On separable Schrödinger-Maxwell equations

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Abstract

We obtain the most general time-dependent potential $V(t, x_1, x_2)$ enabling separating of variables in the (1+2)-dimensional Schrödinger equation. With the use of this result the four classes of separable Schrödinger-Maxwell equations are constructed.

I. Introduction

In the papers [1, 2] we have suggested an efficient direct approach to a problem of variable separation in the (1+2)-dimensional Schrödinger equation (SE) having a time independent potential

$$i\psi_t + \psi_{x_1x_1} + \psi_{x_2x_2} = V(x_1, x_2)\psi. \tag{1}$$

With the help of this approach we have classified potentials $V(x_1, x_2)$ providing separability of Eq.(1) and, furthermore, obtained the full list of coordinate systems such that the corresponding Schrödinger equations are separable. It occurs that this approach can be extended in order to take into consideration the Schrödinger equation with time-dependent potential, namely

$$i\psi_t + \psi_{x_1x_1} + \psi_{x_2x_2} = V(t, x_1, x_2)\psi. \tag{2}$$

This class of SEs has numerous physical applications, in particular, they can be applied to describe the multiphoton ionization problem in the one electron

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atom [3]. Furthermore, the Keldysh type models in two space dimensions can be reduced to the particular form (2) [4].

In the present paper we obtain the complete description of potentials $V(t,x_1,x_2)$ such that SE is solvable by the method of separation of variables. Furthermore, we give the forms of the most general coordinate systems providing separability of the corresponding Schrödinger equations. As an application of thus obtained results we consider a problem of solving the Schrödinger-Maxwell equations by the method of separation of variables. This yields four classes of the Schrödinger-Maxwell equations solvable by the method of separation of variables.

II. Separation of variables in Eq.(2)

SE (2) is solvable in the sense of [2] via the separation Ansatz

$$\psi = Q(t,x_1,x_2)\varphi_0(t)\varphi_1(\omega_1(t,x_1,x_2))\varphi_2(\omega_2(t,x_1,x_2)), \tag{3}$$

if and only if the functions Q,ω_1,ω_2 satisfy the following system of nonlinear partial differential equations:

$$\omega_{1x_1}\omega_{2x_1}+\omega_{1x_2}\omega_{2x_2}=0, \tag{4}$$

$$B_{1a}(\omega_1)(\omega_{1x_1}^2+\omega_{1x_2}^2)+B_{2a}(\omega_2)(\omega_{2x_1}^2+\omega_{2x_2}^2)+R_a(t)=0, \tag{5}$$

$$2(\omega_{ax_1}Q_{x_1}+\omega_{ax_2}Q_{x_2})+Q(i\omega_{at}+\omega_{ax_1x_1}+\omega_{ax_2x_2})=0, \tag{6}$$

$$\begin{aligned}&\Big(B_{01}(\omega_1)(\omega_{1x_1}^2+\omega_{1x_2}^2)+B_{02}(\omega_2)(\omega_{2x_1}^2+\omega_{2x_2}^2)\Big)Q+iQ_t\\&\quad+Q_{x_1x_1}+Q_{x_2x_2}+R_0(t)Q-V(t,x_1,x_2)Q=0,\end{aligned} \tag{7}$$

where B_{1a},B_{2a},R_0,R_a are smooth functions of the given arguments and $a=1,2$.

The general solution of the system of equations (4)–(6) has been constructed in [2]. It splits into the four inequivalent classes given below

