Group classification of linear fourth-order evolution equations [*]

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Abstract

In this paper we solve the problem of group classification of the (1+1)-dimensional fourth-order linear evolution equations of the most general form. We prove that there are three, six and one inequivalent fourth-order linear evolution equations that admit two-, three-, and four-dimensional symmetry algebras, respectively.

Keywords: Fourth-order evolution equation, finite-dimensional Lie group, equivalence transformation.

1. Introduction

In this paper, we obtain the complete solution of the group classification problem for general linear evolution equations of the fourth order

$$u_t = f_1(t,x)u_{xxxx} + f_2(t,x)u_{xxx} + f_3(t,x)u_{xx} + f_4(t,x)u_x + f_5(t,x)u + f_6(t,x). \qquad (1)$$

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Here $f_i,\ (i = 1, 2, \cdots, 6)$ are arbitrary smooth real-valued functions with $f_1 \neq 0$. Hereafter we adopt the following notations, $u = u(t, x),\ u_t = \partial u/\partial t,\ u_x = \partial u/\partial x,\ u_{xx} = \partial^2 u/\partial x^2$ and so on.

The classical approach to tackling group classification of partial differential equations (PDEs) is based on the infinitesimal Lie approach. The latter enables to reduce the problem of group classification of a given PDE to integrating some over-determined system $\mathcal{D}$ of linear PDEs [9–11]. The system $\mathcal{D}$ typically contains arbitrary functions whose specific forms are to be determined from the requirement of its compatibility. However, in case when the arbitrariness is too broad the system $\mathcal{D}$ might become under-determined. As a result the classical Ovsyannikov's approach becomes inefficient (see [2] for the detailed discussion of this matter). To overcome this difficulty Zhdanov and Lahno [14] introduced an alternative approach enabling to handle efficiently the above problem (see, also [7, 8, 12, 13]).

Here we adopt the approach of [4] to describe all inequivalent equations of the form (1) that possess non-trivial Lie symmetry. What is more, we utilize some of the results on group classification of nonlinear fourth-order evolution equations obtained in our recent paper [5]. Note that the results presented below complement those of [5], where only essentially nonlinear equations have been considered.

Our classification algorithm is implemented as three major steps. We start by normalizing the class of PDEs (1) using the most general linear equivalence transformation $u(t, x) \rightarrow V(t, x)u(f(t), g(t, x)) + G(t, x)$ that leaves class (1) invariant. Next, we proceed to calculation of the maximal equivalence group admitted by the normal form of Eq. (1). This yields in addition the most general form of the Lie transformation group admitted by (1) and the corresponding determining equations $\mathcal{D}$. At the last step, we apply the approach of [2, 4] to finalize group classification of the normalized equations of the form (1).

2. Normalization of Eq. (1)

The first step of our classification routine is normalizing Eq. (1).

Making the change of variables $(t,x,u)\to(\tilde t,\tilde x,\tilde u)$, which leaves the class (1) invariant,

$$\tilde t = t,\quad \tilde x = F(t,x),\quad u = V(t,x)\tilde u(\tilde t,\tilde x)+G(t,x), \tag{2}$$

with $V\neq 0$ and $F_x\neq 0$ we obtain an equation of the form

$$\tilde u_{\tilde t} = \tilde f_1(\tilde t,\tilde x)\tilde u_{\tilde x\tilde x\tilde x\tilde x}+\tilde f_2(\tilde t,\tilde x)\tilde u_{\tilde x\tilde x\tilde x}+\tilde f_3(\tilde t,\tilde x)\tilde u_{\tilde x\tilde x}+\tilde f_4(\tilde t,\tilde x)\tilde u_{\tilde x}+\tilde f_5(\tilde t,\tilde x)\tilde u+\tilde f_6(\tilde t,\tilde x),$$

