On preliminary symmetry classification of nonlinear Schrödinger equations with some applications to Doebner-Goldin model

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Abstract

We perform classification of a class of one-dimensional nonlinear Schrödinger equations whose symmetry groups have dimensions $n = 1, 2, 3$. Next, we select from so constructed classes of invariant equations those nonlinear Schrödinger equations that are invariant with respect to the Galilei group and its natural extensions. The results obtained are applied for symmetry classification of complex Galilei-invariant Doebner-Goldin models.

1 Introduction

The idea of adopting a nonlinear Schrödinger equation as a basic equation in quantum mechanics is becoming more and more popular. Beginning with the paper by Bilyanicki-Birulya and Mycielski [1] a number of authors suggested different nonlinear generalizations of the standard Schrödinger equation in order to comply properly with the problems of quantum mechanics that cannot be solved within the framework of the linear theory [2]–[9]. Though the approaches used for choosing one or another nonlinear model were different, the general feature inherent in all of them was that they have non-trivial Lie symmetry. An illustrative example is the Doebner-Goldin model whose symmetry properties were studied in detail in [10]. As shown in our paper [11] for all the cases when the Doebner-Goldin model has an exceptional symmetry it is either linearizable or integrable

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through non-local transformations of variables. This simple observation makes it possible to utilize the powerful methods of the Lie theory of continuous groups in order to develop a systematic procedure for selecting a possible candidates for an admissible basic equation of the nonlinear quantum mechanics.

Our aim is to perform symmetry classification of nonlinear Schrödinger equations belonging to the class of partial differential equations (PDEs)

$$i\psi_t = \psi_{xx} + F(t, x, \psi, \psi^*, \psi_x, \psi_x^*), \tag{1}$$

where $\psi = \psi(t, x)$ is a complex-valued function, the asterisk designates complex conjugation, and F is an arbitrary complex-valued function, the subscripts of the functions are the derivatives with respect to the corresponding arguments.

Following the general Lie's algorithm [12, 13] we will look for an infinitesimal operator of the maximal symmetry group admitted by (1) in the form

$$Q = \xi^1(t, x, \psi, \psi^*)\partial_t + \xi^2(t, x, \psi, \psi^*)\partial_x + r(t, x, \psi, \psi^*)\partial_\psi + r^*(t, x, \psi, \psi^*)\partial_{\psi^*} \tag{2}$$

with ξ^1, ξ^2 being arbitrary real functions and r being an arbitrary complex function. The criterion for equation (1) to be invariant with respect to operator Q (2) reads as:

$$\underset{2}{Q}(i\psi_t - \psi_{xx} - F(t, x, \psi, \psi^*, \psi_x, \psi_x^*))|_{\psi_{xx} = i\psi_t - F(t,x,\psi,\psi^*,\psi_x,\psi_x^*)} = 0, \tag{3}$$

where $\underset{2}{Q}$ is the second prolongation of the operator (2) having the following form:

$$\underset{2}{Q} = Q + r_1\partial_{\psi_t} + r_2\partial_{\psi_x} + r_3\partial_{\psi_{xx}} + r_1^*\partial_{\psi_t^*} + r_2^*\partial_{\psi_x^*} + r_3^*\partial_{\psi_{xx}^*}.$$

Here r_j, $j = 1, 2, 3$ are complex functions determined by the following recurrent formulae:

$$\begin{aligned} r_1 &= D_t(r) - \psi_t D_t(\xi^1) - \psi_x D_t(\xi^2), \\ r_2 &= D_x(r) - \psi_t D_x(\xi^1) - \psi_x D_x(\xi^2), \\ r_3 &= D_x(r_2) - \psi_{tx} D_x(\xi^1) - \psi_{xx} D_x(\xi^2), \end{aligned}$$

D_t and D_x are the total derivatives with respect to t and x correspondingly.

Condition (3) yields the system of determining equations for the coefficients ξ^1, ξ^2, r that contains a classifying equation for the function F. Thus the problem of constructing symmetry group admitted by (1) is equivalent to finding its symmetry algebra through solving an over-determined system of PDEs. Next, the form of a symmetry algebra of (1) is determined by a choice of the function F. So, there arises the problem of describing all the functions F such that the equation under study admits a nontrivial symmetry algebra, i.e., the problem of symmetry classification of the class of nonlinear Schrödinger equations (1). In order to perform symmetry classification of the class of PDEs (1) one could try to utilize the method of equivalence transformations. However, on this way one comes to a problem of classifying all inequivalent subalgebras of some infinite-dimensional Lie algebra L which is a "wild" algebraic problem in a sense that it could not be solved by the methods available by now. A possible way of overcoming this difficulty is based on the fact that all inequivalent abstract Lie algebras of the dimension $n \le 5$ have already been classified [14]–[16]. So one could restrict considerations to low dimensional subalgebras of the algebra L which makes the whole procedure manageable and algorithmic.

Here we describe all the forms of the functions F such that (1) admits one-, two- and three-parameter symmetry groups. Next, we will select from so obtained sets of invariant PDEs those admitting the Lie algebras of the Galilei, extended Galilei and Schrödinger groups.

Our approach is based on the following steps (see also [17] where the underlying ideas are discussed):
(i) we construct a maximal group of transformations leaving class (1) invariant. This group establishes an equivalence relation on the set of PDEs of the form (1).
(ii) by solving the determining equations not containing the function F we find a class of differential operators to which any symmetry of (1) belongs and get realizations of one-, two-, three-dimensional algebras up to the equivalence relations mentioned above.
(iii) the corresponding invariant equations are found by solving the classifying equation for each realization obtained.

The approach in question can be seen as a proper synthesis of the method of equivalence transformations going back to the Ovsjanikov's works [12] (see, also, [18]) and of the approach that is due to Rideau and Winternitz [19]. Note that the latter has been further developed and applied to classifying realizations of Poincaré and Galilei groups and their extensions in the papers [20, 21].

Let us note that through the article we use the following notations: α, α_j, $j = 1, ..., 5$ are arbitrary real constants, c, c_j, $j = 1, 2$ are arbitrary complex constants, m, n are arbitrary real-valued functions, R, r, k are arbitrary complex-valued functions, $\hat{\psi} = \psi - \psi^*$, $\hat{r} = r - r^*$, $\hat{k} = k - k^*$; $\langle Q_1, ...\rangle$ denotes a Lie algebra generated by the operators $Q_1, ...$.

2 Preliminary classification

In this section we describe inequivalent PDEs of the form (1) that are invariant with respect to one-, two- and three-dimensional Lie algebras.

As a direct computation shows, the maximal group of point transformations leaving the class of equations (1) invariant is of the form:

$$\begin{cases} t & \to & \tau & = & T(t), \\ x & \to & \kappa & = & \epsilon x\sqrt{\delta T_t(t)} + S(t), \\ \psi & \to & \varphi & = & C(t, x, \widetilde{\psi}), \end{cases} \qquad \delta T_t(t) > 0, \quad C_{\widetilde{\psi}} \neq 0, \tag{4}$$

where $\epsilon = \pm 1$, $\delta = \pm 1$; $\widetilde{\psi} = ((\delta + 1)\psi - (\delta - 1)\psi^*)/2$, T, S are arbitrary real-valued functions; C is an arbitrary complex-valued function. Next, the most general symmetry algebra admitted by (1) is spanned by the operators of the form:

$$Q = 2\xi(t)\partial_t + (x\xi_t(t) + \gamma(t))\,\partial_x + \eta(x, t, \psi)\partial_\psi + \eta^*(x, t, \psi^*)\partial_{\psi^*}, \tag{5}$$

where ξ, γ are arbitrary real-valued functions, η is an arbitrary complex-valued function and, furthermore, the condition

$$
\begin{aligned}
&i\psi_x(x\xi_{tt}+\gamma_t) - i\eta_t + \eta_{xx} + 2\eta_{x\psi}\psi_x + \eta_{\psi\psi}\psi_x^2 - (\eta_\psi - 2\xi_t)F + 2\xi F_t + (x\xi_t+\gamma)F_x + \\
&+\eta F_\psi + \eta^* F_{\psi^*} + (\eta_x + \eta_\psi\psi_x - \xi_t\psi_x)F_{\psi_x} + (\eta_x^* + \eta_{\psi^*}^*\psi_x^* - \xi_t\psi_x^*)F_{\psi_x^*} = 0
\end{aligned}
\tag{6}
$$

holds.

