Conditional Lie-Bäcklund symmetry and reduction of evolution equations.

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Abstract

We suggest a generalization of the notion of invariance of a given partial differential equation with respect to Lie-Bäcklund vector field. Such generalization proves to be effective and enables us to construct principally new Ansätze reducing evolution-type equations to several ordinary differential equations. In the framework of the said generalization we obtain principally new reductions of a number of nonlinear heat conductivity equations $u_t = u_{xx} + F(u, u_x)$ with poor Lie symmetry and obtain their exact solutions. It is shown that these solutions can not be constructed by means of the symmetry reduction procedure.

1 Introduction

Construction of exact solutions of nonlinear partial differential equations (PDEs) is one of the most important problems of the modern mathematical physics. The most effective and universal method used is the symmetry reduction procedure pioneered by Sophus Lie. But there is a natural restriction on the application of the said procedure: equation under study should have non-trivial Lie symmetry. There exist very important equations (in particular, the ones describing heat conductivity and some nonlinear processes in biology) with very poor Lie symmetry. So it would be desirable to modify the symmetry reduction procedure in such a way that it could be applied to these equations as well. Fortunately, the main idea of the symmetry reduction procedure – the reduction of the equation under study to PDEs having less number of independent variables by means of specially chosen Ansätze – can be applied to some of these if one utilizes their *conditional* symmetry (see

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[5], [7]). The method of conditional symmetries of PDEs is closely connected with the "non-classical reduction" [1] and "direct reduction" [2] methods (see also [12], [13]).

Further possibility of constructing exact solutions of PDEs is to use their Lie-Bäcklund (higher, generalized) symmetry [11]. In this way multi-soliton solutions of the KdV, mKdV, sine-Gordon and cubic Schrödinger equations can be obtained [3]. But the choice of physically significant examples of equations admitting non-trivial Lie-Bäcklund symmetry is very restricted. On the other hand, there are examples due to Galaktionov [10, 16] and Fushchych et al [4, 6] of Ansätze reducing PDEs admitting only trivial Lie-Bäcklund symmetry to systems of ordinary differential equations (ODEs). This facts can be understood within the framework of the *conditional* Lie-Bäcklund symmetry which is introduced below. It will be established that conditional invariance of the equation under study ensures its reducibility and can be applied to construct its exact solutions. Since the class of PDEs conditionally-invariant with respect to some Lie-Bäcklund field is substantially wider than the one of PDEs admitting Lie-Bäcklund symmetry in the classical sense, the said result yields principally new possibilities of reduction of PDEs with poor Lie and Lie-Bäcklund symmetry. We will give several examples of reduction of PDEs to systems of ODEs by means of the Anätze corresponding to their conditional Lie-Bäcklund symmetry and will show that the exact solutions obtained in this way can not be constructed by means of classical symmetry reduction procedure.

Let

$$u_t = F(t,\, x,\, u,\, u_1,\, u_2, \ldots, u_n), \tag{1}$$

where $u \in C^n(\mathbb{R}^2, \mathbb{C}^1)$, $u_k = \partial^k u/\partial x^k$, $1 \le k \le n$, be some evolution-type equation and

$$Q = \eta\partial_u + (D_x\eta)\partial_{u_1} + (D_t\eta)\partial_{u_t} + (D_x^2\eta)\partial_{u_2} + \ldots \tag{2}$$

with

$$\eta = \eta(t,\, x,\, u,\, u_t,\, u_1,\, u_{tt},\, u_{t1}, \ldots) \tag{3}$$

be some smooth Lie-Bäcklund vector field (LBVF).

In the above formulae we denote by the symbols D_t and D_x the total differentiation operators with respect to the variables t and x correspondingly, i.e.

$$\begin{aligned} D_t &= \partial_t + u_t\partial_u + u_{tt}\partial u_t + u_{t1}\partial u_1 + \ldots, \\ D_x &= \partial_x + u_1\partial_u + u_{t1}\partial u_t + u_2\partial u_1 + \ldots. \end{aligned}$$

If the function η is of the form

$$\eta = \tilde{\eta}(t,x,u) - \xi_0(t,x,u)u_t - \xi_1(t,x,u)u_x, \tag{4}$$

then the LBVF (2) is equivalent to the usual Lie vector field and can be represented in the following equivalent form:

$$Q = \xi_0(t,x,u)\partial_t + \xi_1(t,x,u)\partial_x + \tilde{\eta}(t,x,u)\partial_u.$$

Definition 1. We say that Eq.(1) is invariant under the LBVF (2) if the condition

$$Q(u_t - F)\ \Big|_{M} = 0 \tag{5}$$

holds.

