Separation of variables in (1+2)-dimensional Schrödinger equations

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Abstract

Using our classification of separable Schrödinger equations with two space dimensions published in *J.Math.Phys.*, **36**, N 10, 1995 we give an exhaustive description of the coordinate systems providing their separability. Furthermore, we apply these results to separate variables in the heat, Hamilton-Jacobi and Fokker-Plank equations.

I. Introduction

In [1, 2] we have suggested an efficient approach to separation of variables (SV) in linear partial differential equations. Within the framework of this approach we have classified the $(1+2)$-dimensional Schrödinger equations

$$iu_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1, x_2)u \tag{1}$$

admitting SV, i.e. we have described inequivalent (in some sense) potentials $V(x_1, x_2)$ such that Eq.(1) separates at least in one coordinate system. This classification is the principal result of [3]. In the present paper we will obtain a complete description of the coordinate systems providing separability of Eqs.(1) with the above mentioned potentials.

For the well studied cases $V(x_1, x_2) = 0$ and $V(x_1, x_2) = \alpha x_1^{-2} + \beta x_2^{-2}$ we will re-obtain the results on SV in the corresponding Schrödinger equations which are due to Boyer, Kalnins & Miller [4] and Boyer [5].

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The results on SV in the Schrödinger equation can be directly applied to separate variables in the heat equation

$$u_t + u_{x_1x_1} + u_{x_2x_2} = V(x_1, x_2)u. \tag{2}$$

Hence, using the well-known connection between SV in the heat and Hamilton-Jacobi equations (see, e.g. [6]) we will obtain a classification of separable Hamilton-Jacobi equations of the form

$$u_t + u_{x_1}^2 + u_{x_2}^2 = V(x_1, x_2) \tag{3}$$

and coordinate systems providing separability of Eq.(3).

The paper is organized as follows. The second section contains necessary notions, definitions and a brief description of the method. Furthermore, we give the results of classification of potentials $V(x_1, x_2)$ such the Schrödinger equation (1) admits SV. In the next section the full lists of coordinate systems providing separability of the corresponding Schrödinger equations are given. In Sec.IV we apply the results obtained to separate variables in the heat, Hamilton-Jacobi and Fokker-Plank equations. The last section is devoted to discussing some generalizations of the suggested approach to the case of nonlinear partial differential equations.

II. Classification of separable Schrödinger equations

The underlying idea of the method of separation of variables as introduced by Fourier and Euler is quite simple. To separate variables in Eq.(1) with $V = 0$ in, say, polar coordinate system one has to look for a particular solution of the form

$$u(t, x_1, x_2) = \varphi_0(t)\varphi_1\left(\sqrt{x_1^2 + x_2^2}\right)\varphi_2\left(\arctan\frac{x_1}{x_2}\right), \tag{4}$$

where $\varphi_0, \varphi_1, \varphi_2$ are functions of one variable.

Inserting (4) into the initial equation under $V = 0$ and making some simple algebraic manipulations we arrive at three ordinary differential equations for unknown functions $\varphi_0, \varphi_1, \varphi_2$ with respect to the variables t, ω_1, ω_2, correspondingly.

This example indicates clearly two important elements of the method of separation of variables in partial differential equations. The first one is a choice of a special *Ansatz* for a class of particular solutions to be found. The second element is a requirement to have as an output of the whole procedure several ordinary differential equations whose solutions after being substituted into the chosen Ansatz yield a class of exact solutions of the equation under study. One more important point is that thus obtained solutions depend on auxiliary parameters (separation constants).

The principal idea of the approach to separation of variables in partial differential equations suggested in [1, 2] was to postulate these three features in order to develop a regular procedure for obtaining solutions with separated variables.

First, we choose the Ansatz for a solution to be found

$$u(t,x_1,x_2) = Q(t,x_1,x_2)\varphi_0(t)\varphi_1(\omega_1(t,x_1,x_2))\varphi_2(\omega_2(t,x_1,x_2)), \tag{5}$$

where functions Q, ω_1, ω_2 are not fixed *a priori* but chosen in such a way that inserting the expression (5) into Eq.(1) yields three ordinary differential equations for functions $\varphi_0, \varphi_1, \varphi_2$.

Needless to say that only with a special choice of functions Q, ω_1, ω_2 we can meet the above requirement. If this is the case, the Schrödinger equation (1) is called separable in the coordinate system t, ω_1, ω_2 and the corresponding expression (5) is termed as the solution with separated variables.

To get constraints on functions Q, ω_1, ω_2 enabling SV in Eq.(1) we have to formalize a reduction procedure. This is done as follows [3].

Consider a system of three ordinary differential equations

$$i\frac{d\varphi_0}{dt} = U_0(t,\varphi_0;\lambda_1,\lambda_2), \quad \frac{d^2\varphi_a}{d\omega_a^2} = U_a(\omega_a,\varphi_a,\frac{d\varphi_a}{d\omega_a};\lambda_1,\lambda_2), \quad a=1,2, \tag{6}$$

where $U_0,\ U_1,\ U_2$ are some smooth functions of the corresponding arguments, $\{\lambda_1,\lambda_2\} \subset \mathbf{R}$ are arbitrary parameters (separation constants) and what is more

$$\mathrm{rank}\left\| \frac{\partial U_\mu}{\partial \lambda_a} \right\|_{\mu=0}^{2}\,{}_{a=1}^{2} = 2.$$

Now, inserting the expression (5) into (1) we get an equation containing the functions ω_1, ω_2 and their first- and second-order partial derivatives, and the functions $\varphi_0, \varphi_1, \varphi_2$ and their derivatives. Next, we replace the derivatives $\frac{d\varphi_0}{dt},\ \frac{d^2\varphi_1}{d\omega_1^2},\ \frac{d^2\varphi_2}{d\omega_2^2}$ by the corresponding expressions from the right-hand sides of (6). At the last step, we split the obtained equality with respect to the variables $\varphi_0,\ \varphi_1,\ \varphi_2,\ \frac{d\varphi_1}{d\omega_1},\ \frac{d\varphi_2}{d\omega_2},\ \lambda_1,\ \lambda_2$ considered as independent.

As established in [3] system of ordinary differential equations (6) necessarily takes the form

$$i\frac{d\varphi_0}{dt} = \Big(\lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t)\Big)\varphi_0,$$

$$
\begin{aligned}
\frac{d^2\varphi_1}{d\omega_1^2} &= \Big(\lambda_1 B_{11}(\omega_1) + \lambda_2 B_{12}(\omega_1) + B_{01}(\omega_1)\Big)\varphi_1, \\
\frac{d^2\varphi_2}{d\omega_2^2} &= \Big(\lambda_1 B_{21}(\omega_2) + \lambda_2 B_{22}(\omega_2) + B_{02}(\omega_2)\Big)\varphi_2
\end{aligned} \tag{7}
$$

and the functions Q, ω_1, ω_2 have to satisfy the following over-determined system of partial differential equations:

$$
\begin{aligned}
&1.\quad \omega_{1x_1}\omega_{2x_1} + \omega_{1x_2}\omega_{2x_2} = 0, \\
&2.\quad B_{1a}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + B_{2a}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) + R_a(t) = 0, \\
&3.\quad 2(\omega_{ax_1}Q_{x_1} + \omega_{ax_2}Q_{x_2}) + Q(i\omega_{at} + \omega_{ax_1x_1} + \omega_{ax_2x_2}) = 0, \\
&4.\quad \Big(B_{01}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + B_{02}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2)\Big)Q + iQ_t \\
&\qquad\qquad + Q_{x_1x_1} + Q_{x_2x_2} + R_0(t)Q - V(x_1, x_2)Q = 0,
\end{aligned} \tag{8}
$$

where $a = 1, 2$.