A preliminary symmetry classification of (1) is performed in two steps. At the first step, we fix the vector space of the operators (5) as the realization space for the one-, two- and three-dimensional Lie algebras and describe all possible inequivalent realizations of these algebras (two realizations of a Lie algebra are called equivalent if the vector spaces spanned by the basis elements of the first one and by the basis elements of the second one transformed according to (4) are identical). Next, we select from so obtained realizations those being symmetry algebras of (1).

2.1 One-dimensional Lie algebras

All inequivalent PDEs (1) admitting one-dimensional symmetry algebras are given by the following statement.

Theorem 1 *The list of inequivalent one-dimensional Lie algebras having the basis elements (5) is exhausted by the following algebras:*

$$L_{1,1} = \langle \partial_t \rangle,\ L_{1,2} = \langle \partial_x \rangle,\ L_{1,3} = \langle \partial_\psi + \partial_{\psi^*} \rangle.$$

Nonlinear Schrödinger equations invariant with respect to these algebras read as

$$
\begin{aligned}
L_{1,1}:&\quad i\psi_t = \psi_{xx} + R(x, \psi, \psi^*, \psi_x, \psi_x^*),\\
L_{1,2}:&\quad i\psi_t = \psi_{xx} + R(t, \psi, \psi^*, \psi_x, \psi_x^*),\\
L_{1,3}:&\quad i\psi_t = \psi_{xx} + R(t, x, \hat{\psi}, \psi_x, \psi_x^*).
\end{aligned}
$$

Proof. As mentioned above all one-dimensional symmetry algebras of equation (1) are of the form $\langle Q \rangle$, where Q is determined in (5), provided condition (6) holds. Let us show that by making use of transformations (4) operator (5) can be reduced to one of the operators $\partial_\tau, \partial_\kappa$ or $\partial_\varphi + \partial_{\varphi^*}$.

Acting by transformations (4) on (5) we obtain

$$
\begin{aligned}
Q \to \widetilde{Q} = \ & 2\xi T_t\partial_\tau + \left(\epsilon\delta x\frac{\xi T_{tt} + \xi_t T_t}{\sqrt{\delta T_t}} + 2\xi S_t + \epsilon\gamma\sqrt{\delta T_t}\right)\partial_\kappa + \\
&+(2\xi C_t + (2x\xi_t + \gamma)C_x + C_{\widetilde{\psi}}(\eta(\delta+1) - \eta^*(\delta-1))/2)\partial_\varphi + \\
&+(2\xi C_t^* + (2x\xi_t + \gamma)C_x^* + C_{\widetilde{\psi}^*}^*(\eta^*(\delta+1) - \eta(\delta-1))/2)\partial_{\varphi^*}.
\end{aligned}
\tag{7}
$$

Suppose first that $\xi(t) \not\equiv 0$. Then it follows from (7) that choosing $\epsilon = 1$ and functions T, S, C satisfying the system

$$T_t = \frac{1}{2\xi},\ S_t = -\frac{\gamma\sqrt{\delta T_t}}{2\xi},\ 2\xi C_t + (x\xi_t + \gamma)C_x + C_{\widetilde{\psi}}(\eta(\delta+1) - \eta^*(\delta-1))/2 = 0,\ \delta T_t > 0,$$

we reduce operator (5) to the form $\widetilde{Q} = \partial_\tau$.

Analogously, choosing in an appropriate way constants ϵ, δ and functions T, S, C, we reduce (5) to the operator $\widetilde{Q} = \partial_\kappa$, if $\xi(t) \equiv 0$, $\gamma(t) \not\equiv 0$, and to the operator $\widetilde{Q} = \partial_\varphi + \partial_{\varphi^*}$, if $\xi(t) \equiv 0$, $\gamma(t) \equiv 0$.

Thus any one-dimensional algebra having a generator (5) is equivalent to one of the following algebras:

$$\langle \partial_t \rangle, \ \langle \partial_x \rangle, \ \langle \partial_\psi + \partial_{\psi^*} \rangle. \tag{8}$$

And what is more, these algebras are inequivalent, because it is impossible to find such constants δ, ϵ and functions T, S, C that one of the operators (8) can be transformed to another operator from (8), multiplied by a real constant.

The corresponding invariant equations for each of the algebras (8) are obtained by solving the equation (6). The theorem is proved.

Let us give an assertion that is a direct consequence of Theorem 1 and is used in a sequel.

Corollary 1 *Any operator of the form (5) can be reduced by means of transformations (4) to one of the operators ∂_t, ∂_x or $\partial_\psi + \partial_{\psi^*}$.*

3 Two-dimensional Lie algebras

As is well-known, there exist two inequivalent abstract two-dimensional Lie algebras

$$\begin{array}{lll} L_{2,1} & : & [Q_1,\ Q_2] = 0, \quad \text{(commutative Lie algebra)} \\ L_{2,2} & : & [Q_1,\ Q_2] = Q_1 \quad \text{(solvable Lie algebra).} \end{array} \tag{9}$$

So the problem of describing of PDEs (1) admitting two-dimensional symmetry algebras contains as a subproblem the one of solving relations (9) within the class of operators (5) up to the equivalence relation introduced in the beginning of this section. Next, we should solve (6) for each realization obtained. Having done this we get the following assertion.

Theorem 2 *The list of inequivalent realizations of two-dimensional Lie algebras, such that they are symmetry algebras of PDE (1), is exhausted by nine realizations given below*

$$\begin{array}{ll} L_{2,1}: & L_{2,1,1} = \langle \partial_x, \partial_t \rangle, \\ & L_{2,1,2} = \langle \partial_x, m(t)\partial_x + \partial_\psi + \partial_{\psi^*} \rangle, \\ & L_{2,1,3} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t \rangle, \\ & L_{2,1,4} = \langle \partial_\psi + \partial_{\psi^*}, r(x,t)\partial_\psi + r^*(x,t)\partial_{\psi^*} \rangle, \\ L_{2,2}: & L_{2,2,1} = \langle \partial_t, t\partial_t + (1/2)x\partial_x \rangle, \\ & L_{2,2,2} = \langle \partial_x, 2t\partial_t + x\partial_x \rangle, \\ & L_{2,2,3} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t + \psi\partial_\psi + \psi^*\partial_{\psi^*} \rangle, \\ & L_{2,2,4} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x + \psi\partial_\psi + \psi^*\partial_{\psi^*} \rangle, \\ & L_{2,2,5} = \langle \partial_\psi + \partial_{\psi^*}, \psi\partial_\psi + \psi^*\partial_{\psi^*} \rangle. \end{array}$$

The forms of the functions F in the corresponding invariant equations (1) read

$$
\begin{array}{ll}
L_{2,1,1}: & R(\psi,\psi^*,\psi_x,\psi^*_x),\\
L_{2,1,2}: & R(t,\hat\psi,\psi_x,\psi^*_x)-i\psi\psi_x m_t,\\
L_{2,1,3}: & R(x,\hat\psi,\psi_x,\psi^*_x),\\
L_{2,1,4}: & R\left(t,x,\hat r\psi_x-\hat\psi r_x,\hat r\psi^*_x-\hat\psi r^*_x\right)+\hat\psi\hat r^{-1}(ir_t-r_{xx}),\quad \hat r\neq 0,\\
 & R(t,x,\hat\psi,\hat\psi_x)+\psi_x(ir_t-r_{xx})r_x^{-1},\qquad \hat r=0,\ r_x\neq 0,\\
L_{2,2,1}: & x^{-2}R(\psi,\psi^*,x\psi_x,x\psi^*_x),\\
L_{2,2,2}: & t^{-1}R(\psi,\psi^*,\sqrt{t}\psi_x,\sqrt{t}\psi^*_x),\\
L_{2,2,3}: & e^tR(x,e^{-t}\hat\psi,e^{-t}\psi_x,e^{-t}\psi^*_x),\\
L_{2,2,4}: & e^xR(t,e^{-x}\hat\psi,e^{-x}\psi_x,e^{-x}\psi^*_x),\\
L_{2,2,5}: & \hat\psi R\left(t,x,\psi_x\hat\psi^{-1},\psi^*_x\hat\psi^{-1}\right).
\end{array}
$$

Proof. Consider an arbitrary 2-dimensional algebra $\langle Q_1,Q_2\rangle$ with generators being of the form (5). According to Corollary 1, one of these operators, for instance Q_1, can be transformed to become ∂_t, ∂_x or $\partial_\psi+\partial_{\psi^*}$. Therefore, in order to obtain all inequivalent realizations of two-dimensional Lie algebras it suffices to restrict the choice of the operator Q_1 to one of the three forms ∂_t, ∂_x or $\partial_\psi+\partial_{\psi^*}$.