In (3) M is a set of all differential consequences of the equation $u_t - F = 0$.

Definition 2. We say that Eq.(1) is conditionally-invariant under LBVF (2) if the following condition

$$Q(u_t - F)\ \Big|_{M \cap L_x} = 0 \tag{6}$$

holds.

Here the symbol L_x denotes the set of all differential consequences of the equation $\eta = 0$ with respect to the variable x.

Evidently, condition (3) is nothing else than a usual invariance criteria for Eq.(1) under LBVF (2) written in a canonical form (see, e.g. [11]). The most of "soliton equations" like the KdV, mKdV, cubic Schrödinger, sine-Gordon equations admit infinitely many LBVFs which can be obtained from some initial LBVF by applying the recursion operator.

Another important remark is that on the set of solutions of Eq.(1) we can exclude all derivatives with respect to t and thus get the vector field (2) with η of the form

$$\eta = \eta(t, x, u, u_1, u_2, \ldots, u_N). \tag{7}$$

In the following we will consider LBVFs of the form (2), (7) only.

Clearly, if Eq.(1) is invariant under LBVF (2), then it is conditionally-invariant under the said field. But the inverse assertion is not true. This means, in particular, that the Definition 2 is a generalization of the standard definition of invariance of partial differential equation with respect to LBVF. Provided (2) is a Lie vector field, Definition 2 coincides with the one of Q-conditional invariance under the Lie vector field.

One of the important consequences of Q-conditional invariance of a given PDE under the Lie vector field is a possibility to get an Ansatz which reduces this PDE to one PDE with less number of independent variables (see, e.g. [7]). We will show that conditional invariance of the evolution-type equation (1) ensures its reducibility to N ordinary differential equations (ODEs) (N is the order of the highest derivative contained in η from (7)).

2 Reduction theorem

Consider the nonlinear PDE

$$\eta(t,\ x,\ u,\ u_1, \ldots, u_N) = 0 \tag{8}$$

as the N-th order ODE with respect to variable x. Its general integral is written (at least locally) in the form

$$u = f\Big(t,\, x,\, \varphi_1(t),\, \varphi_2(t), \ldots, \varphi_N(t)\Big), \tag{9}$$

where $\varphi_j(t),\ j = \overline{1, N}$ are arbitrary smooth functions. We will call the expression (9) an Ansatz invariant under LBVF (2), (7).

Theorem 1. *Let Eq.(1) be conditionally-invariant under the LBVF (2), (7). Then the Ansatz (9) invariant under LBVF (2), (7) reduces PDE (1) to a system of N ODEs for functions $\varphi_j(t),\ j = \overline{1, N}$.*

P r o o f. We first proof that given the conditions of the theorem the system of PDEs

$$\begin{cases} u_t = F(t,\, x,\, u,\, u_1, \ldots, u_n), \\ \\ \eta(t,\, x,\, u,\, u_1, \ldots, u_N) = 0 \end{cases} \tag{10}$$

is compatible.

Differentiating the first equation from (10) N times with respect to x, the second – one time with respect to t and comparing the derivatives u_{Nt} and u_{tN} we get the equality

$$D_x^N F = -(\eta_{u_N})^{-1}(\eta_t + \eta_u u_t + \eta_{u_1} u_{1t} + \ldots + \eta_{u_{N-1}} u_{N-1t})$$

or

$$D_x^N F = -(\eta_{u_N})^{-1}(\eta_t + \eta_u F + \eta_{u_1} D_x F + \ldots + \eta_{u_{N-1}} D_x^{N-1} F).$$

Consequently, provided the condition

$$(\eta_t + \eta_u F + \eta_{u_1} D_x F + \ldots + \eta_{u_N} D_x^N F) \Big|_{M \cap L} = 0 \tag{11}$$

where L is the set of all differential consequences of the equation $\eta = 0$, holds identically, the system of PDEs (10) is in involution and its general solution contains N arbitrary complex constants $C_1,\ C_2, \ldots, C_N$ [15].