where

$$\begin{aligned}
&\tilde f_1 = f_1F_x^4,\\
&\tilde f_2 = 6f_1F_x^2F_{xx}+f_2F_x^3+4f_1F_x^3V_xV^{-1},\\
&\tilde f_3 = 4f_1F_xF_{xxx}+3f_1F_{xx}^2+3f_2F_xF_{xx}+f_3F_x^2+3V^{-1}(4f_1F_xF_{xx}V_x\\
&\qquad +2f_1F_x^2V_{xx}+f_2F_x^2V_x),\\
&\tilde f_4 = f_1F_{xxxx}+f_2F_{xxx}+f_3F_{xx}+f_4F_x-F_t+V^{-1}(4f_1F_{xxx}V_x+6f_1F_{xx}V_{xx}\\
&\qquad +4f_1F_xV_{xxx}+3f_2F_{xx}V_x+3f_2F_xV_{xx}+2f_3F_xV_x),\\
&\tilde f_5 = V^{-1}(f_1V_{xxxx}+f_2V_{xxx}+f_3V_{xx}+f_4V_x+f_5V-V_t),\\
&\tilde f_6 = V^{-1}(f_1G_{xxxx}+f_2G_{xxx}+f_3G_{xx}+f_4G_x+f_5G+f_6-G_t).
\end{aligned}$$

In what follows we use the notation $\epsilon=\pm1$. Choosing the function F, V and G in (2) so that they satisfy the constraints

$$\begin{aligned}
&f_1F_x^4=\epsilon,\\
&(6f_1F_x^2F_{xx}+f_2F_x^3)V+4f_1F_x^3V_x=0,\\
&G_t=f_1G_{xxxx}+f_2G_{xxx}+f_3G_{xx}+f_4G_x+f_5G+f_6.
\end{aligned}$$

yields

$$\tilde{f}_1 = \epsilon, \quad \tilde{f}_2 = 0, \quad \tilde{f}_6 = 0.$$

Thus Eq. (1) reduces to the following canonical form:

$$u_t = \epsilon u_{xxxx} + A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u.$$

What is more, we can get rid of ϵ by the transformation $\tilde{t} = -t, \tilde{x} = x, \tilde{u} = u$.

Summing up the above considerations we conclude that any PDE from the class (1) is equivalent to the normalized equation

$$u_t = -u_{xxxx} + A(t,x)u_{xx} + B(t,x)u_x + C(t,x)u. \tag{3}$$

Consequently, to obtain complete group classification of the class of evolution equations (1) it suffices to describe all functions $A(t,x)$, $B(t,x)$ and $C(t,x)$ such that (3) admits nontrivial Lie symmetries. Note that under $A = B = C = 0$, Eq. (3) is the non-stationary bi-harmonic equation in one spatial dimension.

$$u_t = -u_{xxxx}.$$

3. Preliminary classification of Eq. (3)

It is a common knowledge that the most general Lie transformation group admitted by Eq. (3) is generated by vector fields of first-order differential operators

$$X = \tau(t,x,u)\partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u,$$

where τ, ξ and η are arbitrary smooth real-valued functions. Using the infinitesimal Lie approach we prove the following statement (see, also [5]).

Theorem 1. *The symmetry group of the linear equation* (3) *is generated by the vector field*

$$X = \tau(t)\partial_t + \Big(\frac{\dot{\tau}}{4}x + \rho(t)\Big)\partial_x + (\alpha(t)u + \beta(t,x))\partial_u,$$

where the functions $\tau(t), \rho(t)$ *and* $\alpha(t)$ *are real-valued functions satisfying the classifying equations*

$$\begin{gathered}(4\rho + \dot{\tau}x)A_x + 4\tau A_t + 2\dot{\tau}A = 0,\\ (4\rho + \dot{\tau}x)B_x + 4\tau B_t + 3\dot{\tau}B + 4\dot{\rho} + \ddot{\tau}x = 0,\\ (4\rho + \dot{\tau}x)C_x + 4\tau C_t + 4\dot{\tau}C - 4\dot{\alpha} = 0,\end{gathered} \tag{4}$$

and $\beta(t,x)$ *is an arbitrary solution of the original equation* (3), *where and hereafter the dot over a symbol denotes differentiation with respect to its argument.*

If functions A, B and C are arbitrary, then the most general symmetry admitted by Eq. (3) reads

$$X = (cu + \beta(t,x))\partial_u,$$

where $\beta(t,x)$ is an arbitrary solution of Eq. (3) and c is an arbitrary constant. Since the one infinite-parameter Lie group generated by the operator $\beta(t,x)\partial_u$ gives no nontrivial information about the solution structure of the equation under study we ignore it and consider the symmetry operators of the form

$$X = \tau(t)\partial_t + \Big(\frac{\dot{\tau}}{4}x + \rho(t)\Big)\partial_x + \alpha(t)u\partial_u. \tag{5}$$