Let us turn to the Lie algebra $L_{2,1}$. Suppose first that $Q_1=\partial_t$. Then the commutation relations of $L_{2,1}$ imply that operator Q_2 is of the form:

$$Q_2=\alpha_1\partial_t+\alpha_2\partial_x+r(x,\psi)\partial_\psi+r^*(x,\psi)\partial_{\psi^*}.$$

Making use of the transformations

$$\tau=t,\ \kappa=x,\ \varphi=C(x,\psi),$$

that belong to the class (4) and leave the operator Q_1 invariant, reduces the algebra $\langle Q_1,Q_2\rangle$ to one of the following forms:

$$\langle\partial_t,\alpha_1\partial_t+\partial_\psi+\partial_{\psi^*}\rangle,\qquad\langle\partial_t,\alpha_1\partial_t+\alpha_2\partial_x\rangle. \tag{10}$$

Similar reasoning when applied to the cases $Q_1=\partial_x$ and $Q_1=\partial_\psi+\partial_{\psi^*}$ yield that all the realizations of the algebra $L_{2,1}$ within the class (5) can be transformed either to realizations (10) or to the following ones:

$$L_{2,1,2},\ \langle\partial_x,\alpha_1\partial_t\rangle,\ \langle\partial_x,m(t)\partial_x\rangle,\ L_{2,1,3},\ L_{2,1,4},\ \langle\partial_\psi+\partial_{\psi^*},\partial_x\rangle. \tag{11}$$

It is easy to see that the basis elements of the realizations (10) are linear combinations of the generators $L_{2,1,3}$ and $L_{2,1,1}$ correspondingly. Similarly, the basis elements of the realizations $\langle\partial_x,\alpha_1\partial_t\rangle$ and $\langle\partial_\psi+\partial_{\psi^*},\partial_x\rangle$ from (11) are linear combinations of the basis elements of the realizations $L_{2,1,1}$ and $L_{2,1,2}$ ($m(t)\equiv 0$). Consequently, any realization of the Lie algebra $L_{2,1}$ within the class (5) is equivalent either to one of the realizations

$$L_{2,1,j},\ j=1,...,4 \tag{12}$$

or to $\langle\partial_x,m(t)\partial_x\rangle$. Checking that the realizations (12) are inequivalent is straightforward.

Applying the same procedure in order to construct realizations of the Lie algebra $L_{2,2}$ yields that all of them can be reduced by means of (4) to one of the realizations

$$L_{2,2,j}, \ j=1,...,5 \tag{13}$$

which are inequivalent.

Integrating (6) for the realizations given in (12) and (13) yields the complete list of nonlinear Schrödinger equations (1) whose symmetry algebras are, at least, two-dimensional. To complete the proof we should only note that there are no equations of the form (1) invariant with respect to the Lie algebra $\langle \partial_x, m(t)\partial_x \rangle$.

3.1 Three-dimensional algebras

Making use of the results of the papers [14]–[16] we give the list of inequivalent abstract three-dimensional Lie algebras

$$\begin{aligned}
L_{3,1} &: \ [Q_1,\ Q_2]=0, \quad [Q_2,\ Q_3]=0, \quad [Q_3,\ Q_1]=0,\\
L_{3,2} &: \ [Q_1,\ Q_2]=Q_1, \quad [Q_2,\ Q_3]=Q_1-Q_3;\\
L_{3,3} &: \ [Q_2,\ Q_3]=Q_1;\\
L_{3,4} &: \ [Q_1,\ Q_3]=aQ_1-Q_2, \quad [Q_2,\ Q_3]=Q_1+aQ_2, \quad (a\geq 0);\\
L_{3,5} &: \ [Q_1,\ Q_2]=Q_1, \quad [Q_1,\ Q_3]=2Q_2, \quad [Q_2,\ Q_3]=Q_3;\\
L_{3,6} &: \ [Q_1,\ Q_2]=Q_1, \quad [Q_2,\ Q_3]=aQ_3, \quad (|a|\leq 1);\\
L_{3,7} &: \ [Q_1,\ Q_2]=Q_3, \quad [Q_2,\ Q_3]=Q_1, \quad [Q_3,\ Q_1]=Q_2;
\end{aligned}$$

where non-vanishing commutation relations are written only.

A complete description of inequivalent realizations of the above Lie algebras together with the corresponding invariant nonlinear Schrödinger equations is given in Theorem 3 below.

Theorem 3 *The list of inequivalent realizations of three-dimensional Lie algebras, such that they are symmetry algebras of PDE (1), is exhausted by the following realizations:*

$$\begin{aligned}
L_{3,1}: \quad & L_{3,1,1}=\langle \partial_x, \partial_\psi+\partial_{\psi^*}, \partial_t\rangle,\\
& L_{3,1,2}=\langle \partial_x, m(t)\partial_x+\partial_\psi+\partial_{\psi^*}, n(t)\partial_x+r(t)\partial_\psi+r^*(t)\partial_{\psi^*}\rangle,\\
& L_{3,1,3}=\langle \partial_\psi+\partial_{\psi^*}, r(x)\partial_\psi+r^*(x)\partial_{\psi^*}, \partial_t\rangle,\\
& L_{3,1,4}=\langle \partial_\psi+\partial_{\psi^*}, r(t,x)\partial_\psi+r^*(t,x)\partial_{\psi^*}, k(t,x)\partial_\psi+k^*(t,x)\partial_{\psi^*}\rangle;
\end{aligned}$$

$$\begin{aligned}
L_{3,2}: \quad & L_{3,2,1}=\langle \partial_x, 2t\partial_t+x\partial_x, (1/2)\ln t\partial_x+(1/\sqrt{t})\partial_\psi+(1/\sqrt{t})\partial_{\psi^*}\rangle,\\
& L_{3,2,2}=\langle \partial_\psi+\partial_{\psi^*}, \partial_t+\psi\partial_\psi+\psi^*\partial_{\psi^*}, e^{-t}\partial_t-(1/2)xe^{-t}\partial_x+t\partial_\psi+t\partial_{\psi^*}\rangle,\\
& L_{3,2,3}=\langle \partial_\psi+\partial_{\psi^*}, \partial_t+\psi\partial_\psi+\psi^*\partial_{\psi^*}, e^{-t}\partial_x+t\partial_\psi+t\partial_{\psi^*}\rangle,\\
& L_{3,2,4}=\langle \partial_\psi+\partial_{\psi^*}, \partial_t+\psi\partial_\psi+\psi^*\partial_{\psi^*}, (r(x)+t)\partial_\psi+(r^*(x)+t)\partial_{\psi^*}\rangle,\\
& L_{3,2,5}=\langle \partial_\psi+\partial_{\psi^*}, \partial_x+\psi\partial_\psi+\psi^*\partial_{\psi^*}, (x+im(t))\partial_\psi+(x-im(t))\partial_{\psi^*}\rangle;
\end{aligned}$$

$$
\begin{aligned}
L_{3,3}: \quad & L_{3,3,1} = \langle \partial_x, \partial_t, t\partial_x + \partial_\psi + \partial_{\psi^*} \rangle, \\
& L_{3,3,2} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t, (t + r(x))\partial_\psi + (t + r^*(x))\partial_{\psi^*} \rangle, \\
& L_{3,3,3} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t, \partial_x + t\partial_\psi + t\partial_{\psi^*} \rangle, \\
& L_{3,3,4} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x, \partial_t + x\partial_\psi + x\partial_{\psi^*} \rangle, \\
& L_{3,3,5} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x, n(t)\partial_x + (x + im(t))\partial_\psi + (x - im(t))\partial_{\psi^*} \rangle;
\end{aligned}
$$

$$
\begin{aligned}
L_{3,4}: \quad & L_{3,4,1} = \langle \partial_\psi + \partial_{\psi^*}, \tan(r(x) - t)\partial_\psi + \tan(r^*(x) - t)\partial_{\psi^*}, \\
& \qquad \partial_t + (a - \tan(r(x) - t))\psi\partial_\psi + (a - \tan(r^*(x) - t))\psi^*\partial_{\psi^*} \rangle, \\
& L_{3,4,2} = \langle \partial_\psi + \partial_{\psi^*}, \tan(im(t) - x)\partial_\psi - \tan(im(t) + x)\partial_{\psi^*}, \\
& \qquad \partial_x + (a - \tan(im(t) - x))\psi\partial_\psi + (a + \tan(im(t) + x))\psi^*\partial_{\psi^*} \rangle;
\end{aligned}
$$