We will prove that the relation (11) follows from (6).

Really, with account of (2) the equality (6) is rewritten in the form

$$D_t \eta - \eta F_u - (D_x \eta) F_{u_1} - \ldots - (D_x^n \eta) F_{u_n} \Big|_{M \cap L_x} = 0$$

or

$$D_t \eta \Big|_{M \cap L_x} = 0.$$

Since $D_t \eta = \eta_t + \eta_u u_t + \eta_{u_1} u_{1t} + \ldots + \eta_{u_N} u_{Nt}$, the above equation reads

$$\eta_t + \eta_u u_t + \eta_{u_1} u_{1t} + \ldots + \eta_{u_N} u_{Nt} \Big|_{M \cap L_x} = 0,$$

whence

$$\eta_t + \eta_u F + \eta_{u_1} D_x F + \ldots + \eta_{u_N} D_x^N F \Big|_{M \cap L_x} = 0. \tag{12}$$

Since the manifold $M \cap L$ is contained in the manifold $M \cap L_x$, relation (11) follows from relation (12).

Next, we consider the determinant

$$\Delta = \begin{vmatrix} \dfrac{\partial f}{\partial \varphi_1} & \dfrac{\partial f}{\partial \varphi_2} & \cdots & \dfrac{\partial f}{\partial \varphi_N} \\ \dfrac{\partial^2 f}{\partial \varphi_1 \partial x} & \dfrac{\partial^2 f}{\partial \varphi_2 \partial x} & \cdots & \dfrac{\partial^2 f}{\partial \varphi_N \partial x} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^N f}{\partial \varphi_1 \partial x^{N-1}} & \dfrac{\partial^N f}{\partial \varphi_2 \partial x^{N-1}} & \cdots & \dfrac{\partial^N f}{\partial \varphi_N \partial x^{N-1}} \end{vmatrix} \tag{13}$$

The above determinant Δ is the Wronsky determinant for functions $y_j = \frac{\partial f}{\partial \varphi_j}$, $j = \overline{1, N}$. We will prove that in the case involved $\Delta \neq 0$.

Let $\Delta = 0$, then due to the properties of the Wronsky determinant the functions y_j are linearly-dependent. Consequently, there exist such $\lambda_j = \lambda_j(t),\ j = \overline{1, N}$ that

$$\sum_{j=1}^{N} \lambda_j(t) y_j = 0.$$

Substituting into the above equality $y_j = \frac{\partial f}{\partial \varphi_j}$ we get

$$\sum_{j=1}^{N} \lambda_j(t) \frac{\partial f}{\partial \varphi_j} = 0. \tag{14}$$

Integrating the first-order PDE (14) we have

$$f = \tilde{f}(t,\, x,\, \omega_1,\, \omega_2, \ldots, \omega_{N-1}),$$

where $\omega_j = \lambda_N \varphi_j - \lambda_j \varphi_N,\ j = \overline{1, N-1}$.

Consequently, in the case $\Delta = 0$ the general solution of the ODE (8) depends not on N but on $N-1$ arbitrary constants $\omega_j(t),\ j = \overline{1, N-1}$. We arrive at the contradiction, a source of which is an assumption that $\Delta = 0$. Hence we conclude that $\Delta \neq 0$.

Substituting (9) into (1) we obtain

$$\sum_{j=1}^{N} \dot{\varphi}_j \frac{\partial f}{\partial \varphi_j} = -f_t + F(t,\, x,\, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}, \ldots, \frac{\partial^n f}{\partial x^n})$$

or

$$\sum_{j=1}^{N} \dot{\varphi}_j \frac{\partial f}{\partial \varphi_j} = G\big(t,\, x,\, \varphi_1(t),\, \varphi_2(t), \ldots, \varphi_N(t)\big). \tag{15}$$

Hereafter, an overdot means differentiation with respect to t.