Thus, the problem of SV in Eq.(1) within the framework of our approach reduces to solving the over-determined system of nonlinear partial differential equations (8). According to [3] the general solution of the first five equations from (8) splits into four inequivalent classes which are presented below

$$
\begin{aligned}
\text{I.}\quad & \omega_1 = A(t)x_1 + W_1(t), \quad \omega_2 = B(t)x_2 + W_2(t), \\
& Q(t, x_1, x_2) = \exp\Big\{-\frac{i}{4}\Big(\frac{A'}{A}x_1^2 + \frac{B'}{B}x_2^2\Big) - \frac{i}{2}\Big(\frac{W_1'}{A}x_1 + \frac{W_2'}{B}x_2\Big)\Big\}; \\
\text{II.}\quad & x_1 = W(t)\mathrm{e}^{\omega_1}\sin\omega_2 + W_1(t), \ x_2 = W(t)\mathrm{e}^{\omega_1}\cos\omega_2 + W_2(t), \\
& Q(t, x_1, x_2) = \exp\{iR(t, x_1, x_2)\}; \\
\text{III.}\quad & x_1 = \frac{1}{2}W(t)(\omega_1^2 - \omega_2^2) + W_1(t), \quad x_2 = W(t)\omega_1\omega_2 + W_2(t), \\
& Q(t, x_1, x_2) = \exp\{iR(t, x_1, x_2)\}; \\
\text{IV.}\quad & x_1 = W(t)\cosh\omega_1\cos\omega_2 + W_1(t), \quad x_2 = W(t)\sinh\omega_1\sin\omega_2 + W_2(t), \\
& Q(t, x_1, x_2) = \exp\{iR(t, x_1, x_2)\}.
\end{aligned} \tag{9}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions and the function $R(t, x_1, x_2)$ is given by the formula

$$
R(t, x_1, x_2) = \frac{W'}{4W}\Big((x_1 - W_1)^2 + (x_2 - W_2)^2\Big) + \frac{1}{2}(W_1'x_1 + W_2'x_2). \tag{10}
$$

Substituting the expressions for the functions Q, ω_1, ω_2 into the last equation from the system (8) yields a functional equation for $V(x_1, x_2)$ and $A(t), B(t), W(t), W_1(t), W_2(t)$. We have succeeded in splitting it with respect to t which gives a number of constraints on the function $V(x_1, x_2)$ in the form of partial differential equations. Their general solutions give rise to all possible potentials $V(x_1, x_2)$ providing separability of the initial equation (1). The corresponding results can be summarized as follows [3]. There are four classes of potentials

$$\begin{aligned}
&1. \quad V(x_1,x_2) = V_1(x_1) + V_2(x_2);\\
&2. \quad V(x_1,x_2) = V_1(x_1^2+x_2^2) + V_2\left(\frac{x_1}{x_2}\right)(x_1^2+x_2^2)^{-1};\\
&3. \quad V(x_1,x_2) = \Big(V_1(\omega_1) + V_2(\omega_2)\Big)(\omega_1^2+\omega_2^2)^{-1},\\
&\qquad \omega_1^2 - \omega_2^2 = 2x_1, \quad \omega_1\omega_2 = x_2;\\
&4. \quad V(x_1,x_2) = \Big(V_1(\omega_1) + V_2(\omega_2)\Big)(\sinh^2\omega_1 + \sin^2\omega_2)^{-1},\\
&\qquad \cosh\omega_1\cos\omega_2 = x_1, \quad \sinh\omega_1\sin\omega_2 = x_2,
\end{aligned} \tag{11}$$

where $V_1,\ V_2$ are arbitrary smooth functions, such that the corresponding Schrödinger equation separates in the Cartesian ($\omega_1 = x_1, \omega_2 = x_2$), polar ($x_1 = \mathrm{e}^{\omega_1}\sin\omega_2, x_2 = \mathrm{e}^{\omega_1}\cos\omega_2$), parabolic ($x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), x_2 = \omega_1\omega_2$) and elliptic ($x_1 = \cosh\omega_1\cos\omega_2, x_2 = \sinh\omega_1\sin\omega_2$) coordinate systems, respectively.

Specifying in a proper way functions V_1, V_2 yields further possibilities for SV in Eq.(1). The list of the corresponding potentials is presented below in Table 1. Note that the potentials $V(x_1, x_2)$ given in Table 1 are not inequivalent in a usual sense. They are distinguished by the form and number of coordinate systems providing separability of the corresponding Schrödinger equations. This distinction is indicated by the last column of the table, where we give the number of inequivalent coordinate systems such that Eq.(1) with the potential adduced in the first column admits SV in these.

III. Coordinate systems providing separability of Eq.(1)

In this section we will obtain full lists of inequivalent coordinate systems providing separability of Eq.(1) with the potentials listed in Table 1. To this end we have to perform the following steps:

1. To insert the potentials listed in Table 1 into the last equation of the system (8) and to split equalities obtained with respect to x_1, x_2 (this gives us systems of nonlinear ordinary differential equations for the functions $A(t)$, $B(t)$, $W(t)$, $W_1(t)$, $W_2(t)$).

2. To solve the systems of ordinary differential equations obtained.

3. To classify coordinate systems within the following equivalence relation (for more detail, see [3]):

   (a) two solutions of Eq.(1) with separated variables

$$\begin{aligned} u &= Q\varphi_0(t;\lambda)\varphi_1(\omega_1;\lambda)\varphi_2(\omega_2;\lambda) \quad \text{and} \\ \tilde{u} &= \tilde{Q}\varphi_0(t;\tilde{\lambda})\varphi_1(\tilde{\omega}_1;\tilde{\lambda})\varphi_2(\tilde{\omega}_2;\tilde{\lambda}) \end{aligned}$$

   are called equivalent if there exist functions $\Psi_a,\ \Omega_a,\ \Lambda_a$ such that the relations

$$\begin{aligned} \tilde{Q} &= Q\Psi_1(\omega_1)\Psi_2(\omega_2), \\ \tilde{\omega}_a &= \Omega_a(\omega_a), \quad a = 1, 2, \\ \tilde{\lambda}_a &= \Lambda_a(\lambda_1, \lambda_2), \quad a = 1, 2 \end{aligned} \tag{12}$$

   hold,

   (b) two solutions of Eq.(1) are called equivalent if they can transformed one into another with the help of a transformation from the Lie transformation group admitted by the corresponding Schrödinger equation.

We will consider in full detail SV in the Schrödinger equation with the potential

$$V(x_1, x_2) = \frac{k_1 x_1}{x_2^2(x_1^2 + x_2^2)^{1/2}} + \frac{k_2}{x_2^2}, \quad k_1 k_2 \neq 0. \tag{13}$$

The general solution of the first five equations of system (8) splits into four inequivalent classes given in (9). Each of these should be considered separately.