$$
\begin{aligned}
L_{3,5}: \quad & L_{3,5,1} = \langle \partial_t, t\partial_t + (1/2)x\partial_x, t^2\partial_t + tx\partial_x + x^2\partial_\psi + x^2\partial_{\psi^*} \rangle, \\
& L_{3,5,2} = \langle \partial_\psi + \partial_{\psi^*}, \psi\partial_\psi + \psi^*\partial_{\psi^*}, \psi^2\partial_\psi + (\psi^*)^2\partial_{\psi^*} \rangle;
\end{aligned}
$$

$$
\begin{aligned}
L_{3,6}: \quad & L_{3,6,1} = \langle \partial_t, t\partial_t + (1/2)x\partial_x, \partial_x \rangle, \ a = -1/2, \\
& L_{3,6,2} = \langle \partial_t, t\partial_t + (1/2)x\partial_x, \partial_x + (1/x)\partial_\psi + (1/x)\partial_{\psi^*} \rangle, \ a = -1/2, \\
& L_{3,6,3} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x + \psi\partial_\psi + \psi^*\partial_{\psi^*}, \partial_t \rangle, \ a = 0, \\
& L_{3,6,4} = \langle \partial_t, t\partial_t + (1/2)x\partial_x, x^{2a}\partial_\psi + x^{2a}\partial_{\psi^*} \rangle, \\
& L_{3,6,5} = \langle \partial_x, 2t\partial_t + x\partial_x, t^{a/2}\partial_\psi + t^{a/2}\partial_{\psi^*} \rangle, \\
& L_{3,6,6} = \langle \partial_x, 2t\partial_t + x\partial_x, t^{(a+1)/2}\partial_x + t^{a/2}\partial_\psi + t^{a/2}\partial_{\psi^*} \rangle, \\
& L_{3,6,7} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t + \psi\partial_\psi + \psi^*\partial_{\psi^*}, e^{at}\partial_t + (a/2)xe^{at}\partial_x \rangle, \ a \neq 0, \\
& L_{3,6,8} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t + \psi\partial_\psi + \psi^*\partial_{\psi^*}, e^{at}\partial_x \rangle, \\
& L_{3,6,9} = \langle \partial_\psi + \partial_{\psi^*}, \partial_t + \psi\partial_\psi + \psi^*\partial_{\psi^*}, r(x)e^{(1+a)t}\partial_\psi + r^*(x)e^{(1+a)t}\partial_{\psi^*} \rangle, \\
& L_{3,6,10} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x + \psi\partial_\psi + \psi^*\partial_{\psi^*}, m(t)\partial_x \rangle, \ a = 0, \\
& L_{3,6,11} = \langle \partial_\psi + \partial_{\psi^*}, \partial_x + \psi\partial_\psi + \psi^*\partial_{\psi^*}, r(t)e^{(1+a)x}\partial_\psi + r^*(t)e^{(1+a)x}\partial_{\psi^*} \rangle, \\
& L_{3,6,12} = \langle \partial_\psi + \partial_{\psi^*}, \psi\partial_\psi + \psi^*\partial_{\psi^*}, \partial_t \rangle, \ a = 0, \\
& L_{3,6,13} = \langle \partial_\psi + \partial_{\psi^*}, \psi\partial_\psi + \psi^*\partial_{\psi^*}, r(t,x)\partial_\psi + r^*(t,x)\partial_{\psi^*} \rangle, \ a = -1.
\end{aligned}
$$

The forms of the functions F in the corresponding invariant equations (1) read

$$
\begin{aligned}
L_{3,1,1}: \quad & R(\hat\psi, \psi_x, \psi_x^*), \\
L_{3,1,2}: \quad & R(t, \psi_x, \psi_x^*) + i\hat\psi\hat r^{-1}(\psi_x(rm_t - n_t) + r_t) - i\psi\psi_x m_t, \ \hat r \neq 0, \\
L_{3,1,3}: \quad & R\left(x, \hat r\psi_x - \hat\psi r_x, \hat r\psi_x^* - \hat\psi r_x^*\right) - r_{xx}\hat\psi\hat r^{-1}, \ \hat r \neq 0, \\
& R(x, \hat\psi, \hat\psi_x) - \psi_x r_{xx} r_x^{-1}, \quad \hat r = 0, \ r_x \neq 0, \\
L_{3,1,4}: \quad & R(t, x, \hat\psi(k_x r_x^* - r_x k_x^*) - \psi_x(r_x^*\hat k - k_x^*\hat r) + \psi_x^*(r_x\hat k - k_x\hat r)) + \\
& + ((ik_t - k_{xx})(r_x\hat\psi - \psi_x\hat r) - (ir_t - r_{xx})(k_x\hat\psi - \psi_x\hat k))(r_x\hat k - k_x\hat r)^{-1}, \\
& k \neq m_1(t)r + m_2(t), \quad m_1, m_2 \textit{ are arbitrary real functions};
\end{aligned}
$$

$$
\begin{aligned}
L_{3,2,1}: \quad & t^{-1}R(\hat\psi, \psi_x\sqrt{t}, \psi_x^*\sqrt{t}) - i\psi(1 + \psi_x\sqrt{t})/(2t), \\
L_{3,2,2}: \quad & e^t R\left(e^t\hat\psi\psi_x^{-2}, \psi_x/\psi_x^*, e^t x/\psi_x\right) + ix\psi_x/2 - 2ie^t\ln(e^{-t}\psi_x), \\
L_{3,2,3}: \quad & e^t R(e^{-t}\hat\psi, \psi_x e^{-t}, \psi_x^* e^{-t}) + ie^t x + ix\psi_x, \\
L_{3,2,4}: \quad & e^t R\left(x, (\hat\psi r_x - \hat r\psi_x)e^{-t}, (\psi_x^* r_x - \psi_x r_x^*)e^{-t}\right) - \psi_x(r_{xx} - i)r_x^{-1}, \ r_x \neq 0, \\
& e^t R(x, \psi_x e^{-t}, \psi_x^* e^{-t}) + i\hat\psi\hat r^{-1}, \ r_x = 0, \hat r \neq 0, \\
L_{3,2,5}: \quad & e^x R\left(t, (\hat\psi - 2i\psi_x m)e^{-x}, \hat\psi_x e^{-x}\right) - \psi_x m_t;
\end{aligned}
$$

$$
\begin{array}{ll}
L_{3,3,1}: & R(\hat{\psi},\psi_x,\psi_x^*) - i\psi\psi_x,\\
L_{3,3,2}: & R\left(x,\psi_x\hat{r}-r_x\hat{\psi},\psi_x^*\hat{r}-r_x^*\hat{\psi}\right)-\hat{\psi}\hat{r}^{-1}(r_{xx}-i),\ \hat{r}\neq 0,\\
 & R(x,\hat{\psi},\hat{\psi}_x)-\psi_x(r_{xx}-i)r_x^{-1},\ \hat{r}=0, r_x\neq 0,\\
L_{3,3,3}: & R(\hat{\psi},\psi_x,\psi_x^*)+ix,\\
L_{3,3,4}: & R(\hat{\psi},t-\psi_x,\hat{\psi}_x),\\
L_{3,3,5}: & R\left(t,2m\psi_x+i\hat{\psi},2m\psi_x^*+i\hat{\psi}\right)-(i\hat{\psi}^2n_t-4m\hat{\psi}(im_t-\psi_x n_t))/(8m^2),\ m\neq 0\\
 & R(t,\hat{\psi},\hat{\psi}_x)-i\psi_x^2 n_t/2,\ m=0;
\end{array}
$$

$$
\begin{array}{ll}
L_{3,4,1}: & e^{at}\cos^{-1}(t-r)R\left(x,\psi_x e^{-at}\sin\hat{r}\cos(t-r)-r_x\hat{\psi}e^{-at}\cos(t-r^*),\right.\\
 & \left.\psi_x^*e^{-at}\sin\hat{r}\cos(t-r^*)-r_x^*\hat{\psi}e^{-at}\cos(t-r)\right)+\\
 & +2r_x\psi_x\tan(t-r)-(i+r_{xx})\hat{\psi}\cos^{-1}(t-r)\cos(t-r^*)\sin^{-1}\hat{r},\ \hat{r}\neq 0,\\
 & \hat{\psi}R\left(x,e^{-at}\cos(t-r)(r_x\hat{\psi}\tan(t-r)+\hat{\psi}_x),\hat{\psi}\cos(t-r)e^{-at}\right)+\\
 & \psi_x(2r_x^2\tan(t-r)-r_{xx}-i)r_x^{-1},\ \hat{r}=0, r_x\neq 0,\\
L_{3,4,2}: & e^{ax}\cos^{-1}(x-im)R\left(t,\psi_x e^{-ax}\sin(2im)\cos(x-im)+\hat{\psi}e^{-ax}\cos(x+im),\right.\\
 & \left.\psi_x^*e^{-ax}\sin(2im)\cos(x+im)+\hat{\psi}e^{-ax}\cos(x-im)\right)-\\
 & -2\psi_x\tan(x-im)-m_t\hat{\psi}\cos(x+im)\cos^{-1}(x-im)\sin^{-1}(2im),\ m\neq 0,\\
 & \hat{\psi}R\left(t,\cos x(\hat{\psi}\tan x-\hat{\psi}_x)e^{-ax},\hat{\psi}\cos x e^{-ax}\right)-2\psi_x\tan x,\quad m=0;
\end{array}
$$