I. $\omega_1=A(t)x_1+W_1(t),\quad \omega_2=B(t)x_2+W_2(t),$

$$Q(t,x_1,x_2)=\exp\Big\{-\frac{i}{4}\Big(\frac{A'}{A}x_1^2+\frac{B'}{B}x_2^2\Big)-\frac{i}{2}\Big(\frac{W_1'}{A}x_1+\frac{W_2'}{B}x_2\Big)\Big\};$$

$$
\begin{aligned}
&\text{II.}\quad x_1 = W(t)\mathrm{e}^{\omega_1}\sin\omega_2 + W_1(t),\ \ x_2 = W(t)\mathrm{e}^{\omega_1}\cos\omega_2 + W_2(t),\\
&\qquad Q(t,x_1,x_2) = \exp\{iR(t,x_1,x_2)\};\\
&\text{III.}\quad x_1 = \frac{1}{2}W(t)(\omega_1^2-\omega_2^2) + W_1(t),\quad x_2 = W(t)\omega_1\omega_2 + W_2(t),\\
&\qquad Q(t,x_1,x_2) = \exp\{iR(t,x_1,x_2)\};\\
&\text{IV.}\quad x_1 = W(t)\cosh\omega_1\cos\omega_2 + W_1(t),\quad x_2 = W(t)\sinh\omega_1\sin\omega_2 + W_2(t),\\
&\qquad Q(t,x_1,x_2) = \exp\{iR(t,x_1,x_2)\}.
\end{aligned}
\tag{8}
$$

Here A, B, W, W_1, W_2 are arbitrary smooth functions and the function $R = R(t,x_1,x_2)$ is given by the formula

$$
R(t,x_1,x_2) = \frac{W'}{4W}\Big((x_1-W_1)^2 + (x_2-W_2)^2\Big) + \frac{1}{2}(W_1'x_1 + W_2'x_2).
$$

Inserting the above formulae into the equation (7) yields the four inequivalent classes of potentials $V(t,x_1,x_2)$ providing separability of SE (2)

$$
\begin{aligned}
\text{I.}\quad V(t,z_1,z_2) &= F_0(t) + A^2F_1(\omega_1) + B^2F_2(\omega_2) + x_1^2\left[\frac{A''}{4A} - \frac{(A')^2}{2A^2}\right] + \\
&+x_2^2\left[\frac{B''}{4B} - \frac{(B')^2}{2B^2}\right] - x_1\left(\frac{A'W_1'}{A^2} - \frac{W_1''}{2A}\right) - x_2\left(\frac{B'W_2'}{B^2} - \frac{W_2''}{2B}\right),
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\text{II.}\quad V(t,x_1,x_2) &= F_0(t) + \mathrm{e}^{-2\omega_1}W^{-2}[F_1(\omega_1) + F_2(\omega_2)]\\
&-\frac{W''}{4W}(x_1^2+x_2^2) + x_1\left(\frac{W_1W''}{2W} - \frac{W_1''}{2}\right) + x_2\left(\frac{W_2W''}{2W} - \frac{W_2''}{2}\right),
\end{aligned}
\tag{10}
$$

$$
\begin{aligned}
\text{III.}\quad V(t,x_1,x_2) &= F_0(t) + (\omega_1^2+\omega_2^2)^{-1}W^{-2}[F_1(\omega_1) + F_2(\omega_2)]\\
&-\frac{W''}{4W}(x_1^2+x_2^2) + x_1\left(\frac{W_1W''}{2W} - \frac{W_1''}{2}\right) + x_2\left(\frac{W_2W''}{2W} - \frac{W_2''}{2}\right),
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\text{IV.}\quad V(t,x_1,x_2) &= F_0(t) + (\sinh^2\omega_1 + \sin^2\omega_2)^{-1}W^{-2}[F_1(\omega_1) + F_2(\omega_2)]\\
&-\frac{W''}{4W}(x_1^2+x_2^2) + x_1\left(\frac{W_1W''}{2W} - \frac{W_1''}{2}\right) + x_2\left(\frac{W_2W''}{2W} - \frac{W_2''}{2}\right),
\end{aligned}
\tag{12}
$$

where the functions ω_1, ω_2 are given by the corresponding formulae I–IV from (8) and F_0, F_1, F_2 are arbitrary functions.