Case I. Inserting the formulae I from (9) and (13) into the last equation of (8) yields the functional-differential equation

$$\begin{aligned} &-\frac{k_2}{x_2^2} - \frac{k_1 x_1}{x_2^2\sqrt{x_1^2 + x_2^2}} + \tilde{R}_0 + A^2 B_{01}(Ax_1 + W_1) + B^2 B_{02}(Bx_2 + W_2) \\ &-\frac{iA'}{2A} - \frac{x_1^2 (A')^2}{2A^2} - \frac{iB'}{2B} - \frac{x_2^2 (B')^2}{2B^2} - \frac{x_1 A' W_1{}'}{A^2} - \frac{(W_1{}')^2}{4A^2} - \frac{x_2 B' W_2{}'}{B^2} \\ &-\frac{(W_2')^2}{4B^2} + \frac{x_1^2 A''}{4A} + \frac{x_2^2 B''}{4B} + \frac{x_1 W_1{}''}{2A} + \frac{x_2 W_2{}''}{2B} = 0. \end{aligned}$$

Multiplying it by x_2^2 and differentiating with respect to the variables x_1, x_2 we get

$$\frac{3k_1 x_2(-12x_1^4 + 21x_1^2 x_2^2 - 2x_2^4)}{(x_1^2 + x_2^2)^{\frac{9}{2}}} = 0.$$

Given the condition $k_1 \neq 0$, the above equation is inconsistent. Hence we conclude that Eq.(1) with the potential (13) is not separable in the coordinate system given by the formulae I from (9).

Case II. Let us denote the equation obtained by substituting the formulae II from (9) and (13) into the last equation of the system (8) as $\mathcal{U} = 0$. Expressing x_1, x_2 via t, ω_1, ω_2 according to the relations

$$x_1 = W(t)\mathrm{e}^{\omega_1} \sin \omega_2 + W_1(t), \quad x_2 = W(t)\mathrm{e}^{\omega_1} \cos \omega_2 + W_2(t) \tag{14}$$

we get an equation containing functions of t, ω_1, ω_2 only. Differentiating it with respect to t and returning back to the variables t, x_1, x_2 give the following functional-differential equation:

$$\begin{aligned}
&\tilde{R}_0 + \frac{k_1(W_1 W' - W W_1{}')}{(x_1^2 + x_2^2)^{\frac{3}{2}}} - \left(\frac{2k_2}{x_2^3} - \frac{k_1 x_1}{x_2(x_1^2 + x_2^2)^{\frac{3}{2}}} - \frac{2k_1 x_1}{x_2^3 \sqrt{x_1^2 + x_2^2}} \right) \\
&\times (W_2 W' - W W_2{}') - \frac{(x_1^2 + x_2^2)(3W'W'' + WW^{(3)})}{4W} \\
&- \frac{x_1}{2W}(3W_1 W'W'' - 3WW'W_1{}'' + W_1 W W^{(3)} - W^2 W_1{}^{(3)}) \\
&- \frac{x_2}{2W}(3W_2 W'W'' - 3WW'W_2{}'' + W_2 W W^{(3)} - W^2 W_2{}^{(3)}) = 0.
\end{aligned} \tag{15}$$

Differentiating (15) with respect to x_1, x_2 we get the equation

$$15k_1 x_1 x_2 (W_1 W' - W W_1{}') + 3k_1(4x_2^2 - x_1^2)(W_2 W' - W W_2{}') = 0,$$

whence we conclude that

$$W_1(t) = C_1 W(t), \quad W_2(t) = C_2 W(t), \tag{16}$$

where C_1, C_2 are arbitrary real constants.

With account of (16) equation (15) is rewritten to become

$$\tilde{R}_0 - \frac{(x_1^2 + x_2^2)(3W'W'' + WW^{(3)})}{4W} = 0. \tag{17}$$

Consequently, there exists a real constant C_0 such that the equality

$$W''W^3 = C_0 \tag{18}$$

holds. Due to (16), (18) the equation $\mathcal{U} = 0$ takes the form

$$\begin{aligned}
&B_{01}(\omega_1) + B_{02}(\omega_2) - \frac{C_0 e^{4\omega_1}}{4} - \frac{C_0 C_2 e^{3\omega_1}\cos\omega_2}{2} - \frac{k_2 e^{2\omega_1}}{(C_2 + e^{\omega_1}\cos\omega_2)^2} \\
&- \frac{k_1 e^{2\omega_1}(C_1 + e^{\omega_1}\sin\omega_2)}{(C_2 + e^{\omega_1}\cos\omega_2)^2 (e^{2\omega_1} + C_1^2 + C_2^2 + 2e^{\omega_1}(C_2\cos\omega_2 + C_1\sin\omega_2))^{\frac{1}{2}}} \\
&- \frac{C_0 C_1 e^{3\omega_1}\sin\omega_2}{2} = 0.
\end{aligned}$$

For the equation obtained to be consistent the sum of the terms beginning from the third should be decomposable into a sum of two functions $\Omega_1(\omega_1)$ and $\Omega_2(\omega_2)$. This requirement impose restrictions on the choice of parameters C_0, C_1, C_2. To obtain these we have to consider differential consequences of the equation $\mathcal{U} = 0$ up to the 28th order. This has been done with the aid of *Mathematica* package and the result is that C_1, C_2 should be equal to zero and C_0 is an arbitrary real constant.

Thus, we have proved that Eq.(1) having the potential (13) admits SV in the coordinate system (14) if and only if $W_1(t) = 0, W_2(t) = 0$ and $W(t)$ is a solution of the ordinary differential equation (18). The general solution of (18) is given by one of the following formulae

$$W_0(t) = \begin{cases} ((c_0 t + c_1)^2 + c_2)^{\frac{1}{2}}, \\ (c_3 - t)^{\frac{1}{2}}, \end{cases}$$

where c_0, c_1, c_2, c_3 are arbitrary real constants.

If $c_0 \neq 0$, then using invariance of the Schrödinger equation with the potential (13) with respect to the one-parameter dilation group

$$t' = t\theta^2, \quad x_1' = x_1\theta, \quad x_2' = x_2\theta, \tag{19}$$

we can scale the parameter c_0 to become 1 and get

$$\begin{aligned}
&x_1 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1}\cos\omega_2; \\
&x_1 = \left((t + C_1)^2 + C_2\right)^{\frac{1}{2}} \mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = \left((t + C_1)^2 + C_2\right)^{\frac{1}{2}} \mathrm{e}^{\omega_1}\cos\omega_2.
\end{aligned} \tag{20}$$

If c_0 vanishes, then $W_0(t) = C, C = \mathrm{const}$ and the coordinate system (14) takes the form

$$x_1 = C\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = C\mathrm{e}^{\omega_1}\cos\omega_2.$$

Making use of the invariance of the Schrödinger equation with the potential (13) with respect to the transformation group (19) we can scale C to become $-\frac{1}{\theta}$. Next, using the fact that the equation under study admits the one-parameter projective transformation group (see, e.g. [7, 8])

$$
\begin{aligned}
&t' = \frac{t}{1-\theta t}, \quad x_1' = \frac{x_1}{1-\theta t}, \\
&u'(t', x_1', x_2') = (1-\theta t) \exp\left\{\frac{i\theta(x_1^2 + x_2^2)}{4(1-\theta t)}\right\} u(t, x_1, x_2)
\end{aligned}
\tag{21}
$$

we reduce the above coordinate system to the form

$$
x_1 = \frac{\theta t - 1}{\theta} \mathrm{e}^{\omega_1} \sin \omega_2, \quad x_2 = \frac{\theta t - 1}{\theta} \mathrm{e}^{\omega_1} \cos \omega_2
$$

thus getting the second coordinate system from (20) with $C_1 = -\frac{1}{\theta}, C_2 = 0$.

