Symmetry and some exact solutions of nonlinear polywave equations

W.I. Fushchych and O.V. Roman

Institute of Mathematics of the Academy of Sciences of Ukraine,

Tereshchenkivska Str.3, 252004 Kiev, Ukraine

R.Z. Zhdanov

Arnold-Sommerfeld Institute for Mathematical Physics,

Leibnitzstraße 10, 38678 Clausthal-Zellerfeld, Germany*

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Abstract

We have studied the maximal symmetry group admitted by the nonlinear polywave equation $\Box^l u = F(u)$. In particular, we establish that equation in question admits the conformal group $C(1,n)$ if and only if $F(u) = \lambda e^u$, $n+1 = 2l$ or $F(u) = \lambda u^{(n+1+2l)/(n+1-2l)}$, $n+1 \neq 2l$. Symmetry reduction for the biwave equation $\Box^2 u = \lambda u^{-3}$ is carried out and some exact solutions are obtained.

Recently a number of works (see, e.g. [1, 2, 5]) have appeared pointing out the possibility to choose linear and nonlinear polywave equations

$$\Box^l u = F(u) \tag{1}$$

as possible mathematical models describing an uncharged scalar particle in quantum field theory.

Here $\Box^l = \Box(\Box^{l-1})$, $l \in \mathrm{N}$; $\quad \Box = \partial^2_{x_0} - \partial^2_{x_1} - \ldots - \partial^2_{x_n}$ is d'Alembertian in $(n+1)$–dimensional pseudo-Euclidean space $R(1,n)$ with metric tensor $g_{\mu\nu} = \mathrm{diag}(1,-1,\ldots,-1)$, $\mu,\nu = \overline{0,n}$; $\quad F(u)$ is an arbitrary smooth function and $u = u(x)$ is a real function (the case $l=1$, $n=1$ has been studied earlier [4], that is why we put $l+n > 2$). In the following, a summation over the repeated indices from 0 to n is understood, rising and lowering of the vector indices is performed by means of the tensor $g_{\mu\nu}$, i.e. $x^\mu = g_{\mu\nu}x_\nu$.

But the fact that the nonlinear partial differential equation (PDE) in question is of high order makes the prospects of studying such a model rather obscure. Using group

*On leave from the Institute of Mathematics of the Academy of Sciences of Ukraine, Tereshchenkivska Str.3, 252004 Kiev, Ukraine properties of equation (1) seems to be the only way to get some non-trivial information about the said equation and its solutions. It occurs that PDE (1) admits wide symmetry group which, in fact, is the same as the one of the standard wave equation

$$\Box u = F(u). \tag{2}$$

The main tool used is the infinitesimal Lie method (see, e.g.[8]). But an application of it to study of symmetry properties of equation (1) is by itself a non-trivial problem in the case $l > 1$. It should be emphasized that because of arbitrariness of the order (l) and of the number of independent variables (n) one can not apply symbolic manipulation programs [6], [7]. We have succeeded in constructing the maximal symmetry group admitted by equation (1) using the remarkable combinatorial properties of the prolongation formulae.
Theorem 1. *Maximal invariance group of PDE (1) with arbitrary smooth function* $F(u)$ *is the Poincaré group* $P(1,n)$ *generated by the operators*

$$P_\mu = \partial_{x_\mu}, \qquad J_{\mu\nu} = x^\mu \partial_{x_\nu} - x^\nu \partial_{x_\mu}, \qquad \mu, \nu = \overline{0,n}. \tag{3}$$

It is established below that the equation of the type (1) admitting the group, which is more extensive than the Poincaré group, is equivalent up to the change of variables to one of the following equations:

$$1.\ \Box^l u = \lambda_1 u^k, \qquad \lambda_1 \neq 0,\ k \neq 0, 1; \tag{4}$$
$$2.\ \Box^l u = \lambda_2 e^u, \qquad \lambda_2 \neq 0; \tag{5}$$
$$3.\ \Box^l u = \lambda_3 u, \qquad \lambda_3 \neq 0; \tag{6}$$
$$4.\ \Box^l u = 0. \tag{7}$$

Here $\lambda_1, \lambda_2, \lambda_3, k$ are arbitrary constants.