Formulae (9)–(12) give the explicit forms of the most general potentials $V = V(t,x_1,x_2)$ such that the corresponding SE is solvable by the separation of variables. However, the potentials obtained can be substantially simplified if we take into account that the class of partial differential equations (2) is invariant with respect to the following coordinate transformations:

$$1)\quad \tilde{t} = t, \quad \tilde{x}_1 = x_1 + a(t), \quad \tilde{x}_2 = x_2 + b(t), \tag{13}$$

$$\tilde{\psi}(\tilde{t}, \tilde{x}_1, \tilde{x}_2) = \psi(t, x_1, x_2) \exp\left\{\tfrac{i}{2}[a'(t)x_1 + b'(t)x_2]\right\}; \tag{14}$$

$$2)\quad \tilde{t} = T(t), \quad \tilde{x}_1 = c(t)x_1, \quad \tilde{x}_2 = c(t)x_2, \tag{15}$$

$$\tilde{\psi}(\tilde{t}, \tilde{x}_1, \tilde{x}_2) = \psi(t, x_1, x_2) \exp\left\{\tfrac{ic'(t)}{4c(t)}(x_1^2 + x_2^2)\right\}; \tag{16}$$

$$3)\quad \tilde{t} = t, \quad \tilde{x}_1 = x_1, \quad \tilde{x}_2 = x_2, \quad \tilde{\psi}(\tilde{t}, \tilde{x}_1, \tilde{x}_2) = \psi(t, x_1, x_2) \exp\{id(t)\}.$$

Here $a(t), b(t), c(t), d(t)$ are arbitrary smooth functions and the function $T(t)$ is determined by the equation $\frac{dT}{dt} = c^2(t)$. Using the above enumerated coordinate transformations we can choose $W_1 = 0, W_2 = 0, B = 1, W = 1$ and get the following expressions for V:

$$1.\quad V(t, x_1, x_2) = A^2 F_1(\omega_1) + F_2(\omega_2) + x_1^2 \left[\frac{A''}{4A} - \frac{(A')^2}{2A^2}\right], \tag{17}$$

$$2.\quad V(t, x_1, x_2) = \mathrm{e}^{-2\omega_1}[F_1(\omega_1) + F_2(\omega_2)], \tag{18}$$

$$3.\quad V(t, x_1, x_2) = (\omega_1^2 + \omega_2^2)^{-1}[F_1(\omega_1) + F_2(\omega_2)], \tag{19}$$

$$4.\quad V(t, x_1, x_2) = (\sinh^2\omega_1 + \sin^2\omega_2)^{-1}[F_1(\omega_1) + F_2(\omega_2)], \tag{20}$$

where the functions F_1, F_2 are arbitrary, the functions ω_1, ω_2 are given by the corresponding relations below

$$1.\quad x_1 = \frac{1}{A}\omega_1, \quad x_2 = \omega_2;$$

$$2.\quad x_1 = \mathrm{e}^{\omega_1}\sin\omega_2, \quad x_2 = \mathrm{e}^{\omega_1}\cos\omega_2;$$

$$3.\quad x_1 = \frac{1}{2}(\omega_1^2 - \omega_2^2), \quad x_2 = \omega_1\omega_2;$$

$$4.\quad x_1 = \cosh\omega_1 \cos\omega_2, \quad x_2 = \sinh\omega_1 \sin\omega_2$$

and $A = A(t)$ is a solution of the nonlinear ordinary differential equation $(4A)^{-1}A'' - (2A)^{-2}(A')^2 + C_1 A + C_2 = 0$ with arbitrary constant C_1, C_2.

Note that the Schrödinger equations with potentials (9)–(12) and (17)–(20), being equivalent from the point of view of the standard theory of partial differential equations, are inequivalent within the context of quantum mechanics. The reason for this situation is that the transformations (14), (16) change the quantity $|\psi(t,x_1,x_2)|$ which is regarded in quantum mechanics as the probability density. This means, in particular, that from the point of view of quantum mechanics time dependence of the potential $V(t,x_1,x_2)$ is essential and cannot be neglected. Nevertheless, in a sequel we restrict our considerations to the simplest case when potentials are of the form (17)–(20).