Case III. Splitting as above the last equation of the system (8) with respect to t we conclude that Eq.(1) with potential (13) admits SV in coordinate system III from (9) if functions $W(t), W_1(t), W_2(t)$ satisfy equations (16) and (18). With account of this fact the last equation of the system (8) takes the form

$$
\begin{aligned}
&B_{01}(\omega_1) + B_{02}(\omega_2) - \frac{C_0(\omega_1^2 + \omega_2^2)^3}{16} - \frac{C_0 C_1 (\omega_1^4 - \omega_2^4)}{4} \\
&- \frac{C_0 C_2 \omega_1 \omega_2 (\omega_1^2 + \omega_2^2)}{2} - \frac{k_2(\omega_1^2 + \omega_2^2)}{(\omega_1\omega_2 + C_2)^2} \\
&- \frac{k_1(\omega_1^2 + \omega_2^2)(\omega_1^2 - \omega_2^2 + 2C_1)}{(\omega_1\omega_2 + C_2)^2 ((\omega_1^2 - \omega_2^2 + 2C_1)^2 + 4(\omega_1\omega_2 + C_2)^2)^{\frac{1}{2}}} = 0.
\end{aligned}
$$

Splitting the above equation with respect to ω_1, ω_2 yields that it is consistent for vanishing C_0, C_1, C_2 only. Hence we get

$$
W_0(t) = c_1 t + c_2, \quad W_1(t) = 0, \quad W_2(t) = 0
$$

with arbitrary real c_1, c_2.

Thus we have arrived at the following coordinate system:

$$
x_1 = \frac{1}{2}(c_1 t + c_2)(\omega_1^2 - \omega_2^2), \quad x_2 = (c_1 t + c_2)\omega_1\omega_2. \tag{22}
$$

Due to the fact that the Schrödinger equation with the potential (13) is invariant under the dilation and projective transformation groups (19), (21)), the coordinate system (22) is equivalent to the standard parabolic coordinate system $x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2)$, $x_2 = \omega_1\omega_2$.

Case IV**.** Splitting the last equation of the system (8) with respect to t we conclude that Eq.(1) with potential (13) admits SV in coordinate system IV from (9) if functions $W(t), W_1(t), W_2(t)$ satisfy equations (16) and (18). With account of this fact the last equation of the system (8) takes the form

$$
\begin{aligned}
&B_{01}(\omega_1) + B_{02}(\omega_2) - \frac{C_0C_1\cos\omega_2\cosh\omega_1(\sin^2\omega_2 + \sinh^2\omega_1)}{2} \\
&-\frac{C_0C_2\sin\omega_2\sinh\omega_1(\sin^2\omega_2 + \sinh^2\omega_1)}{2} - \frac{C_0(\sin^2\omega_2 + \sinh^2\omega_1)^2}{4} \\
&-\frac{k_1(C_1 + \cos\omega_2\cosh\omega_1)(\sin^2\omega_2 + \sinh^2\omega_1)}{(C_2 + \sin\omega_2\sinh\omega_1)^2((C_1 + \cos\omega_2\cosh\omega_1)^2 + (C_2 + \sin\omega_2\sinh\omega_1)^2)^{\frac{1}{2}}} \\
&-\frac{k_2(\sin^2\omega_2 + \sinh^2\omega_1)}{(C_2 + \sin\omega_2\sinh\omega_1)^2} = 0.
\end{aligned}
$$

Splitting the above equation with respect to ω_1, ω_2 yields the following consistency conditions:

$$C_0 = 0, \quad C_1 = \pm 1, \quad C_2 = 0.$$

Consequently, we have proved that the Schrödinger equation with the potential (13) admits SV in the coordinate system IV from (9) if and only if it has the form

$$x_1 = (c_1t + c_2)(\cosh\omega_1\cos\omega_2 \pm 1), \quad x_2 = (c_1t + c_2)\sinh\omega_1\sin\omega_2.$$

As Eq.(1) with the potential (13) is invariant with respect to the groups (19) and (21), the above coordinate system is equivalent to the following one:

$$x_1 = \cosh\omega_1\cos\omega_2 \pm 1, \quad x_2 = \sinh\omega_1\sin\omega_2.$$

Summarizing we conclude that the Schrödinger equation having the potential (13) admits SV in four inequivalent coordinate systems.

The remaining cases 1–15 and 17 from Table 1 are handled in a similar way, the results obtained being listed below.

1. $V(x_1, x_2) = k_1x_1^2 + k_2x_1^{-2} + \tilde{V}(x_2)$

   $(a)\quad k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at), \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases} \qquad x_2 = \omega_2;$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at), \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}, \end{cases} \qquad x_2 = \omega_2.$$

2. $V(x_1, x_2) = k_1 x_1^2 + k_2 x_1^{-2} + k_3 x_2^{-2}$

(a) $k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at), \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at), \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}. \end{cases}$$

3. $V(x_1, x_2) = k_1 x_1^2 + k_2 x_1^{-2}$

(a) $k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at), \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(t + C_4) + C_5(t + C_4)^{-1}, \\ \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at), \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(t+C_4) + C_5(t+C_4)^{-1}, \\ \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t+C_4)^2 + C_5\Big)^{\frac{1}{2}}. \end{cases}$$

4. $V(x_1,x_2) = k_1x_1^2 + k_2x_1^{-2} + k_3x_2^2 + k_4x_2^{-2}$

(a) $k_1 = -a^2/4, \quad k_2 = -b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cosh at + C_2\sinh at), \\ \omega_1(C_1\cosh 2at + C_2\sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4\cosh bt + C_5\sinh bt), \\ \omega_2(C_4\cosh 2bt + C_5\sinh 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = -a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cosh at + C_2\sinh at), \\ \omega_1(C_1\cosh 2at + C_2\sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4\cos bt + C_5\sin bt), \\ \omega_2(C_4\cos 2bt + C_5\sin 2bt + C_6)^{\frac{1}{2}}, \end{cases}$$

(c) $k_1 = a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cos at + C_2\sin at), \\ \omega_1(C_1\cos 2at + C_2\sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4\cos bt + C_5\sin bt), \\ \omega_2(C_4\cos 2bt + C_5\sin 2bt + C_6)^{\frac{1}{2}}, \end{cases}$$

5. $V(x_1,x_2) = k_1x_1^2 + \widetilde{V}(x_2)$

(a) $k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cosh at + C_2\sinh at) \\ \quad + C_3(C_1\cosh at + C_2\sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1\exp\{\mp at\} + C_1\exp\{\pm 3at\}, \\ \omega_1(C_1\cosh 2at + C_2\sinh 2at + C_3)^{\frac{1}{2}}, \end{cases} \qquad x_2 = \omega_2;$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cos at + C_2\sin at) \\ \quad + C_3(C_1\cos at + C_2\sin at)^{-1}, \\ \omega_1(C_1\cos 2at + C_2\sin 2at + C_3)^{\frac{1}{2}}, \end{cases} \qquad x_2 = \omega_2.$$

6. $V(x_1, x_2) = k_1 x_1^2$

(a) $k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cosh at + C_2\sinh at) \\ \qquad + C_3(C_1\cosh at + C_2\sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1\exp\{\mp at\} + C_1\exp\{\pm 3at\}, \\ \omega_1(C_1\cosh 2at + C_2\sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(t + C_4) + C_5(t + C_4)^{-1}, \\ \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cos at + C_2\sin at) \\ \qquad + C_3(C_1\cos at + C_2\sin at)^{-1}, \\ \omega_1(C_1\cos 2at + C_2\sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(t + C_4) + C_5(t + C_4)^{-1}, \\ \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}. \end{cases}$$