It is possible to choose coefficients A, B and C so that Eq. (3) admits larger Lie algebra than the one-dimensional Lie algebra L_1 spanned by the operator $u\partial_u$. Description of all possible specifications of the coefficients A, B and C such that (3) admits an extension of the algebra L_1 is the core of the problem of group classification of Eq. (3). To solve it we need to construct to integrate classifying equation (4). The major difficulty is that (4) is the under-determined system of three PDEs for six unknown functions τ, ρ, α and A, B, C. To overcome this difficulty, we utilize the approach of [2]. The essential element of the latter approach is computation of the maximal equivalence group admitted by Eq. (3). Let

$$\tilde{t} = T(t,x,u), \quad \tilde{x} = Y(t,x,u), \quad \tilde{u} = U(t,x,u), \qquad \frac{D(T,Y,U)}{D(t,x,u)} \neq 0 \tag{6}$$

be an invertible transformation preserving the form of (3). This means that rewriting (3) the new variables $\tilde{t}, \tilde{x}, \tilde{u}$ yields system of PDEs belonging to the class (3) with possibly different A, B, and C

$$\tilde{u}_{\tilde{t}} = -\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \tilde{A}(\tilde{t},\tilde{x})\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{B}(\tilde{t},\tilde{x})\tilde{u}_{\tilde{x}} + \tilde{C}(\tilde{t},\tilde{x})\tilde{u}. \tag{7}$$

Theorem 2. *The maximal equivalence group of Eq.* (3) *is of the form*

$$\tilde{t} = T(t), \quad \tilde{x} = \epsilon \dot{T}^{\frac{1}{4}} x + Y(t), \quad \tilde{u} = V(t)u, \tag{8}$$

where $\dot{T} > 0$ *and* $V \neq 0$.

Proof. Computing u_x according to (6), we get

$$u_x = \frac{T_x \tilde{u}_{\tilde{t}} + Y_x \tilde{u}_{\tilde{x}} - U_x}{U_u - T_u \tilde{u}_{\tilde{t}} - Y_u \tilde{u}_{\tilde{x}}}.$$

As $\tilde{A}\tilde{u}_{\tilde{x}\tilde{x}} + \tilde{B}\tilde{u}_{\tilde{x}} + \tilde{C}\tilde{u}$ is an arbitrary function of $\tilde{t}$, $\tilde{x}$, $\tilde{u}$, $\tilde{u}_{\tilde{x}}$, $\tilde{u}_{\tilde{x}\tilde{x}}$ and does not depend on $\tilde{u}_{\tilde{t}}$, we must have

$$u_x = g(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}})$$

for some function g. This implies that $T_x = T_u = 0$, whence $T = T(t)$ with $\dot{T} \neq 0$.

Next, making the change of variables (6) with $\tilde{t} = T(t)$ yields

$$\begin{aligned} u_t =& \frac{\dot{T}}{(U_u - Y_u \tilde{u}_{\tilde{x}})} \tilde{u}_{\tilde{t}} + \theta_1(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}), \\ u_{xxxx} =& \frac{(Y_x U_u - Y_u U_x)^4}{(U_u - Y_u \tilde{u}_{\tilde{x}})^5} \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \theta_2(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}), \end{aligned}$$

where θ_1 and θ_2 are arbitrary smooth functions. Substituting the expressions above into (3) and taking into account that (3) should be transformed into an equation of the form (7) yield the relation

$$\dot{T}(U_u - Y_u \tilde{u}_{\tilde{x}})^4 - (Y_x U_u - Y_u U_x)^4 = 0.$$

Since functions T, Y and U are independent of u_x, coefficients of $\tilde{u}_{\tilde{x}}$ in the above polynomial must vanish which yields

$$\dot{T} Y_u = 0, \quad \dot{T} U_u^4 - (Y_x U_u - Y_u U_x)^4 = 0.$$

As $\dot{T} \neq 0$, we get $Y_u = 0$ whence

$$(\dot{T} - Y_x^4) U_u^4 = 0.$$

Taking into that $U_u \neq 0$ (otherwise (6) is not invertible) we conclude that $\dot{T} - Y_x^4 = 0$. Consequently, $\dot{T} > 0$ and $Y = \epsilon \dot{T}^{\frac{1}{4}} x + Y(t)$.