Maximal invariance groups of the equations (4) – (7) are described by the following statements.
Theorem 2. *Equation (4) has the following symmetry:*
Case 1. $k \neq (n+1+2l)/(n+1-2l),\ k \neq 0, 1.$
Maximal invariance group of (4) is the extended Poincaré group $\tilde{P}(1,n)$ *generated by the operators (3) and*

$$D = x_\mu \partial_{x_\mu} + \frac{2l}{1-k} u \partial_u;$$

Case 2. $k = (n+1+2l)/(n+1-2l),\ n+1 \neq 2l.$
Maximal invariance group of (4) is the conformal group $C(1,n)$ *generated by the operators (3) and operators*

$$D^{(1)} = x_\mu \partial_{x_\mu} + \frac{(2l-n-1)}{2} u \partial_u, \qquad K^{(1)}_\mu = 2x^\mu D^{(1)} - (x_\nu x^\nu)\partial_{x_\mu}, \qquad \mu, \nu = \overline{0,n}. \tag{8}$$

Theorem 3. *Equation (5) has the following symmetry:*
Case 1. $n \neq 2l - 1$.
Maximal invariance group of (5) is the extended Poincaré group $\widetilde{P}(1,n)$ *generated by the operators (3) and*
$$D^{(2)} = x_\mu \partial_{x_\mu} - 2l\partial_u; \tag{9}$$
Case 2. $n = 2l - 1$.
Maximal invariance group of (5) is the conformal group $C(1,n)$ *generated by the operators (3) and operators*
$$K_\mu^{(2)} = 2x^\mu D^{(2)} - (x_\nu x^\nu)\partial_{x_\mu}, \qquad \mu, \nu = \overline{0, n}. \tag{10}$$
Theorem 4. *Maximal invariance group of the equation (6) is generated by the operators (3) and*
$$Q_\infty = f(x)\partial_u, \qquad I = u\partial_u,$$
where $f(x)$ *is an arbitrary solution of PDE (6).*
Theorem 5. *Maximal invariance group of the equation (7) is generated by the operators (3), (8) and*
$$Q_\infty = q(x)\partial_u, \qquad I = u\partial_u,$$
where $q(x)$ *is an arbitrary solution of PDE (7).*

The proof of the Theorems 1–5 carried out by means of the infinitesimal algorithm of S.Lie [8] requires very cumbersome computations. That is why, we omit it.

An important consequence of the Theorems 1–5 is the following statement.
Theorem 6. *Nonlinear PDE (1) is invariant under the conformal group* $C(1,n)$ *iff it is equivalent to the following*

$$1.\ \Box^l u = \lambda_1 u^{\frac{n+1+2l}{n+1-2l}}, \qquad n+1 \neq 2l; \tag{11}$$
$$2.\ \Box^l u = \lambda_2 e^u, \qquad n+1 = 2l. \tag{12}$$

Remark 1. Conformal invariance of the equation (11) was first ascertained in [9] and that of equation (12) was done in [5] by means of Baker–Campbell–Hausdorff formulae.

Assuming $l = 1$ in (11) we obtain the well-known result [5] that nonlinear wave equation (2) admits the conformal group if it is equivalent to the PDE

$$\Box u = \lambda u^{\frac{n+3}{n-1}} \text{ when } n \neq 1.$$

Remark 2. When $l = 2$ it follows from the Theorem 6 that in the four–dimensional space $R(1,3)$ there is only one $C(1,3)$–invariant equation

$$\Box^2 u = \lambda e^u.$$

One of the important applications of the Lie groups in mathematical physics is the finding exact solutions of nonlinear PDE. To this end one has to construct so called invariant solutions [2, 5, 8] which reduce PDE under study to equations with less number of independent variables (in particular, to ordinary differential equations). Integrating these one gets exact solutions of the initial PDE. A procedure described is called symmetry (or group-theoretical) reduction of differential equations. Here we perform symmetry reduction of the conformally-invariant biwave equation in the two–dimensional space $R(1,1)$:

$$\Box^2 u = \lambda u^{-3}. \tag{13}$$

Making use of inequivalent one–dimensional subalgebras of the conformal algebra $AC(1,1)$ [3] one can obtain the following $C(1,1)$–inequivalent Ansätze which reduce the equation (13) to ordinary differential equations. For each case the reduced equations are given:

1. $u = \varphi(\omega), \qquad \omega = x_0$ or $\omega = x_1$;
$\varphi^{(4)} = \lambda\varphi^{-3}$;
2. $u = \varphi(\omega), \qquad \omega = x_0^2 - x_1^2$;
$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + 2\varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3}$;
3. $u = (x_0 + x_1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 - x_1$;
$\varphi^{(2)} = -\frac{\lambda}{4}\varphi^{-3}$;
4. $u = (x_0 + x_1)^{\frac{1}{\alpha+1}}\varphi(\omega), \qquad \omega = (x_0 - x_1)(x_0 + x_1)^{\frac{\alpha-1}{\alpha+1}}$;
$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \frac{(\alpha-2)(2\alpha-1)}{(\alpha-1)^2}\varphi^{(2)} = \frac{\lambda}{16}\frac{(\alpha+1)^2}{(\alpha-1)^2}\varphi^{-3},\ \alpha > 1$;
5. $u = \exp(x_0 - x_1)\varphi(\omega), \qquad \omega = (x_0 + x_1 + \frac{1}{2})\exp(-2(x_0 - x_1))$;
$\varphi^{(4)}\omega^2 + 4\varphi^{(3)}\omega + \frac{9}{4}\varphi^{(2)} = \frac{\lambda}{64}\varphi^{-3}$;
6. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1$;
$\varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3}$;
7. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1 + \arctan(x_0 - x_1)$;
$\varphi^{(4)} + \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3}$;
8. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega), \qquad \omega = x_0 + x_1 + \frac{1}{2}\ln\frac{1+x_0-x_1}{1-x_0+x_1}$;
$\varphi^{(4)} - \varphi^{(2)} = \frac{\lambda}{16}\varphi^{-3}$;
9. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}(x_0 + x_1)^{\frac{1}{2}}\varphi(\omega)$,
$\omega = \ln(x_0 + x_1) - \beta\arctan(x_1 - x_0)$;
$4\beta^2\varphi^{(4)} + (4 - \beta^2)\varphi^{(2)} - \varphi = \frac{\lambda}{4}\varphi^{-3},\ \beta > 0$;
10. $u = ((x_0 - x_1)^2 + 1)^{\frac{1}{2}}((x_0 + x_1)^2 + 1)^{\frac{1}{2}}\varphi(\omega)$,
$\omega = (\gamma - 1)\arctan(x_0 - x_1) + (\gamma + 1)\arctan(x_0 + x_1)$;
$(\gamma^2 - 1)^2\varphi^{(4)} + 2(\gamma^2 + 1)\varphi^{(2)} + \varphi = \frac{\lambda}{16}\varphi^{-3},\ 0 \le \gamma < 1$.

Integration of the reduced equations gives rise to exact solutions of the nonlinear biwave equation (13). Here we present some exact solutions of this equation obtained with the use of Ansätze 3 and 6:

$$1. \quad u = \pm\lambda^{\frac{1}{4}}\left(x_0^2 - x_1^2\right)^{1/2},$$

$$2. \quad u = \pm\frac{1}{\sqrt{2}}\left(\frac{\lambda}{c_1}\right)^{\frac{1}{4}} |(x_0 - x_1)^2 - c_1|^{1/2}\Big(x_0 + x_1\Big)^{1/2}, \tag{14}$$

$$3. \quad u = \pm\frac{1}{2}\left(\frac{\lambda}{c_2}\right)^{\frac{1}{4}} \Big((x_0 - x_1)^2 + 1\Big)^{1/2} |(x_0 + x_1)^2 + c_2|^{1/2},$$

where $c_1,\ c_2$ are arbitrary constants.

Since the conformal group $C(1,1)$ is a maximal symmetry group of equation (13), formulae 1–10 give "maximal" information about its solutions which can be obtained within the framework of the Lie approach. It means that any solution invariant under a subgroup of the symmetry group of PDE (13) can be reduced by a transformation from the group $C(1,1)$ to one of the Ansätze 1-10.

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