7. $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^2$

(a) $k_1 = -a^2/4, \quad k_2 = -b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1\cosh at + C_2\sinh at) \\ \qquad + C_3(C_1\cosh at + C_2\sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1\exp\{\mp at\} + C_1\exp\{\pm 3at\}, \\ \omega_1(C_1\cosh 2at + C_2\sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4\cosh bt + C_5\sinh bt) \\ \qquad + C_6(C_4\cosh bt + C_5\sinh bt)^{-1}, \ C_4^2 \neq C_5^2, \\ \omega_2\exp\{\mp bt\} + C_4\exp\{\pm 3bt\}, \\ \omega_2(C_4\cosh 2bt + C_5\sinh 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = -a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at) \\ \quad + C_3(C_1 \cosh at + C_2 \sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp at\} + C_1 \exp\{\pm 3at\}, \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos bt + C_5 \sin bt) \\ \quad + C_6(C_4 \cos bt + C_5 \sin bt)^{-1}, \\ \omega_2(C_4 \cos 2bt + C_5 \sin 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(c) $k_1 = a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at) \\ \quad + C_3(C_1 \cos at + C_2 \sin at)^{-1}, \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos bt + C_5 \sin bt) \\ \quad + C_6(C_4 \cos bt + C_5 \sin bt)^{-1}, \\ \omega_2(C_4 \cos 2bt + C_5 \sin 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(d) $k_1 = -a^2, \quad k_2 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh 2at + C_2 \sinh 2at) \\ \quad + C_3(C_1 \cosh 2at + C_2 \sinh 2at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp 2at\} + C_1 \exp\{\pm 6at\}, \\ \omega_1(C_1 \cosh 4at + C_2 \sinh 4at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cosh at + C_5 \sinh at) \\ \quad + C_6(C_4 \cosh at + C_5 \sinh at)^{-1}, \ C_4^2 \neq C_5^2 \\ \omega_2 \exp\{\mp at\} + C_4 \exp\{\pm 3at\}, \\ \omega_2(C_4 \cosh 2at + C_5 \sinh 2at + C_6)^{\frac{1}{2}}; \end{cases}$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2;$$

(e) $k_1 = a^2, \quad k_2 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos 2at + C_2 \sin 2at) \\ \quad + C_3(C_1 \cos 2at + C_2 \sin 2at)^{-1}, \\ \omega_1(C_1 \cos 4at + C_2 \sin 4at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos at + C_5 \sin at) \\ \qquad + C_6(C_4 \cos at + C_5 \sin at)^{-1}, \\ \omega_2(C_4 \cos 2at + C_5 \sin 2at + C_6)^{\frac{1}{2}}; \end{cases}$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2.$$

8. $V(x_1, x_2) = k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}$

(a) $k_1 = -a^2/4, \quad k_2 = -b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at) \\ \qquad + C_3(C_1 \cosh at + C_2 \sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp at\} + C_1 \exp\{\pm 3at\}, \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cosh bt + C_5 \sinh bt), \\ \omega_2(C_4 \cosh 2bt + C_5 \sinh 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = -a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at) \\ \qquad + C_3(C_1 \cosh at + C_2 \sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp at\} + C_1 \exp\{\pm 3at\}, \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos bt + C_5 \sin bt), \\ \omega_2(C_4 \cos 2bt + C_5 \sin 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(c) $k_1 = a^2/4, \quad k_2 = b^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at) \\ \qquad + C_3(C_1 \cos at + C_2 \sin at)^{-1}, \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos bt + C_5 \sin bt), \\ \omega_2(C_4 \cos 2bt + C_5 \sin 2bt + C_6)^{\frac{1}{2}}; \end{cases}$$

(d) $k_1 = -a^2, \quad k_2 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh 2at + C_2 \sinh 2at) \\ \qquad + C_3(C_1 \cosh 2at + C_2 \sinh 2at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp 2at\} + C_1 \exp\{\pm 6at\}, \\ \omega_1(C_1 \cosh 4at + C_2 \sinh 4at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cosh at + C_5 \sinh at), \\ \omega_2(C_4 \cosh 2at + C_5 \sinh 2at + C_6)^{\frac{1}{2}}; \end{cases}$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2;$$

(e) $k_1 = a^2, \quad k_2 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos 2at + C_2 \sin 2at) \\ \quad + C_3(C_1 \cos 2at + C_2 \sin 2at)^{-1}, \\ \omega_1(C_1 \cos 4at + C_2 \sin 4at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 \cos at + C_5 \sin at), \\ \omega_2(C_4 \cos 2at + C_5 \sin 2at + C_6)^{\frac{1}{2}}; \end{cases}$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2.$$

9. $V(x_1, x_2) = k_1 x_1^2 + k_2 x_2^{-2}$

(a) $k_1 = -a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cosh at + C_2 \sinh at) \\ \quad + C_3(C_1 \cosh at + C_2 \sinh at)^{-1}, \ C_1^2 \neq C_2^2, \\ \omega_1 \exp\{\mp at\} + C_1 \exp\{\pm 3at\}, \\ \omega_1(C_1 \cosh 2at + C_2 \sinh 2at + C_3)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}; \end{cases}$$

(b) $k_1 = a^2/4$

$$x_1 = \begin{cases} \omega_1(C_1 \cos at + C_2 \sin at) \\ \quad + C_3(C_1 \cos at + C_2 \sin at)^{-1}, \\ \omega_1(C_1 \cos 2at + C_2 \sin 2at + C_3)^{\frac{1}{2}}; \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_4 - t)^{\frac{1}{2}}, \\ \omega_2\Big((t + C_4)^2 + C_5\Big)^{\frac{1}{2}}. \end{cases}$$

10. $V(x_1, x_2) = k_1 x_1^{-2} + \widetilde{V}(x_2)$

$$x_1 = \begin{cases} \omega_1(C_1 - t)^{\frac{1}{2}}, \\ \omega_1\Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}}, \end{cases} \qquad x_2 = \omega_2.$$

11. $V(x_1,x_2) = k_1x_1^{-2} + k_2x_2^{-2}$

$$x_1 = \begin{cases} \omega_1(C_1-t)^{\frac12}, \\ \omega_1\Big((t+C_1)^2+C_2\Big)^{\frac12}, \end{cases} \qquad x_2 = \begin{cases} \omega_2(C_3-t)^{\frac12}, \\ \omega_2\Big((t+C_3)^2+C_4\Big)^{\frac12}; \end{cases}$$

$$x_1 = (C_1-t)^{\frac12}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = (C_1-t)^{\frac12}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = (C_1-t)^{\frac12}\cosh\omega_1\cos\omega_2, \quad x_2 = (C_1-t)^{\frac12}\sinh\omega_1\sin\omega_2;$$

$$x_1 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\cosh\omega_1\cos\omega_2,$$

$$x_2 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\sinh\omega_1\sin\omega_2;$$

$$x_1 = (C_1-t)^{\frac12}\sinh\omega_1\sin\omega_2, \quad x_2 = (C_1-t)^{\frac12}\cosh\omega_1\cos\omega_2;$$

$$x_1 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\sinh\omega_1\sin\omega_2,$$

$$x_2 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\cosh\omega_1\cos\omega_2;$$