Finally, applying transformation (6) with $\tilde{t} = T(t)$ and $\tilde{x} = \epsilon \dot{T}^{\frac{1}{4}} x + Y(t)$ to Eq. (3), yields the following equation:

$$\tilde{u}_{\tilde{t}} = -\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} - 4\epsilon \frac{\epsilon \dot{T}^{\frac{1}{4}} U_{uu} \tilde{u}_{\tilde{x}} - U_{uu} U_x + U_{xu} U_u}{\dot{T}^{\frac{1}{4}} U_u^2} \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}} + \theta_3(\tilde{t}, \tilde{x}, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_{\tilde{x}\tilde{x}}),$$

where θ_3 is an arbitrary smooth function. Using the equivalence transformation (6) we eliminate coefficients of $\tilde{u}_{\tilde{x}} \tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}$ and $\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}}$. As a result, we get

$$\epsilon \dot{T}^{\frac{1}{4}} U_{uu} = 0, \quad U_{xu} U_u - U_{uu} U_x = 0.$$

This implies that $U_{uu} = 0$ and $U_{xu} = 0$ so that $U = V(t) u + W(t, x), V \neq 0$. Consequently, the transformed equation Eq. (3) takes the form

$$\tilde{u}_{\tilde{t}} = -\tilde{u}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}} + \frac{A}{\dot{T}^{\frac{1}{2}}} \tilde{u}_{\tilde{x}\tilde{x}} + \frac{4\epsilon \dot{T} B - \epsilon \ddot{T} x - 4\dot{T}^{\frac{3}{4}} \dot{Y}}{4\dot{T}^{\frac{7}{4}}} \tilde{u}_{\tilde{x}} + \frac{\dot{V} + VC}{\dot{T} V} \tilde{u}$$
$$+ \frac{V(W_t + W_{xxxx} - AW_{xx} - BW_x - CW) - \dot{V} W}{\dot{T} V}.$$

Choosing $W = 0$ completes the proof of the theorem. □

Corollary 1. *Equivalence transformation* (8) *converts Eq.* (3) *into Eq.* (7), *where the coefficients $\tilde{A}, \tilde{B}$ and $\tilde{C}$ are expressed in terms of the functions A, B, C and their derivatives as follows*

$$\begin{aligned}
&\tilde{A} = \dot{T}^{-\frac{1}{2}} A,\\
&\tilde{B} = \dot{T}^{-1}(\epsilon B \dot{T}^{\frac{1}{4}} - \tfrac{\epsilon}{4}\ddot{T}\dot{T}^{-\frac{3}{4}} x - \dot{Y}),\\
&\tilde{C} = \dot{T}^{-1}(C + \dot{V}V^{-1}).
\end{aligned} \tag{9}$$

4. Group classification of Eq. (3)

Making the change of variables (8) in the vector field (5) yields

$$\tilde{X} = \tau\dot{T}\partial_{\tilde{t}} + \left[\frac{\epsilon}{4}(\tau\ddot{T}\dot{T}^{-\frac{3}{4}} + \dot{T}^{\frac{1}{4}}\dot{\tau}) + \tau\dot{Y} + \epsilon\dot{T}^{\frac{1}{4}}\rho\right]\partial_{\tilde{x}} + (\tau\dot{V} + \alpha V)u\partial_{\tilde{u}}.$$

Consider the cases $\tau \neq 0$ and $\tau = 0$ separately.

Case 1. Suppose $\tau \neq 0$. Choosing in (8) the functions T and Y satisfying

$$\dot{T} = |\tau|^{-1}, \quad Y = -\epsilon\,\mathrm{sign}(\tau)\int^{t} \rho(y)|\tau(y)|^{-\frac{5}{4}}\mathrm{d}y$$

and as taking V a nonzero solution of the equation

$$\tau\dot{V} + \alpha V = 0,$$

we arrive at the operator $\tilde{X} = \epsilon\partial_{\tilde{t}}$.

