mathematical way we have formulated the following scheme of conditional symmetry reduction [7]:

1) the maximal (in Lie sense) invariance group of the equation under study is found by the Lie method.

2) subgroup analysis of the invariance group is carried out, each subgroup giving rise to some Ansatz which reduces PDE in question to an equation having a smaller dimension. As a rule, Ansätze obtained in this way have a quite definite structure which is determined by the representation of the symmetry group.

3) the general form of the invariant Ansatz is obtained. This Ansatz includes several scalar functions $\theta_1, \ldots, \theta_N$ satisfying some compatible over-determined system of nonlinear PDEs (reduction conditions).

4) equations for $\theta_1, \ldots, \theta_N$ are integrated.

In this way we have obtained a generalization of the famous meron and instanton solutions of the nonlinear Dirac equations obtained with the help of the Heisenberg Ansatz [22, 23]. Namely, we have constructed the class of exact solutions involving three arbitrary functions such that choosing these to be equal to zero yield the above mentioned solutions. Furthermore, we have constructed generalizations of the invariant solutions of the wave equation with cubic nonlinearity giving rise to the instanton and meron solutions of the Yang-Mills equations obtainable via the 't Hooft-Corrigan-Fairlie-Wilczek Ansatz (see, e.g. [24]).

However, our previous studies were restricted to investigating non-classical reductions of PDEs either to ordinary differential equations or to PDEs in two independent variables, one of them being parametrical. Saying about the parametrical variable we mean that the corresponding equation contains no derivatives with respect to this variable though the coefficients of the equation may depend on it. *Any* solution of a reduced equation of this kind that involves arbitrary parameters (say, integration constants) will automatically depend on arbitrary functions. In view of a striking progress in integrating of a number of two- and even three-dimensional PDEs by the inverse scattering technique it would be of great interest to utilize our approach to obtain conditional symmetry reductions to differential equations in two independent variables with a third parametrical variable. This might open a possibility to apply the results of the soliton theory to get new exact solutions containing arbitrary functions.

In the present paper we investigate reductions of the nonlinear wave equation

$$\Box u = F(u), \tag{1}$$

to PDEs in three independent variables such that one of the variables is parametrical. Here $\Box = \partial^2/\partial x_0^2 - \Delta$ is the d'Alembert operator, $u = u(x_0, x_1, x_2, x_3)$ is a real-valued function and $F(u)$ is an arbitrary smooth function. Using these reductions we construct new families of exact solutions of equations (1) containing several arbitrary functions of one variable.

2 Conditional symmetry of the nonlinear wave equation

In what follows we will adapt the procedure described in Introduction in order to construct exact solutions of the nonlinear wave equation (1) containing arbitrary functions. The approach is based on the observation that there exist dimensional reductions of Eq.(1) by subgroups of the Poincaré group such that reduced equations do not contain derivatives with respect to one of the new independent variables. The simplest example is the Ansatz

$$u(x) = \varphi(\omega_0 \equiv x_0 + x_3,\ \omega_1 \equiv x_1,\ \omega_2 \equiv x_2). \tag{2}$$

Inserting (2) into Eq.(1) yields reduced PDE:

$$\varphi_{\omega_1\omega_1} + \varphi_{\omega_2\omega_2} = F(\varphi)$$

that contains no derivatives with respect to ω_0. Consequently, any solution of the above PDE containing arbitrary integration constants will automatically include arbitrary functions of ω_0. This is an efficient way for constructing exact solutions of the nonlinear wave equation that contain arbitrary functions.

So it not but natural to generalize the above scheme and to develop a regular procedure for obtaining more general reductions giving rise to PDEs with a parametrical variable. To this end let us consider a general form of the Poincaré-invariant Ansatz

$$u(x) = \varphi(\omega_0, \omega_1, \ldots, \omega_n), \tag{3}$$

where $n = 0, 1, 2$ and ω_μ are some functions of x (see e.g., [3]). We pose the following problem, namely, to describe all the functions $\omega_\mu(x)$ such that inserting Ansatz (3) into Eq.(1) yields PDE that does not contain derivatives with respect to some of the variables ω_μ (this means that they are parametrical variables). As an easy calculation shows each parametric variable have to fulfill the following nonlinear system:

$$\omega_{x_\mu}\omega_{x^\mu} = 0, \quad \Box\omega = 0. \tag{4}$$

Hereafter, the summation convention over the repeated indices is used, raising and lowering the indices being performed with the help of the metric tensor of the Minkowski space $g_{\mu\nu} = \mathrm{diag}\,(1, -1, -1, -1)$. For example, ω_{x^μ} stands for

$$g_{\mu\nu}\omega_{x_\nu} = \begin{cases} \omega_{x_0}, & \mu = 0; \\ -\omega_{x_a}, & \mu = 1, 2, 3. \end{cases}$$