$$
\begin{array}{ll}
L_{3,5,1}: & x^{-2}R(\hat{\psi},x\psi_x-2\psi,x\psi_x^*-2\psi)+\psi x^{-2}(i\psi-ix\psi_x-2),\\
L_{3,5,2}: & \psi_x R\left(t,x,\psi_x\psi_x^*\hat{\psi}^{-2}\right)-2\psi_x^2\hat{\psi}^{-1};
\end{array}
$$

$$
\begin{array}{ll}
L_{3,6,1}: & \psi_x^2 R\left(\psi,\psi^*,\psi_x/\psi_x^*\right),\\
L_{3,6,2}: & e^{2\psi}x^{-2}R(\hat{\psi},e^{-\psi}(x\psi_x-1),e^{-\psi}(x\psi_x^*-1))+x^{-2},\\
L_{3,6,3}: & e^x R(e^{-x}\hat{\psi},e^{-x}\psi_x,e^{-x}\psi_x^*),\\
L_{3,6,4}: & x^{-2}R(\hat{\psi},x\psi_x-2a\psi,x\psi_x^*-2a\psi)+2a\psi(1-2a)x^{-2},\\
L_{3,6,5}: & t^{-1}R(\hat{\psi},\sqrt{t}\psi_x,\sqrt{t}\psi_x^*)+ia\psi/(2t),\\
L_{3,6,6}: & t^{-1}R(\hat{\psi},\sqrt{t}\psi_x,\sqrt{t}\psi_x^*)+i(a\psi-(a+1)\psi\psi_x\sqrt{t})/(2t),\\
L_{3,6,7}: & \psi_x^{\frac{2(1+a)}{2+a}}e^{\frac{-at}{2+a}}R\left(e^{\frac{-at}{2+a}}\psi_x^{\frac{-2}{2+a}}\hat{\psi},\psi_x/\psi_x^*,x(\psi_x e^{-t})^{\frac{a}{2+a}}\right)-iax\psi_x/2,\\
L_{3,6,8}: & e^t R(e^{-t}\hat{\psi},e^{-t}\psi_x,e^{-t}\psi_x^*)-iax\psi_x,\\
L_{3,6,9}: & e^t R\left(x,(r_x\hat{\psi}-\psi_x\hat{r})e^{-t},(r_x\psi_x^*-\psi_x r_x^*)e^{-t}\right)+\psi_x(ir(1+a)-r_{xx})r_x^{-1},\ r_x\neq 0,\\
 & e^t R(x,e^{-t}\psi_x,e^{-t}\psi_x^*)+ir(1+a)\hat{\psi}\hat{r}^{-1},\ r_x=0,\ \hat{r}\neq 0,\\
L_{3,6,10}: & \psi_x R\left(t,\hat{\psi}/\psi_x,\psi_x^*/\psi_x\right)+i\psi_x\ln(\psi_x e^{-x})m_t/m,\\
L_{3,6,11}: & e^x R\left(t,(\hat{\psi}r(1+a)-\psi_x\hat{r})e^{-x},(\psi_x^*r-\psi_x r^*)e^{-x}\right)-\\
 & -\psi_x((1+a)^2r-ir_t)/((1+a)r),\ a\neq -1,\\
 & e^x R(t,\psi_x e^{-x},\psi_x^*e^{-x})+i\hat{\psi}r_t\hat{r}^{-1},\ \hat{r}\neq 0, a=-1,\\
L_{3,6,12}: & \hat{\psi}R(x,\hat{\psi}/\psi_x,\hat{\psi}/\psi_x^*),\\
L_{3,6,13}: & (r_x^*\hat{\psi}-\psi_x^*\hat{r})R\left(t,x,(r_x^*\hat{\psi}-\psi_x^*\hat{r})(r_x\hat{\psi}-\psi_x\hat{r})^{-1}\right)+\hat{\psi}\hat{r}^{-1}(ir_t-r_{xx}),\ \hat{r}\neq 0,\\
 & \hat{\psi}R\left(t,x,\hat{\psi}_x\hat{\psi}^{-1}\right)+\psi_x r_x^{-1}(ir_t-r_{xx}),\ \hat{r}=0, r_x\neq 0.
\end{array}
$$

We skip the proof of Theorem 3 which is analogous to the proofs of Theorems 1,2.

4 Galilei-invariant equations

.

In this section we will pick out of the classes of invariant equations constructed in the previous section the ones invariant with respect to the Lie algebra of the Galilei group and its natural extensions.

Definition 1 *The Lie algebra of the Galilei group in (1+1)-dimensional space-time is the four-dimensional Lie algebra* $g(1) = \langle P_0, P_1, M, G\rangle$ *determined by the following non-vanishing commutation relations:*

$$[P_0, G] = P_1, \quad [P_1, G] = M. \tag{14}$$

Definition 2 *The Lie algebra of the extended Galilei group in (1+1)-dimensional space-time is the five-dimensional Lie algebra* $\tilde{g}(1) = \langle P_0, P_1, M, G, D\rangle$ *determined by commutation relations (14) and*

$$[P_0, D] = 2P_0, \quad [P_1, D] = P_1, \quad [D, G] = G. \tag{15}$$

Definition 3 *The Lie algebra of the Schrödinger group in (1+1)-dimensional space-time is the six-dimensional Lie algebra* $sch(1) = \langle P_0, P_1, M, G, D, K\rangle$ *determined by commutation relations (14), (15) and*

$$[P_0, K] = D, \quad [P_1, K] = G, \quad [D, K] = 2K. \tag{16}$$

For brevity, in a sequel we call the algebras $g(1), \tilde{g}(1), sch(1)$ the Galilei, extended Galilei and Schrödinger algebras.

Note that the operators P_0, P_1 have the physical meaning of time and space translation generators, M is the central element of the algebra, G is the generator of Galilei transformations and D is the generator of scale transformations. The operator K is called the generator of projective or conformal transformations.

A complete description of inequivalent Galilei-invariant equations of the form (1) is given in Theorems 4–6 below. Note that solving this classification problem requires describing all possible realizations of the Galilei, extended Galilei and Schrödinger algebras within the class (5).

Theorem 4 *Any realization of the Galilei algebra* $g(1)$ *by the operators (5), which* *is a symmetry algebra of (1), is equivalent to one of the following realizations:*

$$
\begin{aligned}
&g(1)_1 = \langle \partial_t, \partial_x, \partial_\psi + \partial_{\psi^*}, t\partial_x + (x+i\alpha)\partial_\psi + (x-i\alpha)\partial_{\psi^*}\rangle,\\
&g(1)_2 = \langle t^2/2\partial_x - (t+c)\partial_\psi - (t+c^*)\partial_{\psi^*}, -t\partial_x + \partial_\psi + \partial_{\psi^*}, \partial_x, \partial_t\rangle,\\
&g(1)_3 = \langle -t\partial_x - (t^2/2+i\alpha)\partial_\psi - (t^2/2-i\alpha)\partial_{\psi^*}, \partial_x, \partial_\psi + \partial_{\psi^*}, \partial_t + x\partial_\psi + x\partial_{\psi^*}\rangle,\\
&g(1)_4 = \langle \partial_x, \partial_\psi + \partial_{\psi^*}, \frac{1}{t+i\alpha}\partial_\psi + \frac{1}{t-i\alpha}\partial_{\psi^*}, -\partial_t + \left(\frac{\psi}{t+i\alpha} + x\right)\partial_\psi + \left(\frac{\psi^*}{t-i\alpha} + x\right)\partial_{\psi^*}\rangle,\\
&g(1)_5 = \langle \partial_t, \partial_\psi + \partial_{\psi^*}, \frac{1}{x+i\alpha}\partial_\psi + \frac{1}{x-i\alpha}\partial_{\psi^*}, -\partial_x + \left(t + \frac{\psi}{x+i\alpha}\right)\partial_\psi + \left(t + \frac{\psi^*}{x-i\alpha}\right)\partial_{\psi^*}\rangle,\\
&g(1)_6 = \langle (1/2)(k(x) + (t+r(x))^2)\partial_\psi + (1/2)(k^*(x) + (t+r^*(x))^2)\partial_{\psi^*},\\
&\qquad -(t+r(x))\partial_\psi - (t+r^*(x))\partial_{\psi^*}, \partial_\psi + \partial_{\psi^*}, \partial_t\rangle,\\
&g(1)_7 = \langle (1/2)(r(t) + (x+im(t))^2)\partial_\psi + (1/2)(r^*(t) + (x-im(t))^2)\partial_{\psi^*},\\
&\qquad -(x+im(t))\partial_\psi - (x-im(t))\partial_{\psi^*}, \partial_\psi + \partial_{\psi^*}, \partial_x\rangle.
\end{aligned}
$$