Case 2. If $\tau = 0$, $\rho \neq 0$ in (5), then (8) reduces vector field $X = \rho\partial_x + \alpha u\partial_u$ to the form

$$\tilde{X} = \epsilon\dot{T}^{\frac{1}{4}}\rho\partial_{\tilde{x}} + \alpha\tilde{u}\partial_{\tilde{u}}.$$

Choosing $\dot{T} = (\epsilon\rho)^{-4}$ yields the operator

$$\tilde{X} = \partial_{\tilde{x}} + \alpha\tilde{u}\partial_{\tilde{u}}.$$

Case 3. Provided $\tau = \rho = 0$ and $\dot{\alpha} \neq 0$, we can choose T as $T = \int^t |\dot{\alpha}(y)|\mathrm{d}y$ and thus get

$$\tilde{X} = \epsilon\tilde{t}\tilde{u}\partial_{\tilde{u}}.$$

If, otherwise, $\tau = \rho = \dot{\alpha} = 0$, then trivial symmetry generator $\tilde{X} = \tilde{u}\partial_{\tilde{u}}$ is obtained.

By direct verification we establish that the following operators

$$\partial_x + f(t)u\partial_u \quad (\dot{f} \neq 0), \quad \partial_x, \quad \epsilon\partial_t, \quad \epsilon tu\partial_u$$

are inequivalent within the equivalence transformation (8).

We summarize above results in the following lemma.

Lemma 1. *Vector field* (5) *is equivalent with a point transformation* (8) *to one of the following inequivalent vector fields*

$$\partial_x + f(t)u\partial_u \quad (\dot{f} \neq 0), \quad \partial_x, \quad \epsilon\partial_t, \quad \epsilon tu\partial_u.$$

In what follows we separately analyze each of the operators listed in Lemma 1 and construct explicit forms of all inequivalent A, B and C such that the corresponding Eq. (3) admits an extension of algebra L_1.

4.1. Operator $X_1 = \partial_x + f(t)u\partial_u \quad (\dot{f} \neq 0)$

If equation (3) admits operator X_1, then substituting $\tau = 0, \rho = 1$ and $\alpha = f(t)$ into classifying equation (4) and solving it yield

$$A = A(t), \quad B = B(t), \quad C = \dot{f}(t)x + g(t).$$

Applying equivalence transformation (8) to equation

$$u_t = -u_{xxxx} + A(t)u_{xx} + B(t)u_x + [\dot{f}(t)x + g(t)]u,$$

we arrive at Eq. (7), where

$$\tilde{A} = \dot{T}^{-\frac{1}{2}}A,$$

$$\tilde{B} = \dot{T}^{-1}(\epsilon B\dot{T}^{\frac{1}{4}} - \tfrac{\epsilon}{4}\ddot{T}\dot{T}^{-\frac{3}{4}}x - \dot{Y}),$$

$$\tilde{C} = \dot{T}^{-1}\dot{f}[x + \dot{f}^{-1}V^{-1}(gV + \dot{V})].$$

Choosing $T = 4t, \epsilon = 1$ and taking solutions of

$$2^{\frac{1}{2}}B - \dot{Y} = 0, \quad \dot{f}^{-1}V^{-1}(gV + \dot{V}) = Y$$

as Y and V, gives

$$\tilde{A} = \tilde{A}(\tilde{t}), \quad \tilde{B} = 0, \quad \tilde{C} = \dot{\tilde{f}}(\tilde{t})\tilde{x} \qquad \dot{\tilde{f}} \neq 0.$$

Consequently, we only need to consider the equation

$$u_t = -u_{xxxx} + A(t)u_{xx} + \dot{f}(t)xu, \qquad \dot{f} \neq 0,$$

where

$$\dot{\rho} = 0, \quad \dot{\tau} = 0, \quad \dot{\alpha} = \rho\dot{f},$$

$$4\tau\ddot{f} + 5\dot{\tau}\dot{f} = 0, \quad 2\tau\dot{A} + \dot{\tau}A = 0.$$

Integrating the first three equations we have $\rho = c_1$, $\tau = c_2t + c_3$, $\alpha = c_1f + c_4$, where c_i are arbitrary constants. Inserting $\tau = c_2t + c_3$ into the remaining equations yields

$$4(c_2t + c_3)\ddot{f} + 5c_2\dot{f} = 0, \quad 2(c_2t + c_3)\dot{A} + c_2A = 0. \tag{10}$$

While solving the above equations we need to consider three different cases.