It is one of the geometrical properties of the Minkowski space-time that each two isotropic vectors are linearly dependent. That is why there is no more than one parametrical variable. Furthermore, it is not difficult to show that the case $n = 0$, when the only variable contained in the Ansatz (5) is parametrical, gives rise to the reduction $0 = F(\varphi)$ and therefore is uninteresting. So the only cases when nontrivial results could be expected are $n = 1$ and $n = 2$ and furthermore the variable ω_0 is parametrical. Provided $n = 1$ the problem has been completely solved in [4]. In a sequel, we give its general solution for the case $n = 2$. Namely, we describe all the Ansätze of the form

$$u(x) = \varphi(\omega_0(x), \omega_1(x), \omega_2(x)), \tag{5}$$

such that the following restrictions are satisfied:

- inserting the Ansatz (5) into Eq.(1) yields a PDE for $\varphi(\omega_0, \omega_1, \omega_2)$ with coefficients depending on "new" dependent variables $\omega_0, \omega_1, \omega_2$ only, and
- coefficients of derivatives with respect to ω_0 vanish identically.

A simple computation yields that in order to meet the above requirements the functions $\omega_0, \omega_1, \omega_2$ have to satisfy the following over-determined system of nonlinear PDEs:

$$
\begin{gathered}
\omega_{0x_\mu}\omega_{0x^\mu}=0, \quad \omega_{0x_\mu}\omega_{1x^\mu}=0, \quad \omega_{0x_\mu}\omega_{2x^\mu}=0, \\
\omega_{1x_\mu}\omega_{1x^\mu}=F_1, \quad \omega_{1x_\mu}\omega_{2x^\mu}=F_2, \quad \omega_{2x_\mu}\omega_{2x^\mu}=F_3, \\
\Box\omega_0=0, \quad \Box\omega_1=G_1, \quad \Box\omega_2=G_2.
\end{gathered} \tag{6}
$$

Here F_1, F_2, F_3, G_1, G_2 are arbitrary smooth functions of $\omega_0, \omega_1, \omega_2$.

Provided the above equations are satisfied, we get the following equation for the new unknown function $\varphi(\omega_0, \omega_1, \omega_2)$:

$$
F_1\varphi_{\omega_1\omega_1}+2F_2\varphi_{\omega_1\omega_2}+F_3\varphi_{\omega_2\omega_2}+G_1\varphi_{\omega_1}+G_2\varphi_{\omega_2}=F(\varphi). \tag{7}
$$

At the first sight, the problem of integrating system (6) seems to be hopeless. Indeed, we have to integrate the system of nine nonlinear PDEs in four dimensions. However, the system in question has three remarkable properties, namely,

(a) it is strongly over-determined,

(b) it is compatible, and

(c) it contains as a subsystem the system (4) which is integrable (see. e.g. [4]).

These very properties enable finding an efficient procedure for constructing the general solution of the system of nonlinear PDEs (6). Before formulating the final result we make an important remark. As may be easily verified, the class of equations (6) is invariant with respect to an arbitrary transformation of dependent variables of the form

$$
\omega_0=\Omega_0(\omega_0), \quad \omega_i=\Omega_i(\omega_0,\omega_1,\omega_2), \quad i=1,2. \tag{8}
$$

With the use of this fact we simplify the system under study and choose

$$
\begin{aligned}
&1. \quad F_1=\pm 1, \quad F_3=\mp 1, \quad F_2=0, \ \text{under } \Delta>0; \\
&2. \quad F_1=\pm 1, \quad F_3=\pm 1, \quad F_2=0, \ \text{under } \Delta<0; \\
&3. \quad F_1=\pm 1, \quad F_2=0, \quad F_2=0, \ \text{under } \Delta=0,
\end{aligned} \tag{9}
$$

where Δ stands for $F_2^2-F_1F_3$. Note that the above classification of the right-hand sides of (6) corresponds to transforming PDE (7) to one of the three canonical types, hyperbolic, elliptic and parabolic, respectively.