Differentiation of (15) $N-1$ times with respect to the variable x yields the following result:

$$\sum_{j=1}^{N} \dot{\varphi}_j \frac{\partial^{k+1} f}{\partial \varphi_j \partial x^k} = \frac{\partial^k G}{\partial x^k}, \quad k = \overline{1, N-1}. \tag{16}$$

If we consider Eqs.(15), (16) as a system of linear inhomogeneous algebraic equations for functions $\dot{\varphi}_1,\ \dot{\varphi}_2, \ldots, \dot{\varphi}_N$, then its determinant has the form (13) and, consequently, is not equal to zero. Solving (15), (16) with respect to the functions $\dot{\varphi}_j,\ j = \overline{1, N}$ we get

$$\dot{\varphi}_j = H_j(t,\, x,\, \varphi_1,\, \varphi_2, \ldots, \varphi_N), \quad j = \overline{1, N}. \tag{17}$$

Let us expand the right-hand sides of (17) into a Taylor series with respect to the variable x in the neighbourhood of x_0 and then equate coefficients at $(x - x_0)^k$

$$\dot{\varphi}_j = H_j(t,\, x_0,\, \varphi_1,\, \varphi_2, \ldots, \varphi_N), \quad j = \overline{1, N}, \tag{18}$$

$$0 = \frac{\partial^k H_j}{\partial x^k}(t,\, x_0,\, \varphi_1,\, \varphi_2, \ldots, \varphi_N), \quad j = \overline{1, N}, \quad k \geq 1. \tag{19}$$

Thus, we have established that the system of PDEs (10) is equivalent to the infinite set of Eqs.(18), (19).

Next, we will prove that right-hand sides of Eqs.(19) vanish identically on the solutions of the system of ODEs (18).

Let $\varphi_j = \tilde{\varphi}_j(t,\, C_1,\, C_2, \ldots, C_N),\ j = \overline{1, N}$ where C_j are arbitrary complex constants, be a general solution of the system of ODEs (18). If at least one of the equations is not satisfied identically on the solutions of Eqs.(18), then substituting into it the expressions for φ_j we get a relation of the form $h(C_1, C_2, \ldots, C_N) = 0$. Hence it follows that the general solution of the system of PDEs (10) contains no more than $N-1$ independent constants. We arrive at the contradiction, which proves that the right-hand sides of Eqs.(19) vanish identically on the solutions of system of ODEs (18). Consequently, system (18), (19) is equivalent to system of N ODEs

$$\dot{\varphi}_j = H_j(t,\, x_0,\, \varphi_1,\, \varphi_2, \ldots, \varphi_N) = \tilde{H}_j(t,\, x,\, \varphi_1,\, \varphi_2, \ldots, \varphi_N), \quad j = \overline{1, N}. \tag{20}$$

Thus, given the conditions of the theorem the Ansatz (9), which is invariant under LBVF (2), (7), reduces the equation (1) to system of N ODEs (20). The theorem is proved.

Consequence. *Let Eq.(1) be invariant under the LBVF (2), (7). Then the Ansatz (9) invariant under LBVF (2), (7) reduces PDE (1) to a system of N ODEs for functions $\varphi_j(t),\ j = \overline{1, N}$.*

Proof follows from the fact that if an equation is invariant under LBVF, then it is conditionally-invariant with respect to this LBVF.

3 Some examples

Utilizing the above theorem one can construct principally new exact solutions even for equations with poor Lie symmetry. As an illustration, we give below several examples.

Example 1. Consider the nonlinear heat conductivity equation with a logarithmic-type nonlinearity

$$u_t = u_{xx} + \Big(\alpha + \beta \ln u - \gamma^2 (\ln u)^2\Big) u. \tag{21}$$

We will establish that Eq.(21) is conditionally-invariant with respect to LBVF (2) with

$$\eta = u_{xx} - \gamma u_x - u^{-1} u_x^2. \tag{22}$$

Condition (6) for Eq.(21) reads

$$D_t \eta - D_x^2 \eta - \Big(\alpha + \beta + (\beta - 2\gamma^2) \ln u - \gamma^2 \ln^2 u\Big) \eta \ \Big|_{M \cap L_x} = 0, \tag{23}$$

where M stands for the set of all differential consequences of the equation $u_t = u_{xx} + \Big(\alpha + \beta \ln u - \gamma^2 (\ln u)^2\Big) u$ and L_x stands for the set of all differential consequences of the equation $u_{xx} - \gamma u_x - u^{-1} u_x^2 = 0$ with respect to x.