12. $V(x_1,x_2) = \widetilde{V}(x_2)$

$$x_1 = \begin{cases} \omega_1(t+C_1)+C_2(t+C_1)^{-1}, \\ \omega_1(C_1-t)^{\frac12}, \\ \omega_1\Big((t+C_1)^2+C_2\Big)^{\frac12}, \end{cases} \qquad x_2 = \omega_2;$$

13. $V(x_1,x_2) = k_1x_2^{-2}$

$$x_1 = \begin{cases} \omega_1(t+C_1)+C_2(t+C_1)^{-1}, \\ \omega_1(C_1-t)^{\frac12}, \\ \omega_1\Big((t+C_1)^2+C_2\Big)^{\frac12}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(C_3-t)^{\frac12}, \\ \omega_2\Big((t+C_3)^2+C_4\Big)^{\frac12}; \end{cases}$$

$$x_1 = (C_1-t)^{\frac12}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = (C_1-t)^{\frac12}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = \Big((t+C_1)^2+C_2\Big)^{\frac12}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2) + C_1 t^2, \quad x_2 = \omega_1\omega_2;$$
$$x_1 = (C_1 - t)^{\frac{1}{2}} \cosh\omega_1 \cos\omega_2, \quad x_2 = (C_1 - t)^{\frac{1}{2}} \sinh\omega_1 \sin\omega_2;$$
$$x_1 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \cosh\omega_1 \cos\omega_2,$$
$$x_2 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \sinh\omega_1 \sin\omega_2;$$
$$x_1 = (C_1 - t)^{\frac{1}{2}} \sinh\omega_1 \sin\omega_2, \quad x_2 = (C_1 - t)^{\frac{1}{2}} \cosh\omega_1 \cos\omega_2;$$
$$x_1 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \sinh\omega_1 \sin\omega_2,$$
$$x_2 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \cosh\omega_1 \cos\omega_2;$$

14. $V(x_1, x_2) = (x_1^2 + x_2^2)^{-1} \tilde{V}\left(\frac{x_1}{x_2}\right)$

$$x_1 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \sin\omega_2, \quad x_2 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \cos\omega_2;$$
$$x_1 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \sin\omega_2, \quad x_2 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \cos\omega_2.$$

15. $V(x_1, x_2) = k_1 x_1 x_2^{-2}(x_1^2 + x_2^2)^{-1/2} + k_2 x_2^{-2} + k_3(x_1^2 + x_2^2)^{-1/2}$

$$x_1 = \mathrm{e}^{\omega_1} \sin\omega_2, \quad x_2 = \mathrm{e}^{\omega_1} \cos\omega_2;$$
$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2.$$

16. $V(x_1, x_2) = k_1 x_1 x_2^{-2}(x_1^2 + x_2^2)^{-1/2} + k_2 x_2^{-2}$

$$x_1 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \sin\omega_2, \quad x_2 = (C_1 - t)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \cos\omega_2;$$
$$x_1 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \sin\omega_2, \quad x_2 = \Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}} \mathrm{e}^{\omega_1} \cos\omega_2;$$
$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2;$$
$$x_1 = \cosh\omega_1 \cos\omega_2 \pm 1, \quad x_2 = \sinh\omega_1 \sin\omega_2.$$

17. $V(x_1, x_2) = 0$

$$x_1 = \begin{cases} \omega_1(t + C_1) + C_2(t + C_1)^{-1}, \\ \omega_1(C_1 - t)^{\frac{1}{2}}, \\ \omega_1\Big((t + C_1)^2 + C_2\Big)^{\frac{1}{2}}, \end{cases}$$

$$x_2 = \begin{cases} \omega_2(t+C_3) + C_4(t+C_3)^{-1}, \\ \omega_2(C_3-t)^{\frac{1}{2}}, \\ \omega_2\Big((t+C_3)^2 + C_4\Big)^{\frac{1}{2}}; \end{cases}$$

$$x_1 = (C_1-t)^{\frac{1}{2}}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = (C_1-t)^{\frac{1}{2}}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = \Big((t+C_1)^2 + C_2\Big)^{\frac{1}{2}}\mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = \Big((t+C_1)^2 + C_2\Big)^{\frac{1}{2}}\mathrm{e}^{\omega_1}\cos\omega_2;$$

$$x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2) + C_1 t^2, \quad x_2 = \omega_1\omega_2;$$

$$x_1 = (C_1-t)^{\frac{1}{2}}\cosh\omega_1\cos\omega_2, \quad x_2 = (C_1-t)^{\frac{1}{2}}\sinh\omega_1\sin\omega_2;$$

$$x_1 = \Big((t+C_1)^2 + C_2\Big)^{\frac{1}{2}}\cosh\omega_1\cos\omega_2,$$

$$x_2 = \Big((t+C_1)^2 + C_2\Big)^{\frac{1}{2}}\sinh\omega_1\sin\omega_2.$$

In the above formulae $C_1, \ldots, C_6$ are arbitrary real numbers and a, b are arbitrary positive real numbers.

The results on SV in the Schrödinger equations having the potentials $V(x_1, x_2) = 0$ and $V(x_1, x_2) = k_1 x_1^{-2} + k_2 x_2^{-2}$ are the same as in [4, 5] and are given here for the sake of completeness.

Note that coordinate systems providing separability of the Schrödinger equation with potentials 1–9 depend essentially on the coupling constants a, b which are contained in the corresponding potentials $V(x_1, x_2)$. Comparing these coordinate systems with those making it possible to separate variables in the free Schrödinger equation ($V = 0$) we see that they are essentially different. The free Schrödinger equation does not separate in coordinate systems listed in the above formulae 1–9 if $ab \neq 0$. This is not the case for the potential $V(x_1, x_2) = k_1 x_1^{-2} + k_2 x_2^{-2}$ studied by Boyer [5]. The free Schrödinger equation separates in all coordinate systems providing separability of Eq.(1) having the potential $V(x_1, x_2) = k_1 x_1^{-2} + k_2 x_2^{-2}$.

IV. Separation of variables in the heat and Hamilton-Jacobi equations

The results of the previous sections are straightforwardly applied to the problem of SV in the heat equation (2) if we fix a separation Ansatz in the form (5). The system of nonlinear partial differential equations for the functions Q, ω_1, ω_2 is obtained if we replace in (8) $i\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t}$. By this reason the forms of potentials $V(x_1, x_2)$ and coordinate systems

ω_1, ω_2 are the same as those for the Schrödinger equation. The only difference is the form of the factor $Q(t, x_1, x_2)$

$$
\begin{aligned}
\text{I.} \quad & Q(t,x_1,x_2) = \exp\Big\{-\frac{1}{4}\Big(\frac{A'}{A}x_1^2 + \frac{B'}{B}x_2^2\Big) - \frac{1}{2}\Big(\frac{W_1'}{A}x_1 + \frac{W_2'}{B}x_2\Big)\Big\}; \\
\text{II.} \quad & Q(t,x_1,x_2) = \exp\{R(t,x_1,x_2)\}; \\
\text{III.} \quad & Q(t,x_1,x_2) = \exp\{R(t,x_1,x_2)\}; \\
\text{IV.} \quad & Q(t,x_1,x_2) = \exp\{R(t,x_1,x_2)\},
\end{aligned}
$$

where $R(t, x_1, x_2)$ is given by (10).

Thus, Table 1 gives the full list of the potentials $V(x_1, x_2)$ such that the heat equation (2) admits SV within the framework of our approach. The coordinate systems providing separability of the heat equations are given by the corresponding formulae from Sec.III.