I. Provided $c_2 = c_3 = 0$, Eq. (10) is satisfied identically for arbitrary $A(t)$ and $f(t)$ ($\dot{f} \neq 0$). Thus Eq. (3) with $A(t,x) = A(t)$, $B(t,x) = 0$ and $C(t,x) = \dot{f}(t)x$, $\dot{f} \neq 0$ admits the two-dimensional Abelian algebra

$$\langle \partial_x + f(t)u\partial_u, \quad u\partial_u \rangle.$$

II. Given $c_2 = 0$ and $c_3 \neq 0$, Eq. (10) takes the form

$$\ddot{f} = 0, \quad \dot{A} = 0,$$

whence

$$f(t) = a_1 t + a_2, \quad A(t) = a_3,$$

and a_i are arbitrary constants. Consider the corresponding equation

$$u_t = -u_{xxxx} + a_3 u_{xx} + a_1 x u, \tag{11}$$

where $a_1 \neq 0$ (since, otherwise, $\dot{f} = 0$). Equivalence transformation (8) maps Eq. (11) into an equation of the form (7) where the corresponding $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are given by (9) with

$$A = a_3, \quad B = 0, \quad C = a_1 x,$$

namely,

$$\begin{aligned}
&\tilde{A} = a_3 \dot{T}^{-\frac{1}{2}}, \\
&\tilde{B} = -\dot{T}^{-1}\left(\tfrac{\epsilon}{4}\ddot{T}\dot{T}^{-\frac{3}{4}} x + \dot{Y}\right), \\
&\tilde{C} = \dot{T}^{-1}(a_1 x + \dot{V} V^{-1}).
\end{aligned}$$

We choose V be a nonzero constant, $\epsilon = 1$ and T, Y to satisfy

$$\dot{T} = a_1^{\frac{4}{5}}, \quad Y = 0$$

thus getting

$$\tilde{A} = \tilde{a}_3, \quad \tilde{B} = 0, \quad \tilde{C} = \tilde{x}.$$

Consequently Eq. (11) can be reduced to the form

$$u_t = -u_{xxxx} + a u_{xx} + x u, \qquad a \in \mathcal{R}$$

by a suitable equivalence transformation.

Inserting the coefficients of the equation above into classifying equations (4), we obtain the algebra L_1 is extended by the operators

$$\partial_t, \quad \partial_x + tu\partial_u$$

to form the three-dimensional non-decomposable Lie algebra.

III. When $c_2 \neq 0$, it follows from Eq. (10) that

$$f(t) = a_1 + a_2(t + \frac{c_3}{c_2})^{-\frac{1}{4}}, \quad A(t) = a_3(c_2t + c_3)^{-\frac{1}{2}}.$$

Consider the corresponding PDE

$$u_t = -u_{xxxx} + a_3(c_2t + c_3)^{-\frac{1}{2}}u_{xx} - \frac{a_2}{4}(t + \frac{c_3}{c_2})^{-\frac{5}{4}}xu,$$

where $a_2 \neq 0$. Applying transformation (8) to this equation yields Eq. (7). Next, according to (9) we have

$$\begin{aligned}
\tilde{A} &= a_3\dot{T}^{-\frac{1}{2}}(c_2t + c_3)^{-\frac{1}{2}}, \\
\tilde{B} &= -\dot{T}^{-1}(\tfrac{\epsilon}{4}\ddot{T}\dot{T}^{-\frac{3}{4}}x + \dot{Y}), \\
\tilde{C} &= \dot{T}^{-1}(-\tfrac{a_2}{4}(t + \tfrac{c_3}{c_2})^{-\frac{5}{4}}x + \dot{V}V^{-1}).
\end{aligned}$$

Choosing $T = t + c_3/c_2$, $Y = 0$ and V to be a nonzero constant, we obtain

$$\tilde{A} = a\tilde{t}^{-\frac{1}{2}}, \quad \tilde{B} = 0, \quad \tilde{C} = c\tilde{t}^{-\frac{5}{4}}\tilde{x}, \quad c \neq 0.$$

Inserting $A = at^{-\frac{1}{2}}$, $B = 0$, $C = ct^{-\frac{5}{4}}x$ $(c \neq 0)$ into Eq. (4) and solving the latter yield two additional symmetry operators of Eq. (3)

$$\partial_x - 4ct^{-\frac{1}{4}}u\partial_u, \quad t\partial_t + \frac{x}{4}\partial_x.$$