Substituting into the left-hand side of Eq.(23) expression (22) and transferring to the manifold M (i.e. excluding the derivatives u_t, u_{tx}, u_{txx} with the help of Eq.(21)) we transform it to the form

$$2u^{-1}(u_{xx} - \gamma u_x - u^{-1} u_x^2)^2 + 4\gamma u^{-1} u_x (u_{xx} - \gamma u_x - u^{-1} u_x^2).$$

Evidently, the above expression does not vanish on the manifold M. But on the manifold $M \cap L_x$ it vanishes identically

$$2u^{-1}(u_{xx} - \gamma u_x - u^{-1} u_x^2)^2 + 4\gamma u^{-1} u_x (u_{xx} - \gamma u_x - u^{-1} u_x^2) \ \Big|_{M \cap L_x} = 0.$$

Hence it follows that the nonlinear heat conductivity equation (21) is conditionally-invariant under LBVF (2) with η of the form (22) but not invariant under the said LBVF in the sense of Definition 1. It is also seen from [11], where the results of classification of nonlinear heat conductivity equations $u_t = u_{xx} + F(u)$ admitting LBVF are given. It has been established, in particular, that only linear heat equation admits LBVF which can not represented in the form (2), (4) and, consequently, is not equivalent to a Lie vector field.

Integrating the equation $\eta \equiv u_{xx} - \gamma u_x - u^{-1} u_x^2 = 0$ as an ODE with respect to x we get an Ansatz for $u(t, x)$

$$u(t, x) = \exp\Big(\varphi_1(t) + \varphi_2(t) \exp \gamma x\Big). \tag{24}$$

Substitution of the Ansatz (24) into Eq.(21) gives rise to a system of two ODEs

$$\dot\varphi_1 = \alpha + \beta\varphi_1 - \gamma^2\varphi_1^2, \quad \dot\varphi_2 = (\beta + \gamma^2 - 2\gamma^2\varphi_1)\varphi_2.$$

The general solution of the above system is given by one of the following formulae:
(a) $k = \beta^2 + 4\alpha\gamma^2 > 0$

$$u = C\Big(\cos\frac{k^{1/2}t}{2}\Big)^2 \exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2}\Big(\beta - k^{1/2}\ \tan\frac{k^{1/2}t}{2}\Big);$$

(b) $k = \beta^2 + 4\alpha\gamma^2 < 0$

$$u = C\Big(\cosh\frac{(-k)^{1/2}t}{2}\Big)^2 \exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2}\Big(\beta + (-k)^{1/2}\tanh\frac{(-k)^{1/2}t}{2}\Big);$$

(c) $k = \beta^2 + 4\alpha\gamma^2 = 0$

$$u = Ct^{-2}\exp(\gamma x + \gamma^2 t) + \frac{1}{2\gamma^2 t}(\beta t + 2).$$

Here C is an arbitrary constant.

It is important to emphasize that the above solutions can not be obtained by the symmetry reduction procedure. Really, the maximal local invariance group of Eq.(21) is the two-parameter group of translations [14]

$$t' = t + \theta_1,\ x' = x + \theta_2,\ u' = u.$$

and solutions (a)–(b) are obviously not invariant under the above group.

Example 2. Consider the following nonlinear heat conductivity equation:

$$u_t = u_{xx} + F(u). \tag{25}$$

We will establish that it is conditionally-invariant with respect to the LBVF (2) with $\eta = u_{xx} - A(u)u_x^2$, provided functions $F(u),\ A(u)$ satisfy the following system of ODEs:

$$\ddot A + 4A\dot A + 2A^3 = 0, \quad \ddot F - \dot AF - A\dot F = 0. \tag{26}$$

The equality (6) for Eq.(25) takes the form

$$D_t\eta - D_x^2\eta - \dot F\eta\ \Big|_{M\cap L_x} = 0,$$

where M is the set of all differential consequences of the equation $u_t = u_{xx} + F(u)$ and L_x is the set of all differential consequences of the equation $u_{xx} - A(u)u_x^2 = 0$ with respect to x.