It is known that there exists a deep connection between SV in the heat and Hamilton-Jacobi equations (see, e.g. [6]). The Hamilton-Jacobi equation (3) separates in any coordinate system providing separability of the heat equation (2) and, what is more, the inverse assertion is not true. To reveal the reason for this let us consider in some detail SV in Eq.(3). First we fix the usual form of the separation Ansatz for the Hamilton-Jacobi equation

$$
u(t,x_1,x_2) = \widetilde{Q}(t,x_1,x_2) + \varphi_0(t) + \varphi_1(\omega_1(t,x_1,x_2)) + \varphi_2(\omega_2(t,x_1,x_2)) \tag{23}
$$

and, furthermore, fix the form of the equations for $\varphi_0, \varphi_1, \varphi_2$

$$
\begin{aligned}
\frac{d\varphi_0}{dt} &= \lambda_1 R_1(t) + \lambda_2 R_2(t) + R_0(t), \\
\frac{d\varphi_1}{d\omega_1} &= \Big(\lambda_1 B_{11}(\omega_1) + \lambda_2 B_{12}(\omega_1) + B_{01}(\omega_1)\Big)^{\frac{1}{2}}, \\
\frac{d\varphi_2}{d\omega_2} &= \Big(\lambda_1 B_{21}(\omega_2) + \lambda_2 B_{22}(\omega_2) + B_{02}(\omega_2)\Big)^{\frac{1}{2}}.
\end{aligned} \tag{24}
$$

Then, after inserting the Ansatz (23) into Eq.(3), eliminating the first derivatives of the functions $\varphi_0, \varphi_1, \varphi_2$ with the use of the above equations and splitting by the variables $\varphi_0, \varphi_1, \varphi_2, \lambda_1, \lambda_2$ we arrive at the following system of nonlinear partial differential equations for the functions Q, ω_1, ω_2:

$$
\begin{aligned}
&1. \quad \omega_{1x_1}\omega_{2x_1} + \omega_{1x_2}\omega_{2x_2} = 0, \\
&2. \quad B_{1a}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + B_{2a}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) + R_a(t) = 0,
\end{aligned}
$$

$$
\begin{aligned}
&3. \quad 2(\omega_{ax_1}\widetilde{Q}_{x_1} + \omega_{ax_2}\widetilde{Q}_{x_2}) + \omega_{at} = 0, \\
&4. \quad B_{01}(\omega_1)(\omega_{1x_1}^2 + \omega_{1x_2}^2) + B_{02}(\omega_2)(\omega_{2x_1}^2 + \omega_{2x_2}^2) + \widetilde{Q}_t \\
&\qquad\qquad + \widetilde{Q}_{x_1}^2 + \widetilde{Q}_{x_2}^2 + R_0(t) - V(x_1, x_2) = 0,
\end{aligned}
\tag{25}
$$

where $a = 1, 2$.

It is not difficult to become convinced of that the functions

$$
\widetilde{Q}(t, x_1, x_2) = \ln Q(t, x_1, x_2, x_3), \quad \omega_1(t, x_1, x_2), \quad \omega_2(t, x_1, x_2)
$$

satisfy the system (25) if and only if Q, ω_1, ω_2 is a solution of (8), where one should replace $i\frac{\partial}{\partial t}$ by $\frac{\partial}{\partial t}$. Indeed, the general solution $\{\omega_1, \omega_2\}$ of the first three equations (which are the same for both systems) can be reduced with the help of an equivalence transformation (12) to such a form that it satisfy the Laplace equation [3]

$$
\omega_{1x_1x_1} + \omega_{1x_2x_2} = 0, \quad \omega_{2x_1x_1} + \omega_{2x_2x_2} = 0.
$$

Hence it follows that the general solution of the first five equations of the system (25) is given by the formulae (9) and $\widetilde{Q} = R(t, x_1, x_2)$ with $R(t, x_1, x_2)$ of the form (10). Now if we make in the last equation of the system (9) the change $Q = \exp\{\widetilde{Q}\}$, and multiply the equality obtained by $\exp\{-\widetilde{Q}\}$, and take into account the relation

$$
\widetilde{Q}_{x_1x_1} + \widetilde{Q}_{x_2x_2} \equiv R_{x_1x_1} + R_{x_2x_2} = \frac{W'(t)}{W(t)},
$$

then we get the last equation from the system (24).

Summarizing we conclude that Eq.(3) separates for any potential given in Table 1 and, furthermore, all coordinate systems listed in Sec.III provide separability of the corresponding Hamilton-Jacobi equation.

Let us note that some more details on separation of variables in the Hamilton-Jacobi equation (3) with specific potentials $V(x_1, x_2)$ can be found in [9]–[13].

We finish this section with one more equation that can be solved with the help of the method of separation of variables within the approach suggested above. Consider the $(1+2)$-dimensional Fokker-Plank equation of the following special form:

$$
u_t + u_{x_1x_1} + u_{x_2x_2} + (a_1 u)_{x_1} + (a_2 u)_{x_2} = 0, \tag{26}
$$

where a_1, a_2 are some smooth functions of x_1, x_2. In addition, we require for Eq.(26) to be reducible to the form (2). It is a common knowledge that this is possible if and only if the relation $a_{1x_2} = a_{2x_1}$ holds. Consequently, we can represent the coefficients of Eq.(26) in the form

$$a_i = -2\frac{\partial f}{\partial x_i} f^{-1}, \quad i = 1, 2 \tag{27}$$

with a sufficiently smooth function $f = f(x_1, x_2)$.

Given the conditions (27), the change of the dependent variable

$$u(t, x_1, x_2) = U(t, x_1, x_2) f(x_1, x_2)$$

transforms the Fokker-Plank equation (26) to become

$$U_t + U_{x_1x_1} + U_{x_2x_2} - V(x_1, x_2)U = 0, \tag{28}$$

where we have denoted

$$V(x_1, x_2) = (f_{x_1x_1} + f_{x_2x_2}) f^{-1}. \tag{29}$$

Table 1 gives the full list of potentials V such that Eq.(26) admits SV. Solving for each of them the equation (29) we get classes of separable Fokker-Plank equations. Note that the equation $f_{x_1x_1} + f_{x_2x_2} = V(x_1, x_2)f$ can be solved by the method of separation of variables for each potential $V(x_1, x_2)$ from Table 1.

Furthermore, if

$$U = Q(t, x_1, x_2)\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2),$$

is a solution of Eq.(28) with separated variables, then the formula

$$u = f(x_1, x_2)Q(t, x_1, x_2)\varphi_0(t)\varphi_1(\omega_1)\varphi_2(\omega_2),$$

where $f = f(x_1, x_2)$ is a solution of Eq.(29), yields a solution of the initial Fokker-Plank equation (26) with separated variables.