The corresponding invariant equations read as

$$
\begin{aligned}
&g(1)_1: \quad i\psi_t = \psi_{xx} - i\psi_x^2/2 + R(\hat{\psi}_x, 2i\alpha\psi_x - \hat{\psi}),\\
&g(1)_2: \quad i\psi_t = \psi_{xx} + i(\hat{\psi} + \psi_x(c\psi^* - c^*\psi))\hat{c}^{-1} + R(\psi_x, \psi_x^*),\ c \neq c^*,\\
&g(1)_3: \quad i\psi_t = \psi_{xx} + \hat{\psi}(t-\psi_x)/(2\alpha) + R(\hat{\psi}_x, t-\psi_x),\ \alpha \neq 0,\\
&g(1)_4: \quad i\psi_t = \psi_{xx} + \hat{\psi}(t-i\alpha)(2\alpha(t+i\alpha))^{-1} +\\
&\qquad +(t+i\alpha)^{-1}R\left(2\psi_x(t+i\alpha) + (t+i\alpha)^2, 2\psi_x^*(t-i\alpha) + (t-i\alpha)^2\right),\\
&g(1)_5: \quad i\psi_t = \psi_{xx} - i(x+i\alpha)/2 + 2\psi_x/(x+i\alpha) +\\
&\qquad +(x+i\alpha)^{-1}R(2\alpha i\psi_x(x+i\alpha) - \hat{\psi}(x-i\alpha), \psi_x(x+i\alpha) - \psi_x^*(x-i\alpha) + \hat{\psi}),\\
&g(1)_6: \quad i\psi_t = \psi_{xx} + R\left(x, \hat{\psi}(r_x k_x^* - k_x r_x^* - 2r_x r_x^*\hat{r}) + 2\mathrm{Re}[\psi_x(r_x^*\hat{k} - k_x^*\hat{r} + r_x^*\hat{r}^2)]\right) +\\
&\qquad +((i - r_{xx})(\psi_x(\hat{k} - \hat{r}^2) - k_x\hat{\psi}) + (2r_x^2 + k_{xx})(\psi_x\hat{r} - r_x\hat{\psi}))(r_x\hat{k} - k_x\hat{r} - r_x\hat{r}^2)^{-1},\\
&\qquad k \neq \alpha_2 - (r+\alpha_1)^2,\ r \neq \alpha_1,\\
&g(1)_7: \quad i\psi_t = \psi_{xx} + R\left(t, 4i\hat{\psi}m + \psi_x(4m^2 - \hat{r}) + \psi_x^*(4m^2 + \hat{r})\right) +\\
&\qquad +(r_t + 2i)(i\hat{\psi} + 2m\psi_x)(4m^2 + \hat{r})^{-1} - \psi_x m_t, \qquad 4m^2 + \hat{r} \neq 0.
\end{aligned}
$$

Theorem 5 *Any realization of the extended Galilei algebra $\tilde{g}(1)$ by the operators (5), which is a symmetry algebra of (1), is equivalent to one of the following realizations:*

$$
\begin{aligned}
&\tilde{g}(1)_1 = \langle \partial_t, \partial_x, \partial_\psi + \partial_{\psi^*}, t\partial_x + x\partial_\psi + x\partial_{\psi^*}, 2t\partial_t + x\partial_x + i\alpha\partial_\psi - i\alpha\partial_{\psi^*}\rangle,\\
&\tilde{g}(1)_2 = \langle (1/2)(c_2x^4 + t(t + 2c_1x^2))\partial_\psi + (1/2)(c_2^*x^4 + t(t + 2c_1^*x^2))\partial_{\psi^*},\\
&\qquad -(t + c_1x^2)\partial_\psi - (t + c_1^*x^2)\partial_{\psi^*}, \partial_\psi + \partial_{\psi^*}, \partial_t, -2t\partial_t - x\partial_x\rangle,\\
&\tilde{g}(1)_3 = \langle (1/2)(ct + (x + i\alpha\sqrt{t})^2)\partial_\psi + (1/2)(c^*t + (x - i\alpha\sqrt{t})^2)\partial_{\psi^*},\\
&\qquad -(x + i\alpha\sqrt{t})\partial_\psi - (x - i\alpha\sqrt{t})^2)\partial_{\psi^*}, \partial_\psi + \partial_{\psi^*}, \partial_x, -t\partial_t - (1/2)x\partial_x\rangle.
\end{aligned}
$$

The corresponding invariant equations read as:

$$\tilde g(1)_1: \quad i\psi_t=\psi_{xx}-i\psi_x^2/2+\hat\psi_x^2R(\hat\psi+2i\alpha\log\hat\psi_x),$$

$$\begin{aligned}\tilde g(1)_2: \quad i\psi_t=\psi_{xx}+&\frac{x\psi_x(2\hat c_1(6c_2-ic_1)+\hat c_2(i-2c_1))-4\hat\psi(i(c_2-c_1^2)+4c_1c_2)}{2x^2(c_1\hat c_2-2c_2\hat c_1)}+\\ &+x^{-2}R\left(\mathrm{Re}[x\psi_x(c_1^*\hat c_2-2c_2^*\hat c_1)]+2\hat\psi(c_1c_2^*-c_1^*c_2)\right), \quad c_1,c_2\neq 0,\end{aligned}$$

$$\begin{aligned}\tilde g(1)_3: \quad i\psi_t=\psi_{xx}+&t^{-1}R\left(4i\alpha\hat\psi+\sqrt{t}\psi_x(4\alpha^2-\hat c)+\sqrt{t}\psi_x^*(4\alpha^2+\hat c)\right)-\\ &-\alpha\psi_xt^{-1/2}/2+t^{-1}(2i+c)(i\hat\psi+2\alpha\sqrt{t}\psi_x)(4\alpha^2+\hat c)^{-1}, \qquad 4\alpha^2+c-c^*\neq 0.\end{aligned}$$

Theorem 6 *Any realization of the Schrödinger algebra $sch(1)$ by the operators (5), which is a symmetry algebra of (1), can be reduced to the form:*

$$\begin{aligned}sch_1=\langle&\partial_t,\partial_x,\partial_\psi+\partial_{\psi^*},t\partial_x+x\partial_\psi+x\partial_{\psi^*},2t\partial_t+x\partial_x-i\partial_\psi+i\partial_{\psi^*},\\ &t^2\partial_t+tx\partial_x+(x^2/2-it)\partial_\psi+(x^2/2+it)\partial_{\psi^*}\rangle.\end{aligned}$$

The corresponding invariant equations have the following structure:

$$i\psi_t=\psi_{xx}-i\psi_x^2/2+\hat\psi_x^2R(\hat\psi-2i\log\hat\psi_x).$$

Proof of Theorems 4-6. The Galilei algebra, the extended Galilei algebra and the Schrödinger algebra contain a three-dimensional subalgebra $\langle M,P_1,G\rangle$ belonging to the class $L_{3,3}$. One can prove that operators M,P_1,G can be transformed by (4) to the operators forming one of the following realizations of $L_{3,3}$:

$$\begin{aligned}
L_{3,3,5}&=\langle\partial_\psi+\partial_{\psi^*},\partial_x,n(t)\partial_x+(x+im(t))\partial_\psi+(x-im(t))\partial_{\psi^*}\rangle,\\
L_{3,3,6}&=\langle\partial_x,\alpha_1\partial_t,\alpha_2\partial_t+(t/\alpha_1)\partial_x+\partial_\psi+\partial_{\psi^*}\rangle,\\
L_{3,3,7}&=\langle\partial_\psi+\partial_{\psi^*},\partial_t,\alpha_2\partial_t+(t+r(x))\partial_\psi+(t+r^*(x))\partial_{\psi^*}\rangle,\\
L_{3,3,8}&=\langle\partial_\psi+\partial_{\psi^*},\partial_t,\alpha_2\partial_t+\alpha_1\partial_x+t\partial_\psi+t\partial_{\psi^*}\rangle,\\
L_{3,3,9}&=\langle\partial_x,-(t/\alpha_1)\partial_x+\partial_\psi+\partial_{\psi^*},\alpha_1\partial_t\rangle,\\
L_{3,3,10}&=\langle\partial_\psi+\partial_{\psi^*},\partial_x,\alpha_1\partial_t+x\partial_\psi+x\partial_{\psi^*}\rangle,\\
L_{3,3,11}&=\langle\partial_\psi+\partial_{\psi^*},-(t+r(x))\partial_\psi-(t+r^*(x))\partial_{\psi^*},\partial_t\rangle,\\
L_{3,3,12}&=\langle\partial_\psi+\partial_{\psi^*},-(x+im(t))\partial_\psi-(x-im(t))\partial_{\psi^*},\partial_x\rangle.
\end{aligned}$$

We consider the above enumerated cases separately.