Thus without loss of generality we can consider instead of the general system (6) its three particular forms given by relations (9).

Theorem 1 *The general solution of the system of PDEs (6), where right-hand sides are given by one of the formulae (9), splits into two inequivalent classes which are presented below*

1. $\omega_0(x) = \theta_\mu x^\mu, \quad \omega_1(x) = b_\mu x^\mu, \quad \omega_2(x) = c_\mu x^\mu,$

2. $\omega_0(x)$ *is defined in an implicit way*

$$A_\mu(\omega_0)x^\mu + B(\omega_0) = 0,$$

where A_μ, B *are arbitrary sufficiently smooth functions satisfying the relation* $A_\mu A^\mu = 0$, *and*

$$\begin{aligned} \omega_1(x) &= (-\dot{A}_\mu \dot{A}^\mu)^{-\frac{1}{2}}(\dot{A}_\nu x^\nu + \dot{B}), \\ \omega_2(x) &= (-\dot{A}_\mu \dot{A}^\mu)^{-\frac{3}{2}}\epsilon_{\mu\nu\alpha\beta}A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta. \end{aligned}$$

Here θ_μ, b_μ, c_μ *are arbitrary real constants satisfying the constraints*

$$\theta_\mu\theta^\mu = \theta_\mu b^\mu = \theta_\mu c^\mu = b_\mu c^\mu = 0, \quad b_\mu b^\mu = c_\mu c^\mu = -1,$$

a dot over the symbol means differentiation with respect to ω_0 *and*

$$\epsilon_{\mu\nu\alpha\beta} = \begin{cases} 1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(0,1,2,3); \\ -1, & (\mu,\nu,\alpha,\beta) = \text{cycle}\,(1,0,2,3); \\ 0, & \text{in other cases.} \end{cases}$$

We do not present here the proof which is rather involved and cumbersome. The technique used to integrate system (9) is based on a proper modification of the hodograph transformation suggested in [4].

Thus we get an exhaustive description of inequivalent Ansätze (5) reducing the nonlinear wave equation (1) to PDE (7). According to Theorem 1, there are two inequivalent classes of such Ansätze, namely

$$\begin{aligned} u(x) &= \varphi(\theta_\mu x^\mu,\ b_\mu x^\mu,\ c_\mu x^\mu), &(10)\\ u(x) &= \varphi\left(\omega_0,\ \frac{\dot{A}_\nu x^\nu + \dot{B}}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}}, \frac{\epsilon_{\mu\nu\alpha\beta}A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{3}{2}}}\right). &(11) \end{aligned}$$

The Ansatz (10) is invariant with respect to the translation group having the generator

$$Q_1 = \theta_\mu \frac{\partial}{\partial x_\mu}$$

and, consequently, can be obtained by the symmetry reduction routine. The Ansatz (11) is an essentially new one. It corresponds to the following conditional symmetry of the nonlinear wave equation:

$$Q_2 = A_\mu(\omega_0)\frac{\partial}{\partial x_\mu}.$$

The fact that (11) is the general solution of PDE $Q_2 u(x) = 0$ is established by a direct computation. Next, acting with the second prolongation $\tilde{Q}_2$ of the operator Q_2 on Eq.(1) yields

$$\tilde{Q}_2(\Box u - F(u)) = 2(\dot{A}_\mu + \dot{B})^{-1}\ \dot{A}_\nu \frac{\partial}{\partial x_\nu}\ Q_2 u.$$

Hence it immediately follows that the system of PDEs

$$\Box u = F(u), \quad Q_2 u = 0$$

is invariant with respect to the one-parameter Lie transformation group having the generator Q_2.