4.2. Operator $X_2 = \partial_x$

If Eq. (3) is invariant under ∂_x, then $A = A(t)$, $B = B(t)$, $C = C(t)$. By properly choosing T, Y and V in transformation (8) we can reduce equation

$$u_t = -u_{xxxx} + A(t)u_{xx} + B(t)u_x + C(t)u,$$

to Eq. (7) with

$$\tilde{A} = \tilde{A}(\tilde{t}), \quad \tilde{B} = \tilde{C} = 0.$$

So that we can restrict our considerations to the equation

$$u_t = -u_{xxxx} + A(t)u_{xx}.$$

Inserting the coefficients of the above equation into the classifying equations (4) we obtain

$$4\dot{\rho} + \ddot{\tau}x = 0, \quad \dot{\alpha} = 0,$$

$$2\tau\dot{A} + \dot{\tau}A = 0.$$

Integrating the first two equations gives

$$\tau = c_1 t + c_2, \quad \rho = c_3, \quad \alpha = c_4.$$

So that the system under study reduces to a single equation

$$2(c_1 t + c_2)\dot{A} + c_1 A = 0.$$

Analyzing this ordinary differential equation shows that there are three cases of extension of symmetry algebra L_1, namely,

i). $A = B = C = 0$ and the additional symmetry operators are of the form

$$\partial_x, \quad \partial_t, \quad t\partial_t + \frac{x}{4}\partial_x.$$

ii). $A=\epsilon,\ B=C=0$ and the additional symmetry operators are of the form

$$\partial_x, \quad \partial_t.$$

iii). $A=at^{-\frac{1}{2}}\ (a\neq 0),\ B=C=0$ and the additional symmetry operators are of the form

$$\partial_x, \quad t\partial_t+\frac{x}{4}\partial_x.$$

4.3. Operator $X_3=\epsilon\partial_t$

In this case the system of classifying equations reduces to

$$A_t=B_t=C_t=0.$$

Consequently, $A=A(x),\ B=B(x),\ C=C(x)$. With these A, B and C, Eq. (4) becomes

$$(4\rho+\dot{\tau}x)A_x+2\dot{\tau}A=0,$$

$$(4\rho+\dot{\tau}x)B_x+3\dot{\tau}B+4\dot{\rho}+\ddot{\tau}x=0,$$

$$(4\rho+\dot{\tau}x)C_x+4\dot{\tau}C-4\dot{\alpha}=0.$$

Analysis of the above system of ordinary differential equations yields that extension of the symmetry algebra for Eq. (3) with $A=A(x),\ B=B(x),\ C=C(x)$ is only possible when

i). $A=ax^{-2}, \quad B=bx^{-3}, \quad C=cx^{-4}, \qquad a,b,c\in\mathcal{R}, \quad a^2+b^2+c^2\neq 0,$

ii). $A=B=C=0,$

iii). $A=C=0, \quad B=cx(a^3x^3+4a^2bx^2+6ab^2x+4b^3)(ax+b)^{-3}, \qquad a,b,c\in\mathcal{R}, \quad abc\neq 0.$

The second case has already been considered. In the first case the algebra L_1 is extended by the operators ∂_t and $t\partial_t+\frac{x}{4}\partial_x$. In the third case the corresponding Eq. (3) admits the three-dimensional decomposable symmetry algebra

$$\langle\partial_t, \mathrm{e}^{-4ct}\partial_t-c\mathrm{e}^{-4ct}(x+\frac{b}{a})\partial_x\rangle\oplus\langle u\partial_u\rangle.$$

4.4. Operator $X_4 = \epsilon t u \partial_u$

Inserting the coefficients of this operator

$$\tau = 0, \quad \rho = 0, \quad \alpha = \epsilon t,$$

into Eq. (4) leads to contradiction

$$\epsilon = 0,$$

whence it follows that there are no equations of the form (3) admitting the operator $\epsilon t u \partial_u$.

We summarize the above classification results in the following assertion.