Excluding from the left-hand side of the above equality the derivatives $u_t,\ u_{tx},\ u_{txx}$ and grouping in a proper way terms in the expression obtained we have

$$2A\eta^2 + 4(\dot{A} + A^2)\eta + (\ddot{A} + 4A\dot{A} + 2A^3)u_x^4 + (\ddot{F} - \dot{A}F - A\dot{F})u_x^2 \Big|_{M\cap L_x} = 0$$

or with account of Eqs.(26)

$$2A\eta^2 + 4(\dot{A} + A^2)\eta \Big|_{M\cap L_x} = 0. \tag{27}$$

Evidently, the left-hand side of Eq.(27) does not vanish on the manifold M but it does vanish on the manifold $M \cap L_x$. Consequently, the nonlinear heat equation (25) is conditionally-invariant with respect to LBVF (2) with $\eta = u_{xx} - A(u)u_x^2$ iff Eqs.(26) hold.

Thus, conditions of the Theorem 1 are satisfied and we can reduce Eq.(25) to two ODEs with the help of the Ansatz (9) invariant under the above mentioned LBVF.

Let the function $\theta(u)$ be determined by the following equality:

$$\int\limits_0^{\theta(u)} (\ln\tau)^{-1/2} d\tau = \alpha u + \beta,$$

where α, β are arbitrary real constants. Then the Ansatz

$$\int\limits_0^{u(t,x)} \frac{d\tau}{\theta(\tau)} = x\varphi_1(t) + \varphi_2(t)$$

reduce the nonlinear equation (25) with

$$F(u) = \left(\lambda_1 + \lambda_2 \int\limits_0^{u} \frac{d\tau}{\theta(\tau)}\right)\theta(u) \tag{28}$$

to the following system of ODEs:

$$\dot{\varphi}_1 = \left(\frac{\alpha^2}{2}\varphi_1^2 + \lambda_2\right)\varphi_1, \quad \dot{\varphi}_2 = \left(\frac{\alpha^2}{2}\varphi_1^2 + \lambda_2\right)\varphi_2 + \dot{\theta}(0)\varphi_1^2 + \lambda_1.$$

Here λ_1, λ_2 are arbitrary real constants and $\dot{\theta}(0)$ is a value of the first derivative of the function $\dot{\theta}(u)$ in the point $x = 0$.

The above system of ODEs is integrated in quadratures thus giving rise to a family of exact solutions of the nonlinear PDE (25) with rather exotic nonlinearity (28). The solutions obtained are also non-invariant with respect to the two-parameter group of translations with respect to t, x, which is the maximal local invariance group of Eq.(25), (28).

Example 3. Here, we will perform the reduction of a nonlinear PDE of the form (21)

$$u_t = u_{xx} + a(\ln^2 u)u, \quad a \in \mathbb{R}^1 \tag{29}$$

to systems of three ODEs.

By a rather cumbersome computation one can check that Eq.(29) is conditionally-invariant with respect to the LBVF (2) with

$$\eta = u^2 u_{xxx} - 3uu_x u_{xx} + 2u_x^3 + au_x u^2. \tag{30}$$

Integrating the third-order ODE $\eta = 0$ we obtain the following Ansätze for the function $u(t,x)$:

1) under $a = \alpha^2 > 0$

$$u(t,x) = \exp\{\varphi_1(t) + \varphi_2(t)\cos\alpha x + \varphi_3(t)\sin\alpha x\}, \tag{31}$$

2) under $a = -\alpha^2 < 0$

$$u(t,x) = \exp\{\varphi_1(t) + \varphi_2(t)\cosh\alpha x + \varphi_3(t)\sinh\alpha x\}, \tag{32}$$

where $\varphi_1,\ \varphi_2,\ \varphi_3$ are arbitrary smooth functions.

Substitution of the expressions (31), (32) into PDE (29) gives rise to the systems of nonlinear ODEs for the functions $\varphi_1,\ \varphi_2,\ \varphi_3$

1) under $a = \alpha^2 > 0$

$$\begin{aligned}\dot\varphi_1 &= \alpha^2(\varphi_1^2 + \varphi_2^2 + \varphi_3^2),\\ \dot\varphi_2 &= \alpha^2(2\varphi_1 - 1)\varphi_2,\\ \dot\varphi_3 &= \alpha^2(2\varphi_1 - 1)\varphi_3;\end{aligned}$$

2) under $a = -\alpha^2 < 0$

$$\begin{aligned}\dot\varphi_1 &= \alpha^2(\varphi_3^2 - \varphi_2^2 - \varphi_1^2),\\ \dot\varphi_2 &= \alpha^2(1 - 2\varphi_1)\varphi_2,\\ \dot\varphi_3 &= \alpha^2(1 - 2\varphi_1)\varphi_3.\end{aligned}$$