3 Exact solutions

Inserting the Ansatz (11) into Eq.(1) yields the following PDE for the function $\varphi(\omega_0, \omega_1, \omega_2)$:

$$\varphi_{\omega_1\omega_1} + \varphi_{\omega_2\omega_2} + \frac{2}{\omega_1}\varphi_{\omega_1} = -F(\varphi). \tag{12}$$

Remarkably, if the initial equation is the linear Klein-Gordon-Fock equation, i.e. if $F(u) = -m^2 u,\ m = \text{const}$, then the above equation is reduced to the Helmholtz equation

$$\phi_{\omega_1\omega_1} + \phi_{\omega_2\omega_2} = m^2\phi,$$

with the help of the change of the dependent variable

$$\varphi(\omega_0, \omega_1, \omega_2) = \omega_1^{-1}\phi(\omega_0, \omega_1, \omega_2).$$

The Helmholtz equation can be integrated by the method of separation of variables [25]. In particular, if $m = 0$ (which means that Eq.(1) is the d'Alembert equation), it reduces to the Laplace equation which general solution reads

$$\phi = U(\omega_0, z) + U(\omega_0, z^*).$$

Here U is an arbitrary function analytic with respect to the variable $z = \omega_1 + i\omega_2$ and z^* stands for the complex conjugate of z. Using this result we get the following class of exact solutions of the d'Alembert equation:

$$\begin{aligned} u(x) &= \frac{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}}{\dot{A}_\nu x^\nu + \dot{B}} \left[U\left(\omega_0, \frac{\dot{A}_\nu x^\nu + \dot{B}}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}} + i \frac{\epsilon_{\mu\nu\alpha\beta} A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{3}{2}}}\right) \right. \\ &\qquad \left. + U\left(\omega_0, \frac{\dot{A}_\nu x^\nu + \dot{B}}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}} - i \frac{\epsilon_{\mu\nu\alpha\beta} A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{3}{2}}}\right) \right]. \end{aligned}$$

It is not difficult to become convinced of the fact that nonlinear Eq.(12) is conditionally-invariant with respect to the one-parameter group generated by the operator $Q = \frac{\partial}{\partial \omega_1}$. Inserting an Ansatz $\varphi = \varphi(\omega_0, \omega_2)$ invariant with respect to this group into (12) yields the two-dimensional PDE

$$\varphi_{\omega_2 \omega_2} = -F(\varphi)$$

that contains the variable ω_0 as a parameter. Consequently, the above equation can be treated as an ordinary differential equation with respect to ω_1. Its general solution after being inserted into the formula (11) gives the following class of exact solutions of the nonlinear wave equation (1):

$$u(x) = \varphi\left(\omega_0,\ \frac{\epsilon_{\mu\nu\alpha\beta} A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{3}{2}}}\right), \tag{13}$$

where the function $\varphi(\omega_0, \omega_2)$ is given by the quadrature

$$\int^{\varphi(\omega_0,\ \omega_2)} \left[f(\omega_0) - 2\int^{t} F(\tau) d\tau \right]^{-1/2} dt = \omega_2 + g(\omega_0).$$

In a similar way using classical and conditional symmetries of PDE (12) we have obtained new exact solutions of the nonlinear d'Alembert equation with the power nonlinearity

$$\Box u = \lambda u^k, \quad k \neq 0, 1 \tag{14}$$

which are listed below

1. k is an arbitrary real number

$$u(x) = \left[\frac{4(k-2)}{\lambda(k-1)^2}\right]^{\frac{1}{k-1}} \left(-\frac{(\epsilon_{\mu\nu\alpha\beta}A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta)^2}{(\dot{A}_\mu \dot{A}^\mu)^3}\right. \tag{15}$$

$$\left. - \frac{(\dot{A}_\nu x^\nu + \dot{B})^2}{\dot{A}_\mu \dot{A}^\mu}\right)^{\frac{1}{1-k}}, \quad k \neq 2, \tag{16}$$

$$u(x) = \left[\frac{2(k-3)}{\lambda(k-1)^2}\right]^{\frac{1}{k-1}} \left(\frac{\dot{A}_\nu x^\nu + \dot{B}}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}}\right)^{\frac{2}{1-k}}, \quad k \neq 3. \tag{17}$$