Theorem 3. *There are three, six and one inequivalent equations from the class* (3) *admitting two-, three-, and four-dimensional symmetry algebras, respectively.*

In Table 1, we present all inequivalent forms of the functions A, B, C and the finite-dimensional parts of their maximal symmetry algebras for the corresponding equations (3). Note that the all two-dimensional algebras are Abelian algebras and the four-dimensional algebra is decomposable solvable Lie algebra. All three-dimensional symmetry algebras are solvable Lie algebras, as well.

5. Discussion and conclusions

In this paper, we obtain the complete solution of the classical problem of group classification of the fourth-order linear evolution equations of the most general form (3). All inequivalent classes of invariant equation (3) are constructed together with their maximal symmetry algebras.

We prove that there are three, six and one inequivalent equations from the class (3) admitting two-, three-, and four-dimensional symmetry algebras, respectively.

One of the applications of these results is classification of nonlinear fourth-order evolution equations that are linearizable by nonlocal transformations using the technique developed in our recent paper [6].

Another interesting application is applying the results of this paper to perform symmetry analysis of nonlinear evolution equations with small coupling constant. To this end one can utilize the concept of approximate Lie symmetry developed by Ibragimov *et al* [1] and Fushchych *et al* [3]. The basic idea of their approach is using a symmetry, S, of linear PDE as the first approximation of the Lie symmetry of its nonlinear generalization. In this way it's possible to expand the class of nonlinear partial differential equations that can be handled by Lie group approach.

These problems are under investigation now and will be reported in our future publications.

Interestingly, the symmetry group admitted by linear PDE (3) is four-parameter at most, while there are nonlinear equation whose symmetry algebras are higher dimensional [5, 6]. For example, the equation

$$u_t = -u_{xxxx} + \frac{u_{xx}u_{xxx}}{u_x}$$

admits the five-dimensional symmetry algebra

$$\langle \partial_t,\ \partial_x,\ t\partial_t + \frac{x}{4}\partial_x \rangle \oplus \langle \partial_u,\ u\partial_u \rangle.$$

This result is in contrast to the second-order evolution equations

$$u_t = F(t,x,u,u_x)u_{xx} + F(t,x,u,u_x), \quad F \neq 0,$$

where the linear one $u_t = u_{xx}$ possesses the maximal symmetry algebra.

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Table 1: Symmetry classification of Eq. (3)

N	A	B	C	finite-dimensional symmetry algebra
1	$A(x)$	$B(x)$	$C(x)$	$\langle\partial_t,\ u\partial_u\rangle$
2	$A(t)$	0	$\dot f(t)x, \quad \dot f \neq 0$	$\langle\partial_x + f(t)u\partial_u,\ u\partial_u\rangle$
3	$A(t)$	0	0	$\langle\partial_x,\ u\partial_u\rangle$
4	ϵ	0	0	$\langle\partial_x\rangle \oplus \langle\partial_t\rangle \oplus \langle u\partial_u\rangle$
5	$at^{-\frac{1}{2}},\ a \neq 0$	0	0	$\langle\partial_x,\ t\partial_t + \frac{x}{4}\partial_x\rangle \oplus \langle u\partial_u\rangle$
6	a	0	x	$\langle\partial_x + tu\partial_u,\ \partial_t,\ u\partial_u\rangle$
7	$at^{-\frac{1}{2}}$	0	$ct^{-\frac{5}{4}}x,\ c \neq 0$	$\langle\partial_x - 4ct^{-\frac{1}{4}}u\partial_u,\ t\partial_t + \frac{x}{4}\partial_x\rangle \oplus \langle u\partial_u\rangle$
8	ax^{-2}	bx^{-3}	$cx^{-4},\ a^2+b^2+c^2 \neq 0$	$\langle\partial_t,\ t\partial_t + \frac{x}{4}\partial_x\rangle \oplus \langle u\partial_u\rangle$
9	0	$\frac{cx(a^3x^3+4a^2bx^2+6ab^2x+4b^3)}{(ax+b)^3}$, $abc \neq 0$	0	$\langle\partial_t,\ \mathrm{e}^{-4ct}\partial_t - c\mathrm{e}^{-4ct}(x+\frac{b}{a})\partial_x\rangle \oplus \langle u\partial_u\rangle$
10	0	0	0	$\langle\partial_x,\ \partial_t,\ t\partial_t + \frac{x}{4}\partial_x\rangle \oplus \langle u\partial_u\rangle$

Here $a, b, c \in \mathcal{R}$.