Making the change of the dependent variable $u = \exp v$ we rewrite Eq.(29) in the form

$$v_t = v_{xx} + v_x^2 + av^2 \tag{33}$$

and what is more, the Ansätze (31), (32) take the form

1) under $a = \alpha^2 > 0$

$$v(t,x) = \varphi_1(t) + \varphi_2(t)\cos\alpha x + \varphi_3(t)\sin\alpha x, \tag{34}$$

2) under $a = -\alpha^2 < 0$

$$v(t,x) = \varphi_1(t) + \varphi_2(t)\cosh\alpha x + \varphi_3(t)\sinh\alpha x. \tag{35}$$

If we choose in formulae (34), (35) $\varphi_3 = 0$, then the well-known Galaktionov's Ansätze are obtained [10, 16]. These Ansätze were used to study blow-up solutions of the nonlinear

PDE (33). It should be said that all solutions of the nonlinear heat conductivity equations obtained in [10] can be constructed within the framework of our approach.

Example 4. Let us describe all PDEs of the form

$$u_t = u_{xx} + R(u, u_x) \tag{36}$$

which are conditionally-invariant under the LBVF (2) with $\eta = u_{xx} - au,\ a \in \mathbb{R}^1$.

Acting by the operator (2) on the equation (36) and transferring to the manifold $M \cap L_x$ we obtain the determining equation for the function R

$$a^2u^2R_{u_xu_x} + 2auu_xR_{uu_x} + u_x^2R_{uu} + auR_u + au_xR_{u_x} + aR = 0.$$

The above PDE is rewritten in the form

$$(J^2 + a)R = 0,$$

where $J = u_x\partial_u + au\partial_{u_x}$. With this remark it is easily integrated and its general solution reads

$$R = f_1(u_x^2 - au^2)u_x + f_2(u_x^2 - au^2)u.$$

Here $f_1,\ f_2$ are arbitrary smooth functions.

Thus, the most general PDE of the form (36) conditionally-invariant with respect to LBVF (2) with $\eta = u_{xx} - au$ is as follows

$$u_t = u_{xx} + f_1(u_x^2 - au^2)u_x + f_2(u_x^2 - au^2)u. \tag{37}$$

Solving the equation $\eta \equiv u_{xx} - au = 0$ we obtain the Ansätze for $u(t,x)$

1) under $a = -\alpha^2 < 0$

$$u(t,x) = \varphi_1(t)\cos\alpha x + \varphi_2(t)\sin\alpha x,$$

2) under $a = \alpha^2 > 0$

$$u(t,x) = \varphi_1(t)\cosh\alpha x + \varphi_2(t)\sinh\alpha x$$

which reduce PDE (37) to systems of two ODEs for functions $\varphi_1(t),\ \varphi_2(t)$

$$1)\quad \dot\varphi_1 = -\alpha^2\varphi_1 + \alpha f_1^+\varphi_2 + f_2^+\varphi_1, \quad \dot\varphi_2 = -\alpha^2\varphi_2 - \alpha f_1^+\varphi_1 + f_2^+\varphi_2,$$
$$2)\quad \dot\varphi_1 = \alpha^2\varphi_1 + \alpha f_1^-\varphi_2 + f_2^-\varphi_1, \quad \dot\varphi_2 = \alpha^2\varphi_2 + \alpha f_1^-\varphi_1 + f_2^-\varphi_2,$$

where $f_i^\pm = f_i(\alpha^2(\varphi_2^2 \pm \varphi_1^2))$.

4 Conclusion

In the papers [8, 9] we have constructed a number of Ansätze (9) which reduce the nonlinear heat equation $u_t = [a(u)u_x]_x + f(u)$ to several ODEs. The basic technique used was the anti-reduction method. The present paper provides symmetry interpretation of the said results. It is important to emphasize that there exist non-evolution equations which also admit anti-reduction. In particular, in [4, 6, 17] an anti-reduction of the nonlinear acoustics equation, of the equation for short waves in gas dynamics and of the nonlinear wave equation is carried out. It would be of interest to extend the Theorem 1 in order to include into consideration these equations.

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