2. $k = 3$

$$u(x) = \left[-\frac{(\epsilon_{\mu\nu\alpha\beta}A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta)^2}{(\dot{A}_\mu \dot{A}^\mu)^3} - \frac{(\dot{A}_\nu x^\nu + \dot{B})^2}{\dot{A}_\mu \dot{A}^\mu}\right]^{-\frac{1}{2}} \tag{18}$$

$$\times \varphi\left(\omega_0,\, \frac{1}{2}\ln\left(-\frac{(\epsilon_{\mu\nu\alpha\beta}A^\mu \dot{A}_\nu \ddot{A}^\alpha x^\beta)^2}{(\dot{A}_\mu \dot{A}^\mu)^3} - \frac{(\dot{A}_\nu x^\nu + \dot{B})^2}{\dot{A}_\mu \dot{A}^\mu}\right)\right),$$

where $\varphi(\omega_0, y)$ is given by the following quadrature:

$$\int^{\varphi(\omega_0,y)} \left(-\frac{\lambda}{2}t^4 + t^2 + f(\omega_0)\right)^{-\frac{1}{2}} dt = y + g(\omega_0).$$

3. $k = 5$

$$u(x) = -\frac{\dot{A}_\mu \dot{A}^\mu}{\dot{A}_\nu x^\nu + \dot{B}} \varphi\left(\omega_0,\, \ln\left(\frac{\dot{A}_\nu x^\nu + \dot{B}}{(-\dot{A}_\mu \dot{A}^\mu)^{\frac{1}{2}}}\right)\right), \tag{19}$$

where $\varphi(\omega_0, y)$ is given by the following quadrature:

$$\int^{\varphi(\omega_0,y)} \left(-\frac{\lambda}{3}t^4 + \frac{1}{4}t^2 + f(\omega_0)\right)^{-\frac{1}{2}} dt = y + g(\omega_0).$$

We remind that in the formulae (13), (16)–(19) the function $\omega_0 = \omega_0(x)$ is determined implicitly $A_\mu(\omega_0)x^\mu + B(\omega_0) = 0$ and $A_\mu(\omega_0), B(\omega_0)$ are arbitrary sufficiently smooth functions satisfying the equality $A_\mu A^\mu = 0$. Furthermore, f, g are arbitrary sufficiently smooth functions of ω_0. Provided, arbitrary functions are appropriately fixed, namely,

$$A_0(\omega_0) = 1, \quad A_1(\omega_0) = (\omega_0), \quad A_2(\omega_0) = \sqrt{1-\omega_0^2}, \quad A_3(\omega_0) = 0,$$

and $f=\mathrm{const}$, $g=\mathrm{const}$, the above solutions reduce to the already known ones. Indeed, with this choice of arbitrary functions the variables $\omega_0, \omega_1, \omega_2$ take the form

$$\omega_0 = \frac{x_1 x_0 + x_2\sqrt{x_1^2+x_2^2-x_0^2}}{x_1^2+x_2^2}, \quad \omega_1 = \sqrt{x_1^2+x_2^2-x_0^2}, \quad \omega_2 = x_3. \qquad (20)$$

Given this form of $\omega_\mu, \mu = 0, 1, 2$ formulae (13), (16), (17) yield exact solutions of Eq.(14) constructed by the symmetry reduction method in [5] and the formulae (18), (19) under $f=\mathrm{const}$, $g=\mathrm{const}$ give solutions of Eq.(14) obtained by the same method in [26]. Note that provided $f=g=0$ and relations (20) hold, the formula (18) yields two particular classes of exact solutions of the cubic wave equation which in its turn give rise to the famous instanton and meron solutions of the Yang-Mills equations obtainable via the 't Hooft-Corrigan-Fairlie-Wilczek Ansatz (see, e.g. [24]).

4 Concluding Remarks

Thus even for such a well studied model as the nonlinear wave equation it is possible to construct broad families of principally new exact solutions. It is a proper use of conditional symmetries that enables us to generalize the well-known and widely used exact solutions in such a way that the new ones include arbitrary functions. However, a further progress in this direction depends strongly on developing new more powerful symbolic computations routines for integrating over-determined multi-dimensional systems of nonlinear PDEs.

The method suggested in the paper relies heavily upon an information about the structure of invariant solutions given by the specific representation of the Poincaré group realized on the set of solutions of the nonlinear wave equation. However it can be easily modified in order to take into account the structure of solutions invariant with respect to natural extensions of the Poincaré group, namely, with respect to the similitude and conformal groups. Furthermore, it is also possible to apply a similar approach for the sake of obtaining exact solutions of complex wave equations containing arbitrary functions.

On the other hand, the fact that PDEs considered are of hyperbolic type is crucial. The problem of adapting of the approach suggested in the present paper to parabolic (evolution) type equations is completely open.

All these problems are under study now and will be reported in our future publications.

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