- the case of the realization $L_{3,3,5}$

In this case

$$M=\partial_\psi+\partial_{\psi^*}, \ P_1=\partial_x, \ G=n(t)\partial_x+(x+im(t))\partial_\psi+(x-im(t))\partial_{\psi^*}.$$

Furthermore, P_0,D,K are of the form (5) and have to obey together with M,P_1,G commutation relations (14) for the Galilei algebra, (14), (15) for the extended Galilei algebra and (14)–(16) for the Schrödinger algebra. Solving these commutation relations and simplifying the operators P_0,P_1,M,G,D,K by means of transformations (4) we obtain that

$$P_0=\alpha_1\partial_t, \ P_1=\partial_x, \ M=\partial_\psi+\partial_{\psi^*}, \ G=(t/\alpha_1)\partial_x+(x+i\alpha)\partial_\psi+(x-i\alpha)\partial_{\psi^*},$$

for the Galilei algebra and

$$P_0=\alpha_1\partial_t,\ P_1=\partial_x,\ M=\partial_\psi+\partial_{\psi^*},\ G=(t/\alpha_1)\partial_x+x\partial_\psi+x\partial_{\psi^*}$$

$$D=2t\partial_t+x\partial_x+c\partial_\psi+c^*\partial_{\psi^*},$$

$$K=(t^2/\alpha_1)\partial_t+(tx/\alpha_1)\partial_x+(x^2/2+ct/\alpha_1)\partial_\psi+(x^2/2+c^*t/\alpha_1)\partial_{\psi^*}$$

for the extended Galilei algebra and the Schrödinger algebra.

The realization of the Galilei algebra $\langle P_0, P_1, M, G\rangle$ obtained above is equivalent to the realization $g(1)_1$. Indeed, making use of the transformations from the class (4)

$$t\to\tau=t/\alpha_1^2;\ x\to\kappa=x/\alpha_1,\ \psi\to\varphi=\psi/\alpha_1$$

we reduce the operators P_0, P_1, M, G to become

$$\tilde{P}_0=\alpha_1^3\partial_t,\ \tilde{P}_1=\alpha_1\partial_x,\ \widetilde{M}=\alpha_1\partial_\psi+\alpha_1\partial_{\psi^*},\ \tilde{G}=t\partial_x+(x+i\tilde{\alpha})\partial_\psi+(x-i\tilde{\alpha})\partial_{\psi^*},$$

($\tilde{\alpha}=\alpha\alpha_1$) and the latter operators are linear combination of basis elements of $g(1)_1$.

Similarly, the realization of the extended Galilei algebra obtained above is equivalent to the realization $\tilde{g}(1)_1$. At last, the above obtained realization of the Schrödinger algebra is equivalent to the following realization:

$$\begin{aligned}&\langle\partial_t,\partial_x,\partial_\psi+\partial_{\psi^*},t\partial_x+x\partial_\psi+x\partial_{\psi^*},2t\partial_t+x\partial_x+c\partial_\psi+c^*\partial_{\psi^*},\\&t^2\partial_t+tx\partial_x+(x^2/2+ct)\partial_\psi+(x^2/2+c^*t)\partial_{\psi^*}\rangle.\end{aligned}\tag{17}$$

Solving the equation (6) for each of the operators listed in (17) yields that there exists an invariant equation of the form (1) admitting the symmetry algebra (17) iff $c=-i$. Therefore, (17) necessarily takes the form of the realization $sch(1)_1$ given in Theorem 6.

- the case of realizations $L_{3,3,6}$, $L_{3,3,7}$ or $L_{3,3,8}$

There is no operator P_0 of the form (5) such that the operators M, P_1, G, that are basis elements of the realizations $L_{3,3,6}$, $L_{3,3,7}$ or $L_{3,3,8}$, together with P_0 satisfy commutation relations (14). Therefore, there are no realizations of the Galilei algebra and its extensions this case.

- the case of realization $L_{3,3,9}$

In analogy to to the case $L_{3,3,5}$ we show that the operators P_0, P_1, M, G satisfying (14), where M, P_1, G form the realization of the algebra $L_{3,3,9}$, can be reduced by means of (4) to the operators

$$\left(\frac{t^2}{2\alpha_1^2}+\alpha_2\right)\partial_x-\left(\frac{t}{\alpha_1}+c\right)\partial_\psi-\left(\frac{t}{\alpha_1}+c^*\right)\partial_{\psi^*},\ -\frac{t}{\alpha_1}\partial_x+\partial_\psi+\partial_{\psi^*},\ \partial_x,\ \alpha_1\partial_t,$$

which are basis elements of the realization equivalent to $g(1)_2$. What is more, there are no realizations of the extended Galilei and Schrödinger algebras in this case.

- the case of realization $L_{3,3,10}$

The operators P_0, P_1, M, G can be reduced to become

$$\alpha_2\partial_t - \frac{t}{\alpha_1}\partial_x - \left(\frac{t^2}{2\alpha_1^2} + c\right)\partial_\psi - \left(\frac{t^2}{2\alpha_1^2} + c^*\right)\partial_{\psi^*},\ \partial_x,\ \partial_\psi + \partial_{\psi^*},\ \alpha_1\partial_t + x\partial_\psi + x\partial_{\psi^*}.$$

The above operators form a realization equivalent to $g(1)_1$, $(\alpha_2 \neq 0)$ or $g(1)_3$, $(\alpha_2 = 0)$.

What is more, there are no realizations of the extended Galilei and Schrödinger algebras in this case.

- the case of the realization $L_{3,3,11}$

The operators P_0, P_1, M, G can be reduced either to the basis elements of the realization $g(1)_6$ or to the following ones:

$$\alpha_1\partial_t + \alpha_2\partial_x + (1/2)(t - (\alpha_1/\alpha_2)x + c)^2\partial_\psi + (1/2)(t - (\alpha_1/\alpha_2)x + c^*)^2\partial_{\psi^*},$$

$$-(t - (\alpha_1/\alpha_2)x + c)\partial_\psi - (t - (\alpha_1/\alpha_2)x + c^*)\partial_{\psi^*},\ \partial_\psi + \partial_{\psi^*},\ \partial_t,$$

where the latter form the realization equivalent either to $g(1)_4$, $(\alpha_1 = 0)$ or to $g(1)_5$, $(\alpha_1 \neq 0)$.

Now the operators P_0, P_1, M, G, D can be reduced to those forming the realization $\tilde{g}(1)_2$.

There are no realizations of the Schrödinger algebra in this case.

- $L_{3,3,12}$

The operators P_0, P_1, M, G can be reduced either to those forming the realization of the algebra $g(1)_7$ or to the following operators:

$$\alpha_1\partial_t + (x + i\alpha)^2/2\partial_\psi + (x - i\alpha)^2/2\partial_{\psi^*},\ -(x + i\alpha)\partial_\psi - (x - i\alpha)\partial_{\psi^*},\ \partial_\psi + \partial_{\psi^*},\ \partial_x,$$

$(\alpha_1 \neq 0)$, that form a realization equivalent to $g(1)_5$.

The operators P_0, P_1, M, G, D can be reduced to the ones forming the realization $\tilde{g}(1)_3$.

There are no realizations of the Schrödinger algebra in this case.

The proof of the fact that the realizations of the Galilei and extended Galilei algebras are inequivalent is performed by direct (and rather cumbersome) computations.

Solving (6) for all the realizations obtained yields the corresponding invariant nonlinear Schrödinger equations which completes the proof of Theorems 4–6.

At the end of this section we apply the above results to symmetry classification of complex Galilei-invariant Doebner-Goldin equations.