III. Separation of variables in the Schrödinger-Maxwell system

Now we apply the above results to separate variables in the Schrödinger-Maxwell system

$$
\begin{aligned}
&(p_0 - eA_0)\,\psi - (\vec{p} - e\vec{A})^2\,\psi = 0,\\
&\Box A_0 - \frac{\partial}{\partial t}\,(A_{0t} + \operatorname{div}\vec{A}) = 0,\\
&\Box \vec{A} + \vec{\operatorname{grad}}\,(A_{0t} + \operatorname{div}\vec{A}) = \vec{0}.
\end{aligned}
$$

Here $p_0 = i\partial/\partial t,\ p_a = i\partial/\partial x_a,\ a = 1,2,3,\ e = \mathrm{const},\ \Box = \partial^2/\partial t^2 - \Delta$ is the d'Alembert operator.

To this end we consider the expressions (17)–(20) as the Ansätze for the vector-potential of a special form

$$
A = \left(A_0(t,\vec{x}) \equiv e^{-1}V(t,x_1,x_2), \vec{0}\right). \tag{21}
$$

If we insert the above expression into the vacuum Maxwell equations without currents then the three equations for the function V

$$
V_{x_1x_1} + V_{x_2x_2} = 0, \quad V_{tx_a} = 0, \quad a = 1,2
$$

are obtained. Hence we get restrictions on the choice of the functions F_1, F_2 in the form of ordinary differential equations. After solving these we arrive at the following expressions for the functions $V(t,x_1,x_2)$:

1. $V = C_1(x_1^2 - x_2^2) + C_2x_1 + C_3x_2,$ (22)
2. $V = C_1\ln(x_1^2 + x_2^2) + (x_1^2 + x_2^2)^{-2}\left[C_2(x_1^2 - x_2^2) + C_3x_1x_2\right],$ (23)
3. $V = C_1x_1 + \dfrac{1}{\sqrt{x_1^2 + x_2^2}}\left[C_2\sqrt{\sqrt{x_1^2 + x_2^2} + x_1}\right.$

$$+C_3\sqrt{\sqrt{x_1^2+x_2^2}-x_1}\,\Bigg], \tag{24}$$

$$4.\quad V = \frac{f^2}{f^4+x_2^2}\Bigg[(C_2+C_1\operatorname{arcsinh} f)\, f\sqrt{1+f^2}$$

$$+\left(C_3+C_1\arcsin\frac{x_2}{f}\right)\frac{x_2\sqrt{f^2-x_2^2}}{f^2}\Bigg], \tag{25}$$

where

$$f^2 = \tfrac{1}{2}\left[x_1^2+x_2^2-1+\sqrt{(x_1^2+x_2^2)^2-2(x_1^2-x_2^2)+1}\right]$$

and C_1, C_2, C_3 are integration constants.

SE having the potential (22) separates in Cartesian coordinates into two harmonic oscillator one-dimensional stationary Schrödinger equations and, consequently, is exactly solvable. Given the potentials (23), (25) with $C_1 = 0$, SE (2) separates into the Hill and Matieu equations. Hence we conclude that its solutions are represented as products of the degenerated hypergeometric and Matieu functions [5]. At last, SE having the potential (24) separates in parabolic coordinates into two quasi-exactly solvable one-dimensional stationary Schrödinger equations and, consequently, is quasi-exactly solvable. This means that the finite part of the spectrum of the operator $-\Delta + V(x_1, x_2)$ can be computed by purely algebraic methods (for more details about quasi-exactly solvable models of quantum mechanics, see [6]).

IV. Concluding Remarks

The variable separation technique used in the present paper can be directly applied to the Schrödinger equations with electro-magnetic fields of more general configurations than (21). Namely it is possible to classify vector-potentials of the electro-magnetic field of the form

$$A = (A_0(t, x_1, x_2),\ A_1(t, x_1, x_2),\ A_2(t, x_1, x_2),\ 0)\,.$$

such that the corresponding SEs are solvable by the method of separation of variables. Furthermore, it is also possible to modify this technique in order to take into consideration the Schrödinger equation in $(1 + 3)$-dimensions. Both problems are under investigation now and will be a topic of our future publications.

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