V. Some generalizations.

Generally speaking, construction of particular solutions of a special form is only the first step of the method of separation of variables. The next step is to use the solutions obtained as the basis functions to expand an arbitrary smooth solution of the equation under study in a properly chosen Hilbert space [14]. Consequently, to be able to implement the method of separation of variables in a full generality we have to restrict our considerations to *linear* partial differential equations. But as a tool for obtaining particular solutions the method is applicable to *nonlinear* equations as well. The well-known example is the Laplace equation with a logarithmic nonlinearity

$$u_{xx} + u_{yy} + \lambda u \ln u = 0, \quad \lambda = \text{const},$$

which is separable in the Cartesian coordinate system via usual separation Ansatz $u = \varphi_1(x)\varphi_2(y)$ (see, e.g. [6]). The second example which is due to Osborn and Stuart [15] is more peculiar. They have shown that the sine-Gordon equation

$$u_{tt} - u_{xx} = \lambda \sin u, \quad \lambda = \text{const}$$

can be separated into two first-order ordinary differential equations if one makes use of the following Ansatz:

$$u(t,x) = 4\arctan(\varphi_0(t)\varphi_1(x)).$$

The standard Ansatz (5) cannot be applied to separate variables in the sine-Gordon equation. The reason is that it does not take into account a specific structure of nonlinearity of the equation in question.

Thus, to be able to separate variables in nonlinear partial differential equations we have to make a separation Ansatz more flexible and not to fix *a priori* a form of dependence on the new unknown functions $\varphi_0, \varphi_1, \varphi_2$.

Being motivated by this idea we have suggested in [16] the following generalization of the Ansätze (5), (23):

$$\omega_{m+1} = f(\varphi_1(\omega_1), \ldots, \varphi_m(\omega_m)), \quad m < n, \tag{30}$$

where $\omega_1, \ldots, \omega_{m+1}$ are functions of $x_1, \ldots x_n, u$ such that

$$\text{rank} \left\| \frac{\partial \omega_i}{\partial x_j} \right\|_{i=1}^{m+1}\,{}_{j=0}^{n} = m + 1,$$

where $x_{n+1} \equiv u$.

The form of the function f is determined from the requirement that substitution of the Ansatz (30) into a given kth order partial differential equation should yield a system of ordinary differential equations for the functions $\varphi_i(\omega_i)$, $i = 1, \ldots, n$ of the form

$$\frac{d^{k_i}\varphi_i}{d\omega_i^{k_i}} = F_i\left(\omega_i, \varphi_i, \frac{d\varphi_i}{d\omega_i}, \ldots, \frac{d^{k_i-1}\varphi_i}{d\omega_i^{k_i-1}}; \lambda_1, \ldots, \lambda_N\right), \tag{31}$$

where $i = 1, \ldots, m$, $k_i \le k$.

The key point of this approach is a proper formalization of what a *reduction* is. This is done in the same way as in the linear case (see Sec.II). After inserting the Ansatz (30) into the equation under study we eliminate all the derivatives of the functions $\varphi_i(\omega_i)$ of the order higher than $k_i - 1$ using equations (31) and their differential consequences. Next, we split the equality obtained by $\varphi_i, \frac{d\varphi_i}{d\omega_i}, \ldots, \frac{d^{k_i-1}\varphi_i}{d\omega_i^{k_i-1}}, i = 1, \ldots, m, \lambda_1, \ldots, \lambda_N$ considered as new independent variables. This gives us, on the one hand, the form of the Ansatz (30) and, on the other hand, an over-determined system of partial differential equations for functions ω_i. A solution of the latter gives rise to an Ansatz providing separability of the initial equation into m ordinary differential equations.

The reduction scheme outlined above is very general and contains as particular cases

- symmetry reduction, provided $m = 1$, $f = \varphi_1(\omega_1)$ and ω_1, ω_2 are invariants of a Lie transformation group admitted by an equation under consideration (see, e.g. [17, 18]),

- conditional symmetry (non-classical) reduction, provided $m = 1$, $f = \varphi_1(\omega_1)$ and at least one of the functions ω_1, ω_2 is not an invariant of a Lie transformation group admitted by an equation under consideration [19]–[24],

- anti-reduction, provided f is independent of parameters $\lambda_1, \ldots, \lambda_N$ [25]–[27] (see also [28, 29]),

- multiplicative separation of variables, provided $m = n, N = n - 1$, and

  $$\omega_i = \omega_i(x), \quad \omega_{n+1} = \frac{u}{Q(x)}, \quad f = \varphi_1 \times \cdots \times \varphi_n,$$

  where $i = 1, \ldots, n$,

- additive separation of variables, provided $m = n, N = n - 1$, and

  $$\omega_i = \omega_i(x), \quad \omega_{n+1} = u - Q(x), \quad f = \varphi_1 + \cdots + \varphi_n,$$

  where $i = 1, \ldots, n$.

The most important is that the generalized method of separation of variables as introduced above can be efficiently applied to construct exact solutions of *nonlinear* equations. In particular, we have succeeded in classifying nonlinear wave equations $u_{tt} - u_{xx} = F(u)$ which can be separated into two first-order ordinary differential equations with the aid of the Ansatz $u = f(\varphi_0(t) + \varphi_1(x))$ [30]. There are strong evidences that a similar idea should work for nonlinear Schrödinger-like equations as well. Study of this problem is is progress now and the corresponding results will be published elsewhere.

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Table 1. **Separable potentials for the Schrödinger equation.**

N	Potential $V(x_1, x_2)$	Constraints for parameters	Number of CS allowing SV
1	$k_1x_1^2 + k_2x_1^{-2} + \widetilde{V}(x_2)$		2
2	$k_1x_1^2 + k_2x_1^{-2} + k_3x_2^{-2}$		4
3	$k_1x_1^2 + k_2x_1^{-2}$		6
4	$k_1x_1^2 + k_2x_1^{-2} + k_3x_2^2 + k_4x_2^{-2}$		4
5	$k_1x_1^2 + \widetilde{V}(x_2)$	$k_1 < 0$	3
		$k_1 > 0$	2
6	$k_1x_1^2$	$k_1 < 0$	9
		$k_1 > 0$	6
7	$k_1x_1^2 + k_2x_2^2$	$k_1 < 0,\ k_2 < 0$	9
		$k_1 < 0,\ k_2 > 0$	6
		$k_1 < 0,\ k_2 < 0$	4
		$4k_1 = k_2 < 0$	10
		$4k_1 = k_2 > 0$	5
8	$k_1x_1^2 + k_2x_2^2 + k_3x_2^{-2}$	$k_1 < 0,\ k_2 < 0$	6
		$k_1 < 0,\ k_2 > 0$	6
		$k_1 < 0,\ k_2 < 0$	4
		$4k_1 = k_2 < 0$	7
		$4k_1 = k_2 > 0$	5
9	$k_1x_1^2 + k_2x_2^{-2}$	$k_1 < 0$	6
		$k_1 > 0$	4

Continuation of the Table 1.

N	Potential $V(x_1,x_2)$	Constraints for parameters	Number of CS allowing SV
10	$k_1x_1^{-2}+\tilde{V}(x_2)$		2
11	$k_1x_1^{-2}+k_2x_2^{-2}$	$k_1\neq k_2$ $k_1=k_2$	10 8
12	$\tilde{V}(x_2)$		3
13	$k_1x_2^{-2}$		13
14	$(x_1^2+x_2^2)^{-1}\tilde{V}\left(\frac{x_1}{x_2}\right)$		2
15	$k_1x_1x_2^{-2}(x_1^2+x_2^2)^{-1/2}+k_2x_2^{-2}$ $+k_3(x_1^2+x_2^2)^{-1/2}$	$k_3\neq 0$	2
16	$k_1x_1x_2^{-2}(x_1^2+x_2^2)^{-1/2}+k_2x_2^{-2}$		4
17	0		14

Here k_1,k_2,k_3,k_4 are arbitrary parameters, $\tilde{V}$ is an arbitrary smooth function and CS is the abbreviation for coordinate system.