The nonlinear Schrödinger equation, which was derived by Doebner and Goldin [2, 3] can be written as:

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\Delta\psi + V(t, x)\psi + \frac{i\hbar}{2}\nu R_2[\psi]\psi + \hbar\nu'\sum_{j=1}^{5}\mu_j R_j[\psi]\psi, \tag{18}$$

where $\psi(t,x)$ is a quantum-mechanical wave function of a particle of mass m having the probability density $\rho = \psi\psi^*$ and the current $J = \hbar/(2mi)(\psi^*\nabla\psi - (\nabla\psi^*)\psi)$, V is a potential energy and $R_j[\psi]$, $j = 1, ..., 5$, are real-valued functionals

$$R_1 = \frac{\nabla J'}{\rho}, \quad R_2 = \frac{\nabla^2 \rho}{\rho}, \quad R_3 = \frac{J'J'}{\rho^2}, \quad R_4 = \frac{J'\nabla\rho}{\rho^2}, \quad R_5 = \frac{(\nabla\rho)^2}{\rho^2}, \tag{19}$$

with $J' = Jm/\hbar$.

It is supposed in (18) that ν, ν' are real constants having the dimension of diffusion coefficients. Symmetry classification of the Doebner-Goldin equations for the case of real coefficients μ_j has been done in [10].

By making use of the results from the previous section we construct here new classes of Galilei-invariant nonlinear Doebner-Goldin equations (18) with complex coefficients μ_j.

We restrict our considerations to subfamilies of the Doebner-Goldin equations with $V(x) = 0$. What is more, it is supposed that nonlinear terms do not contain the second-order derivatives. As result, we have

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + \hbar\nu'(\mu_3 R_3[\psi] + \mu_4 R_4[\psi] + \mu_5 R_5[\psi])\psi, \tag{20}$$

where μ_3, μ_4, μ_5 are arbitrary complex parameters.

The results of symmetry classification of equation (20) are given below without proof.

Theorem 7 *The nonlinear Doebner-Goldin equation (20) is invariant under the Galilei algebra iff its coefficients satisfy the following conditions:*

$$\begin{aligned} \mu_3 &= -b(\beta+\beta^*)^2 - \hbar/(2m\nu')(i\lambda/(2\beta)+1), \\ \mu_4 &= -ib(\beta^2-\beta^{*2}) + i\hbar/(2m\nu')(i\lambda/(2\beta)+1), \\ \mu_5 &= (b/4)(\beta-\beta^*)^2 + \hbar/(8m\nu')(i\lambda/(2\beta)+1), \end{aligned} \tag{21}$$

where b, β are arbitrary complex constants and $\lambda = 0, 1$. In this case the realizations of Galilei algebra $\langle P_0, P_1, M, G\rangle$ read as

$$\begin{aligned} g(1)_8 = \langle &\partial_t, \partial_x, -(2m/\hbar)\beta\psi\partial_\psi - (2m/\hbar)\beta^*\psi^*\partial_{\psi^*}, \\ &t\partial_x - (2m/\hbar)\beta(x+i\alpha)\psi\partial_\psi - (2m/\hbar)\beta^*(x-i\alpha)\psi^*\partial_{\psi^*}\rangle, \ \lambda = 1; \end{aligned}$$

$$\begin{aligned} g(1)_9 = \langle &-(2m/\hbar)\partial_t + \beta(x^2/2 + i\hbar t/(2m))\psi\partial_\psi + \beta^*(x^2/2 - i\hbar t/(2m))\psi^*\partial_{\psi^*}, \\ &\beta(x+i\alpha)\psi\partial_\psi + \beta^*(x-i\alpha)\psi^*\partial_{\psi^*}, \beta\psi\partial_\psi + \beta^*\psi^*\partial_{\psi^*}, \\ &-\partial_x - \alpha\beta(x^2/2 + i\hbar t/(2m))\psi\partial_\psi - \alpha\beta^*(x^2/2 - i\hbar t/(2m))\psi^*\partial_{\psi^*}\rangle, \ \lambda = 0; \end{aligned}$$

$$\begin{aligned} g(1)_{10} = \langle &\beta(x^2/2 + i\hbar t/(2m))\psi\partial_\psi + \beta^*(x^2/2 - i\hbar t/(2m))\psi^*\partial_{\psi^*}, \\ &-\beta(x-i\alpha)\psi\partial_\psi - \beta^*(x+i\alpha)\psi^*\partial_{\psi^*}, \beta\psi\partial_\psi + \beta^*\psi^*\partial_{\psi^*}, \\ &-(2m/\hbar)\alpha\partial_t + \partial_x\rangle, \ \lambda = 0, \end{aligned}$$

where α is an arbitrary real constant.

It should be noted that, provided condition (21) holds true, equation (20) can be rewritten in a more compact form

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\psi_{xx} + \frac{\hbar^2}{2m}\left(\frac{i\lambda}{2\beta}+1\right)\frac{\psi_x^2}{\psi} + \hbar\nu' b\left(\beta^*\frac{\psi_x}{\psi} - \beta\frac{\psi_x^*}{\psi^*}\right)^2\psi.$$

Now it is easy to see that if $b=0$, then the above equation is linearized by means of the following change of variables:

$\psi \to e^{\psi}\,(\lambda=0)$ and $\psi \to \psi^{2i\beta}\,(\lambda=1)$.

This is the reason why we do not distinguish the case $b=0$ when formulating Theorem 7.

Theorem 8 *The nonlinear Doebner-Goldin equation (20) is invariant under the extended Galilei algebra iff its coefficients satisfy conditions (21). In this case the realizations of the extended Galilei algebras read as*

$$\begin{aligned}\tilde{g}(1)_4 = \langle &\partial_t, \partial_x, -(2m/\hbar)\beta\psi\partial_\psi - (2m/\hbar)\beta^*\psi^*\partial_{\psi^*},\\ &t\partial_x - (2m/\hbar)\beta x\psi\partial_\psi - (2m/\hbar)\beta^* x\psi^*\partial_{\psi^*}, 2t\partial_t + x\partial_x + i\alpha\psi\partial_\psi - i\alpha\psi^*\partial_{\psi^*}\rangle,\ \lambda=1;\\ \tilde{g}(1)_5 = \langle &\beta(x^2/2 + i\hbar t/(2m))\psi\partial_\psi + \beta^*(x^2/2 - i\hbar t/(2m))\psi^*\partial_{\psi^*},\\ &-\beta x\psi\partial_\psi - \beta^* x\psi^*\partial_{\psi^*}, \beta\psi\partial_\psi + \beta^*\psi^*\partial_{\psi^*}, \partial_x, -t\partial_t - (1/2)x\partial_x\rangle,\ \lambda=0.\end{aligned}$$

Theorem 9 *The nonlinear Doebner-Goldin equation (20) is invariant under the Schrödinger algebra iff the coefficients satisfy conditions (21) with $\lambda = 1$. In this case the realization of the Schrödinger algebra reads as*

$$\begin{aligned}sch(1)_2 = \langle &\partial_t, \partial_x, -(2m/\hbar)\beta\psi\partial_\psi - (2m/\hbar)\beta^*\psi^*\partial_{\psi^*},\\ &t\partial_x - (2m/\hbar)\beta x\psi\partial_\psi - (2m/\hbar)\beta^* x\psi^*\partial_{\psi^*}, 2t\partial_t + x\partial_x - i\beta\psi\partial_\psi + i\beta^*\psi^*\partial_{\psi^*},\\ &t^2\partial_t + tx\partial_x - (x^2 m/\hbar + it)\beta\psi\partial_\psi - (x^2 m/\hbar - it)\beta^*\psi^*\partial_{\psi^*}\rangle.\end{aligned}$$

5 Concluding Remarks

One of the evident conclusions is that the choice of possible nonlinear generalizations of the Schrödinger is very extensive, even if we restrict them to be of the special form (1). Specifying these equations may be achieved if we impose a requirement that they should admit the wider symmetry group (this is the case for the Schrödinger algebra). On this way, it might be possible to obtain a complete symmetry classification, namely to describe all possible forms of the functions F that (5) admits a non-trivial symmetry group. However, the full solution of this problem requires more powerful algebraic techniques utilizing fine properties of the Lie groups (like the Levi-Maltsev theorem, properties of the simple, semi-simple and solvable Lie groups). We are postponing investigation of this challenging problem to our future publications.

Since the subgroup structure of low dimensional continuous groups is known, the results obtained might be utilized for symmetry reduction of the invariant equations obtained in the paper. We intend to devote one of our future publications to a detailed study of this problem.

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